A MEAN VALUE RESULT FOR A PRODUCT OF $GL(2)$ AND $GL(3)$ L-FUNCTIONS

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Abstract. In this paper various analytic techniques are combined in order to study the average of a product of a Hecke L-function and a symmetric square L-function at the central point in the weight aspect. The evaluation of the second main term relies on the theory of Maaß forms of half-integral weight and the Rankin-Selberg method. The error terms are bounded using the Liouville-Green approximation.

Contents

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1. Introduction

The asymptotic evaluation of moments of L-functions is not only an important tool to solve problems in number theory and arithmetic geometry, but is also a subject of independent interest. Various conjectures predict the shape of the main terms for all moments of L-functions within a family of certain type of symmetry (see e.g. [5, 7]). Even though exact results for moments are till now known only for small values this is already sufficient for many applications. See, for example, [12]. In this regard, the quality of the asymptotic error estimates plays a crucial role.

In general, there are three main techniques used for evaluating the moments: we can apply the approximate functional equation, the Rankin-Selberg method or the method of analytic continuation, each of which has certain advantages. The method of the approximate functional equation which allows the bypassing of convergence problems is the most common approach. The method of analytic continuation and the Rankin-Selberg method however reveal the structure of the mean values and yield exact formulas for the moments. Consequently, the theory of special functions can be used in order to prove sharp error estimates.

The problem we consider here requires the combination of all three methods. We study the asymptotic behaviour of the first moment of the product of the Hecke L-function $L(f,1/2)$ and the symmetric square L-function $L(\mathrm{sym}^2 f,1/2)$ in the weight aspect on average, where f runs over the space H_{4k} of primitive forms of weight $4k$, $k\in\mathbf{N}$.

The most interesting phenomenon of this moment is the presence of the non-diagonal main term, which is smaller in size and depends on the special value of the double Dirichlet series

$$L_g^-(s)=\frac{\Gamma(3/4)}{4\sqrt{\pi}}\sum_{n<0}\frac{\mathscr{L}_{4n}(1/2)}{|n|^{s+1/2}}, \tag{1.1}$$

$$\mathscr{L}_n(z)=\frac{\zeta(2z)}{\zeta(z)}\sum_{q=1}^{\infty}\frac{1}{q^z}\left(\sum_{1\le t\le 2q;\, t^2\equiv n \pmod{4q}}1\right). \tag{1.2}$$

Isolating this non-diagonal main term and applying the Liouville-Green method for error estimates, we prove an asymptotic formula with an arbitrary power saving error term.

Before stating the main theorem we introduce some notation. Let $\langle f,f\rangle_1$ be the Petersson inner product on the space of level 1 holomorphic modular forms and the standard harmonic weight we denote by

$$\omega(f) := \frac{\Gamma(2k-1)}{(4\pi)^{2k-1}\langle f, f\rangle_1}. \tag{1.3}$$

Then we have the following asymptotic formula.

Theorem 1.1. *Let $h \in C_0^\infty(\mathbf{R}^+)$ be a non-negative, compactly supported function on the interval $[\theta_1, \theta_2]$, $\theta_2 > \theta_1 > 0$, and*

$$\|h^{(n)}\|_1 \ll 1 \text{ for all } n \geq 0. \tag{1.4}$$

Then, for any fixed $A > 0$, we have

(1.5)

$$\sum_k h\left(\frac{4k}{K}\right) \sum_{f\in H_{4k}} \omega(f)L(f,1/2)L(\mathrm{sym}^2 f,1/2) =$$

$$\frac{HK}{4}\zeta(3/2)\left(2\log K - 3\log 2\pi - 2\log 2 + \frac{\pi}{2} + 3\gamma + \frac{2\zeta'(3/2)}{\zeta(3/2)} + \frac{2H_1}{H}\right)$$

$$+ P_h(1/K) + L_g^-(1/4)\sqrt{K}Q_h(1/K) + O\left(K^{-A}\right),$$

where $\zeta(s)$ is the Riemann zeta function and γ denotes the Euler constant. Furthermore,

$$H := \int_0^\infty h(y)dy, \quad H_1 := \int_0^\infty h(y)\log y dy$$

and $P_h(x)$, $Q_h(x)$ are polynomials in x of degree $A-1$ and A, respectively, with coefficients depending on the function h.

This is an example of mixed moments that were previously investigated, for example, in [15, 18, 19, 23].

