Submitted exclusively to the London Mathematical Society
doi:10.1112/0000/000000

Mixed moment of $GL(2)$ and $GL(3)$ L-functions

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Abstract

Let $\mathfrak{f}$ run over the set H_{4k} of primitive cusp forms of level one and weight $4k$, $k \in \mathbf{N}$. We prove an explicit formula for the mixed moment of the Hecke L-function $L(\mathfrak{f}, 1/2)$ and the symmetric square L-function $L(\mathrm{sym}^2 \mathfrak{f}, 1/2)$, relating it to the dual mixed moment of a double Dirichlet series and the Riemann zeta function weighted by the ${}_3F_2$ hypergeometric function. Analysing the corresponding special functions by means of the Liouville-Green approximation followed by the saddle point method, we prove that the initial mixed moment is bounded above by $\log^3 k$.

1. *Introduction*

The asymptotic evaluation of moments of L-functions is a central problem of analytic number theory with a rich variety of methods employed. The main terms for moments of all orders can be predicted from conjectures based either on random matrix theory [**15**] or on the theory of multiple Dirichlet series [**18**]. The structure of moments of L-functions can also be described by identities and explicit formulae relating different moments to each other of which there exist several types in literature.

The first type, called the reciprocity law, expresses a moment of twisted L-functions as the same moment such that the twist and another parameter are interchanged. The first identity of this kind was discovered by Conrey [**14**] for the twisted second moment of Dirichlet L-functions. This result was refined by Young [**33**] and Bettin [**5**] and extended to the case of rational function fields by Djankovic [**17**]. Much activity of this type is currently being pursued for different families of L-functions, including Rankin-Selberg L-functions [**1**], cusp form L-functions [**9**, **10**] and triple product L-functions over a number field [**36**].

The second type of explicit formulae provides a relationship between a moment of one family of L-functions and a dual moment of another family. One such formula is the expression of the fourth moment of the Riemann zeta function in terms of the third moments of automorphic L-functions as given by Motohashi [**22**]. Another example is due to Petrow who refined the estimate of Conrey and Iwaniec [**16**] for the cubic moment of central L-values of level q cusp forms twisted by quadratic characters of conductor q and showed that the role of the dual moment is played by the weighted fourth moment of Dirichlet L-functions [**27**, Theorems 1,2]. See also [**28**, **29**, **34**] for related recent results.

The main result of the present paper is a new explicit formula of the second type. More precisely, we prove an expression for the mixed moment of symmetric square L-functions and Hecke L-functions relating it to the dual mixed moment of the Riemann zeta function and the

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2000 *Mathematics Subject Classification* 11F12, 11F37, 11M32 (primary), 33C20 (secondary).

The first author was supported by the Russian Science Foundation under grant [19-11-00065]. The second author acknowledges the support of the Labex CEMPI. The third author was supported by the Foundation for the Advancement of Theoretical Physics and Mathematics BASIS. The fourth author acknowledges the support of the Labex CEMPI.

following double Dirichlet series:

$$L_f^-(s) := \frac{\Gamma(3/4)}{2\sqrt{\pi}} \sum_{n<0} \frac{\mathscr{L}_n(1/2)}{|n|^{s+1/2}}, \quad L_g^-(s) := \frac{\Gamma(3/4)}{4\sqrt{\pi}} \sum_{n<0} \frac{\mathscr{L}_{4n}(1/2)}{|n|^{s+1/2}},$$

where

$$\mathscr{L}_n(s) := \frac{\zeta(2s)}{\zeta(s)} \sum_{q=1}^{\infty} \frac{1}{q^s} \left(\sum_{1\le t\le 2q; t^2\equiv n \pmod{4q}} 1 \right). \tag{1.1}$$

Note that the subscripts f and g in L_f^- and L_g^- indicate that these series are associated to two different transforms of Maaß-Eisenstein series of half-integral weight (see Section 3 for details).

The appearance of $\mathscr{L}_n(s)$ and the associated double Dirichlet series as a part of the dual moments turns out to be a specific characteristic of symmetric square L-functions. A similar phenomenon for the second moment of symmetric square L-functions in the level aspect was discovered by Iwaniec-Michel [**20**] and Blomer [**8**]. Furthermore, it is expected that by refining the asymptotic formula of Munshi-Sengupta [**24**] for the mixed moment in the level aspect it should be possible to obtain a second main term of size $q^{-1/2}$ which would involve special values of a similar double Dirichlet series.

In order to state our results rigorously we introduce some notation. Let H_{2k} be the normalised Hecke basis for the space of holomorphic cusp forms of even weight $2k \ge 2$ with respect to the full modular group. Every function $\mathfrak{f} \in H_{2k}$ has a Fourier expansion of the form

$$\mathfrak{f}(z) = \sum_{n\ge 1} \lambda_{\mathfrak{f}}(n) n^{k-1/2} \exp(2\pi i n z), \quad \lambda_{\mathfrak{f}}(1) = 1. \tag{1.2}$$

Consider the mixed moment at the critical point:

$$\mathcal{M}(0,0) := \sum_{\mathfrak{f}\in H_{4k}} \omega(\mathfrak{f}) L(\mathfrak{f}, 1/2) L(\operatorname{sym}^2 \mathfrak{f}, 1/2), \quad \omega(\mathfrak{f}) := \frac{12\zeta(2)}{(4k-1)L(\operatorname{sym}^2 \mathfrak{f}, 1)}, \tag{1.3}$$

where the corresponding L-functions are defined for $\Re s > 1$ as

$$L(\mathfrak{f}, s) := \sum_{n=1}^{\infty} \frac{\lambda_{\mathfrak{f}}(n)}{n^s}, \quad L(\operatorname{sym}^2 \mathfrak{f}, s) := \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_{\mathfrak{f}}(n^2)}{n^s}, \tag{1.4}$$

and admit an analytic continuation to the whole complex plane. Note that we consider only weights divisible by 4 in (1.3) because otherwise $L(\mathfrak{f}, 1/2)$ is identically zero.

The mixed moment (1.3) *with an extra smooth average over weight* was studied in [**3**] by combining an explicit formula for the first moment of symmetric square L-functions and an approximate functional equation for the Hecke L-function. This approach along with the Liouville-Green method appeared to be quite effective, producing an asymptotic formula with an arbitrary power saving error term. However, the same problem *without the extra smooth averaging* is much more difficult since in this case L-functions are averaged over the family of size k instead of k^2 which means that the non-diagonal terms require more involved analysis. For this reason we modify the methods of [**3**], relying now entirely on analytic continuation. More precisely, we prove an explicit formula for the mixed moment (1.3), which contains the diagonal main term of size $\log k$, the non-diagonal main term of size $k^{-1/2}$ and dual mixed moments weighted by ${}_3F_2$ hypergeometric functions.

Theorem 1.1. *For any $\epsilon > 0$ the following formula holds*

$$\mathcal{M}(0,0) = 2\mathcal{M}^D(0,0) + 2\mathcal{M}^{ND}(0,0) + \frac{1}{2\pi i} \int_{(0)} G_{2k}(0,s) ds + O\left(\frac{k^{\epsilon}}{k}\right), \tag{1.5}$$

where

$$\mathcal{M}^D(0,0) = \frac{\zeta(3/2)}{2}\left(\frac{\pi}{2} - 3\log 2\pi + 3\gamma + 2\frac{\zeta'(3/2)}{\zeta(3/2)} + \psi(2k-1/4) + \psi(2k+1/4)\right)$$
$$= \zeta(3/2)\log k + \frac{\zeta(3/2)}{2}\left(\frac{\pi}{2} - 3\log 2\pi + 3\gamma + 2\frac{\zeta'(3/2)}{\zeta(3/2)} + 2\log 2\right) + O(k^{-1}), \quad (1.6)$$

$$\mathcal{M}^{ND}(0,0) = \frac{2^{3/2}\pi}{\Gamma(3/4)}\frac{\Gamma(2k-1/4)}{\Gamma(2k+1/4)}L_g^-(1/4) = \frac{2\pi L_g^-(1/4)}{\Gamma(3/4)}\frac{1}{\sqrt{k}} + O(k^{-3/2}), \quad (1.7)$$

$$G_{2k}(0,s) = \frac{2^{5/2}}{\Gamma^2(3/4)}\Gamma(2k-1/4)\Gamma(3/4-2k)\Gamma(1/2+s)\Gamma(1/4-s)\zeta(1/2-2s)$$
$$\times\left(\left(1-2^{2s-1/2}\right)L_f^-(s) - \frac{\left(1-2^{2s+1/2}\right)}{2^{2s}}L_g^-(s)\right){}_3F_2\left(2k-\frac{1}{4}, \frac{3}{4}-2k, \frac{1}{4}-s; \frac{1}{2}, \frac{3}{4}; 1\right). \quad (1.8)$$

Remark 1. *Note that the integral in* (1.5) *is absolutely convergent (see* (6.33) *and Lemma 5.8).*

Remark 2. *The error term in formula* (1.5) *is stated as* $O\left(k^{-1+\epsilon}\right)$ *for simplicity but it can be replaced by a completely explicit expression. See* (6.2)*, Lemma 6.5 and Lemma 6.10 for details.*

The "recipe" of Conrey, Farmer, Keating, Rubinstein and Snaith [**15**] confirms that $\mathcal{M}^D(0,0)$ is indeed the main term, but the evaluation of the second main term $\mathcal{M}^{ND}(0,0)$ of size $k^{-1/2}$ is beyond the precision of "recipe conjectures". The analysis of the third term given by the integral of $G_{2k}(0,s)$ is the core of this paper and we show that this can be bounded by $\log^3 k$. Consequently, we derive from Theorem 1.1 an upper bound for the mixed moment.

Theorem 1.2. *The following upper bound holds*

$$\mathcal{M}(0,0) \ll \log^3 k. \quad (1.9)$$

The estimate of Theorem 1.2 is at the edge of current technology. However, we expect that the integral of $G_{2k}(0,s)$ is very small because of the oscillatory behaviour of the corresponding ${}_3F_2$ hypergeometric function. The proof of Theorem 1.2 consists in obtaining a sharp upper bound for ${}_3F_2$ and estimating the dual moment by absolute value. In order to improve this result and to derive an asymptotic formula for the mixed moment, it is required to replace absolute value estimates by a direct evaluation of the mixed moment with the oscillating multiple given by the ${}_3F_2$ hypergeometric function. Results of this paper yield a uniform approximation of ${}_3F_2$ in terms of simpler functions which may be useful for further study of this problem.

We now sketch the main ideas of the proof of Theorems 1.1 and 1.2. In order to obtain (1.5) we use the series representation for the Hecke L-function in (1.3) which allows us to reduce the problem to the evaluation of the twisted first moment of symmetric square L-functions. Applying the explicit formula proved in [**4**] to the latter moment we obtain sums of the following shape:

$$\sum_{n,m\geq 1}\mathscr{L}_{-m}(1/2)g_{2k}(0;0;m/n^2), \quad (1.10)$$

where $g_{2k}(0;0;m/n^2)$ is a multiple of the ${}_2F_1$ hypergeometric function and some rational function. In the next step we apply the Mellin inversion in order to separate the variables m and n. This yields the term

$$\frac{1}{2\pi i}\int_{(0)} G_{2k}(0,s)ds \tag{1.11}$$

in (1.5). Equivalently, (1.11) can be written as the following weighted mixed moment:

$$\int_{-\infty}^{\infty} L^{-}_{f,g}(ir)\zeta(1/2-2ir)\hat{g}_{2k}(0,0;ir)dr, \tag{1.12}$$

where $\hat{g}_{2k}$ is the Mellin transform of g_{2k}, which is up to some Gamma multiples given by the ${}_3F_2$ hypergeometric function. The proof of Theorem 1.2 consists of a careful analysis of (1.12). The contribution of $|r|>3k$ is negligibly small because in this range the function $\hat{g}_{2k}(0,0;ir)$ is of rapid decay (see Lemma 5.8). In the remaining range, we write $\hat{g}_{2k}(0,0;ir)$ as an integral of the ${}_2F_1$ hypergeometric function, for which we can apply the Liouville-Green approximation in terms of Y_0 and J_0 Bessel functions. Consequently, we prove (see Lemma 5.9 for details) that for

$$\mathbf{k} := 4k-1 \tag{1.13}$$

the following formula

$$\hat{g}_{2k}(0,0;ir) \sim -2^{3/2}\pi^{1/2}\int_0^{\pi/2}\frac{(\tan x)^{2ir}}{(\sin(2x))^{1/2}}Y_0(\mathbf{k}x)x^{1/2}dx \tag{1.14}$$

holds uniformly for $|r|\ll k$ as $k\to\infty$. We remark that taking the absolute values to estimate the integrals in (1.14) and using standard estimates for the Bessel functions yields $\hat{g}_{2k}(0,0;ir)\ll k^{-1/2}$, and consequently $\mathcal{M}(0,0)\ll k^{1/2+\epsilon}$. In order to improve these bounds, we analyse the integrals in (1.14) further by making the partition of unity and replacing the Bessel functions with their asymptotic formulas. Consequently, it is required to study the oscillating integral (see Lemma 5.11)

$$\int_0^{\pi/2}\frac{\beta(x)\exp(i\mathbf{k}h(x))}{(\sin(2x))^{1/2}}dx, \tag{1.15}$$

where $\beta=\beta(x)$ is a smooth characteristic function vanishing at the end points, and

$$h(x) = -x + \frac{2r}{\mathbf{k}}\log(\tan x).$$

A possible approach to estimate the integral (1.15) is the saddle point method. However, as $4r\to k$ we encounter the problem of two coalescing saddle points. It is known that in this case the considered integral has different behaviour in three different ranges where r is small, r is near $k/4$ and r is large. As the standard saddle point method cannot be applied in such a situation, we follow instead [**7**, Section 9.2], which describes the method that was originally developed by Chester, Friedman and Ursell [**12**], with some additional ideas due to Bleistein [**6**]. As a result, we obtain a uniform expansion of (1.15) in terms of the Airy function (see (5.89)) which yields the following result.

Lemma 1.3. *Let δ be some fixed constant such that $0<\delta<1/4$. For $\mathbf{k}^{1/2-\delta}<r\le\mathbf{k}$ we have*

$$\hat{g}_{2k}(0,0;ir) \ll \frac{1}{\mathbf{k}^{5/6}}\min\left(1,\frac{\mathbf{k}^{1/12}}{|\mathbf{k}-4r|^{1/4}}\right)+\frac{k^{-1/4-3\delta}+k^{-1/2}}{r}. \tag{1.16}$$

For the proof of Lemma 1.3 see Section 5.3.

The paper is organised as follows. Section 2 contains background information on special functions. Section 3 is devoted to the generalised Dirichlet L-functions and the associated double Dirichlet series. In Section 4 we recall the explicit formula for the twisted first moment of symmetric square L-functions. Section 5 is the core of the paper containing all required estimates for special functions. Finally, in Section 6 we prove Theorem 1.1 and Theorem 1.2.

2. *Background on special functions*

This section contains material required for this paper included only for the convenience of the reader. In particular, we describe some properties of the Gamma function, the Bessel functions, the Gauss hypergeometric function and the generalised hypergeometric function, most of which can be found, for example in the NIST handbook [**25**], [**19**] or in [**2**], respectively. These formulae will be used later in the paper.

Euler's reflection formula [**25**, (5.5.3)] for the Gamma function can be stated as

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)}. \tag{2.1}$$

In the special case that $z=\frac{1}{2}+it$ this formula becomes

$$\Gamma(1/2+it)\Gamma(1/2-it)=\frac{\pi}{\cosh(\pi t)} \tag{2.2}$$

(see also [**25**, (5.4.4)]). Another important property of the Gamma function is the Legendre duplication formula (see [**25**, (5.5.5)])

$$\Gamma(2z)=\pi^{-1/2}2^{2z-1}\Gamma(z)\Gamma(z+1/2). \tag{2.3}$$

Furthermore, by [**25**, (5.6.6)] the following inequality holds

$$|\Gamma(x+it)|\leq|\Gamma(x)| \tag{2.4}$$

and Stirling's formula (see [**25**, (5.11.3)]) provides us with the relation

$$\Gamma(z)\sim\sqrt{2\pi}e^{-z}z^{z-1/2}. \tag{2.5}$$

Another function that shows up in this paper is the polygamma function that is given by

$$\psi(z)=\frac{\Gamma'}{\Gamma}(z)=\log(z)-\frac{1}{2z}+O(z^{-2}) \tag{2.6}$$

(see [**25**, (5.11.2)]). For $n\in\mathbf{Z}$ it satisfies the identity

$$\psi(3/4-n)=\psi(1/4+n)+\pi \tag{2.7}$$

(see [**19**, (8.365.10)]). According to [**25**, (5.5.2)]

$$\psi(x+1)=\frac{1}{x}+\psi(x). \tag{2.8}$$

The Gamma function also appears when calculation certain integrals. Namely, it follows from [**25**, (5.12.1)] that

$$\int_0^1 x^{a-1}(1-x)^{b-1}dx=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \tag{2.9}$$

and from [**25**, (5.12.3)] that

$$\int_0^1\frac{x^{a-1}}{(1+x)^{a+b}}dx=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{2.10}$$

Now let us turn to the Bessel functions. As $x \to 0$ we have the following asymptotic relations for J and Y-Bessel functions of order 0 (see [**25**, (10.7.1)]):

$$J_0(x) \sim 1, \quad Y_0(x) \sim \frac{2}{\pi} \log x. \tag{2.11}$$

Moreover, as $x \to \infty$ we have ([**19**, (8.451.1)])

$$J_0(x) = \left(\frac{2}{\pi z}\right)^{1/2} \left(\cos(z - \pi/4)\mathcal{C}_1(x) - \sin(z - \pi/4)\mathcal{C}_2(x)\right) \tag{2.12}$$

and

$$Y_0(x) = \left(\frac{2}{\pi z}\right)^{1/2} \left(\sin(z - \pi/4)\mathcal{C}_1(x) + \cos(z - \pi/4)\mathcal{C}_2(x)\right), \tag{2.13}$$

where

$$\mathcal{C}_1(x) := 1 + \sum_{k=1}^{n-1} \frac{a_k}{x^{2k}} + O(x^{-2n}), \quad \mathcal{C}_2(x) := \sum_{k=0}^{n-1} \frac{b_k}{x^{2k+1}} + O(x^{-1-2n}). \tag{2.14}$$

Here a_k and b_k are some constants independent of x. Using [**25**, (10.25.3)], we find that as $x \to \infty$ the following asymptotic relation holds for the K-Bessel function

$$K_\nu(x) \sim \left(\frac{\pi}{2z}\right)^{1/2} e^{-x}. \tag{2.15}$$

As a consequence of [**25**, (10.30.2)] and [**25**, (10.30.3)] we get the following asymptotic behaviour for $x \to 0$ and $\Re\nu > 0$

$$K_\nu(x) \sim \frac{2^{\nu-1}\Gamma(\nu)}{x^\nu}, \quad K_0(x) \sim -\log x. \tag{2.16}$$

Although hypergeometric functions can be defined more generally we are only interested in the so-called Gauß hypergeometric function ${}_2F_1$ and the generalised hypergeometric function ${}_3F_2$ in this paper. A good reference for the theory of hypergeometric functions is [**2**]. We start by introducing the Pochhammer symbol: For $a \in \mathbf{R}$ and $j \in \mathbf{N}$ it is given by

$$(a)_j := \frac{\Gamma(a+j)}{\Gamma(a)} = a(a+1)\ldots(a+j-1)$$

and we set $(a)_0 := 1$. Then for $|z| < 1$ the Gauß hypergeometric function is defined by the series

$${}_2F_1(a, b, c; z) := \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{j!(c)_j} z^j = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{j=0}^{\infty} \frac{\Gamma(a+j)\Gamma(b+j)}{\Gamma(c+j)} \frac{z^j}{j!} = 1 + \frac{ab}{c} z + \ldots \tag{2.17}$$