The proof of Theorem 1.1 consists of several steps. We start by combining the exact formula for the twisted first moment of symmetric square L-functions and the approximate functional equation for the Hecke L-function. Consequently, the average

$$\sum_{f\in H_{4k}} \omega(f)L(f,1/2)L(\mathrm{sym}^2 f,1/2)$$

splits into a diagonal main term, a non-diagonal main term plus a smaller contribution expressed in terms of the special functions

$$\Psi_k(x) := x^k \frac{\Gamma(k-1/4)\Gamma(k+1/4)}{\Gamma(2k)} {}_2F_1\left(k-\frac{1}{4}, k+\frac{1}{4}; 2k; x\right), \tag{1.6}$$

$$\Phi_k(x) := \frac{\Gamma(k-1/4)\Gamma(3/4-k)}{\Gamma(1/2)} {}_2F_1\left(k-\frac{1}{4}, \frac{3}{4}-k; 1/2; x\right), \tag{1.7}$$

where $\Gamma(s)$ is the Gamma function and ${}_2F_1(a,b;c;x)$ is the Gauss hypergeometric function.

The diagonal main term is evaluated in Corollary 4.3. The Rankin-Selberg method serves to isolate the non-diagonal main term, as shown in Corollary 5.2. Lemma 6.1 provides an estimate for the error term of the first type. Using the Liouville-Green method, we approximate the error term of the second type by the series in Y_0 and J_0 Bessel functions, average the result over k and, as a consequence, prove estimate (6.5). To sum up, we obtain the full asymptotic expansion with an arbitrary power saving error term.

Note that the second main term phenomenon does not appear in Khan's [14] evaluation of the moment

$$\sum_k h\left(\frac{2k}{K}\right)\sum_{f\in H_{2k}}\omega(f)L^2(\operatorname{sym}^2 f,1/2). \tag{1.8}$$

Namely, Khan computed the diagonal main term of size $K\log^3 K$ and estimated the remaining terms as $O(K^{\epsilon})$ for any $\epsilon>0$. This fact can be easily explained by the exact formula (see [2, Theorem 2.1]) for the twisted first moment of symmetric square L-functions associated to H_{2k}. The off-diagonal term in this formula is

$$(-1)^k\frac{\sqrt{2\pi}}{2l^{1/2}}\frac{\Gamma(k-1/4)}{\Gamma(k+1/4)}\mathscr{L}_{-4l^2}(1/2). \tag{1.9}$$

To obtain asymptotics for (1.8), it is required to average (1.9) over all values of k. Consequently, the contribution of this term is rather small due to the oscillating multiple $(-1)^k$. In the case of the mixed moment

$$\sum_k h\left(\frac{4k}{K}\right)\sum_{f\in H_{4k}}\omega(f)L(f,1/2)L(\operatorname{sym}^2 f,1/2), \tag{1.10}$$

the summation is taken only over even values of k because $L(f,1/2)$ is identically zero for $f\in H_{2k}$ with odd k. As a result, the oscillating multiple $(-1)^k$ disappears and the summand corresponding to (1.9) becomes the second main term.

Note that the absolute value estimate for the non-diagonal main term in (1.10) is $K^{3/4+\theta}$, where θ denotes the subconvexity exponent for the Dirichlet series (1.2). Using the best subconvexity result due to Conrey and Iwaniec [6] gives a power saving bound $K^{11/12+\epsilon}$. The Rankin-Selberg method allows removing the dependence on θ and improving the estimate to $K^{3/4}$. We finally observe that there are some cancellations between the diagonal and the non-diagonal main terms. This can be shown by direct computations but the most convenient way is to choose a suitable form of the approximate functional equation for $L(f,s)$. Accordingly, we obtain the second main term of size $K^{1/2}$.

In the level aspect similar results for the Hecke congruence subgroup of prime level q were obtained by Munshi and Sengupta [18] with the error estimate $O(q^{-1/8})$. The authors of [18] isolate only the diagonal main term. Our expectation is that there is a second main term of size $q^{-1/2}$.

We remark that the additional average over k in (1.5) is only required to estimate the error term of the second type, which involves the highly oscillatory special function $\Phi_k(x)$. It might be possible to smooth out the oscillations of $\Phi_k(x)$ by using instead the average over l in (6.2), and, hence, prove an asymptotic formula for

$$\sum_{f\in H_{4k}}\omega(f)L(f,1/2)L(\text{sym}^2 f,1/2). \tag{1.11}$$

A consequence of this result is the simultaneous non-vanishing of the corresponding L-functions.

Finally, the methods of the present work shed some light on the structure of the second moment of symmetric square L-functions. In the level aspect the upper bound

$$\sum_{f\in H_{2k}(N)}L^2(\text{sym}^2 f,1/2)\ll N^{1+\epsilon} \text{ for any } \epsilon>0 \text{ and fixed } k, \tag{1.12}$$

was proved by Iwaniec and Michel [11] in 2001. In the weight aspect this is a challenging unsolved problem in the theory of L-functions. It is believed that finding asymptotic formula or even an upper bound for

$$\sum_{f\in H_{2k}}\omega(f)L^2(\text{sym}^2 f,1/2) \text{ as } k\to\infty, \tag{1.13}$$

is out of reach by standard tools. Our results suggest the following approach. Combining exact formula (3.15) and the approximate functional equation for $L(\text{sym}^2 f,1/2)$, one can isolate the diagonal and the off-diagonal main terms in (1.13). The main difference with (1.11) is the presence of the off-off-diagonal main terms in sums (6.1) and (6.2). Technically, this can be explained as follows. The approximate functional equation for symmetric square L-functions is longer and, therefore, the required range includes a transition point at which the special functions $\Phi_k(x)$ and $\Psi_k(x)$ change their behaviour. This produces the third off-off-diagonal main term and creates additional technical difficulties in estimating error terms.

2. Double Dirichlet series

The goal of this subsection is to obtain functional equations for double Dirichlet series of the type (1.1). To this end, we use the results of Müller [17].

2.1. **Some transformation properties.**

As the first step, we prove certain transformation properties. The main references for this subsection are [13, 22].

For a Hecke congruence subgroup $\Gamma_0(4)$ of level 4, we denote by $\mathscr{F}(\Gamma_0(4), \chi, 1/2, \lambda)$ the space of all non-holomorphic automorphic forms of weight 1/2, multiplier system χ and Laplace eigenvalue $\lambda = 1/4 - \rho^2$, $\Re\rho \geq 0$. Any $f \in \mathscr{F}(\Gamma_0(4), \chi, 1/2, \lambda)$ has the Fourier-Whittaker expansion of the form

$$f(z) = A_0(y) + \sum_{n\neq 0} a_n W_{\mathrm{sgn}(n)/4,\rho}(4\pi|n|y)\exp(2\pi i n x), \tag{2.1}$$

where $z = x + iy \in \mathbf{H}$ and $W_{\nu,\mu}(y)$ is the classical Whittaker function (see, for example, [17, Section 3]) normalized such that $W_{\nu,\mu}(y) \sim e^{-y/2}y^{\nu}$ as $y \to \infty$. Assume that

$$a_n = 0 \text{ for } n \equiv 2, 3 \pmod 4. \tag{2.2}$$

Furthermore, we define the following operators

$$(f|U)(z) = \frac{1}{\sqrt{2}}\left(f\left(\frac{z}{4}\right) + f\left(\frac{z+2}{4}\right)\right), \tag{2.3}$$

$$(f|W)(z) = \left(\frac{-iz}{|z|}\right)^{-1/2} f\left(\frac{-1}{4z}\right). \tag{2.4}$$

These operators leave the space $\mathscr{F}(\Gamma_0(4), \chi, 1/2, \lambda)$ stable and preserve the property (2.2). If f has the expansion (2.1), then the function

$$g(z) := \frac{1}{\sqrt{2}}(f|U)(z) \tag{2.5}$$

satisfies

$$g(z) = A_0(y/4) + \sum_{n\neq 0} a_{4n} W_{\mathrm{sgn}(n)/4,\rho}(4\pi|n|y)\exp(2\pi i n x). \tag{2.6}$$

By [22, Prop. 4.1] we have

$$(f|U|W)(z) = f(z). \tag{2.7}$$

Lemma 2.1. *Let f and g be as above. For $z = x + iy \in \mathbf{H}$ we have*

$$\exp(\pi i/4)\left(f\big|_{1/2}J\right)(z) = \sqrt{2}g(z/4), \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.8}$$

where the slash operator is defined by the relation

$$(f\big|_{1/2}M)(z) = \left(\frac{cz+d}{|cz+d|}\right)^{-1/2} f(Mz), \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.9}$$

Proof. Applying equations (2.4), (2.7) and (2.9), we find

$$\sqrt{2}g(z/4) = (f\big|U)(z/4) = i^{1/2}\left(\frac{z}{|z|}\right)^{-1/2}(f\big|U\big|W)(-1/z) =$$
$$= i^{1/2}\left(\frac{z}{|z|}\right)^{-1/2} f(-1/z) = \exp(\pi i/4)\left(f\big|_{1/2}J\right)(z).$$

□

2.2. **Eisenstein series of weight** $1/2$ **and level** 4. The connection between quadratic Dirichlet L-functions and Fourier coefficients of half-integral weight forms was first discovered by Maaß in 1937 [16]. Similar results were obtained by Shimura [24], Shintani [25], Cohen [4], Goldfeld-Hoffstein [8], and others.