(see ([**25**, (15.2.1)])) and it admits an analytic continuation to $z \in \mathbf{C} \setminus [1, +\infty)$. Note that ${}_2F_1(a, b, c; z)$ is analytic as a function of a and b and is meromorphic as a function of c with poles at $c = 0, -1, -2, \ldots$. If $\Re(c - a - b) > 0$ then the series given in (2.17) also converges for $z = 1$ and we can evaluate ${}_2F_1(a, b, c; 1)$ explicitly in terms of Gamma functions

$${}_2F_1(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{2.18}$$

(see [**25**, (15.4.20)]). One property that is useful when working with Gauß hypergeometric functions is that we know how these functions transform if we permute their arguments. We have, e.g.,

$${}_2F_1(a, b, c; z) = (1-z)^{c-a-b} {}_2F_1(c-a, c-b, c; z). \tag{2.19}$$

(see [**19**, (9.131.1)]). Furthermore, the Gauss hypergeometric function satisfies the so-called Gauß's contiguous relations, one of them being

$$c \, {}_2F_1(a, b, c; z) - c \, {}_2F_1(a, b+1, c; z) + az \, {}_2F_1(a+1, b+1, c+1; z) = 0. \tag{2.20}$$

(see [**19**, (9.137.11)]). The definition of the Gauß hypergeometric function can be generalised and for $|z|<1$ the generalised hypergeometric function ${}_3F_2$ is defined similarly to the Gauß hypergeometric function by the series

$$\begin{aligned} {}_3F_2(a_1,a_2,a_3;b_1,b_2;z) &:= \frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}\sum_{j=0}^{\infty}\frac{\Gamma(a_1+j)\Gamma(a_2+j)\Gamma(a_3+j)}{\Gamma(b_1+j)\Gamma(b_2+j)}\frac{z^j}{j!} \\ &= 1+\frac{a_1a_2a_3}{b_1b_2}z+\dots \end{aligned} \tag{2.21}$$

(see ([**25**, (16.2.1)]) and it admits the analytic continuation to $z\in\mathbf{C}\setminus[1,+\infty)$. Note that ${}_3F_2(a_1,a_2,a_3;b_1,b_2;z)$ is analytic as a function of a_1,a_2,a_3 and is meromorphic as a function of b_1,b_2 with poles at $b_i=0,-1,-2,\dots$, $i=1,2$. A useful fact when working with hypergeometric functions is that they can be simplified if one of the parameters a_j, $j=1,2,3$, is equal to one of the b_i, $i=1,2$. Namely, by the definition of the hypergeometric function we get

$${}_3F_2(a_1,a_2,a_3;b_1,a_3;z)={}_2F_1(a_1,a_2,b_1;z). \tag{2.22}$$

3. *Generalised Dirichlet L-functions*

In this section, we gather various results related to generalised Dirichlet L-functions that are required for the evaluation of the non-diagonal terms.

Consider

$$\mathscr{L}_n(s)=\frac{\zeta(2s)}{\zeta(s)}\sum_{q=1}^{\infty}\frac{1}{q^s}\left(\sum_{1\le t\le 2q;t^2\equiv n \pmod{4q}}1\right),\quad \Re s>1.$$

It follows from [**35**, Proposition 3] that $\mathscr{L}_n(s)$ has a meromorphic continuation to the whole complex plane. The completed L-function

$$\mathscr{L}^*_n(s)=(\pi/|n|)^{-s/2}\Gamma(s/2+1/4-\operatorname{sgn} n/4)\mathscr{L}_n(s)$$

satisfies the functional equation (see [**35**, Proposition 3, p. 130])

$$\mathscr{L}^*_n(s)=\mathscr{L}^*_n(1-s). \tag{3.1}$$

According to [**11**, Section 1], the function $\mathscr{L}_n(s)$ considered as a function of s does not identically vanish only if $n\equiv 0,1 \pmod 4$. This is due to the fact that $t^2\equiv n \pmod 4$ does not have any solutions if $n\not\equiv 0,1 \pmod 4$.

For any $\epsilon>0$ we have (see [**4**, Lemma 4.2])

$$\mathscr{L}_n(1/2)\ll |n|^{\theta+\epsilon}, \tag{3.2}$$

where $\theta=1/6$ is the best known subconvexity exponent for Dirichlet L-functions obtained by Conrey and Iwaniec in [**16**]. It follows from (3.2) and the Phragmen-Lindelöf principle that for any $\epsilon>0$ and $\Re u>0$ the following upper bound holds

$$\mathscr{L}_n(1/2+u)\ll |n|^{\max(\theta(1-2\Re u),0)+\epsilon}. \tag{3.3}$$

The generalised Dirichlet L-function $\mathscr{L}_n(s)$ shows up in the Fourier expansion of a linear combination of half-integral weight Eisenstein series. To be more precise we briefly summarise the principal arguments of [**3**, Section 2]: let $\Gamma_0(4)$ be the Hecke congruence subgroup of level 4 and ν be the weight $1/2$ multiplier system related to the theta series

$$\theta(z):=y^{1/4}\sum_{m\in\mathbf{Z}}e^{2\pi i m^2 z}.$$

It is well-know that $\Gamma_0(4)$ has three cusps which we denote by $\mathfrak{a}_1=\infty$, $\mathfrak{a}_2=0$ and $\mathfrak{a}_3=1/2$. Then for a cusp $\mathfrak{a}$ of $\Gamma_0(4)$ we define $E_{\mathfrak{a}}(z;s;1/2)$ to be the Eisenstein series of weight $1/2$ for the group $\Gamma_0(4)$ at the cusp $\mathfrak{a}$ with respect to the multiplier system ν. Then $\mathscr{L}_n(1/2)$ appears in the Fourier expansion of the following linear combination of the Maaß-Eisenstein series of weight1/2 and level 4 at the cusps ∞ and 0:

$$f = f(z;s) := \zeta(4s-1)\left(E_\infty(z;s;1/2) + \frac{1+i}{4^s}E_0(z;s;1/2)\right). \tag{3.4}$$

Namely,

$$f(z;s) = \frac{1}{2}y^{1/2}\log y + (\gamma - \log 4\pi)y^{1/2} + \frac{1}{2\sqrt{\pi}}\sum_{n\neq 0}\frac{\mathscr{L}_n(1/2)}{|n|^{1/2}}\Gamma\left(\frac{1}{2} - \frac{\operatorname{sgn} n}{4}\right)W_{\operatorname{sgn} n/4,0}\left(4\pi|n|y\right)\exp\left(2\pi inx\right).$$

Furthermore, we define g by

$$g = g(z;s) := \frac{1}{2}\left(f\left(\frac{z}{4};s\right) + f\left(\frac{z+2}{4};s\right)\right). \tag{3.5}$$

The properties of $\mathscr{L}_n(s)$ imply that the function f belongs to the Kohnen plus space, as defined in [**30**] for Maaß forms, so that g has the following Fourier expansion

$$g(z;s) = \frac{1}{4}y^{1/2}\log y + \frac{1}{2}(\gamma - \log 8\pi)y^{1/2} + \frac{1}{4\sqrt{\pi}}\sum_{n\neq 0}\frac{\mathscr{L}_{4n}(1/2)}{|n|^{1/2}}\Gamma\left(\frac{1}{2} - \frac{\operatorname{sgn} n}{4}\right)W_{\operatorname{sgn} n/4,0}\left(4\pi|n|y\right)\exp\left(2\pi inx\right).$$

As usual we can associate to the functions f and g the double Dirichlet series

$$L_f^+(s) := \frac{\Gamma(1/4)}{2\sqrt{\pi}}\sum_{n>0}\frac{\mathscr{L}_n(1/2)}{n^{s+1/2}}, \quad L_f^-(s) := \frac{\Gamma(3/4)}{2\sqrt{\pi}}\sum_{n<0}\frac{\mathscr{L}_n(1/2)}{|n|^{s+1/2}}, \tag{3.6}$$

$$L_g^+(s) := \frac{\Gamma(1/4)}{4\sqrt{\pi}}\sum_{n>0}\frac{\mathscr{L}_{4n}(1/2)}{n^{s+1/2}}, \quad L_g^-(s) := \frac{\Gamma(3/4)}{4\sqrt{\pi}}\sum_{n<0}\frac{\mathscr{L}_{4n}(1/2)}{|n|^{s+1/2}}, \tag{3.7}$$

where the plus and minus sign indicate the sign of n. Their analytic properties as well as functional equations result from the fact that f and g satisfy the important transformation property

$$\exp\left(\pi i/4\right)\left(f\big|_{1/2}J\right)(z;s) = \sqrt{2}g(z/4;s), \quad J = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \tag{3.8}$$

where the slash operator is given by

$$\left(f\big|_{1/2}M\right)(z;s) = \left(\frac{cz+d}{|cz+d|}\right)^{-1/2}f(Mz), \quad M = \begin{pmatrix}a & b\\ c & d\end{pmatrix}.$$

Using this transformation property as well as the Rankin-Selberg method yields:

THEOREM 3.1. *The functions $L_f^{\pm}(s)$ and $L_g^{\pm}(s)$ have a meromorphic continuation to the whole complex plane and satisfy the functional equations*

$$L_g^+(s) = \frac{-\pi^{2s+2}}{\sqrt{2}\Gamma^2(1/2+s)\sin^2\pi s}\left(\frac{\sin\pi(-s-1/4)}{\pi}L_f^+(-s) - \frac{L_f^-(-s)}{\Gamma^2(3/4)}\right), \tag{3.9}$$

$$L_g^-(s) = \frac{\pi^{2s+2}}{\sqrt{2}\Gamma^2(1/2+s)\sin^2\pi s}\left(-\frac{\sin\pi(-s+1/4)}{\pi}L_f^-(-s) + \frac{L_f^+(-s)}{\Gamma^2(1/4)}\right). \tag{3.10}$$

Furthermore, $L_f^{\pm}(s)$ and $L_g^{\pm}(s)$ are holomorphic in $\mathbf{C}$ *except for a double pole at $s = 1/2$.*

Proof. See [**3**, Theorem 2.3]. □

THEOREM 3.2. *Writing the Laurent expansion of $L^{\pm}_{f,g}$ as*

$$L^{\pm}_{f,g}(s+1/2) = \frac{c^{\pm}_{f,g}(-2)}{s^2} + \frac{c^{\pm}_{f,g}(-1)}{s} + O(1) \tag{3.11}$$

we obtain that the coefficients $c^{\pm}_f$, $c^{\pm}_g$ satisfy the following identities:

$$\frac{c^{+}_{f}(-2)}{\Gamma(1/4)} - \frac{c^{-}_{f}(-2)}{\Gamma(3/4)} = 0, \tag{3.12}$$

$$\frac{c^{+}_{f}(-1)}{\Gamma(1/4)} - \frac{c^{-}_{f}(-1)}{\Gamma(3/4)} + \frac{c^{-}_{f}(-2)\pi}{\Gamma(3/4)} = 0, \tag{3.13}$$

$$\frac{16\sqrt{\pi}}{\Gamma(3/4)}\left(c^{-}_{f}(-2)(1-\sqrt{2}) + c^{-}_{g}(-2)(\sqrt{2}-1/2)\right) = 1. \tag{3.14}$$

REMARK 3. *The identities of Theorem 3.2 will be used in Section 6 to combine the main terms coming from Lemma 4.1.*

Proof. The proof the the theorem consists of three steps. First, we investigate certain linear combinations of hypergeometric functions. These results are then used to write down the Laurent expansion for the different double Dirichlet series. Finally, the identities given in the theorem result from these expansions.

As in [**23**] we define

$$\Gamma_\alpha(s) := 2^\alpha \frac{\Gamma^2(s+1/2)}{\Gamma(s+1-\alpha)} {}_2F_1\left(1/2-\alpha, 1/2-\alpha; s+1-\alpha; 1/2\right) \tag{3.15}$$

and set

$$F(s+1/2) := \frac{1}{\Gamma^2(s+1)}\left(\Gamma_{3/4}(s+1/2) - \frac{1}{4}\Gamma_{-1/4}(s+1/2)\right). \tag{3.16}$$

Using the definition of Γ_α and a Gauß contiguous relation (see (2.20)) this expression can be simplified to

$$\begin{aligned} &F(s+1/2) \\ &= \frac{2^{3/4}}{\Gamma(s+3/4)}\left({}_2F_1\left(-1/4,-1/4;s+3/4;1/2\right) - \frac{1}{8(s+3/4)}\,{}_2F_1\left(3/4,3/4;s+7/4;1/2\right)\right) \\ &= \frac{2^{3/4}}{\Gamma(s+3/4)}\,{}_2F_1\left(-1/4,3/4;s+3/4;1/2\right). \end{aligned} \tag{3.17}$$

This expression has the advantage that, as $s \to 0$, the second and the third argument of the hypergeometric function become equal so that the hypergeometric function simplifies and can be calculated as a binomial series. Furthermore, it is advantageous to permute the arguments of the first hypergeometric function appearing in (3.17). This will enable us later to see directly the first terms of the Laurent expansion. From (2.17) we infer

$$
\begin{aligned}
{}_2F_1\left(-1/4, 3/4; s+3/4; 1/2\right) &= \left(\frac{1}{2}\right)^{s+1/4} {}_2F_1\left(s+1, s; s+3/4; 1/2\right) \\
&= \left(\frac{1}{2}\right)^{s+1/4} \left(1 + \frac{\Gamma(s+3/4)}{\Gamma(s+1)\Gamma(s)} \sum_{n=0}^{\infty} \frac{\Gamma(s+2+n)\Gamma(s+1+n)}{(n+1)!\Gamma(s+7/4+n)} \left(\frac{1}{2}\right)^{n+1}\right),
\end{aligned}
$$

and therefore,

$$
F(s+1/2) = \frac{2^{1/2-s}}{\Gamma(s+3/4)} + \frac{2^{-(s+1/2)}}{\Gamma(s)\Gamma(s+1)} \sum_{n=0}^{\infty} \frac{\Gamma(s+2+n)\Gamma(s+1+n)}{(n+1)!\Gamma(s+7/4+n)} \left(\frac{1}{2}\right)^{n}. \tag{3.18}
$$

Another linear combination that appears when treating our double Dirichlet series is the following:

$$
G(s+1/2) := \frac{1}{\Gamma^2(s+\frac{1}{2})} \left(\frac{1}{4}\Gamma_{-3/4}(s+1/2) + \Gamma_{1/4}(s+1/2)\right). \tag{3.19}
$$

Using the definition of Γ_α and a Gauß contiguous relation (see (2.20)) as before this expression can be simplified to

$$
\begin{aligned}
G(s+1/2) &= \frac{2^{1/4}}{\Gamma(s+9/4)} \left(\frac{1}{8}{}_2F_1\left(5/4, 5/4; s+9/4; 1/2\right) + (s+5/4){}_2F_1\left(1/4, 1/4; s+5/4; 1/2\right)\right) \\
&= \frac{2^{1/4}}{\Gamma(s+5/4)} {}_2F_1\left(1/4, 5/4; s+5/4; 1/2\right).
\end{aligned} \tag{3.20}
$$

Using, as before, (2.17) we get

$$
\begin{aligned}
G(s+1/2) &= \frac{2^{1/2-s}}{\Gamma(s+5/4)} {}_2F_1\left(s+1, s; s+5/4; 1/2\right) \\
&= \frac{2^{1/2-s}}{\Gamma(s+5/4)} + \frac{2^{-1/2-s}}{\Gamma(s)\Gamma(s+1)} \sum_{n=0}^{\infty} \frac{\Gamma(s+2+n)\Gamma(s+1+n)}{(n+1)!\Gamma(s+9/4+n)} \left(\frac{1}{2}\right)^{n}.
\end{aligned} \tag{3.21}
$$

Furthermore, we define

$$
M(f,s) := \int_0^{\infty} \left(f(iy) - A_0(f,y)\right) y^{s-1} dy, \tag{3.22}
$$

where $A_{\infty,0}(y)$ denotes the zeroth Fourier coefficient in the Fourier expansion of the automorphic form f. This means if f is the function defined in (3.4) we have

$$
A_0(f,y) = (\gamma - \log 4\pi) y^{1/2} + \frac{1}{2} y^{1/2} \log y
$$

which in order to simplify the notation we write in the form

$$
A_0(f,y) = a_0 y^{1/2} + b_0 y^{1/2} \log y.
$$

Here we set $a_0 = \gamma - \log 4\pi$ and $b_0 = 1/2$. Furthermore, for the function g defined in (3.5) we have

$$
A_0(g,y) = \frac{1}{2}(\gamma - \log 8\pi) y^{1/2} + \frac{1}{4} y^{1/2} \log y = \widehat{a}_0 y^{1/2} + \widehat{b}_0 y^{1/2} \log y,
$$

where $\widehat{a}_0 := (\gamma - \log 8\pi)/2$ and $\widehat{b}_0 = 1/4$. Replacing the automorphic form f by its Fourier expansion in (3.22) we see that this integral equals a double Dirichlet series multiplied by a hypergeometric function. This function appears as the Mellin transform of the Whittaker function. According to [**23**, Eq. 52] the Laurent expansion of $M(f, s+1/2)$ about 0 is

$$
M(f, s+1/2) = \frac{1}{2^{5/2} s^2} + \frac{2\widehat{a}_0 - \log 2}{2^{3/2} s} + O(1). \tag{3.23}
$$

Moreover, the Laurent expansion of $M(E_{1/2}f, s+1/2)$ about 0 equals

$$M(E_{1/2}f, s+1/2) = \frac{1}{2^{9/2}s^2} + \frac{1}{2^{7/2}s}\left(2\widehat{a}_0 - \log 2 + 2\right) + O(1). \tag{3.24}$$

Here $E_{1/2}$ denotes the Maaß lowering operator which is given by

$$E_{1/2} = y\left(i\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) + \frac{1}{4} \tag{3.25}$$

The Laurent expansions of $M(f, -(s+1/2))$ and $M(E_{1/2}\, f, -(s+1/2))$ about 0 are given by

$$M(f, -(s+1/2)) = \frac{1}{2s^2} + \frac{a_0}{s} + O(1) \tag{3.26}$$

and

$$M(E_{1/2}f, -(s+1/2)) = -\frac{1}{2^3s^2} - \frac{a_0+2}{4s} + O(1) \tag{3.27}$$

(see [**23**, Eq. 52, 53]).