We study the more general Dirichlet series (see [3] and [26] for details)

$$\mathscr{L}_n(s) = \frac{\zeta(2s)}{\zeta(s)} \sum_{q=1}^{\infty} \frac{1}{q^s} \left(\sum_{1 \le t \le 2q;\, t^2 \equiv n \pmod{4q}} 1 \right). \tag{2.10}$$

Only if $n \equiv 0, 1 \pmod 4$ the function $\mathscr{L}_n(s)$ considered as a function of s does not vanish. We can think of $\mathscr{L}_n(s)$ as a certain generalization of the Riemann zeta function and quadratic Dirichlet L-functions. Indeed,

$$\mathscr{L}_0(s) = \zeta(2s-1).$$

If n is a fundamental discriminant, then

$$\mathscr{L}_n(s) = \sum_{q=1}^{\infty} \frac{\chi_n(q)}{q^s},$$

where χ_n is a primitive quadratic character mod $|n|$. According to [2, Lemma 4.2] for any $\epsilon > 0$

$$\mathscr{L}_n(1/2) \ll |n|^{\theta+\epsilon}, \tag{2.11}$$

where θ is a subconvexity exponent for Dirichlet L-functions. The best known result $\theta = 1/6$ is due to Conrey and Iwaniec [6]. The Lindelöf hypothesis asserts that $\theta = 0$. The completed L-function

$$\mathscr{L}_n^*(s) = (\pi/|n|)^{-s/2}\Gamma(s/2 + 1/4 - \operatorname{sgn} n/4)\mathscr{L}_n(s)$$

satisfies the functional equation

$$\mathscr{L}_n^*(s) = \mathscr{L}_n^*(1-s). \tag{2.12}$$

The function $\mathscr{L}_n^*(s)$ appears in the Fourier-Whittaker expansion of the combination of the Maaß-Eisenstein series of weight $1/2$ and level 4, namely

$$E^{(1/2)}_{(0;\infty)}(z;s) = \zeta(4s-1)\left(E_\infty(z;s;1/2) + \frac{1+i}{4^s}E_0(z;s;1/2)\right). \tag{2.13}$$

where for a cusp α the series $E_\alpha(z;s;k)$ is defined in [21, Section 3]. Note that according to [21, Section 3A]

$$E^{(1/2)}_{(0;\infty)}(z;s) \in \mathscr{F}(\Gamma_0(4), \nu, 1/2, 1/4 - s^2),$$

where ν is the weight $1/2$ multiplier system related to the theta series (see [21, p. 1570]) such that

$$\nu(\gamma) = \left(\frac{c}{d}\right)\epsilon_d^{-1} \text{ for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$$

with $\left(\frac{c}{d}\right)$ being the Jacobi symbol and $\epsilon_d = (\chi_{-4}(d))^{1/2}$ for the quadratic character $\chi_{-4}(d) = (-1)^{(d-1)/2}$.

Lemma 2.2. *We have*

$$E^{(1/2)}_{(0;\infty)}(z;s) = \zeta(4s-1)y^s + \frac{\sqrt{\pi}\Gamma(2s-1)\zeta(4s-2)}{4^{2s-1}\Gamma(2s-1/2)}y^{1-s} + \frac{\pi^{2s-3/4}}{4^s\Gamma(2s-1/2)} \times \sum_{n\neq 0}\frac{\mathscr{L}_n^*(2s-1/2)}{|n|^{3/4}}W_{\operatorname{sgn} n/4, s-1/2}\left(4\pi|n|y\right)\exp\left(2\pi i n x\right).$$

Proof. The Fourier-Whittaker expansion for $E_\infty(z;s;1/2)$ is given in Section $3A$ of [21]. Computations for $E_0(z;s;1/2)$ are similar. □

Computing the limit as $s \to 1/2$, we find

$$E^{(1/2)}_{(0;\infty)}(z;1/2) = \frac{1}{2}y^{1/2}\log y + (\gamma - \log 4\pi)y^{1/2} + \frac{1}{2\sqrt{\pi}} \times \sum_{n\neq 0}\frac{\mathscr{L}_n(1/2)}{|n|^{1/2}}\Gamma\left(\frac{1}{2} - \frac{\operatorname{sgn} n}{4}\right)W_{\operatorname{sgn} n/4, 0}\left(4\pi|n|y\right)\exp\left(2\pi i n x\right). \tag{2.14}$$

Now we can apply the results of Section 2.1. Note that condition (2.2) is satisfied since $\mathscr{L}_n(1/2)$ vanishes for $n \equiv 2, 3 \pmod 4$. For

$$f(z) := E^{(1/2)}_{(0;\infty)}(z; 1/2) \in \mathscr{F}(\Gamma_0(4), \nu, 1/2, 0)$$

define $g(z)$ by equation (2.5). Then g has the expansion

$$g(z) = \frac{1}{4}y^{1/2}\log y + \frac{1}{2}(\gamma - \log 8\pi)y^{1/2} + \frac{1}{4\sqrt{\pi}}\times \times \sum_{n\neq 0} \frac{\mathscr{L}_{4n}(1/2)}{|n|^{1/2}}\Gamma\left(\frac{1}{2} - \frac{\operatorname{sgn} n}{4}\right) W_{\operatorname{sgn} n/4,0}(4\pi|n|y)\exp(2\pi i n x). \tag{2.15}$$

2.3. **Functional equations for the double Dirichlet series.** Now we consider the associated Dirichlet series:

$$L_f^+(s) = \frac{\Gamma(1/4)}{2\sqrt{\pi}}\sum_{n>0}\frac{\mathscr{L}_n(1/2)}{n^{s+1/2}}, \quad L_f^-(s) = \frac{\Gamma(3/4)}{2\sqrt{\pi}}\sum_{n<0}\frac{\mathscr{L}_n(1/2)}{|n|^{s+1/2}}, \tag{2.16}$$

$$L_g^+(s) = \frac{\Gamma(1/4)}{4\sqrt{\pi}}\sum_{n>0}\frac{\mathscr{L}_{4n}(1/2)}{n^{s+1/2}}, \quad L_g^-(s) = \frac{\Gamma(3/4)}{4\sqrt{\pi}}\sum_{n<0}\frac{\mathscr{L}_{4n}(1/2)}{|n|^{s+1/2}}. \tag{2.17}$$

Theorem 2.3. *The functions $L_f^\pm(s)$ and $L_g^\pm(s)$ have a meromorphic continuation to the whole complex plane and satisfy the functional equations*

$$L_g^+(s) = \frac{-\pi^{2s+2}}{\sqrt{2}\Gamma^2(1/2+s)\sin^2\pi s}\times \left(\frac{\sin\pi(-s-1/4)}{\pi}L_f^+(-s) - \frac{L_f^-(-s)}{\Gamma^2(3/4)}\right), \tag{2.18}$$

$$L_g^-(s) = \frac{\pi^{2s+2}}{\sqrt{2}\Gamma^2(1/2+s)\sin^2\pi s}\times \left(-\frac{\sin\pi(-s+1/4)}{\pi}L_f^-(-s) + \frac{L_f^+(-s)}{\Gamma^2(1/4)}\right). \tag{2.19}$$

Furthermore, $L_f^\pm(s)$ and $L_g^\pm(s)$ are holomorphic in $\mathbf{C}$ except for a double pole at $s = 1/2$.

Proof. Note that according to Lemma 2.1 the first condition of [17, Theorem 4.1] is satisfied by taking $k = 1/2$, $C = \sqrt{2}$, $\gamma = 1/4$. Therefore, the statement of the theorem is a direct consequence of [17, Theorem 4.1] for $\rho = 0$ with a small correction that the functions $L_f^\pm(s)$ and $L_g^\pm(s)$ have poles of order two at $s = 1/2$. □

3. Preliminary evaluation

Let $\mathbf{H}$ be the Poincare upper half-plane and denote by H_{2k} the normalized Hecke basis for the space of holomorphic cusp forms of even weight $2k \geq 2$ with respect to the full modular group. If the function $f \in H_{2k}$ has the Fourier expansion

$$f(z) = \sum_{n\geq 1} \lambda_f(n) n^{k-1/2} \exp(2\pi i n z), \quad \lambda_f(1) = 1, \tag{3.1}$$

the associated Hecke L-function is defined by

$$L(f,s) = \sum_{n\geq 1} \frac{\lambda_f(n)}{n^s}, \quad \Re s > 1. \tag{3.2}$$

Let $\Gamma(s)$ be the Gamma function. Then the completed L-function

$$\Lambda(f,s) = \left(\frac{1}{2\pi}\right)^s \Gamma\left(s + \frac{2k-1}{2}\right) L(f,s) \tag{3.3}$$

satisfies the functional equation

$$\Lambda(f,s) = \epsilon_f \Lambda(f, 1-s), \quad \epsilon_f = i^{2k}, \tag{3.4}$$

and can be analytically continued to the entire complex plane. Note that by equation (3.4) we have $L_f(1/2) = 0$ for odd k.

For $\Re s > 1$ the symmetric square L-function is defined by

$$L(\text{sym}^2 f, s) = \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_f(n^2)}{n^s}. \tag{3.5}$$

Let

$$L_\infty(s) := \pi^{-3s/2} \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{s-1}{2} + k\right) \Gamma\left(\frac{s}{2} + k\right). \tag{3.6}$$

The completed L-function

$$\Lambda(\text{sym}^2 f, s) := L_\infty(s) L(\text{sym}^2 f, s)$$

is entire and satisfies the functional equation

$$\Lambda(\text{sym}^2 f, s) = \Lambda(\text{sym}^2 f, 1-s). \tag{3.7}$$

In this section we combine the techniques of analytic continuation and the approximate functional equation in order to express the average

$$\sum_{f \in H_{4k}} \omega(f) L(f, 1/2) L(\text{sym}^2 f, 1/2) \tag{3.8}$$

as a sum of three parts.

First, we use the exact formula for the twisted moment of symmetric square L-functions shifted from the critical point by a small parameter

$u \neq 0$. The role of the shift u is to simplify the evaluation of the diagonal main term in Section 4.