Now we have all the necessary prerequisites for determining the different Laurent expansions. By [**23**, Eq. 30, 48] we obtain

$$L_f^+(s+1/2) = \frac{(2\pi)^{s+1/2}}{2\Gamma^2(s+1)}\left(\Gamma_{3/4}(s+1/2)M(f, s+1/2) - \Gamma_{-1/4}(s+1/2)M(E_{1/2}f, s+1/2)\right). \tag{3.28}$$

Using (3.28), (3.23) and (3.24) we obtain

$$\begin{aligned}\frac{2^{3/2}}{(2\pi)^{s+1/2}}L_f^+(s+1/2) &= \frac{1}{4s^2}F(s+1/2)\\ &+ \frac{1}{s}\left(\left(\widehat{a}_0 - \frac{1}{2}\log 2\right)F(s+1/2) - \frac{1}{2^{9/4}\Gamma(s+7/4)}{}_2F_1\left(3/4, 3/4; s+7/4; 1/2\right)\right) + O(1).\end{aligned} \tag{3.29}$$

with F being the function defined in (3.16). Using (3.18) we infer

$$\begin{aligned}L_f^+(s+1/2) = {} & \frac{\pi^{s+1/2}}{2^{5/2}\Gamma(s+3/4)}\frac{1}{s^2} + \frac{1}{s}\left(\frac{\pi^{s+1/2}}{2^{7/2}\Gamma^2(s+1)}\sum_{n=0}^{\infty}\frac{\Gamma(s+2+n)\Gamma(s+1+n)}{(n+1)!\Gamma(s+7/4+n)}\left(\frac{1}{2}\right)^n\right.\\ &\left. + \frac{\pi^{s+1/2}\left(\widehat{a}_0 - \frac{1}{2}\log 2\right)}{2^{1/2}\Gamma(s+3/4)} - \frac{(2\pi)^{s+1/2}}{2^{15/4}\Gamma(s+7/4)}{}_2F_1\left(3/4, 3/4; s+7/4; 1/2\right)\right)\\ & + O(1).\end{aligned} \tag{3.30}$$

This yields the Laurent expansion of $L_f^+(s+1/2)$ about $s=0$ once we have determined the coefficients of $1/s^2$ and $1/s$ in (3.30). For the coefficient of $1/s^2$ we get $\pi^{1/2}/\left(2^{5/2}\Gamma(s+3/4)\right)$. In order to obtain the coefficient of $1/s$ we note that

$$\frac{\pi^{s+1/2}}{\Gamma(s+3/4)} = \frac{\pi^{1/2}}{\Gamma(3/4)} + \frac{\pi^{1/2}}{\Gamma(3/4)}\left(\log\pi - \psi(3/4)\right)s + O\left(s^2\right)$$

and that by (2.19)

$$\begin{aligned}\lim_{s\to 0}\frac{\pi^{s+1/2}}{2^{7/2}\Gamma^2(s+1)}\sum_{n=0}^{\infty}\frac{\Gamma(s+2+n)\Gamma(s+1+n)}{(n+1)!\Gamma(s+7/4+n)}\left(\frac{1}{2}\right)^n &= \frac{\pi^{1/2}}{2^{7/2}}\sum_{n=0}^{\infty}\frac{\Gamma(1+n)\Gamma(1+n)}{n!\Gamma(7/4+n)}\left(\frac{1}{2}\right)^n\\ &= \frac{\pi^{1/2}}{2^{7/2}\Gamma(7/4)}{}_2F_1\left(1, 1; 7/4; 1/2\right)\\ &= \frac{1}{2^{13/4}\Gamma(7/4)}{}_2F_1\left(3/4, 3/4; 7/4; 1/2\right).\end{aligned}$$

Thus the series and the hypergeometric series appearing in (3.30) do not contribute to the coefficient of $1/s$ and we finally infer

$$\begin{aligned} L_f^+(s+1/2) = {} & \frac{\pi^{1/2}}{2^{5/2}\Gamma(3/4)}\frac{1}{s^2} \\ & + \frac{\pi^{1/2}}{2^{1/2}\Gamma(3/4)}\left(\frac{1}{4}\left(\log\pi - 2\log 2 - \psi(3/4)\right) + \widehat{a}_0\right)\frac{1}{s} + O(1). \end{aligned} \tag{3.31}$$

Determining the Laurent expansion of $L_f^-(s+1/2)$ will follow the same ideas. First, we remark that [**23**, Eq. 30, 49] gives

$$\begin{aligned} & L_f^-(s+1/2) = \\ & \frac{(2\pi)^{s+1/2}}{2\Gamma^2(s+1)}\left((1/4)^2\Gamma_{-3/4}(s+1/2)M(f,s+1/2) + \Gamma_{1/4}(s+1/2)M(E_{1/2}f,s+1/2)\right). \end{aligned} \tag{3.32}$$

By (3.23) and (3.24) this implies

$$\begin{aligned} \frac{2^{3/2}}{(2\pi)^{s+1/2}}L_f^-(s+1/2) = {} & \frac{1}{4^2s^2}G(s+1/2) + \frac{\widehat{a}_0 - \frac{1}{2}\log 2}{4s}G(s+1/2) \\ & + \frac{1}{2^{7/4}\Gamma(s+5/4)s}\,{}_2F_1\left(1/4,1/4;s+5/4;1/2\right) + O(1), \end{aligned}$$

where the function G is defined in (3.19). Using (3.21) gives

$$\begin{aligned} L_f^-(s+1/2) = {} & \frac{\pi^{s+1/2}}{2^{9/2}\Gamma(s+5/4)}\frac{1}{s^2} + \frac{1}{s}\left(\frac{\pi^{s+1/2}}{2^{11/2}\Gamma^2(s+1)}\sum_{n=0}^{\infty}\frac{\Gamma(s+2+n)\Gamma(s+1+n)}{(n+1)!\Gamma(s+9/4+n)}\left(\frac{1}{2}\right)^n\right. \\ & \left. + \frac{\pi^{s+1/2}\left(\widehat{a}_0 - \frac{1}{2}\log 2\right)}{2^{5/2}\Gamma(s+5/4)} + \frac{(2\pi)^{s+1/2}}{2^{13/4}\Gamma(s+5/4)}\,{}_2F_1\left(1/4,1/4;s+5/4;1/2\right)\right) \\ & + O(1). \end{aligned} \tag{3.33}$$

In order to simplify this expression we note that by (2.19) and (2.20)

$$\begin{aligned} \lim_{s\to 0}\frac{1}{\Gamma^2(s+1)}\sum_{n=0}^{\infty}\frac{\Gamma(s+2+n)\Gamma(s+1+n)}{(n+1)!\Gamma(s+9/4+n)}\left(\frac{1}{2}\right)^n & = \frac{1}{\Gamma(9/4)}\,{}_2F_1\left(1,1;9/4;1/2\right) \\ & = \frac{1}{2^{1/4}\Gamma(9/4)}\,{}_2F_1\left(5/4,5/4;9/4;1/2\right) \\ & = \frac{2^{11/4}}{\Gamma(5/4)}\left({}_1F_0\left(1/4;1/2\right) - {}_2F_1\left(1/4,1/4;5/4;1/2\right)\right). \end{aligned}$$

Furthermore, we note that the hypergeometric function ${}_1F_0$ is related to a binomial series and can therefore be explicitly evaluated. Namely, we have

$${}_1F_0\left(1/4;1/2\right) = \left(\frac{1}{2}\right)^{-1/4} = 2^{1/4}$$

so that

$$\begin{aligned} L_f^-(s+1/2) = {} & \frac{\pi^{1/2}}{2^{5/2}\Gamma(1/4)}\frac{1}{s^2} \\ & + \frac{\pi^{1/2}}{2^{5/2}\Gamma(1/4)}\left(\log\pi - \psi(5/4) + 4 + 4\widehat{a}_{\infty,0} - 2\log 2\right)\frac{1}{s} + O(1). \end{aligned} \tag{3.34}$$

In order to determine the Laurent expansions of $L_g^{\pm}(s+1/2)$ it is necessary to know the Laurent expansions of $M(f,-(s+1/2))$ and $M(E_{1/2}f,-(s+1/2))$ about $s=0$ as we have according to [**23**, Eq. 54]

$$M(g, s+1/2) = \frac{1}{2^{3/2+2s}} M(f, -(s+1/2)) \tag{3.35}$$

and

$$M(E_{1/2}\, g, s+1/2) = \frac{1}{2^{3/2+2s}} M(E_{1/2}\, f, -(s+1/2)). \tag{3.36}$$

Then, as for $L_f^+(s+1/2)$ and $L_f^-(s+1/2)$ (see (3.28) and (3.32)) we get

$$L_g^+(s+1/2) = \frac{(2\pi)^{s+1/2}}{2^{5/2+2s}\Gamma^2(s+1)} \Big(\Gamma_{3/4}(s+1/2) M(f, -(s+1/2)) + \Gamma_{-1/4}(s+1/2) M(E_\kappa f, -(s+1/2)) \Big)$$

and

$$L_g^-(s+1/2) = \frac{(2\pi)^{s+1/2}}{2^{5/2+2s}\Gamma^2(s+1)} \Bigg(\left(\frac{1}{4}\right)^2 \Gamma_{-3/4}(s+1/2) M(f, -(s+1/2)) - \Gamma_{1/2}(s+1/2) M(E_{1/2}\, f, -(s+1/2) \Bigg).$$

Let us first look at $L_g^+(s)$. By (3.26), (3.27) and (3.18) we infer

$$L_g^+(s+1/2) = \frac{(2\pi)^{1/2}}{2^3\Gamma(3/4)} \frac{1}{s^2} + \frac{1}{s} \frac{(2\pi)^{1/2}}{2^3\Gamma(3/4)} \left(\log(\pi/4) - \psi(3/4) + 2a_0\right) + O(1) \tag{3.37}$$

using the same arguments that led to the Laurent expansion of $L_f^+(s+1/2)$. Similarly, the same arguments that gave the Laurent expansion of $L_f^-(s+1/2)$ together with (3.26), (3.27) and (3.21) imply

$$L_g^-(s+1/2) = \frac{(2\pi)^{1/2}}{2^3\Gamma(1/4)} \frac{1}{s^2} + \frac{1}{s} \frac{(2\pi)^{1/2}}{2^3\Gamma(1/4)} \left(\log(\pi/4) - \psi(1/4) + 2a_0\right) + O(1). \tag{3.38}$$

Having determined the Laurent expansions of $L_f^\pm(s+1/2)$, and $L_g^\pm(s+1/2)$ we now prove the various identities of Theorem 3.2. The identity (3.12) follows immediately from the expansion (3.31) and (3.34) for $L_f^+(s+1/2)$ and $L_f^-(s+1/2)$. Furthermore, (3.31) and (3.34) also imply

$$\begin{aligned} \frac{c_f^+(-1)}{\Gamma(1/4)} - \frac{c_f^-(-1)}{\Gamma(3/4)} + \frac{c_f^-(-2)\pi}{\Gamma(3/4)} &= \frac{(2\pi)^{1/2}}{2^3\Gamma(1/4)\Gamma(3/4)} \left(\pi - \psi(3/4) - (-\psi(5/4) + 4)\right) \\ &= \frac{(2\pi)^{1/2}}{2^3\Gamma(1/4)\Gamma(3/4)} \left(\pi - \psi(3/4) + \psi(1/4)\right) = 0, \end{aligned}$$

where we used (2.8) and (2.7). Finally, (3.34), (3.38) and the reflection formula (2.1) yield

$$\begin{aligned} &\frac{16\sqrt{\pi}}{\Gamma(3/4)} \left(c_f^-(-2)\left(1-\sqrt{2}\right) + c_g^-(-2)\left(\sqrt{2}-1/2\right)\right) \\ &= \frac{16\sqrt{\pi}}{\Gamma(3/4)} \left(\frac{(2\pi)^{1/2}}{2^3\Gamma(1/4)} \left(1-\sqrt{2}\right) + \frac{(2\pi)^{1/2}}{2^3\Gamma(1/4)} \left(\sqrt{2}-1/2\right) \right) \\ &= \frac{2^{1/2}\pi}{\Gamma(3/4)\Gamma(1/4)} \\ &= 1. \end{aligned}$$

□

THEOREM 3.3. *The following estimates hold*

$$\int_0^T |L_g^{\pm}(it)|^2 dt \ll T(\log T)^4, \quad \int_0^T |L_f^{\pm}(it)|^2 dt \ll T(\log T)^4. \tag{3.39}$$

Proof. We recall that the series $L_f^{\pm}$ and $L_g^{\pm}$ are associated to the automorphic forms f and g that satisfy the transformation property (3.8). Thus the theorem is a direct consequence of [**23**, Theorem 5.1 (iv)]. □

4. *Explicit formula for the twisted first moment of symmetric square L-functions*

For $0 < x < 1$ and $0 \le \Re u < 2k - 3/2$ let

$$\Psi_k(u;x) := x^k \frac{\Gamma(k-1/4-u/2)\Gamma(k+1/4-u/2)}{\Gamma(2k)} {}_2F_1\left(k-\frac{1}{4}-\frac{u}{2}, k+\frac{1}{4}-\frac{u}{2}; 2k; x\right), \tag{4.1}$$

$$\Phi_k(u;x) := \frac{\Gamma(k-1/4-u/2)\Gamma(3/4-k-u/2)}{\Gamma(1/2)} {}_2F_1\left(k-\frac{1}{4}-\frac{u}{2}, \frac{3}{4}-k-\frac{u}{2}; 1/2; x\right), \tag{4.2}$$

where ${}_2F_1(a,b;c;x)$ is the Gauss hypergeometric function. For simplicity, let us introduce the following notation

$$\Psi_k(x) := \Psi_k(0;x), \quad \Phi_k(x) := \Phi_k(0;x). \tag{4.3}$$

For our purpose, it is required to evaluate certain integrals involving $\Psi_k(x)$ and $\Phi_k(x)$. To this end, it is convenient to use the Mellin-Barnes representation for these functions. For $\Re u > 0$, $1-2k < \Delta < 1/2 - \Re u$ let

$$I_k(u;x) := \frac{1}{2\pi i}\int_{(\Delta)} \frac{\Gamma(k-1/2+w/2)}{\Gamma(k+1/2-w/2)}\Gamma(\frac{1}{2}-u-w)\sin\left(\pi\frac{1/2+u+w}{2}\right) x^w dw. \tag{4.4}$$

According to [**4**, (5.3)], for $x > 2$ we have

$$I_k(u;x) = (-1)^k \frac{\cos(\pi(1/4+u/2))}{2^{1/2+u}\pi^{1/2}} x\Psi_k\left(u; \frac{4}{x^2}\right). \tag{4.5}$$

According to [**4**, (5.5)], for $0 < x < 2$ we have

$$I_k(u;x) = (-1)^k \frac{\sin(\pi(1/4+u/2))}{\pi^{1/2}} x^{1/2-u}\Phi_k\left(u; \frac{x^2}{4}\right). \tag{4.6}$$

Note that Equations (4.5) and (4.6) provide the analytic continuation of $I_k(u;x)$ to $\Re u = 0$.
Using (4.4), we find that $\Psi_k(u;x) \sim x^k$ and $\Phi_k(u;x) \sim 1$ as $x \to 0$.
Now we are ready to state the explicit formula for the twisted first moment of symmetric square L-functions.

LEMMA 4.1. *For $0 \le \Re u < 4k - 3/2$ we have*

$$\sum_{\mathfrak{f}\in H_{4k}} \omega(\mathfrak{f})\lambda_{\mathfrak{f}}(l)L(\mathrm{sym}^2 \mathfrak{f}, 1/2+u) = M^D(u,l)\delta_{l=\square} + M^{ND}(u,l) + ET_1(u,l) + ET_2(u,l),$$

where

$$\delta_{l=\square} = \begin{cases} 1 & \text{if } l \text{ is a full square,} \\ 0 & \text{otherwise,} \end{cases} \tag{4.7}$$

$$M^D(u,l^2)=\frac{\zeta(1+2u)}{l^{1/2+u}}+\sqrt{2}(2\pi)^{3u}\cos\pi(1/4+u/2)\times \\ \frac{\zeta(1-2u)}{l^{1/2-u}}\frac{\Gamma(2k-1/4-u/2)\Gamma(2k+1/4-u/2)\Gamma(1-2u)}{\Gamma(2k+1/4+u/2)\Gamma(2k-1/4+u/2)\Gamma(1-u)}, \quad (4.8)$$

$$M^{ND}(u,l)=\frac{(2\pi)^{1/2+u}}{2l^{1/4-u/2}}\frac{\Gamma(2k-1/4-u/2)}{\Gamma(2k+1/4+u/2)}\mathscr{L}_{-4l}(1/2+u), \quad (4.9)$$

$$ET_1(u,l)=(2\pi)^{1/2+u}\sum_{1\leq n<2\sqrt{l}}\frac{\mathscr{L}_{n^2-4l}(1/2+u)}{n^{1/2-u}}I_{2k}\left(u;\frac{n}{l^{1/2}}\right), \quad (4.10)$$

$$ET_2(u,l)=(2\pi)^{1/2+u}\sum_{n>2\sqrt{l}}\frac{\mathscr{L}_{n^2-4l}(1/2+u)}{n^{1/2-u}}I_{2k}\left(u;\frac{n}{l^{1/2}}\right). \quad (4.11)$$

Proof. See [**4**, (2.9), (5.6)]. □

REMARK 4. *The role of the shift u is to guarantee the absolute convergence of the integral* (4.4).

5. *Estimates on special functions*

In this section we introduce two special functions that appear in the study of the mixed moment in Section 6 and prove various estimates on the Mellin transforms of these functions. For a function $h(x)$, we denote its Mellin transform by

$$\hat{h}(s)=\int_0^\infty h(x)x^{s-1}dx. \quad (5.1)$$

Let us define for $0<x<1$

$$f_{2k}(u,v;x):=\frac{x^{1/2+v}}{(1-x)^{1/2+v}}I_{2k}\left(u;\frac{2}{(1-x)^{1/2}}\right), \quad (5.2)$$

and $f_{2k}(u,v;x):=0$ for $x>1$.

For $0<x<\infty$ let

$$g_{2k}(u,v;x):=\frac{x^{1/2+v}}{(1+x)^{1/2+v}}I_{2k}\left(u;\frac{2}{(1+x)^{1/2}}\right). \quad (5.3)$$

Now we analyse the Mellin transforms of the functions $f_{2k}(u,v;x)$ and $g_{2k}(u,v;x)$.

5.1. *Mellin transform of f_{2k}*

LEMMA 5.1. *For $\Re s>-1/2-\Re v$ and $\Re v<2k$, $0\leq\Re u<4k-1$, the Mellin transform of the function* (5.2) *can be written in three different ways:*

$$\hat{f}_{2k}(u,v;s)=\frac{2\cos(\pi(1/4+u/2))}{2^{1/2+u}\pi^{1/2}}\int_0^1\frac{(1-x)^{s+v-1/2}}{x^{1+v}}\Psi_{2k}(u;x)\,dx, \quad (5.4)$$

$$\hat{f}_{2k}(u,v;s)=\Gamma(1/2+s+v)\frac{1}{2\pi i}\int_{(\Delta)}\frac{\Gamma(2k-1/2+w/2)}{\Gamma(2k+1/2-w/2)} \\ \times\Gamma(\frac{1}{2}-u-w)\sin\left(\pi\frac{1/2+u+w}{2}\right)\frac{\Gamma(1/2-v-w/2)}{\Gamma(1+s-w/2)}2^w dw, \quad (5.5)$$

where $1-4k<\Delta<\min(1-2\Re v, 1/2-\Re u)$, *and*

$$\hat{f}_{2k}(u,v;s)=\frac{2^{1/2-u}\sin(\pi(3/4+u/2))}{\pi^{1/2}}\Gamma(1/2+s+v)\times\frac{\Gamma(2k-1/4-u/2)\Gamma(2k+1/4-u/2)\Gamma(2k-v)}{\Gamma(4k)\Gamma(2k+1/2+s)}\times{}_3F_2\left(2k-\frac{1}{4}-\frac{u}{2},2k+\frac{1}{4}-\frac{u}{2},2k-v;4k,2k+1/2+s;1\right). \quad (5.6)$$

Proof. It follows from the definition of the Mellin transform (5.1) that

$$\hat{f}_{2k}(u,v;s)=\int_0^1\frac{x^{s+v-1/2}}{(1-x)^{1/2+v}}I_{2k}\left(u;\frac{2}{(1-x)^{1/2}}\right)dx. \quad (5.7)$$

Substituting (4.5) into (5.7) we obtain (5.4).