Lemma 3.1. *The following formula holds*

$$\sum_{f\in H_{4k}} \omega(f)\lambda_f(l)L(\mathrm{sym}^2 f, 1/2+u) = M^D(u,l)\delta_{l=\square}+M^{ND}(u,l)+ET(u,l),$$

where

$$\delta_{l=\square} = \begin{cases} 1 & \text{if } l \text{ is a full square,} \\ 0 & \text{otherwise,} \end{cases} \tag{3.9}$$

$$M^D(u,l^2) = \frac{\zeta(1+2u)}{l^{1/2+u}} + \sqrt{2}(2\pi)^{3u}\cos\pi(1/4+u/2)\times \frac{\zeta(1-2u)}{l^{1/2-u}}\frac{\Gamma(2k-1/4-u/2)\Gamma(2k+1/4-u/2)\Gamma(1-2u)}{\Gamma(2k+1/4+u/2)\Gamma(2k-1/4+u/2)\Gamma(1-u)}, \tag{3.10}$$

$$M^{ND}(0,l) = \frac{\sqrt{2\pi}}{2l^{1/4}}\frac{\Gamma(2k-1/4)}{\Gamma(2k+1/4)}\mathscr{L}_{-4l}(1/2), \tag{3.11}$$

$$ET(0,l) = \frac{1}{l^{1/4}}\sum_{1\leq n<2\sqrt{l}} \mathscr{L}_{n^2-4l}(1/2)\Phi_{2k}\left(\frac{n^2}{4l}\right) + \frac{1}{\sqrt{2}l^{1/2}}\sum_{n>2\sqrt{l}} \mathscr{L}_{n^2-4l}(1/2)\sqrt{n}\Psi_{2k}\left(\frac{4l}{n^2}\right), \tag{3.12}$$

where the functions $\Phi_{2k}(x)$ and $\Psi_{2k}(x)$ are defined by (1.7) and (1.6), respectively.

Proof. See [2, Eq. 2.9, 5.6]. □

Second, we obtain an approximate functional equation for the Hecke L-function at the central point. Let

$$g(s,u) := \frac{(s^2-(-1/4-u/2)^2)\,(s^2-(-1/4+u/2)^2)}{(1/4+u/2)^2(1/4-u/2)^2}\exp(s^2).$$

Lemma 3.2. *For $f \in H_{4k}$ we have*

$$L(f,1/2) = 2\sum_{l=1}^{\infty}\frac{\lambda_f(l)}{\sqrt{l}}V_k(l,u), \tag{3.13}$$

where

$$V_k(l,u) = \frac{1}{2\pi i}\int_{(\sigma)} g(s,u)\frac{\Gamma(2k+s)}{\Gamma(2k)}\frac{ds}{(2\pi l)^s s}, \ \sigma > 0. \tag{3.14}$$

Proof. Consider

$$I(s)=\int_{(\sigma)}g(s,u)\Lambda(f,1/2+s)\frac{ds}{s},\quad \sigma>0.$$

Note that $g(s,u)$ is an even function and $g(0,u)=1$. Therefore, moving the contour of integration from σ to $-\sigma$, applying the functional equation (3.4) and picking up a simple pole at $s=0$, we conclude that

$$2I(s)=\Lambda(f,1/2).$$

The assertion follows. □

As a consequence of Lemma 3.1 and Lemma 3.2 we get the following decomposition:

(3.15)

$$\sum_{f\in H_{4k}}\omega(f)L(f,1/2)L(\mathrm{sym}^2 f,1/2+u)=2\sum_{l=1}^{\infty}\frac{V_k(l,u)}{\sqrt{l}}M^D(u,l)\delta_{l=\square}$$
$$+2\sum_{l=1}^{\infty}\frac{V_k(l,u)}{\sqrt{l}}M^{ND}(u,l)+2\sum_{l=1}^{\infty}\frac{V_k(l,u)}{\sqrt{l}}ET(u,l).$$

4. Diagonal main term

In this section we evaluate asymptotically the diagonal term in (3.15), namely

$$M^D:=2\sum_{l=1}^{\infty}\frac{V_k(l,0)}{\sqrt{l}}M^D(0,l)\delta_{l=\square}.$$

Lemma 4.1. *For any $\epsilon>0$ and any real fixed number $0<a<k$ we have*

(4.1)
$$2\sum_{l=1}^{\infty}\frac{V_k(l,u)}{\sqrt{l}}M^D(u,l)\delta_{l=\square}=2\Bigg(\zeta(1+2u)\zeta(3/2+u)+$$
$$\sqrt{2}(2\pi)^{3u}\cos\pi(1/4+u/2)\zeta(1-2u)\zeta(3/2-u)\times$$
$$\frac{\Gamma(2k-1/4-u/2)}{\Gamma(2k+1/4+u/2)}\frac{\Gamma(2k+1/4-u/2)\Gamma(1-2u)}{\Gamma(2k-1/4+u/2)\Gamma(1-u)}\Bigg)+O(k^{-a+\epsilon}).$$