Assuming first that $\Re u>0$, we substitute (4.4) to (5.7). For $\Re u>0$, $\Re w<1-2\Re v$, $\Re s>-1/2-\Re v$, the resulting double integral converges absolutely. Changing the order of integration and using (2.9), namely

$$\int_0^1\frac{x^{s+v-1/2}}{(1-x)^{1/2+w/2+v}}dx=\Gamma(1/2+s+v)\frac{\Gamma(1/2-v-w/2)}{\Gamma(1+s-w/2)},$$

we obtain (5.5). Note that the integral on the right-hand side of (5.5) converges absolutely provided that $\Re s>-1/2-\Re u-\Re v$.

Moving the line of integration in (5.5) to the left and crossing the poles at $w=1-4k-2j$, we finally prove (5.6) by applying (2.21). □

LEMMA 5.2. *For* $-1/4<\Re v<2k$ *we have*

$$\hat{f}_{2k}(0,v;-1/4)=\frac{\Gamma^2(1/4+v)}{\pi^{1/2}}\frac{\Gamma(2k-1/4)\Gamma(2k-v)}{\Gamma(2k+1/4)\Gamma(2k+v)}. \quad (5.8)$$

Proof. Rewriting (5.6) for $u=0$ and using (2.22), we obtain

$$\hat{f}_{2k}(0,v;-1/4)=\frac{\Gamma(1/4+v)}{\pi^{1/2}}\frac{\Gamma(2k-1/4)\Gamma(2k-v)}{\Gamma(4k)}{}_2F_1\left(2k-\frac{1}{4},2k-v;4k;1\right).$$

Then (5.8) follows by applying (2.18). □

LEMMA 5.3. *The following estimates hold*

$$\hat{f}_{2k}(0,0;1/2),\quad \frac{\partial}{\partial s}\hat{f}_{2k}(0,0;s)\Big|_{s=1/2}\ll\frac{k^{\epsilon}}{k^2}. \quad (5.9)$$

Proof. To prove (5.9) we apply (5.4) together with the Liouville-Green approximation of the function $\Psi_{2k}(x)$ obtained in [**4**]. More precisely, using [**4**, (6.58), (6.62), (6.64), (6.68)], we have that for $0<\xi<\infty$ the following asymptotic formula holds

$$\Psi_{2k}\left(\frac{1}{\cosh^2\sqrt{\xi}/2}\right)\left(\xi\sinh^2\sqrt{\xi}\right)^{1/4}=C(\mathbf{k})\left(\sqrt{\xi}K_0\left(\frac{\mathbf{k}\sqrt{\xi}}{2}\right)-\frac{2\xi}{\mathbf{k}}K_1\left(\frac{\mathbf{k}\sqrt{\xi}}{2}\right)B(0;\xi)\right)+O\left(\sqrt{\xi}K_0\left(\frac{\mathbf{k}\sqrt{\xi}}{2}\right)\mathbf{k}^{-3}\min\left(\sqrt{\xi},\frac{1}{\xi}\right)\right), \quad (5.10)$$

where $\mathbf{k}$ is defined by (1.13), $C(\mathbf{k})$ is a function independent of ξ for which we have the asymptotic formula $C(\mathbf{k}) = 2 + O(\mathbf{k}^{-1})$, and

$$B(0;\xi) = \frac{1}{16}\left(\frac{\coth\sqrt{\xi/4}}{\sqrt{\xi}} - \frac{2}{\xi}\right).$$

Note that there is a typo in the formula [**4**, (6.58)] for $B(0;\xi)$. Instead of $\coth\sqrt{\xi}$ there should be $\coth\sqrt{\xi/4}$. It follows from (5.10) and the standard bounds on the K-Bessel functions (2.15), (2.16) that

$$\Psi_{2k}\left(\frac{1}{\cosh^2\sqrt{\xi}/2}\right)\left(\xi\sinh^2\sqrt{\xi}\right)^{1/4} \ll \sqrt{\xi}K_0\left(\frac{\mathbf{k}\sqrt{\xi}}{2}\right). \tag{5.11}$$

Applying (5.4) and making the change of variable $x = \cosh^{-2}\sqrt{\xi}/2$, we obtain

$$\hat{f}_{2k}(0,0;1/2) \ll \int_0^1 x^{-1}\Psi_{2k}(x)\,dx \ll \int_0^\infty \Psi_{2k}\left(\frac{1}{\cosh^2\sqrt{\xi}/2}\right)\frac{\sinh\sqrt{\xi}/2}{\cosh\sqrt{\xi}/2}\frac{d\xi}{\xi^{1/2}}.$$

Then according to (5.11) we have

$$\hat{f}_{2k}(0,0;1/2) \ll \int_0^\infty \left|K_0\left(\frac{\mathbf{k}\sqrt{\xi}}{2}\right)\right|\frac{\tanh\sqrt{\xi}/2}{\sinh^{1/2}\sqrt{\xi}}\frac{d\xi}{\xi^{1/4}}. \tag{5.12}$$

Estimating the K-Bessel function by the means of (2.15), (2.16) completes the proof of the first estimate in (5.9). The derivative of $\hat{f}_{2k}(0,0;s)$ can be estimated similarly since it follows from (5.4) that

$$\frac{\partial}{\partial s}\hat{f}_{2k}(0,0;s)\Big|_{s=1/2} \ll \int_0^1 \frac{\log(1-x)}{x}\Psi_{2k}(x)\,dx.$$

□

Lemma 5.4. *For $r \in \mathbf{R}$ the following estimate holds*

$$\hat{f}_{2k}(0,0;ir) \ll \frac{k^\epsilon}{k(1+|r|)^2}. \tag{5.13}$$

Proof. For $|r| \ll 1$ we estimate (5.4) trivially:

$$\hat{f}_{2k}(0,0;ir) \ll \int_0^1 \frac{(1-x)^{-1/2}}{x}\Psi_{2k}(x)\,dx.$$

Repeating the arguments of Lemma 5.3, we obtain

$$\hat{f}_{2k}(0,0;ir) \ll \int_0^\infty \left|K_0\left(\frac{\mathbf{k}\sqrt{\xi}}{2}\right)\right|\frac{d\xi}{\xi^{1/4}\sinh^{1/2}\sqrt{\xi}}.$$

Using (2.15), (2.16) we prove (5.13).

Now let us consider the case $|r| \gg 1$. Introducing the notation

$$T_{2k}(x) := (1-x)^{1/2}\Psi_{2k}(x), \tag{5.14}$$

we have

$$\hat{f}_{2k}(0,0;ir) = \frac{1}{\pi^{1/2}}\int_0^1 \frac{(1-x)^{ir-1}}{x}T_{2k}(x)dx. \tag{5.15}$$

Integrating (5.15) by parts three times, we obtain

$$\hat{f}_{2k}(0,0;ir) \ll \frac{1}{(1+|r|)^3}\int_0^1 (1-x)^{ir+2}\left(T_{2k}(x)x^{-1}\right)'''\,dx. \tag{5.16}$$

According to [**4**, (6.47)], the function $T_{2k}(x)$ satisfies the differential equation

$$T''_{2k}(x) - ((\mathbf{k}/2)^2\alpha(x) + \beta(x))T_{2k}(x) = 0, \tag{5.17}$$

where $\mathbf{k}$ is given by (1.13) and

$$\alpha(x) := \frac{1}{x^2(1-x)}, \quad \beta(x) := -\frac{1}{4x^2(1-x)^2} + \frac{3}{16x(1-x)}. \tag{5.18}$$

Differentiating (5.17) yields

$$T'''_{2k}(x) = ((\mathbf{k}/2)^2\alpha'(x) + \beta'(x))T_{2k}(x) + ((\mathbf{k}/2)^2\alpha(x) + \beta(x))T'_{2k}(x).$$

Consequently,

$$\begin{aligned}\left(T_{2k}(x)x^{-1}\right)''' &= \left(\frac{(\mathbf{k}/2)^2\alpha(x) + \beta(x)}{x} + \frac{6}{x^3}\right)T'_{2k}(x)\\ &\quad + \left(\frac{(\mathbf{k}/2)^2\alpha'(x) + \beta'(x)}{x} - 3\frac{(\mathbf{k}/2)^2\alpha(x) + \beta(x)}{x^2} - \frac{6}{x^4}\right)T_{2k}(x).\end{aligned} \tag{5.19}$$

Substituting (5.19) into (5.16), we have

$$\begin{aligned}\hat{f}_{2k}(0,0;ir) &\ll \frac{1}{(1+|r|)^3}\\ &\times \int_0^1 (1-x)^2 \left(\frac{(\mathbf{k}/2)^2|\alpha'(x)| + |\beta'(x)|}{x} + \frac{(\mathbf{k}/2)^2|\alpha(x)| + |\beta(x)|}{x^2} + \frac{1}{x^4}\right)|T_{2k}(x)|dx\\ &\quad + \frac{1}{(1+|r|)^3}\left|\int_0^1 (1-x)^{2+ir}\left(\frac{(\mathbf{k}/2)^2\alpha(x) + \beta(x)}{x} + \frac{6}{x^3}\right)T'_{2k}(x)dx\right|.\end{aligned} \tag{5.20}$$

To estimate the second integral in (5.20), we integrate it by parts, getting

$$\begin{aligned}&\frac{1}{(1+|r|)^3}\int_0^1 (1-x)^{2+ir}\left(\frac{(\mathbf{k}/2)^2\alpha(x) + \beta(x)}{x} + \frac{6}{x^3}\right)T'_{2k}(x)dx\\ &\qquad \ll \frac{1}{(1+|r|)^2}\int_0^1 (1-x)\left(\frac{(\mathbf{k}/2)^2|\alpha(x)| + |\beta(x)|}{x} + \frac{1}{x^3}\right)|T_{2k}(x)|dx +\\ &\int_0^1 \frac{(1-x)^2}{(1+|r|)^3}\left(\frac{(\mathbf{k}/2)^2|\alpha'(x)| + |\beta'(x)|}{x} + \frac{(\mathbf{k}/2)^2|\alpha(x)| + |\beta(x)|}{x^2} + \frac{1}{x^4}\right)|T_{2k}(x)|dx.\end{aligned} \tag{5.21}$$

Note that various constants are omitted since we are using the $\ll$ sign. Substituting (5.21) into (5.20), we obtain

$$\begin{aligned}\hat{f}_{2k}(0,0;ir) &\ll \frac{1}{(1+|r|)^3}\\ &\times \int_0^1 (1-x)^2 \left(\frac{(\mathbf{k}/2)^2|\alpha'(x)| + |\beta'(x)|}{x} + \frac{(\mathbf{k}/2)^2|\alpha(x)| + |\beta(x)|}{x^2} + \frac{1}{x^4}\right)|T_{2k}(x)|dx\\ &\quad + \frac{1}{(1+|r|)^2}\int_0^1 (1-x)\left(\frac{(\mathbf{k}/2)^2|\alpha(x)| + |\beta(x)|}{x} + \frac{1}{x^3}\right)|T_{2k}(x)|dx.\end{aligned} \tag{5.22}$$

Consider the second integral in (5.22). Using (5.14), (5.18) and making the change of variable $x = \cosh^{-2}\sqrt{\xi}/2$, we show that

$$\frac{1}{(1+|r|)^2}\int_0^1 (1-x)\left(\frac{(\mathbf{k}/2)^2|\alpha(x)|+|\beta(x)|}{x}+\frac{1}{x^3}\right)|T_{2k}(x)|dx \\ \ll \frac{1}{(1+|r|)^2}\int_0^1 (1-x)^{1/2}\left(\frac{(\mathbf{k}/2)^2}{x^3}+\frac{1}{x^3(1-x)}\right)|\Psi_{2k}(x)|dx \\ \ll \frac{1}{(1+|r|)^2}\int_0^\infty \left|\Psi_{2k}\left(\frac{1}{\cosh^2\sqrt{\xi}/2}\right)\right|\left((\mathbf{k}/2)^2\cosh^2\frac{\sqrt{\xi}}{2}\sinh^2\frac{\sqrt{\xi}}{2}+\cosh^4\frac{\sqrt{\xi}}{2}\right)\frac{d\xi}{\xi^{1/2}}. \quad (5.23)$$

Applying (5.11) and estimating the K-Bessel function using (2.15), (2.16), we obtain

$$\frac{1}{(1+|r|)^2}\int_0^1 (1-x)\left(\frac{(\mathbf{k}/2)^2|\alpha(x)|+|\beta(x)|}{x}+\frac{1}{x^3}\right)|T_{2k}(x)|dx \\ \ll \frac{1}{(1+|r|)^2}\int_0^\infty \frac{|K_0((\mathbf{k}/2)\sqrt{\xi})|}{\sinh^{1/2}\sqrt{\xi}}\left((\mathbf{k}/2)^2\cosh^2\frac{\sqrt{\xi}}{2}\sinh^2\frac{\sqrt{\xi}}{2}+\cosh^4\frac{\sqrt{\xi}}{2}\right)\frac{d\xi}{\xi^{1/4}} \\ \ll \frac{k^\epsilon}{k(1+|r|)^2}. \quad (5.24)$$

Consider the first integral in (5.22). Using (5.18), we have for $0 < x < 1$

$$\alpha'(x) \ll \frac{1}{x^3(1-x)}+\frac{1}{x^2(1-x)^2}, \quad \beta'(x) \ll -\frac{1}{x^3(1-x)^2}+\frac{1}{x^2(1-x)^3}. \quad (5.25)$$

Using (5.18), (5.25), (5.14) and making the change of variable $x = \cosh^{-2}\sqrt{\xi}/2$, we obtain

$$\frac{1}{(1+|r|)^3}\int_0^1 (1-x)^2\left(\frac{(\mathbf{k}/2)^2|\alpha'(x)|+|\beta'(x)|}{x}+\frac{(\mathbf{k}/2)^2|\alpha(x)|+|\beta(x)|}{x^2}+\frac{1}{x^4}\right)|T_{2k}(x)|dx \\ \ll \frac{1}{(1+|r|)^3}\int_0^1 (1-x)^{1/2}\left(\frac{(\mathbf{k}/2)^2(1-x)}{x^4}+\frac{(\mathbf{k}/2)^2}{x^3}+\frac{1}{x^4}+\frac{1}{x^3(1-x)}\right)|\Psi_{2k}(x)|\,dx \\ \ll \frac{1}{(1+|r|)^3}\int_0^\infty \left|\Psi_{2k}\left(\frac{1}{\cosh^2\sqrt{\xi}/2}\right)\right|\left((\mathbf{k}/2)^2\cosh^2\frac{\sqrt{\xi}}{2}\sinh^4\frac{\sqrt{\xi}}{2}+\right. \\ \left.+(\mathbf{k}/2)^2\cosh^2\frac{\sqrt{\xi}}{2}\sinh^2\frac{\sqrt{\xi}}{2}+\cosh^4\frac{\sqrt{\xi}}{2}\sinh^2\frac{\sqrt{\xi}}{2}+\cosh^4\frac{\sqrt{\xi}}{2}\right)\frac{d\xi}{\xi^{1/2}}. \quad (5.26)$$

Applying (5.11) and the standard bounds on the K-Bessel function (2.15), (2.16), we have

$$\frac{1}{(1+|r|)^3}\int_0^1 (1-x)^2\left(\frac{(\mathbf{k}/2)^2|\alpha'(x)|+|\beta'(x)|}{x}+\frac{(\mathbf{k}/2)^2|\alpha(x)|+|\beta(x)|}{x^2}+\frac{1}{x^4}\right)|T_{2k}(x)|dx \\ \ll \frac{k^\epsilon}{k(1+|r|)^2}. \quad (5.27)$$

Substituting (5.27) and (5.24) into (5.22), we complete the proof of (5.13). □

5.2. *Mellin transform of g_{2k}*

LEMMA 5.5. *Assume that $-1/2-\Re v < \Re s < 1/4 - \Re u/2$ and $0 \le \Re u < 4k-1/2$. Then the Mellin transform of the function $g_{2k}(u,v;x)$ can be written as follows:*

$$\hat{g}_{2k}(u,v;s) = \frac{2^{1/2-u}\sin(\pi(1/4+u/2))}{\pi^{1/2}}\int_0^1 \frac{(1-x)^{s+v-1/2}}{x^{s+u+3/4}}\Phi_{2k}(u;x)\,dx, \quad (5.28)$$

$$\hat{g}_{2k}(u,v;s) = \Gamma(1/2+s+v)\frac{1}{2\pi i}\int_{(\Delta)} \frac{\Gamma(2k-1/2+w/2)}{\Gamma(2k+1/2-w/2)}\Gamma(\frac{1}{2}-u-w) \times \sin\left(\pi\frac{1/2+u+w}{2}\right)\frac{\Gamma(w/2-s)}{\Gamma(1/2+v+w/2)}2^w dw, \quad (5.29)$$

where $\max(1-4k, 2\Re s) < \Delta < 1/2 - \Re u$,

$$\hat{g}_{2k}(u,v;s) = \frac{2^{1/2-u}\sin(\pi(1/4+u/2))}{\pi^{1/2}}\Gamma(1/2+s+v) \times \frac{\Gamma(2k-1/4-u/2)\Gamma(3/4-2k-u/2)\Gamma(1/4-u/2-s)}{\Gamma(1/2)\Gamma(3/4+v-u/2)} \times {}_3F_2\left(2k-\frac{1}{4}-\frac{u}{2}, \frac{3}{4}-2k-\frac{u}{2}, \frac{1}{4}-\frac{u}{2}-s; \frac{1}{2}, \frac{3}{4}+v-\frac{u}{2}; 1\right). \quad (5.30)$$

Proof. It follows from (5.1) and (5.3) that

$$\hat{g}_{2k}(u,v;s) = \int_0^\infty \frac{x^{s+v-1/2}}{(1+x)^{1/2+v}} I_{2k}\left(u; \frac{2}{(1+x)^{1/2}}\right) dx. \quad (5.31)$$

Applying (4.6) to evaluate (5.31), we obtain (5.28). Assuming that $\Re u > 0$, we substitute (4.4) to (5.31). For $\Re u > 0$, $\Re w > 2\Re s$, $\Re s > -1/2 - \Re v$, the resulting double integral converges absolutely. Changing the order of integration and applying (2.10), namely

$$\int_0^\infty \frac{x^{s+v-1/2}}{(1+x)^{1/2+w/2+v}}dx = \Gamma(1/2+s+v)\frac{\Gamma(w/2-s)}{\Gamma(1+v+w/2)},$$

we prove (5.29). Note that the integral on the right-hand side of (5.29) converges absolutely provided that $\Re s > -1/2 - \Re u - \Re v$. Moving the line of integration in (5.29) to the right and crossing the poles at $w = 1/2 - u + j$, we obtain (5.30) by applying (2.21).

□

LEMMA 5.6. *For* $\Re v > -1/4$ *the following equality holds*

$$\hat{g}_{2k}(0,v;-1/4) = -\sqrt{2}\sin(\pi v)\frac{\Gamma^2(1/4+v)}{\pi^{1/2}}\frac{\Gamma(2k-1/4)\Gamma(2k-v)}{\Gamma(2k+1/4)\Gamma(2k+v)}. \quad (5.32)$$

In particular,

$$\hat{g}_{2k}(0,0;-1/4) = 0. \quad (5.33)$$

Proof. According to (5.30) and (2.22) we have

$$\hat{g}_{2k}(0,v;-1/4) = \frac{\Gamma(1/4+v)\Gamma(2k-1/4)\Gamma(3/4-2k)}{\Gamma(3/4+v)\pi^{1/2}}{}_2F_1\left(2k-\frac{1}{4}, \frac{3}{4}-2k; \frac{3}{4}+v; 1\right).$$

Applying (2.18), this expression simplifies to

$$\hat{g}_{2k}(0,v;-1/4) = \frac{\Gamma^2(1/4+v)}{\pi^{1/2}}\frac{\Gamma(2k-1/4)\Gamma(3/4-2k)}{\Gamma(1+v-2k)\Gamma(2k+v)}.$$

Finally, using the reflection formula (2.1) we obtain (5.32). □

LEMMA 5.7. *For* $v \to 1/4$ *the following asymptotic formulas hold*

$$\hat{g}_{2k}(0,v;1/2-v) = \frac{2^{3/2}}{2v-1/2}\frac{\Gamma(2k-1/4)}{\Gamma(2k+1/4)} + O(1), \quad (5.34)$$

$$\frac{\partial}{\partial s}\hat{g}_{2k}(0,v;s)\Big|_{s=1/2-v} = \frac{2^{5/2}}{(2v-1/2)^2}\frac{\Gamma(2k-1/4)}{\Gamma(2k+1/4)} + O(1). \tag{5.35}$$

Furthermore,

$$\hat{g}_{2k}(0,0;1/2) = 2^{3/2}\Gamma(-1/2) + O(k^{-1+\epsilon}), \tag{5.36}$$

$$\frac{\partial}{\partial s}\hat{g}_{2k}(0,0;s)\Big|_{s=1/2} = -2^{5/2}\Gamma(-1/2)\psi(-1/2) \\ + 2^{3/2}\Gamma(-1/2)\left(2\psi(2k) + 2\log 2 - \pi\right) + O(k^{-1+\epsilon}). \tag{5.37}$$

Proof. For $u = 0$, $0 \le \Re v \le 1/2$ and $0 < \Re s < 1/4$, we move the line of integration in (5.29) to $-2 + 2\Re s < \Delta < 2\Re s$ crossing the pole at $w = 2s$. Hence

$$\hat{g}_{2k}(0,v;s) = 2^{2s+1}\frac{\Gamma(2k-1/2+s)}{\Gamma(2k+1/2-s)}\Gamma(\frac{1}{2}-2s)\sin\left(\pi\frac{1/2+2s}{2}\right) \\ + \Gamma(1/2+s+v)\frac{1}{2\pi i}\int_{(\Delta)}\frac{\Gamma(2k-1/2+w/2)}{\Gamma(2k+1/2-w/2)}\Gamma(\frac{1}{2}-u-w) \\ \times \sin\left(\pi\frac{1/2+u+w}{2}\right)\frac{\Gamma(w/2-s)}{\Gamma(1/2+v+w/2)}2^w dw, \tag{5.38}$$

where $-2 + 2\Re s < \Delta < \min(2\Re s, 1/2)$. Therefore, (5.38) is now valid for $\Re s < 5/4$. Choosing $\Delta = 0$, we obtain

$$\hat{g}_{2k}(0,v;1/2-v) = 2^{2-2v}\frac{\Gamma(2k-v)}{\Gamma(2k+v)}\Gamma(2v-\frac{1}{2})\sin(3\pi/4-\pi v) + \\ + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Gamma(2k-1/2+ir)}{\Gamma(2k+1/2-ir)}\Gamma(\frac{1}{2}-2ir)\frac{\sin(\pi/4+\pi ir)2^{2ir}dr}{(ir-1/2+v)}. \tag{5.39}$$

Estimating the integral above trivially using Stirling's formula, we conclude the proof of (5.36). Another direct consequence of the representation (5.39) is (5.34). Finally, the formulas (5.35) and (5.37) can also be derived from (5.38) by taking the derivative with respect to s. □

LEMMA 5.8. *For $r \in R$ such that $|r| > 3k$ and any $A > 0$ we have*

$$\hat{g}_{2k}(0,0;ir) \ll \frac{1}{|r|^A}. \tag{5.40}$$

Proof. Using the representation (5.30), we obtain

$$\hat{g}_{2k}(0,0;ir) = \frac{\Gamma(1/2+ir)}{\pi^{1/2}}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\frac{\Gamma(2k-1/4+j)\Gamma(3/4-2k+j)\Gamma(1/4-ir+j)}{\Gamma(1/2+j)\Gamma(3/4+j)}.$$

It follows from (2.2), (2.4) that

$$|\Gamma(1/2+ir)| \ll \exp(-\pi|r|/2) \text{ and } |\Gamma(1/4-ir+j)| \le \Gamma(1/4+j).$$

Consequently,

$$\hat{g}_{2k}(0,0;ir) \ll \exp(-\pi|r|/2)\sum_{j=0}^{\infty}\frac{|\Gamma(3/4-2k+j)|\Gamma(2k-1/4+j)\Gamma(1/4+j)}{\Gamma(1+j)\Gamma(1/2+j)\Gamma(3/4+j)}.$$

According to Euler's reflection formula (2.1) we have $|\Gamma(3/4-2k+j)| = \pi\sqrt{2}|\Gamma(2k+1/4-j)|^{-1}$. Furthermore,

$$\frac{\Gamma(1/4+j)}{\Gamma(1+j)\Gamma(1/2+j)\Gamma(3/4+j)} \ll \frac{1}{\Gamma^2(1+j)}.$$

As a result,

$$\hat{g}_{2k}(0,0;ir) \ll \exp(-\pi|r|/2)\sum_{j=0}^{2k-1}\frac{\Gamma(2k-1/4+j)}{\Gamma(2k+1/4-j)\Gamma^2(1+j)}+ \\ \exp(-\pi|r|/2)\sum_{j=2k}^{\infty}\frac{\Gamma(2k-1/4+j)\Gamma(j+3/4-2k)}{\Gamma^2(1+j)}. \quad (5.41)$$

Using Stirling's formula (2.5) we obtain that for $0<j<2k-1$

$$\frac{\Gamma(2k-1/4+j)}{\Gamma(2k+1/4-j)\Gamma^2(1+j)} \asymp \frac{(2k+j)^{-1/4}}{(2k-j)^{1/4}j^2}\frac{\Gamma(2k+j)}{\Gamma(2k-j)\Gamma^2(j)} \asymp \\ \frac{(2k+j)^{-3/4}}{(2k-j)^{-1/4}j}\frac{(2k+j)^{2k+j}}{(2k-j)^{2k-j}j^{2j}} = \frac{(2k-j)^{1/4}}{(2k+j)^{3/4}j}\exp(s_1(k,j)),$$

where

$$s_1(k,j) := (2k+j)\log(2k+j)-(2k-j)\log(2k-j)-2j\log j.$$

The function $s_1(k,j)$ attains its maximum at the point $j=k\sqrt{2}$, and therefore,

$$\exp(-\pi|r|/2)\sum_{j=0}^{2k-1}\frac{\Gamma(2k-1/4+j)}{\Gamma(2k+1/4-j)\Gamma^2(1+j)} \ll \\ \exp(-\pi|r|/2)\sum_{j=1}^{2k-1}\frac{(2k-j)^{1/4}}{(2k+j)^{3/4}j}\exp(s_1(k,j)) \ll \\ \exp(-\pi|r|/2+s_1(k,k\sqrt{2}))k^{1/2} \ll \frac{1}{|r|^A} \quad (5.42)$$

for $|r|>3k$. In the same way we show that for $j>2k$

$$\frac{\Gamma(2k-1/4+j)\Gamma(j+3/4-2k)}{\Gamma^2(1+j)} \asymp \frac{(j-2k)^{1/4}}{(2k+j)^{3/4}j}\exp(s_2(k,j)),$$

where

$$s_2(k,j) := (2k+j)\log(2k+j)-(j-2k)\log(j-2k)-2j\log j.$$

The function $s_2(k,j)$ is decreasing. For $j=2k$ it follows from the duplication formula (2.3) that

$$\frac{\Gamma(2k-1/4+j)\Gamma(j+3/4-2k)}{\Gamma^2(1+j)} \asymp \frac{\Gamma(4k-1/4)}{\Gamma^2(2k+1)} \ll \frac{2^{4k}}{k^{7/4}}.$$

Finally, we obtain

$$\exp(-\pi|r|/2)\sum_{j=2k}^{\infty}\frac{\Gamma(2k-1/4+j)\Gamma(j+3/4-2k)}{\Gamma^2(1+j)} \ll \exp(-\pi|r|/2)\sum_{j=2k}^{\infty}\frac{2^{4k}}{j^{3/2}} \ll \frac{1}{|r|^A} \quad (5.43)$$

for $|r|>3k$. Substituting (5.42) and (5.43) into (5.41), we prove the lemma. □

To investigate the behaviour of the function $\hat{g}_{2k}(0,0;ir)$ for $|r|\le 3k$ we use the formula (5.28).

LEMMA 5.9. *For* $\mathbf{k} = 4k-1$ *we have*

$$\hat{g}_{2k}(0,0;ir) = -2^{3/2}\pi^{1/2}\int_0^{\pi/2}\frac{(\tan x)^{2ir}}{(\sin(2x))^{1/2}}Y_0(\mathbf{k}x)x^{1/2}dx- \\ - 2^{3/2}\pi^{1/2}\int_0^{\pi/2}\frac{(\tan x)^{2ir}}{(\sin(2x))^{1/2}}J_0(\mathbf{k}x)x^{1/2}dx + O(k^{-3/2}), \quad (5.44)$$

$$\hat{g}_{2k}(0,0;ir) \ll \frac{1}{k^{1/2}}. \quad (5.45)$$

Proof. It follows from (5.28) that

$$\hat{g}_{2k}(0,0;ir) = \frac{1}{\pi^{1/2}}\int_0^{\pi^2/4}\frac{(\tan\sqrt{\xi})^{2ir}}{(\cos\sqrt{\xi})^{1/2}}\Phi_{2k}\left(\cos^2\sqrt{\xi}\right)\frac{d\xi}{\sqrt{\xi}}. \quad (5.46)$$

In order to prove (5.44), we apply the following approximation of the function Φ_{2k} (see [**4**, Theorems 6.5 and 6.10, Corollary 6.9] for details):

$$\Phi_{2k}(\cos^2\sqrt{\xi}) = \frac{-\pi}{\xi^{1/4}(\sin\sqrt{\xi})^{1/2}}\left[\sqrt{\xi}Y_0(\mathbf{k}\sqrt{\xi}) + \sqrt{\xi}J_0(\mathbf{k}\sqrt{\xi}) + O\left(\frac{1}{k}\left|\sqrt{\xi}Y_0(\mathbf{k}\sqrt{\xi})\right|\right)\right]. \quad (5.47)$$

Substituting (5.47) into (5.46) and estimating the error term using standard estimates on the Y-Bessel function (2.11), (2.12), (2.13), we obtain (5.44). Applying (2.11), (2.12), (2.13) to estimate the integrals in (5.44) we prove (5.45). □

The estimate (5.45) is sufficiently good for our purposes only if $r \ll \mathbf{k}^{1/2-\delta}$. For $r \gg \mathbf{k}^{1/2-\delta}$, it is required to analyse (5.44) more carefully. We consider further only the first integral in (5.44), as the second integral can be treated similarly.

The idea is to replace the Y-Bessel function in (5.44) by its asymptotic formula (2.13). To this end, we first make the following partition of unity:

$$\alpha_1(x) + \beta(x) + \alpha_2(x) = 1 \quad \text{for} \quad 0 \le x \le \frac{\pi}{2}, \quad (5.48)$$

where $\alpha_{1,2}(x)$, $\beta(x)$ are smooth infinitely differentiable functions such that for some small $\varepsilon > 0$ (to be chosen later), we have

$$\alpha_1(x) = 1 \text{ for } 0 \le x \le \varepsilon, \quad \alpha_1(x) = 0 \text{ for } x \ge 2\varepsilon, \quad (5.49)$$

$$\alpha_2(x) = 1 \text{ for } \frac{\pi}{2}-\varepsilon \le x \le \frac{\pi}{2}, \quad \alpha_2(x) = 0 \text{ for } 0 \le x \le \frac{\pi}{2}-2\varepsilon, \quad (5.50)$$

$$\beta(x) = 1 \text{ for } 2\varepsilon \le x \le \frac{\pi}{2}-2\varepsilon, \quad \beta(x) = 0 \text{ for } 0 \le x \le \varepsilon, \frac{\pi}{2}-\varepsilon \le x \le \frac{\pi}{2}, \quad (5.51)$$

and $\alpha_{1,2}^{(j)}(x) \ll \varepsilon^{-j}$, $\beta^{(j)}(x) \ll \varepsilon^{-j}$.

LEMMA 5.10. *For* $r \in \mathbf{R}$ *such that* $|r| > 1$ *the following holds*

$$\hat{g}_{2k}(0,0;ir) \ll \left|\int_0^{\pi/2}\frac{\beta(x)(\tan x)^{2ir}}{(\sin(2x))^{1/2}}Y_0(\mathbf{k}x)x^{1/2}dx\right| + \\ \left|\int_0^{\pi/2}\frac{\beta(x)(\tan x)^{2ir}}{(\sin(2x))^{1/2}}J_0(\mathbf{k}x)x^{1/2}dx\right| + \frac{k^{-1+\epsilon}+k^{1/2}\varepsilon^{3/2}}{r}. \quad (5.52)$$

Proof. As in the first step, we use the partition of unity (5.48) to rewrite the integrals in (5.44). Then to prove the lemma, it is required to estimate the contribution of integrals with $\alpha_{1,2}(x)$. All these integrals can be analysed similarly. Therefore, we consider only

$$I_1 := \int_0^{\pi/2} \frac{\alpha_1(x)Y_0(\mathbf{k}x)x^{1/2}}{(\sin(2x))^{1/2}}(\tan x)^{2ir}dx. \tag{5.53}$$

Integrating by parts we obtain

$$I_1 \ll \frac{1}{r}\int_0^{\pi/2} \frac{\partial}{\partial x}\left[\alpha_1(x)Y_0(\mathbf{k}x)x^{1/2}(\sin(2x))^{1/2}\right](\tan x)^{2ir}dx. \tag{5.54}$$

Evaluating the derivative and estimating the integral trivially with the use of (2.11), (2.13), we complete the proof of (5.52). □

For simplicity, let us assume further that $r > 0$. The case $r < 0$ can be treated in the same way.

LEMMA 5.11. *Let δ be some fixed constant such that $0 < \delta < 1/4$. Then for $r > 1$ and $\varepsilon = k^{-1/2-2\delta}$ we have*

$$\hat{g}_{2k}(0,0;ir) \ll \frac{1}{k^{1/2}}\left|\int_0^{\pi/2} \frac{\beta(x)\exp(i\mathbf{k}h(x))}{(\sin(2x))^{1/2}}dx\right| + \frac{k^{-1/4-3\delta}+k^{-1/2}}{r} + \frac{1}{k^{5/4-\delta}}, \tag{5.55}$$

where

$$h(x) = -x + \frac{2r}{\mathbf{k}}\log(\tan x). \tag{5.56}$$

Proof. We substitute the asymptotic formulas for Bessel functions (2.12), (2.13) into (5.52) and estimate the error terms by its absolute value, obtaining

$$\hat{g}_{2k}(0,0;ir) \ll \frac{1}{k^{1/2}}\sum_{\pm}\left|\int_0^{\pi/2} \frac{\beta(x)\exp(ih_{\pm}(x))}{(\sin(2x))^{1/2}}dx\right| + \frac{k^{-1/4-3\delta}}{r} + \frac{1}{k^{5/4-\delta}},$$

where

$$h_{\pm}(x) = \pm\mathbf{k}x + 2r\log(\tan x). \tag{5.57}$$

Consider the integral with h_+. Splitting the interval of integration into the parts where the function $\beta(x)(\sin(2x))^{-1/2}/h_+'(x)$ is monotonic and applying [**31**, Lemma 4.3], we have

$$\frac{1}{k^{1/2}}\left|\int_0^{\pi/2} \frac{\beta(x)\exp(ih_+(x))}{(\sin(2x))^{1/2}}dx\right| \ll \frac{1}{k^{1/2}}\max_{0<x<\pi/2}\frac{\beta(x)(\sin(2x))^{1/2}}{\mathbf{k}\sin(2x)+4r} \ll \frac{1}{rk^{1/2}}.$$

□

The classical approach to estimate the integral on the right-hand side of (5.55) is the saddle point method (also called the method of steepest descent). Another possibility is the stationary phase method, which is in some sense (see discussion in [**7**, pp. 276–279]) an analogue of the saddle point method for Fourier-type integrals. The first step in all these methods is to determine the so-called saddle points of the function $h(x)$ defined as zeros of $h'(x) = 0$. Using (5.56) we find that

$$h'(x) = -1 + \frac{4r}{\mathbf{k}}\frac{1}{\sin(2x)}. \tag{5.58}$$

It is convenient to introduce two new parameters ϑ and μ such that:

$$\sin(2\vartheta) = \frac{4r}{\mathbf{k}}, \quad 0 < \vartheta < \frac{\pi}{4} \quad \text{if} \quad 4r \le \mathbf{k}, \tag{5.59}$$

$$\cosh(2\mu) = \frac{4r}{\mathbf{k}}, \quad \mu > 0 \quad \text{if} \quad 4r > \mathbf{k}. \tag{5.60}$$

Then the saddle points of the function $h(x)$ are

$$x_1 = \vartheta, \quad x_2 = \frac{\pi}{2} - \vartheta \quad \text{if} \quad 4r \le \mathbf{k}, \tag{5.61}$$

$$x_3 = \frac{\pi}{4} - i\mu, \quad x_4 = \frac{\pi}{4} + i\mu \quad \text{if} \quad 4r > \mathbf{k}. \tag{5.62}$$

We consider only the case $4r \le \mathbf{k}$ since the second case can be analysed similarly. Note that the condition $r \gg \mathbf{k}^{1/2-\delta}$ implies that

$$\vartheta \gg \mathbf{k}^{-1/2-\delta} > k^{-1/2-2\delta} = \varepsilon. \tag{5.63}$$

Thus both saddle points belong to the interval of integration.

An important observation is that as $4r \to \mathbf{k}$ *the saddle points coalesce*. It is known that in this case the integral has a different behaviour in three different ranges:

- r is small,
- r is near $\mathbf{k}/4$,
- r is large.

The case of coalescing saddle points is usually described in books, see [**7**, Section 9.2] and [**32**, Section 7.4]. It is well known that the standard saddle point method does not work in this situation and a more refined analysis is required. Therefore, we mainly follow [**7**, Section 9.2]. This approach was originally developed by Chester, Friedman and Ursell [**12**], with some additional ideas due to Bleistein [**6**].