Proof. Consider

$$2\sum_{l=1}^{\infty}\frac{V_k(l,u)}{\sqrt{l}}M^D(u,l)\delta_{l=\square}=\frac{2}{2\pi i}\int_{(\sigma)}\frac{\Gamma(2k+s)}{\Gamma(2k)}\frac{g(s,u)}{(2\pi)^s}\sum_{l=1}^{\infty}\frac{M^D(u,l^2)}{l^{2s+1}}\frac{ds}{s}.$$

According to (3.10) this is equal to

$$\frac{2}{2\pi i}\int_{(\sigma)}\frac{\Gamma(2k+s)}{\Gamma(2k)}g(s,u)\Bigg[\frac{\zeta(1+2u)\zeta(3/2+u+2s)}{2}+ \sqrt{2}(2\pi)^{3u}\cos\pi(1/4+u/2)\zeta(1-2u)\zeta(3/2-u+2s)\times \frac{\Gamma(2k-1/4-u/2)\Gamma(2k+1/4-u/2)\Gamma(1-2u)}{\Gamma(2k+1/4+u/2)\Gamma(2k-1/4+u/2)\Gamma(1-u)}\Bigg]\frac{ds}{(2\pi)^s s}.$$

The integrand has poles at

$$s=0,\quad s=-k-j,\quad j=0,1,\ldots$$

Note that the poles at $s=-1/4\pm u/2$ are compensated by the zeros of $g(s,u)$. Crossing the pole at $s=0$, we can move the line of integration to any real number $-a$ such that $0<a<k$. Consequently, the resulting integral is bounded by $O(k^{-a+\epsilon})$. The assertion follows by calculating the residue at $s=0$. □

Computing the limit as $u\to 0$, we evaluate the main term at the central point.

Corollary 4.2. *For any $\epsilon>0$ and any real fixed number $0<a<k$ we have*

$$M^D=\zeta(3/2)\Bigg(-3\log 2\pi+\frac{\pi}{2}+3\gamma+2\frac{\zeta'(3/2)}{\zeta(3/2)} + \psi(2k-1/4)+\psi(2k+1/4)\Bigg)+O(k^{-a+\epsilon}), \tag{4.2}$$

where $\psi(x)$ is the logarithmic derivative of the Gamma function.

Finally, the diagonal main term can be averaged over the weight k with a suitable test function.

Corollary 4.3. *For any fixed $A>0$ we have*

$$\sum_k h\left(\frac{4k}{K}\right)M^D=\frac{HK}{4}\zeta(3/2)\times \left(2\log K-3\log 2\pi-2\log 2+\frac{\pi}{2}+3\gamma+2\frac{\zeta'(3/2)}{\zeta(3/2)}+2\frac{H_1}{H}\right) + P_h(1/K)+O(K^{-A}), \tag{4.3}$$

where

$$H=\int_0^\infty h(y)dy, \quad H_1=\int_0^\infty h(y)\log y dy \tag{4.4}$$

and $P_h(x)$ is a polynomial in x of degree $A-1$ and with coefficients depending on the function h.

Proof. This is derived from Corollary 4.2 by using the asymptotic formula

$$\psi(2k-1/4)+\psi(2k+1/4)=2\log k+2\log 2+P(1/k)+O\left(\frac{1}{k^{A+1}}\right),$$

where $P(x)$ is a polynomial of degree A such that $P(0)=0$, and results of [1, Section 7], namely

$$\sum_k h\left(\frac{4k}{K}\right)=\frac{HK}{4}+O\left(\frac{1}{K^b}\right)$$

and

$$\sum_k h\left(\frac{4k}{K}\right)\log k=\frac{HK}{4}(\log K-\log 4)+\frac{H_1K}{4}+O\left(\frac{1}{K^b}\right)$$

for any $b>0$. ☐

5. Non-diagonal main term

Consider the non-diagonal term

$$M^{ND}:=2\sum_{l=1}^\infty \frac{V_k(l,0)}{\sqrt{l}}M^{ND}(0,l)$$
$$=\sqrt{2\pi}\frac{\Gamma(2k-1/4)}{\Gamma(2k+1/4)}\sum_{l=1}^\infty \frac{V_k(l,0)}{l^{3/4}}\mathscr{L}_{-4l}(1/2).$$

Lemma 5.1. *For any fixed $A>0$ we have*

$$M^{ND}=\frac{4\pi\sqrt{2}}{\Gamma(3/4)}L_g^-(1/4)\frac{\Gamma(2k-1/4)}{\Gamma(2k+1/4)}+O(k^{-A}). \tag{5.1}$$

Proof. Consider

$$\sum_{l=1}^\infty \frac{V_k(l,0)}{l^{3/4}}\mathscr{L}_{-4l}(1/2)=\frac{1}{2\pi i}\int_{(\sigma_1)}\frac{\Gamma(2k+s)}{\Gamma(2k)}\sum_{l=1}^\infty \frac{\mathscr{L}_{-4l}(1/2)}{l^{s+3/4}}\frac{g(s,0)ds}{s(2\pi)^s}.$$