The main idea of the method is to change the variable of integration such that the integral can be written in terms of the Airy function, which has for real x the following representation:

$$Ai(ax^{2/3}) = \frac{x^{1/3}}{2\pi} \int_{-\infty}^{\infty} \exp\left(ix\left(\frac{y^3}{3} + ay\right)\right) dy. \tag{5.64}$$

For simplicity, let us denote

$$g(x) := \frac{\beta(x)}{(\sin(2x))^{1/2}}. \tag{5.65}$$

Our goal is to estimate the integral (see (5.55))

$$I = \int_0^{\pi/2} g(x) \exp(i\mathbf{k}h(x))dx, \quad h(x) = -x + \frac{\sin 2\vartheta}{2} \log(\tan x). \tag{5.66}$$

To this end, following [**7**, (9.2.6)] we define a new variable t such that:

$$\frac{t^3}{3} - \gamma^2 t + \rho = h(x), \tag{5.67}$$

where the constants γ and ρ are chosen such that the point $x = x_1$ corresponds to $t = -\gamma$ and the point $x = x_2$ corresponds to $t = \gamma$. These conditions yield

$$h(x_1) = \frac{2\gamma^3}{3} + \rho, \quad h(x_2) = -\frac{2\gamma^3}{3} + \rho,$$

and therefore,

$$\frac{4\gamma^3}{3} = h(x_1) - h(x_2), \quad 2\rho = h(x_1) + h(x_2). \tag{5.68}$$

Evaluating $h(x_{1,2})$ we find that $\rho=-\pi/4$ and

$$\frac{4\gamma^3}{3}=\frac{\pi}{2}-2\vartheta+\sin(2\vartheta)\log(\tan\vartheta). \tag{5.69}$$

Note that for $0<\vartheta<\pi/4$ the right-hand side of (5.69) is positive, and that for $\vartheta=\pi/4$ we obtain $\gamma=0$. It follows from (5.69) that there are three choices for γ. In order to determine γ uniquely we apply [**7**, (9.2.17)], getting

$$\gamma=\left(\frac{3}{4}\left(\frac{\pi}{2}-2\vartheta+\sin(2\vartheta)\log(\tan\vartheta)\right)\right)^{1/3}. \tag{5.70}$$

Changing the variable x in the integral (5.66) by t defined by (5.67), we obtain an analogue of [**7**, (9.2.18), (9.2.19)], namely

$$I=\exp(i\mathbf{k}\rho)\int_{-\infty}^{\infty}G_0(t,\vartheta)\exp\left(i\mathbf{k}(t^3/3-\gamma^2t)\right)dt, \tag{5.71}$$

where

$$G_0(t,\vartheta)=g(x(t))\frac{dx}{dt}. \tag{5.72}$$

Following [**7**, (9.2.21),(9.2.22)]), we define

$$a_0:=\frac{G_0(\gamma,\vartheta)+G_0(-\gamma,\vartheta)}{2},\quad a_1:=\frac{G_0(\gamma,\vartheta)-G_0(-\gamma,\vartheta)}{2\gamma}. \tag{5.73}$$

Further, using [**7**, (9.2.20)] we define $H_0(t,\vartheta)$ via the following equation

$$G_0(t,\vartheta)=a_0+a_1t+(t^2-\gamma^2)H_0(t,\vartheta). \tag{5.74}$$

Note that a_0 and a_1 are chosen such that the function $H_0(t,\vartheta)$ has a finite derivative at points $t=\pm\gamma$.

To evaluate $a_{0,1}$, as well as to analyse the properties of $H_0(t,\vartheta)$, we need some preliminary results.

Lemma 5.12. *For $\vartheta<\pi/4$ we have*

$$\left.\frac{dx}{dt}\right|_{t=\pm\gamma}=\sqrt{\gamma\tan(2\vartheta)}, \tag{5.75}$$

$$\left.\frac{d^2x}{dt^2}\right|_{t=\pm\gamma}=\mp\frac{1}{3}\left(4\gamma+2\gamma\tan^2(2\vartheta)-\sqrt{\frac{\tan(2\vartheta)}{\gamma}}\right). \tag{5.76}$$

For $\vartheta=\pi/4$ we have

$$\left.\frac{dx}{dt}\right|_{t=0}=2^{-1/3},\quad \left.\frac{d^2x}{dt^2}\right|_{t=0}=0,\quad \left.\frac{d^3x}{dt^3}\right|_{t=0}=-1. \tag{5.77}$$

Proof. First, consider the case $\vartheta<\pi/4$, $t=-\gamma$. We can write

$$x-\vartheta=\sum_{n=0}^{\infty}b_n(t+\gamma)^n,\quad h'(x)=\sum_{n=0}^{\infty}c_n(t+\gamma)^n. \tag{5.78}$$

Let us compute b_i, c_i for $i=0,1,2$. Note that $b_0=0$ since the point $x=\vartheta$ corresponds to $t=-\gamma$. We have

$$h'(x)=-1+\frac{\sin(2\vartheta)}{\sin(2x)}=-\frac{2(x-\vartheta)}{\tan(2\vartheta)}+(x-\vartheta)^2\left(\frac{4}{\tan^2(2\vartheta)}+2\right)+O((x-\vartheta)^3). \tag{5.79}$$

Substituting the expansion for $(x-\vartheta)$ from (5.78) into (5.79), we show that

$$c_0 = 0, \quad c_1 = -\frac{2b_1}{\tan(2\vartheta)}, \quad c_2 = -\frac{2b_2}{\tan(2\vartheta)} + b_1^2\left(\frac{4}{\tan^2(2\vartheta)} + 2\right). \tag{5.80}$$

It follows from (5.67) that

$$h'(x)\frac{dx}{dt} = t^2 - \gamma^2 = -2\gamma(t+\gamma) + (t+\gamma)^2. \tag{5.81}$$

Substituting (5.78) into (5.81) yields

$$c_1 b_1 = -2\gamma, \quad c_2 b_1 + 2c_1 b_2 = 1. \tag{5.82}$$

Using (5.82) and (5.80), we obtain

$$b_1 = \sqrt{\gamma\tan(2\vartheta)}, \quad b_2 = \frac{1}{6}\left(4\gamma + 2\gamma\tan^2(2\vartheta) - \sqrt{\frac{\tan(2\vartheta)}{\gamma}}\right).$$

This proves (5.75) and (5.76) for $t=-\gamma$. The case $t=\gamma$ is similar.

Second, consider $\vartheta = \pi/4$. In that case $\gamma = 0$. We can write

$$x - \frac{\pi}{4} = \sum_{n=0}^{\infty} d_n t^n, \quad h'(x) = \sum_{n=0}^{\infty} e_n t^n. \tag{5.83}$$

We proceed to compute d_i, e_i for $i = 0,1,2,3$. Note that $d_0 = 0$. Furthermore, we have

$$h'(x) = -1 + \frac{1}{\sin(2x)} = 2\left(x - \frac{\pi}{4}\right)^2 + \frac{10}{3}\left(x - \frac{\pi}{4}\right)^4 + O\left(\left(x - \frac{\pi}{4}\right)^6\right). \tag{5.84}$$

Substituting the expansion for $(x-\pi/4)$ from (5.83) into (5.84), we show that

$$e_0 = e_1 = 0, \quad e_2 = 2d_1^2, \quad e_3 = 4d_1d_2, \quad e_4 = 2d_2^2 + 4d_1d_3 + \frac{10}{3}d_1^4. \tag{5.85}$$

Substituting (5.83) into (5.81) gives

$$e_2 d_1 = 1, \quad 2e_2d_2 + e_3d_1 = 0, \quad 3e_2d_3 + 2e_3d_2 + e_4d_1 = 0. \tag{5.86}$$

Using (5.86) and (5.85) we finally show that

$$d_1 = 2^{-1/3}, \quad d_2 = 0, \quad d_3 = -\frac{1}{6}.$$

This completes the proof of (5.77). ☐

LEMMA 5.13. *For $\vartheta < \pi/4$ we have*

$$a_0 = \sqrt{\frac{\gamma}{\cos(2\vartheta)}}, \quad a_1 = 0, \tag{5.87}$$

and for $\vartheta = \pi/4$ we have

$$a_0 = 2^{-1/3}, \quad a_1 = 0. \tag{5.88}$$

Proof. Consider the case $\vartheta < \pi/4$. It follows from (5.73), (5.72), (5.75) and (5.65) that

$$a_0 = \frac{\beta(\vartheta) + \beta(\pi/2-\vartheta)}{2}\sqrt{\frac{\gamma}{\cos(2\vartheta)}}, \quad a_1 = \frac{\beta(\pi/2-\vartheta) - \beta(\vartheta)}{2\gamma}\sqrt{\frac{\gamma}{\cos(2\vartheta)}}.$$

As a consequence of (5.63) we obtain (5.87).

Consider the case $\vartheta = \pi/4$. It follows from (5.73), (5.72), (5.77) and (5.65) that $a_0 = 2^{-1/3}$ and

$$a_1 = \frac{d}{dt}G_0\left(t, \frac{\pi}{4}\right)\Big|_{t=0} = g'(\pi/4)\left(\frac{dx}{dt}\Big|_{t=0}\right)^2 + g(\pi/4)\frac{d^2x}{dt^2}\Big|_{t=0} = \frac{d^2x}{dt^2}\Big|_{t=0} = 0.$$

This proves (5.88). □

Substituting (5.74) into (5.71) and using Lemma 5.13, we obtain the following representation for our integral

$$I = \exp(i\mathbf{k}\rho)a_0\int_{-\infty}^{\infty}\exp\left(i\mathbf{k}(t^3/3-\gamma^2 t)\right)dt + \frac{\exp(i\mathbf{k}\rho)}{i\mathbf{k}}\int_{-\infty}^{\infty}H_0(t,\vartheta)d\exp\left(i\mathbf{k}(t^3/3-\gamma^2 t)\right).$$

Using (5.64) and integrating by parts yields

$$I = \exp(i\mathbf{k}\rho)\frac{2a_0\pi}{\mathbf{k}^{1/3}}Ai(-\gamma^2\mathbf{k}^{2/3}) - \frac{\exp(i\mathbf{k}\rho)}{i\mathbf{k}}\int_{-\infty}^{\infty}\frac{d}{dt}\left(H_0(t,\vartheta)\right)\exp\left(i\mathbf{k}(t^3/3-\gamma^2 t)\right)dt.$$

Since the function $H_0(t,\vartheta)$ has a finite number of intervals of monotonicity, we can estimate the integral in the formula above by its absolute value, getting

$$I = \exp(i\mathbf{k}\rho)\frac{2a_0\pi}{\mathbf{k}^{1/3}}Ai(-\gamma^2\mathbf{k}^{2/3}) + O\left(\frac{1}{\mathbf{k}}\max_t|H_0(t,\vartheta)|\right). \tag{5.89}$$

The final step is to estimate the function $H_0(t,\vartheta)$.

Lemma 5.14. *For $\vartheta \le \pi/4$ satisfying* (5.63) *we have*

$$\max_t|H_0(t,\vartheta)| \ll \frac{1}{\sqrt{\vartheta}}. \tag{5.90}$$

Proof. Since the function $H_0(t,\vartheta)$ is continuous and piecewise smooth, it attains the extreme values at critical points. Consequently, it is required to analyse the behaviour of this function at the points $t = \pm\gamma$ and as $t \to \pm\infty$. Note that for $\vartheta = \pi/4$, the critical points $t = \gamma$ and $t = -\gamma$ coincide, and therefore, this case should be treated separately. Furthermore, since we aim to obtain an estimate uniform in ϑ, the case when $\vartheta \to 0$ should also be studied separately. To sum up, we consider three different cases: $\vartheta \to \frac{\pi}{4}$, $\vartheta \to 0$ and the remaining case when ϑ is some fixed number.

First, let us assume that ϑ is some fixed number. Then γ (see (5.69)) is also some fixed number. We start by estimating $H_0(t,\vartheta)$ near the points $t = \pm\gamma$. According to [**7**, (9.2.24)] and since $a_1 = 0$ by Lemma 5.13, we have

$$\lim_{t\to\pm\gamma}H_0(t,\vartheta) = \pm\frac{1}{2\gamma}\frac{d}{dt}G_0(t,\vartheta)\Big|_{t=\pm\gamma}. \tag{5.91}$$

Using (5.65), (5.72) and Lemma 5.12, we prove that $H_0(t,\vartheta)$ is bounded near the points $t = \pm\gamma$. Other critical points of $H_0(t,\vartheta)$ are $t \to \pm\infty$. In this case we use (5.65), (5.74) and (5.81), getting

$$H_0(t,\vartheta) = \frac{g(x(t))}{h'(x(t))} - \frac{a_0}{t^2-\gamma^2} = \frac{\beta(x(t))\sqrt{\sin(2x(t))}}{\sin(2\vartheta)-\sin(2x(t))} - \frac{a_0}{t^2-\gamma^2}. \tag{5.92}$$

Consequently, for all t that do not belong to a neighbourhood of $\pm\gamma$, the function $H_0(t,\vartheta)$ is trivially bounded by a constant.

Second, consider the case $\vartheta \to \frac{\pi}{4}$. It is enough to prove (5.90) for $\vartheta = \pi/4$. In this case we have $\gamma = 0$ and (see (5.91))

$$\lim_{t\to 0} H_0(t,\pi/4) = \frac{1}{2}\frac{d^2}{dt^2}G_0(t,\pi/4)\Big|_{t=0}. \tag{5.93}$$

It follows from (5.72) that

$$\frac{d^2}{dt^2}G_0(t,\pi/4) = g''(x(t))\left(\frac{dx}{dt}\right)^3 + 3g'(x(t))\frac{dx}{dt}\frac{d^2x}{dt^2} + g(x(t))\frac{d^3x}{dt^3}. \tag{5.94}$$

Using the fact that all derivatives of $g(x)$ at $x = \pi/4$ are finite and applying Lemma 5.12, we conclude that the limit in (5.93) is finite. For t outside of a neighbourhood of 0, the function $H_0(t,\pi/4)$ is trivially bounded by a constant using (5.92).

Third, consider the case $\vartheta \to 0$. Let us estimate the right-hand side of (5.91). We remark that γ is a constant for small ϑ . Therefore, it is only required to estimate the derivative of $G_0(t,\vartheta)$. It follows from (5.72) that

$$\frac{d}{dt}G_0(t,\vartheta) = g'(x(t))\left(\frac{dx}{dt}\right)^2 + g(x(t))\frac{d^2x}{dt^2}.$$

Using (5.65) and Lemma 5.12 we obtain the estimate

$$\frac{d}{dt}G_0(t,\vartheta)\Big|_{t=\pm\gamma} \ll \frac{\tan(2\vartheta)}{\sin^{3/2}(2\vartheta)} + \frac{1}{\sin^{1/2}(2\vartheta)} \ll \frac{1}{\vartheta^{1/2}}, \tag{5.95}$$

which completes the proof of (5.90). □

5.3. *Proof of Lemma 1.3*

Substituting (5.90) into (5.89) we obtain for $\mathbf{k}^{1/2-\delta} < r \le \mathbf{k}/4$ (see (5.63)) that

$$I = \exp(i\mathbf{k}\rho)\frac{2a_0\pi}{\mathbf{k}^{1/3}}Ai(-\gamma^2\mathbf{k}^{2/3}) + O\left(\frac{1}{(r\mathbf{k})^{1/2}}\right). \tag{5.96}$$

According to [**25**, (9.7.9), (9.4.1)] we have $Ai(-x) \ll \min(1, x^{-1/4})$, and therefore

$$I \ll \frac{1}{\mathbf{k}^{1/3}}\min\left(1, \frac{1}{\gamma^{1/2}\mathbf{k}^{1/6}}\right) + \frac{1}{(r\mathbf{k})^{1/2}}. \tag{5.97}$$

It follows from (5.59) and (5.69) that

$$\gamma = 2^{1/3}\left(\frac{\pi}{4} - \vartheta\right) + O\left((\pi/4-\vartheta)^5\right) = 2^{-2/3}\arccos\frac{4r}{\mathbf{k}} + O\left(\arccos^5\frac{4r}{\mathbf{k}}\right) =$$
$$= 2^{-1/6}\left(1 - \frac{4r}{\mathbf{k}}\right)^{1/2} + O\left((1-4r/\mathbf{k})^{3/2}\right).$$

Consequently,

$$I \ll \frac{1}{\mathbf{k}^{1/3}}\min\left(1, \frac{1}{(1-4r/\mathbf{k})^{1/4}\mathbf{k}^{1/6}}\right) + \frac{1}{(r\mathbf{k})^{1/2}}. \tag{5.98}$$

Using (5.98) to estimate the integral in (5.55), we finally prove Lemma 1.3.

6. *Explicit formula for the mixed moment*

This section is devoted to proving an explicit formula for the mixed moment

$$\sum_{\mathfrak{f}\in H_{4k}}\omega(\mathfrak{f})L(\mathfrak{f},1/2)L(\text{sym}^2\mathfrak{f},1/2).$$

To this end, we introduce two complex variables u, v with sufficiently large real parts and consider the shifted moment

$$\mathcal{M}(u,v)=\sum_{\mathfrak{f}\in H_{4k}}\omega(\mathfrak{f})L(\mathfrak{f},1/2+v)L(\mathrm{sym}^2\,\mathfrak{f},1/2+u). \tag{6.1}$$

This enables us to use the technique of analytic continuation.

Let us assume for simplicity that $0<\Re u<1$ and $\Re v>3/4+\Re u/2$. Using (1.4) and Lemma 4.1 we obtain

$$\mathcal{M}(u,v)=\mathcal{M}^D(u,v)+\mathcal{M}^{ND}(u,v)+\mathcal{ET}_1(u,v)+\mathcal{ET}_2(u,v), \tag{6.2}$$

where

$$\mathcal{M}^D(u,v)=\sum_{l=1}^{\infty}\frac{M^D(u,l^2)}{l^{1+2v}},\quad \mathcal{M}^{ND}(u,v)=\sum_{l=1}^{\infty}\frac{M^{ND}(u,l)}{l^{1/2+v}}, \tag{6.3}$$

$$\mathcal{ET}_1(u,v)=\sum_{l=1}^{\infty}\frac{ET_1(u,l)}{l^{1/2+v}},\quad \mathcal{ET}_2(u,v)=\sum_{l=1}^{\infty}\frac{ET_2(u,l)}{l^{1/2+v}}. \tag{6.4}$$

As a consequence of (4.8) and (6.3) we obtain

$$\begin{aligned}\mathcal{M}^D(u,v)&=\zeta(3/2+2v+u)\zeta(1+2u)\\&\quad+\zeta(3/2+2v-u)\zeta(1-2u)\sqrt{2}(2\pi)^{3u}\cos\pi(\frac{1}{4}+\frac{u}{2})\\&\quad\times\frac{\Gamma(2k-1/4-u/2)\Gamma(2k+1/4-u/2)\Gamma(1-2u)}{\Gamma(2k+1/4+u/2)\Gamma(2k-1/4+u/2)\Gamma(1-u)},\end{aligned} \tag{6.5}$$

$$\begin{aligned}\mathcal{M}^D(0,v)=\frac{\zeta(3/2+2v)}{2}\Biggl(&-3\log 2\pi+\frac{\pi}{2}+3\gamma\\&+2\frac{\zeta'(3/2+2v)}{\zeta(3/2+2v)}+\psi(2k-1/4)+\psi(2k+1/4)\Biggr).\end{aligned} \tag{6.6}$$

Similarly, it follows from (3.7), (4.9) and (6.3) that

$$\mathcal{M}^{ND}(u,v)=\frac{(2\pi)^{1/2+u}}{2}\frac{\Gamma(2k-1/4-u/2)}{\Gamma(2k+1/4+u/2)}\sum_{l=1}^{\infty}\frac{\mathscr{L}_{-4l}(1/2+u)}{l^{3/4+v-u/2}}, \tag{6.7}$$

$$\mathcal{M}^{ND}(0,v)=\frac{2^{3/2}\pi}{\Gamma(3/4)}\frac{\Gamma(2k-1/4)}{\Gamma(2k+1/4)}L_g^-(1/4+v). \tag{6.8}$$

We remark that a part of the main term is also contained in $\mathcal{ET}_1(u,v)$ and $\mathcal{ET}_2(u,v)$, which we analyse in detail in the next two subsections.

6.1. *Analysis of $\mathcal{ET}_2(u,v)$*

LEMMA 6.1. *For $0<\Re u<1$, $\Re v>3/4+\Re u/2+\max(\theta(1-2\Re u),0)$ we have*

$$\begin{aligned}\mathcal{ET}_2(u,v)&=(2\pi)^{1/2+u}2^{1+2v}\frac{1}{2\pi i}\\&\times\int_{(\sigma)}\Biggl(\sum_{\substack{n\ge1\\n\equiv0(2)}}\sum_{\substack{m\ge1\\m\equiv0(4)}}+\sum_{\substack{n\ge1\\n\equiv1(2)}}\sum_{\substack{m\ge1\\m\equiv1(4)}}\Biggr)\frac{\mathscr{L}_m(1/2+u)}{m^{1/2+v+s}n^{1/2-u-2s}}\hat{f}_{2k}(u,v;s)ds,\end{aligned} \tag{6.9}$$

where $1/2-\Re v+\max(\theta(1-2\Re u),0)<\sigma<-1/4-\Re u/2$.

Proof. Substituting (4.11) into (6.4) we obtain

$$\mathcal{E}\mathcal{T}_2(u,v)=\sum_{l=1}^{\infty}\frac{(2\pi)^{1/2+u}}{l^{1/2+v}}\sum_{n>2\sqrt{l}}\frac{\mathscr{L}_{n^2-4l}(1/2+u)}{n^{1/2-u}}I_{2k}\left(u;\frac{n}{l^{1/2}}\right). \tag{6.10}$$

It follows from (4.4) that $I_{2k}\left(u;x\right)\sim x^{1-4k}$ as $x\to\infty$. Thus using (3.3) we have

$$\sum_{l=1}^{\infty}\frac{1}{l^{1/2+v}}\sum_{n>2\sqrt{l}}\frac{\mathscr{L}_{n^2-4l}(1/2+u)}{n^{1/2-u}}I_{2k}\left(u;\frac{n}{l^{1/2}}\right)\ll\sum_{l=1}^{\infty}\frac{l^{1/4+\Re u/2+\max(\theta(1-2\Re u),0)}}{l^{1/2+\Re v}}.$$

And we see that the double series on the right-hand side of (6.10) converges absolutely provided that $\Re v>3/4+\Re u/2+\max(\theta(1-2\Re u),0)$. Changing the order of summation in (6.10) and making the change of variables $m=n^2-4l$, we obtain

$$\mathcal{E}\mathcal{T}_2(u,v)=(2\pi)^{1/2+u}\sum_{n\geq1}\sum_{\substack{0<m<n^2\\ m\equiv n^2(4)}}\frac{\mathscr{L}_m(1/2+u)2^{1+2v}}{(n^2-m)^{1/2+v}n^{1/2-u}}I_{2k}\left(u;\frac{2n}{(n^2-m)^{1/2}}\right).$$

Rewriting this using (5.2) yields

$$\mathcal{E}\mathcal{T}_2(u,v)=(2\pi)^{1/2+u}2^{1+2v}\left(\sum_{\substack{n\geq1\\ n\equiv0(2)}}\sum_{\substack{m\geq1\\ m\equiv0(4)}}+\sum_{\substack{n\geq1\\ n\equiv1(2)}}\sum_{\substack{m\geq1\\ m\equiv1(4)}}\right)\times\frac{\mathscr{L}_m(1/2+u)}{m^{1/2+v}n^{1/2-u}}f_{2k}\left(u,v;\frac{m}{n^2}\right). \tag{6.11}$$

Applying the Mellin inversion formula for $f_{2k}\left(u,v;m/n^2\right)$ completes the proof. □

LEMMA 6.2. *For $\Re v>3/4$ we have*

$$\mathcal{E}\mathcal{T}_2(0,v)=\frac{1}{2\pi i}\int_{(\sigma)}F_{2k}(v,s)ds, \tag{6.12}$$

where $1/2-\Re v<\sigma<-1/4$ and

$$F_{2k}(v,s)=(2\pi)^{1/2}2^{1+2v}\Bigg(\left(1-2^{2s-1/2}\right)\frac{2\sqrt{\pi}}{\Gamma(1/4)}L_f^+(s+v)-\\ -\left(1-2^{2s+1/2}\right)\frac{4\pi^{1/2}}{2^{1+2v+2s}\Gamma(1/4)}L_g^+(s+v)\Bigg)\zeta(1/2-2s)\hat{f}_{2k}(0,v;s). \tag{6.13}$$

Proof. We first let $u=0$ in (6.9). Then for $\Re v>3/4+\theta$ the following formula holds

$$\mathcal{E}\mathcal{T}_2(0,v)=(2\pi)^{1/2}2^{1+2v}\frac{1}{2\pi i}\int_{(\sigma)}\left(\sum_{\substack{n\geq1\\ n\equiv0(2)}}\sum_{\substack{m\geq1\\ m\equiv0(4)}}+\sum_{\substack{n\geq1\\ n\equiv1(2)}}\sum_{\substack{m\geq1\\ m\equiv1(4)}}\right)\times\frac{\mathscr{L}_m(1/2)}{m^{1/2+v+s}n^{1/2-2s}}\hat{f}_{2k}(0,v;s)ds, \tag{6.14}$$

where $1/2-\Re v+\theta<\sigma<-1/4$. It follows from (3.7) that

$$\sum_{\substack{m\geq1\\ m\equiv0(4)}}\frac{\mathscr{L}_m(1/2)}{m^{1/2+v+s}}=\frac{4\pi^{1/2}}{4^{1/2+v+s}\Gamma(1/4)}L_g^+(s+v). \tag{6.15}$$

Since $\mathscr{L}_n(s)$ vanishes if $n \equiv 2, 3 \pmod 4$, we obtain using (3.6) and (6.15) that

$$\sum_{\substack{m\geq 1\\ m\equiv 1(4)}} \frac{\mathscr{L}_m(1/2)}{m^{1/2+v+s}} = \sum_{m\geq 1} \frac{\mathscr{L}_m(1/2)}{m^{1/2+v+s}} - \sum_{\substack{m\geq 1\\ m\equiv 0(4)}} \frac{\mathscr{L}_m(1/2)}{m^{1/2+v+s}}$$

$$= \frac{2\sqrt{\pi}}{\Gamma(1/4)} L_f^+(s+v) - \frac{4\pi^{1/2}}{4^{1/2+v+s}\Gamma(1/4)} L_g^+(s+v). \quad (6.16)$$

Furthermore,

$$\sum_{\substack{n\geq 1\\ n\equiv 0(2)}} \frac{1}{n^{1/2-2s}} = \frac{\zeta(1/2-2s)}{2^{1/2-2s}}, \quad (6.17)$$

$$\sum_{\substack{n\geq 1\\ n\equiv 1(2)}} \frac{1}{n^{1/2-2s}} = \left(1 - \frac{1}{2^{1/2-2s}}\right)\zeta(1/2-2s). \quad (6.18)$$

Substituting (6.15), (6.16), (6.17) and (6.18) into (6.14) we prove (6.12). □

LEMMA 6.3. *For $1/2 < \Re v < 3/4$ we have*

$$\mathcal{ET}_2(0,v) = \frac{1}{2\pi i}\int_{(0)} F_{2k}(v,s)ds +$$
$$\sqrt{2\pi}2^{2v}\frac{\Gamma^2(1/4+v)}{\Gamma(1/4)}\frac{\Gamma(2k-1/4)\Gamma(2k-v)}{\Gamma(2k+1/4)\Gamma(2k+v)} L_f^+(v-1/4), \quad (6.19)$$

where $F_{2k}(v,s)$ is defined by (6.13).

Proof. The function $F_{2k}(v,s)$ has a simple pole at $s=-1/4$ coming from $\zeta(1/2-2s)$. Moving the line of integration in (6.12) to $\sigma = 0$ we cross this pole, obtaining

$$\mathcal{ET}_2(0,v) = -\operatorname{res}_{s=-1/4} F_{2k}(v,s) + \frac{1}{2\pi i}\int_{(0)} F_{2k}(v,s)ds. \quad (6.20)$$

The right-hand side of (6.20) shows that $\mathcal{ET}_2(0,v)$ can be continued to the region $\Re v > 1/2$. To prove (6.19) it remains to evaluate the residue. Using (6.13) we have

$$\operatorname{res}_{s=-1/4} F_{2k}(v,s) = \frac{2^{1/2+2v}\pi}{\Gamma(1/4)} \hat{f}_{2k}(0,v;-1/4) L_f^+(v-1/4).$$

Applying (5.8) we prove the lemma. □

LEMMA 6.4. *For $0 \leq \Re v < 1/2$ we have*

$$\mathcal{ET}_2(0,v) = \operatorname{res}_{s=1/2-v} F_{2k}(v,s) + \frac{1}{2\pi i}\int_{(0)} F_{2k}(v,s)ds$$
$$+ \sqrt{2\pi}2^{2v}\frac{\Gamma^2(1/4+v)}{\Gamma(1/4)}\frac{\Gamma(2k-1/4)\Gamma(2k-v)}{\Gamma(2k+1/4)\Gamma(2k+v)} L_f^+(v-1/4), \quad (6.21)$$

where $F_{2k}(v,s)$ is defined by (6.13).

Proof. The function $F_{2k}(v,s)$ has a double pole at $s = 1/2 - v$ coming from $L_{f,g}^+(s+v)$. To prove the analytic continuation of $\mathcal{ET}_2(0,v)$ to the region $\Re v < 1/2$, we first change the contour of integration to

$$\gamma_1 = (-i\infty, -i\Im v - i\epsilon) \cup C_\epsilon \cup (-i\Im v + i\epsilon, i\infty), \tag{6.22}$$

where C_ϵ is a semicircle in the right half-plane of radius ϵ. Consequently, we obtain

$$\mathcal{ET}_2(0, v) = -\operatorname{res}_{s=-1/4} F_{2k}(v, s) + \frac{1}{2\pi i} \int_{\gamma_1} F_{2k}(v, s) ds. \tag{6.23}$$

The right-hand side of (6.23) provides the analytic continuation of $\mathcal{ET}_2(0, v)$ to the region $1/2 - \epsilon < \Re v < 1/2$. Next, we decompose the contour γ_1 as the sum of the line $\Re s = 0$ and the contour $\gamma_2 = C_\epsilon \cup (-i\Im v + i\epsilon, -i\Im v - i\epsilon)$. Since

$$\frac{1}{2\pi i} \int_{\gamma_2} F_{2k}(v, s) ds = \operatorname{res}_{s=1/2-v} F_{2k}(v, s), \tag{6.24}$$

we show that for $1/2 - \epsilon < \Re v < 1/2$

$$\mathcal{ET}_2(0, v) = \operatorname{res}_{s=1/2-v} F_{2k}(v, s) - \operatorname{res}_{s=-1/4} F_{2k}(v, s) + \frac{1}{2\pi i} \int_{(0)} F_{2k}(v, s) ds. \tag{6.25}$$

This concludes the proof. □

Lemma 6.5. *The following formula holds*

$$\mathcal{ET}_2(0, 0) = \mathcal{M}^{ND}(0, 0) + \operatorname{res}_{s=1/2-v} F_{2k}(v, s) + \frac{1}{2\pi i} \int_{(0)} F_{2k}(v, s) ds. \tag{6.26}$$

Proof. Comparing (6.8) and (6.21), we find that in order to prove (6.26), it is required to show that

$$\sqrt{2\pi}\Gamma(1/4) L_f^+(-1/4) = \frac{2^{3/2}\pi}{\Gamma(3/4)} L_g^-(1/4).$$

Since $\Gamma(1/4)\Gamma(3/4) = \pi\sqrt{2}$ we need to verify that

$$\sqrt{\pi} L_f^+(-1/4) = 2^{1/2} L_g^-(1/4),$$

and this follows from (3.10). □

Lemma 6.6. *For any $\epsilon > 0$ we have*

$$\mathcal{ET}_2(0, 0) = \mathcal{M}^{ND}(0, 0) + O\left(\frac{k^\epsilon}{k}\right), \tag{6.27}$$

$$\mathcal{ET}_2(0, 0) \ll k^{-1/2}. \tag{6.28}$$

Proof. This follows immediately from (6.26), (6.13), Lemma 5.3, Theorem 3.3 and the estimate (5.13). □

6.2. *Analysis of* $\mathcal{ET}_1(u,v)$

LEMMA 6.7. *Assume that* $0<\Re u<1$ *and* $\Re v>3/4+\Re u/2+\max(\theta(1-2\Re u),0)$. *Then the following formula holds*

$$\mathcal{ET}_1(u,v)=(2\pi)^{1/2+u}2^{1+2v}\times$$
$$\frac{1}{2\pi i}\int_{(\sigma)}\left(\sum_{\substack{n\geq1\\ n\equiv0(2)}}\sum_{\substack{m\geq1\\ m\equiv0(4)}}+\sum_{\substack{n\geq1\\ n\equiv1(2)}}\sum_{\substack{m\geq1\\ m\equiv1(4)}}\right)\frac{\mathscr{L}_{-m}(1/2+u)}{m^{1/2+v+s}n^{1/2-u-2s}}\hat{g}_{2k}(u,v;s)ds, \quad (6.29)$$

where $1/2-\Re v+\max(\theta(1-2\Re u),0)<\sigma<-1/4-\Re u/2$.

Proof. Substituting (4.10) into (6.4) we show that

$$\mathcal{ET}_1(u,v)=\sum_{l=1}^{\infty}\frac{(2\pi)^{1/2+u}}{l^{1/2+v}}\sum_{0<n<2\sqrt{l}}\frac{\mathscr{L}_{n^2-4l}(1/2+u)}{n^{1/2-u}}I_{2k}\left(u;\frac{n}{l^{1/2}}\right). \quad (6.30)$$

It follows from (4.4) that $I_{2k}(u;x)\sim x^{1/2-\Re u}$ as $x\to0$. Using this fact and applying (3.3), we obtain

$$\sum_{l=1}^{\infty}\frac{1}{l^{1/2+v}}\sum_{0<n<2\sqrt{l}}\frac{\mathscr{L}_{n^2-4l}(1/2+u)}{n^{1/2-u}}I_{2k}\left(u;\frac{n}{l^{1/2}}\right)\ll\sum_{l=1}^{\infty}\frac{1}{l^{1/2+\Re v}}\frac{l^{1/2+\max(\theta(1-2\Re u),0)+\epsilon}}{l^{1/4-\Re u/2}}.$$

Therefore, the double series on the right-hand side of (6.30) converges absolutely provided that $\Re v>3/4+\Re u/2+\max(\theta(1-2\Re u),0)$. Changing the order of summation in (6.30) and making the change of variables $-m=n^2-4l$, we have

$$\mathcal{ET}_1(u,v)=(2\pi)^{1/2+u}\sum_{n\geq1}\sum_{\substack{m\geq1\\ m+n^2\equiv0(4)}}\frac{\mathscr{L}_{-m}(1/2+u)2^{1+2v}}{(n^2+m)^{1/2+v}n^{1/2-u}}I_{2k}\left(u;\frac{2n}{(n^2+m)^{1/2}}\right).$$

Applying (5.3), we obtain

$$\mathcal{ET}_1(u,v)=(2\pi)^{1/2+u}2^{1+2v}\left(\sum_{\substack{n\geq1\\ n\equiv0(2)}}\sum_{\substack{m\geq1\\ m\equiv0(4)}}+\sum_{\substack{n\geq1\\ n\equiv1(2)}}\sum_{\substack{m\geq1\\ m\equiv-1(4)}}\right)$$
$$\times\frac{\mathscr{L}_{-m}(1/2+u)}{m^{1/2+v}n^{1/2-u}}g_{2k}\left(u,v;\frac{m}{n^2}\right). \quad (6.31)$$

Using the Mellin inversion formula for $g_{2k}\left(u,v;m/n^2\right)$ we prove the lemma. □

LEMMA 6.8. *For* $\Re v>3/4$ *the following representation takes place*

$$\mathcal{ET}_1(0,v)=\frac{1}{2\pi i}\int_{(\sigma)}G_{2k}(v,s)ds, \quad (6.32)$$

where $1/2-\Re v<\sigma<-1/4$ *and*

$$G_{2k}(v,s)=(2\pi)^{1/2}2^{1+2v}\Biggl(\left(1-2^{2s-1/2}\right)\frac{2\sqrt{\pi}}{\Gamma(3/4)}L_f^-(s+v)-$$
$$-\left(1-2^{2s+1/2}\right)\frac{4\pi^{1/2}}{2^{1+2v+2s}\Gamma(3/4)}L_g^-(s+v)\Biggr)\zeta(1/2-2s)\hat{g}_{2k}(0,v;s). \quad (6.33)$$

Proof. Letting $u=0$ in (6.29), we obtain for $\Re v > 3/4+\theta$

$$\mathcal{E}\mathcal{T}_1(0,v) = (2\pi)^{1/2}2^{1+2v}\times$$
$$\frac{1}{2\pi i}\int_{(\sigma)}\left(\sum_{\substack{n\geq 1\\ n\equiv 0(2)}}\sum_{\substack{m\geq 1\\ m\equiv 0(4)}}+\sum_{\substack{n\geq 1\\ n\equiv 1(2)}}\sum_{\substack{m\geq 1\\ m\equiv -1(4)}}\right)\frac{\mathscr{L}_{-m}(1/2)}{m^{1/2+v+s}n^{1/2-2s}}\hat{g}_{2k}(0,v;s)ds, \quad (6.34)$$

where $1/2-\Re v+\theta<\sigma<-1/4$. It follows from (3.7) that

$$\sum_{\substack{m\geq 1\\ m\equiv 0(4)}}\frac{\mathscr{L}_{-m}(1/2)}{m^{1/2+v+s}} = \frac{4\pi^{1/2}}{4^{1/2+v+s}\Gamma(3/4)}L_g^-(s+v). \quad (6.35)$$

Recall that $\mathscr{L}_n(s)$ vanishes for $n\equiv 2,3 \pmod 4$. Consequently, using (3.6) and (6.35) we show that

$$\sum_{\substack{m\geq 1\\ m\equiv -1(4)}}\frac{\mathscr{L}_{-m}(1/2)}{m^{1/2+v+s}} = \sum_{m\geq 1}\frac{\mathscr{L}_{-m}(1/2)}{m^{1/2+v+s}} - \sum_{\substack{m\geq 1\\ m\equiv 0(4)}}\frac{\mathscr{L}_{-m}(1/2)}{m^{1/2+v+s}}$$
$$= \frac{2\sqrt{\pi}}{\Gamma(3/4)}L_f^-(s+v) - \frac{4\pi^{1/2}}{4^{1/2+v+s}\Gamma(3/4)}L_g^-(s+v). \quad (6.36)$$

Applying (6.17), (6.18), (6.35) and (6.36) to evaluate (6.34), we prove (6.32). □

LEMMA 6.9. *For $1/2<\Re v<3/4$ the following formula holds*

$$\mathcal{E}\mathcal{T}_1(0,v) = \frac{1}{2\pi i}\int_{(0)}G_{2k}(v,s)ds$$
$$-\sqrt{\pi}2^{1+2v}\sin(\pi v)\frac{\Gamma^2(1/4+v)}{\Gamma(3/4)}\frac{\Gamma(2k-1/4)\Gamma(2k-v)}{\Gamma(2k+1/4)\Gamma(2k+v)}L_f^-(v-1/4), \quad (6.37)$$

where $G_{2k}(v,s)$ is defined by (6.33).

Proof. The function $G_{2k}(v,s)$ has a simple pole at $s=-1/4$ from $\zeta(1/2-2s)$. Moving the line of integration in (6.32) to $\sigma=0$ we cross this pole, getting

$$\mathcal{E}\mathcal{T}_1(0,v) = -\operatorname{res}_{s=-1/4}G_{2k}(v,s)+\frac{1}{2\pi i}\int_{(0)}G_{2k}(v,s)ds. \quad (6.38)$$

The right-hand side of (6.38) proves the analytic continuation of $\mathcal{E}\mathcal{T}_1(0,v)$ to the region $\Re v>1/2$. Then to complete the proof of (6.37), it remains to evaluate the residue. Using (6.33) we have

$$\operatorname{res}_{s=-1/4}G_{2k}(v,s) = \frac{2^{1/2+2v}\pi}{\Gamma(1/4)}\hat{g}_{2k}(0,v;-1/4)L_f^-(v-1/4).$$

The lemma follows by applying (5.32). □

LEMMA 6.10. *For $0\leq\Re v<1/2$ we have*

$$\mathcal{E}\mathcal{T}_1(0,v) = \operatorname{res}_{s=1/2-v}G_{2k}(v,s)+\frac{1}{2\pi i}\int_{(0)}G_{2k}(v,s)ds$$
$$-\sqrt{\pi}2^{1+2v}\sin(\pi v)\frac{\Gamma^2(1/4+v)}{\Gamma(1/4)}\frac{\Gamma(2k-1/4)\Gamma(2k-v)}{\Gamma(2k+1/4)\Gamma(2k+v)}L_f^+(v-1/4), \quad (6.39)$$

where $G_{2k}(v,s)$ is defined by (6.33) *and*

$$\mathrm{res}_{s=1/2-v}\, G_{2k}(v,s) = M_1(v) + M_2(v), \tag{6.40}$$

$$M_1(v) = \frac{2^{5/2+2v}\pi}{\Gamma(3/4)}\Bigg[c_f^-(-1)\zeta(2v-1/2)\hat{g}_{2k}(0,v;1/2-v)\left(1-2^{1/2-2v}\right) + c_f^-(-2)\Bigg(\zeta(2v-1/2)\left(1-2^{1/2-2v}\right)\frac{\partial}{\partial s}\hat{g}_{2k}(0,v;s)\Big|_{s=1/2-v} - 2\zeta'(2v-1/2)\hat{g}_{2k}(0,v;1/2-v)\left(1-2^{1/2-2v}\right) - \zeta(2v-1/2)\hat{g}_{2k}(0,v;1/2-v)2^{3/2-2v}\log 2\Bigg)\Bigg], \tag{6.41}$$

$$M_2(v) = \frac{8\pi}{\Gamma(3/4)}\Bigg[c_g^-(-1)\zeta(2v-1/2)\hat{g}_{2k}(0,v;1/2-v)\left(1-2^{2v-3/2}\right) + c_g^-(-2)\Bigg(\zeta(2v-1/2)\left(1-2^{2v-3/2}\right)\frac{\partial}{\partial s}\hat{g}_{2k}(0,v;s)\Big|_{s=1/2-v} - 2\zeta'(2v-1/2)\hat{g}_{2k}(0,v;1/2-v)\left(1-2^{2v-3/2}\right) + \zeta(2v-1/2)\hat{g}_{2k}(0,v;1/2-v)2^{2v-1/2}\log 2\Bigg)\Bigg]. \tag{6.42}$$

Proof. The function $G_{2k}(v,s)$ has a double pole at $s=1/2-v$ from $L^-_{f,g}(s+v)$. To prove the analytic continuation of $\mathcal{ET}_1(0,v)$ to the region $\Re v<1/2$ we apply [**13**, Corollary 2.4.2, p. 55]. Consequently, for $\Re v<1/2$ we have

$$\mathcal{ET}_1(0,v) = \mathrm{res}_{s=1/2-v}\, G_{2k}(v,s) - \mathrm{res}_{s=-1/4}\, G_{2k}(v,s) + \frac{1}{2\pi i}\int_{(0)} G_{2k}(v,s)ds. \tag{6.43}$$

Then it follows from (6.33) that

$$\mathrm{res}_{s=1/2-v}\, G_{2k}(v,s) = \frac{2^{5/2+2v}\pi}{\Gamma(3/4)}\mathrm{res}_{s=1/2-v}\left(\zeta(1/2-2s)\hat{g}_{2k}(0,v;s)\left(1-2^{2s-1/2}\right)L_f^-(s+v)\right) + \frac{8\pi}{\Gamma(3/4)}\mathrm{res}_{s=1/2-v}\left(\zeta(1/2-2s)\hat{g}_{2k}(0,v;s)\left(1-2^{-2s-1/2}\right)L_g^-(s+v)\right). \tag{6.44}$$

Let $H(s)$ be an arbitrary function that is holomorphic at $s=1/2-v$. Using (3.11), we obtain the Laurent series

$$H(s)L^-_{f,g}(s+v) = \frac{c^-_{f,g}(-2)H(1/2-v)}{(s+v-1/2)^2} + \frac{c^-_{f,g}(-1)H(1/2-v)+c^-_{f,g}(-2)H'(1/2-v)}{s+v-1/2} + O(1). \tag{6.45}$$

Applying (6.45) to evaluate (6.44), we prove (6.40). ∎

6.3. *Analytic continuation*

Finally, we obtain the following decomposition for the mixed moment.

THEOREM 6.11. *For $\Re v \geq 0$ we have*

$$\mathcal{M}(0,v) = \mathcal{M}^D(0,v) + \mathcal{M}^{ND}(0,v) + \mathcal{ET}_1(0,v) + \mathcal{ET}_2(0,v), \tag{6.46}$$

where $\mathcal{M}^D(0,v)$ is defined by (6.6) and $\mathcal{M}^{ND}(0,v)$ by (6.8). Furthermore, the terms $\mathcal{ET}_1(0,v)$ and $\mathcal{ET}_2(0,v)$ are given by (6.32) and (6.12) for $\Re v > 3/4$, by (6.37) and (6.19) for $1/2 < \Re v \leq 3/4$ and by (6.39) and (6.21) for $0 \leq \Re v < 1/2$.

Proof. In order to prove the theorem, it remains to show that the right-hand side of (6.46) is holomorphic for $\Re v \geq 0$. More precisely, we need to consider points $v = 3/4$ and $v = 1/4$. The only summands on the right-hand side of (6.46) that are not holomorphic at $v = 3/4$ come from (6.37) and (6.19), namely:

$$\begin{aligned}
&\sqrt{2\pi}2^{2v}\frac{\Gamma^2(1/4+v)}{\Gamma(1/4)}\frac{\Gamma(2k-1/4)\Gamma(2k-v)}{\Gamma(2k+1/4)\Gamma(2k+v)}L_f^+(v-1/4)-\\
&\quad-\sqrt{\pi}2^{1+2v}\sin(\pi v)\frac{\Gamma^2(1/4+v)}{\Gamma(3/4)}\frac{\Gamma(2k-1/4)\Gamma(2k-v)}{\Gamma(2k+1/4)\Gamma(2k+v)}L_f^-(v-1/4)=\\
&\quad=\sqrt{\pi}2^{2v}\Gamma^2(1/4+v)\frac{\Gamma(2k-1/4)\Gamma(2k-v)}{\Gamma(2k+1/4)\Gamma(2k+v)}\times\\
&\qquad\left(\frac{\sqrt{2}}{\Gamma(1/4)}L_f^+(v-1/4)-\frac{2\sin(\pi v)}{\Gamma(3/4)}L_f^-(v-1/4)\right).
\end{aligned}\tag{6.47}$$

Therefore, to prove that the right-hand side of (6.46) is holomorphic at $v = 3/4$, it is sufficient to show that

$$\frac{\sqrt{2}}{\Gamma(1/4)}L_f^+(v-1/4)-\frac{2\sin(\pi v)}{\Gamma(3/4)}L_f^-(v-1/4) \tag{6.48}$$

is holomorphic at $v = 3/4$. Using Theorem 3.2 and the asymptotic formula

$$\sin(\pi v) = \frac{1}{\sqrt{2}} - \frac{\pi(v-3/4)}{\sqrt{2}} + O((v-3/4)^2),$$

we obtain

$$\begin{aligned}
&\frac{\sqrt{2}}{\Gamma(1/4)}L_f^+(v-1/4)-\frac{2\sin(\pi v)}{\Gamma(3/4)}L_f^-(v-1/4)=\\
&\quad=\frac{1}{(v-3/4)^2}\left(\frac{c_f^+(-2)\sqrt{2}}{\Gamma(1/4)}-\frac{c_f^-(-2)\sqrt{2}}{\Gamma(3/4)}\right)+\\
&\quad+\frac{1}{v-3/4}\left(\frac{c_f^+(-1)\sqrt{2}}{\Gamma(1/4)}-\frac{c_f^-(-1)\sqrt{2}}{\Gamma(3/4)}+\frac{c_f^-(-2)\pi\sqrt{2}}{\Gamma(3/4)}\right)+O(1)=O(1).
\end{aligned}\tag{6.49}$$

Thus the right-hand side of (6.46) is holomorphic at $v = 3/4$.

The only summands on the right-hand side of (6.46) that are not holomorphic at $v = 1/4$ come from (6.8) and (6.40), namely

$$\mathcal{M}^{ND}(0,v) + \operatorname{res}_{s=1/2-v} G_{2k}(v,s).$$

Let us consider the $M_1(v)$ part of $\operatorname{res}_{s=1/2-v} G_{2k}(v,s)$ given by (6.41). Using (5.34) and (5.35), we obtain a Laurent series for $M_1(v)$ at the point $v=1/4$:

$$M_1(v)=\frac{2^{5/2}\pi}{\Gamma(3/4)}c_f^-(-2)\Biggl(\zeta(2v-1/2)\left(2^{2v}-2^{1/2}\right)\frac{\partial}{\partial s}\hat{g}_{2k}(0,v;s)\Bigr|_{s=1/2-v} - \frac{2^3\zeta(0)\log 2}{2v-1/2}\frac{\Gamma(2k-1/4)}{\Gamma(2k+1/4)}\Biggr)+O(1). \quad (6.50)$$

Using (5.35) and the fact that

$$\zeta(2v-1/2)\left(2^{2v}-2^{1/2}\right)=(2v-1/2)\zeta(0)2^{1/2}\log 2+O((2v-1/2)^2),$$

we show that $M_1(v)=O(1)$ as $v\to 1/4$.

Let us consider the $M_2(v)$ part of $\operatorname{res}_{s=1/2-v} G_{2k}(v,s)$ given by (6.42). Applying (5.34), we obtain

$$M_2(v)=\frac{8\pi}{\Gamma(3/4)}c_g^-(-2)\zeta(2v-1/2)\left(1-2^{2v-3/2}\right)\frac{\partial}{\partial s}\hat{g}_{2k}(0,v;s)\Bigr|_{s=1/2-v} + \frac{1}{2v-1/2}\frac{8\pi}{\Gamma(3/4)}\frac{\Gamma(2k-1/4)}{\Gamma(2k+1/4)}\Biggl(c_g^-(-1)\zeta(0)2^{1/2} + c_g^-(-2)\left(-2^{3/2}\zeta'(0)+\zeta(0)2^{3/2}\log 2\right)\Biggr). \quad (6.51)$$

In order to evaluate a Laurent series for the remaining term we use (5.35) together with the following formula

$$\zeta(2v-1/2)\left(1-2^{2v-3/2}\right)=\frac{\zeta(0)}{2}+\frac{\zeta'(0)-\zeta(0)\log 2}{2}(2v-1/2)+O((2v-1/2)^2).$$

Consequently,

$$M_2(v)=\frac{8\pi}{\Gamma(3/4)}\frac{\Gamma(2k-1/4)}{\Gamma(2k+1/4)}\frac{c_g^-(-2)2^{3/2}\zeta(0)}{(2v-1/2)^2}+ + \frac{8\pi}{\Gamma(3/4)}\frac{\Gamma(2k-1/4)}{\Gamma(2k+1/4)}\frac{c_g^-(-1)2^{1/2}\zeta(0)}{2v-1/2}+O(1). \quad (6.52)$$

It follows from (6.8) and (3.11) that

$$\mathcal{M}^{ND}(0,v)=\frac{2^{7/2}\pi}{\Gamma(3/4)}\frac{\Gamma(2k-1/4)}{\Gamma(2k+1/4)}\frac{c_g^-(-2)}{(2v-1/2)^2}+ + \frac{2^{5/2}\pi}{\Gamma(3/4)}\frac{\Gamma(2k-1/4)}{\Gamma(2k+1/4)}\frac{c_g^-(-1)}{2v-1/2}+O(1). \quad (6.53)$$

Since $M_1(v)=O(1)$, applying (6.52) and (6.53) we conclude that the sum

$$\mathcal{M}^{ND}(0,v)+\operatorname{res}_{s=1/2-v} G_{2k}(v,s),$$

and consequently the right-hand side of (6.46), are holomorphic at $v=1/4$. □

Lemma 6.12. *The following asymptotic formula holds*

$$\operatorname{res}_{s=1/2} G_{2k}(0,s)=\mathcal{M}^D(0,0)+O(k^{-2}). \quad (6.54)$$

Proof. We compare the leading terms. It follows from (2.6) that

$$2\psi(2k) = \psi(2k-1/4) + \psi(2k+1/4) + O(k^{-2}). \tag{6.55}$$

Therefore, (6.6) implies that the leading term of $\mathcal{M}^D(0,0)$ is equal to

$$\zeta(3/2)\psi(2k). \tag{6.56}$$

Let us compute the leading term of $\operatorname{res}_{s=1/2} G_{2k}(0,s)$. It follows from (5.34) and (5.35) that the leading term is

$$2\psi(2k)\frac{16\pi}{\Gamma(3/4)}\Bigg(c_f^-(-2)(1-\sqrt{2})\zeta(-1/2)\Gamma(-1/2)+ \\ + c_g^-(-2)(\sqrt{2}-1/2)\zeta(-1/2)\Gamma(-1/2)\Bigg). \tag{6.57}$$

Using the functional equation for the Riemann zeta function, we have

$$\zeta(-1/2)\Gamma(-1/2) = \frac{\zeta(3/2)}{2\sqrt{\pi}}.$$

Consequently, the leading term of $\operatorname{res}_{s=1/2} G_{2k}(0,s)$ is as follows:

$$\psi(2k)\zeta(3/2)\frac{16\sqrt{\pi}}{\Gamma(3/4)}\left(c_f^-(-2)(1-\sqrt{2}) + c_g^-(-2)(\sqrt{2}-1/2)\right). \tag{6.58}$$

Finally, applying (3.14), we find that (6.58) is equal to (6.56). □

6.4. *Proof of main theorems*

Proof of Theorem 1.1. Asymptotic formula (1.5) is a direct consequence of (6.46) for $v=0$. More precisely, we replace $\mathcal{ET}_2(0,v)$ by (6.27), $\mathcal{ET}_1(0,v)$ by (6.39), and apply (6.54). □

Proof of Theorem 1.2. Consider (1.5) and note that all summands except the integral can be trivially bounded by $\log k$. The final step is to show that

$$\frac{1}{2\pi i}\int_{(0)} G_{2k}(0,s)ds \ll \log k^3. \tag{6.59}$$

In view of (6.33), it is required to estimate

$$I := \frac{1}{2\pi i}\int_{(0)} L^-_{f,g}(s)\zeta(1/2-2s)\hat{g}_{2k}(0,0;s)ds. \tag{6.60}$$

Let $r = \Im s$. By Lemma 5.8 the contribution of $|r| > 3k$ is negligible.

Consider $|r| \le 3k$. Let δ be some fixed constant such that $0<\delta<1/4$. For $|r| < \mathbf{k}^{1/2-\delta}$ we use the trivial bound (5.45) showing that

$$I = \frac{1}{2\pi}\int_{\mathbf{k}^{1/2-\delta}<|r|<3k} L^-_{f,g}(ir)\zeta(1/2-2ir)\hat{g}_{2k}(0,0;ir)dr + o(1). \tag{6.61}$$

Next, we apply Lemma 1.3, Theorem 3.3, the following estimate for the second moment of the Riemann zeta function

$$\int_T^{T+H} |\zeta(1/2+ir)|^2 dr \ll H\log T \tag{6.62}$$

over short intervals $H \gg T^{1/3}$, and the Cauchy-Schwarz inequality. Consequently, we prove that the contribution of the second summand on the right-hand side of (1.16) to (6.61) is negligibly small and

$$I \ll \int_{\mathbf{k}^{1/2-\delta}}^{3k} |L_{f,g}^{-}(ir)\zeta(1/2-2ir)| \frac{1}{\mathbf{k}^{5/6}} \min\left(1, \frac{\mathbf{k}^{1/12}}{|\mathbf{k}-4r|^{1/4}}\right) dr + 1. \tag{6.63}$$

Opening the minimum we obtain three integrals:

$$\int_{r_1}^{r_2} \frac{|L_{f,g}^{-}(ir)\zeta(1/2-2ir)|}{\mathbf{k}^{3/4}|\mathbf{k}-4r|^{1/4}} dr + \int_{r_2}^{r_3} |L_{f,g}^{-}(ir)\zeta(1/2-2ir)| \frac{dr}{\mathbf{k}^{5/6}} + \int_{r_3}^{r_4} \frac{|L_{f,g}^{-}(ir)\zeta(1/2-2ir)|}{\mathbf{k}^{3/4}|\mathbf{k}-4r|^{1/4}} dr, \tag{6.64}$$

where $r_1 = \mathbf{k}^{1/2-\delta}$, $r_2 = \mathbf{k}/4 - \mathbf{k}^{1/3}$, $r_3 = \mathbf{k}/4 + \mathbf{k}^{1/3}, r_4 = 3k$. To estimate the second integral we apply the Cauchy-Schwarz inequality, Theorem 3.3 and (6.62), getting

$$\int_{r_2}^{r_3} |L_{f,g}^{-}(ir)\zeta(1/2-2ir)| \frac{dr}{\mathbf{k}^{5/6}} \ll \frac{\log^{5/2} k}{k^{1/6}}. \tag{6.65}$$

Let us now consider the first integral in (6.64). Applying the Cauchy-Schwarz inequality and Theorem 3.3, we obtain

$$\int_{r_1}^{r_2} \frac{|L_{f,g}^{-}(ir)\zeta(1/2-2ir)|}{\mathbf{k}^{3/4}|\mathbf{k}-4r|^{1/4}} dr \ll \frac{\log^2 k}{\mathbf{k}^{1/4}} \left(\int_{r_1}^{r_2} \frac{|\zeta(1/2-2ir)|^2}{|\mathbf{k}-4r|^{1/2}} dr\right)^{1/2}. \tag{6.66}$$

Making the change of variable $\mathbf{k} - 4r = x$, and then performing a dyadic partition of unity, we prove using (6.62) that

$$\int_{r_1}^{r_2} \frac{|\zeta(1/2-2ir)|^2}{|\mathbf{k}-4r|^{1/2}} dr \ll k^{1/2} \log^2 k, \tag{6.67}$$

where one of the logarithms comes from the partition of unity. Substituting (6.67) into (6.66) we obtain

$$\int_{r_1}^{r_2} \frac{|L_{f,g}^{-}(ir)\zeta(1/2-2ir)|}{\mathbf{k}^{3/4}|\mathbf{k}-4r|^{1/4}} dr \ll \log^3 k. \tag{6.68}$$

Finally, the third integral in (6.64) can be estimated in the same way as the first one. This completes the proof. □

Acknowledgements. We would like to thank the referee for careful reading of our manuscript and many valuable comments and suggestions.

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