We assume that $\sigma_1>1$ to justify the change of order of summation and integration. Equation (2.17) implies that

$$\frac{\Gamma(3/4)}{4\sqrt{\pi}}\sum_{l=1}^\infty \frac{\mathscr{L}_{-4l}(1/2)}{l^{s+3/4}}=L_g^-(s+1/4).$$

According to Theorem 2.3 the function $L_g^-(s+1/4)$ is holomorphic in $\mathbf{C}$ except for a double pole at $s=1/4$, which is compensated by the zeros of $g(s,0)$. Therefore, moving the contour of integration from σ_1 to $\sigma_2=-A$ for any $A>0$, we cross only a simple pole at $s=0$. Consequently,

$$\sum_{l=1}^{\infty}\frac{V_k(l,0)}{l^{3/4}}\mathscr{L}_{-4l}(1/2)=\operatorname{res}_{s=0}F(s)+\frac{1}{2\pi i}\int_{(-A)}F(s)ds,$$

where

$$F(s):=\frac{4\sqrt{\pi}}{\Gamma(3/4)}g(s,0)\frac{\Gamma(2k+s)}{\Gamma(2k)}\frac{L_g^-(s+1/4)}{(2\pi)^s s}.$$

The functional equation (see (2.19))

$$L_g^-(s+1/4)=\frac{\pi^{2s+5/2}}{\sqrt{2}\Gamma^2(s+3/4)\sin^2\pi(s+1/4)}\times \left(\frac{1}{\Gamma^2(1/4)}L_f^+(-s-1/4)+\frac{\sin\pi s}{\pi}L_f^-(-s-1/4)\right)$$

and the estimate

$$\frac{\Gamma(2k+s)}{\Gamma(2k)}\ll\begin{cases}k^{\Re s}, & |\Im s|\le k^{1+\epsilon}\\ k^{\Re s}\exp((-\pi/2+\epsilon_1)|\Im s|), & |\Im s|>k^{1+\epsilon}\end{cases}$$

imply that

$$\int_{(-A)}\frac{\Gamma(2k+s)}{\Gamma(2k)}L_g^-(s+1/4)\frac{g(s,0)ds}{(2\pi)^s s}\ll k^{-A}.$$

The residue at the origin is equal to

$$\operatorname{res}_{s=0}F(s)=\frac{4\sqrt{\pi}}{\Gamma(3/4)}L_g^-(1/4).$$

Therefore, for any $A>0$ we have

$$\sum_{l=1}^{\infty}\frac{V_k(l,0)}{l^{3/4}}\mathscr{L}_{-4l}(1/2)=\frac{4\sqrt{\pi}}{\Gamma(3/4)}L_g^-(1/4)+O(k^{-A}).$$

The assertion follows.

□

Corollary 5.2. *For any fixed $A>0$ we have*

$$\sum_k h\left(\frac{4k}{K}\right)M^{ND}=L_g^-(1/4)\sqrt{K}Q_h(1/K)+O(K^{-A}), \tag{5.2}$$

where $Q_h(x)$ is a polynomial in x of degree A with coefficients depending on the function h.

Proof. The statement follows from Lemma 5.1 and [20, Eq 5.11.13]. □

6. Error terms

Finally, we estimate the last term appearing in (3.15) which we split into two terms

$$ET := 2\sum_{l=1}^{\infty}\frac{V_k(l,0)}{\sqrt{l}}ET(0,l) = ET_1 + ET_2,$$

where (see (3.12))

$$ET_1 := \sqrt{2}\sum_{l=1}^{\infty}\frac{V_k(l,0)}{l}\sum_{n>2\sqrt{l}}\mathscr{L}_{n^2-4l}(1/2)\sqrt{n}\Psi_{2k}\left(\frac{4l}{n^2}\right), \tag{6.1}$$

$$ET_2 := 2\sum_{l=1}^{\infty}\frac{V_k(l,0)}{l^{3/4}}\sum_{n<2\sqrt{l}}\mathscr{L}_{n^2-4l}(1/2)\Phi_{2k}\left(\frac{n^2}{4l}\right). \tag{6.2}$$

Note that if $l > k^{1+\epsilon}$, then

$$V_k(l,0) \ll \left(\frac{k}{l}\right)^A \quad \text{for any } A > 0. \tag{6.3}$$

Using inequality (6.3) and estimating the sums over n in (6.1) and (6.2) by Lemmas 7.1 and 7.3 in [2], we can assume that $l < k^{1+\epsilon}$ in (6.1) and (6.2) at the cost of a negligible error term.

6.1. **Error term of the first type.**

Lemma 6.1. *For any $A > 0$ we have*

$$\sum_k h\left(\frac{4k}{K}\right)ET_1 \ll K^{-A}. \tag{6.4}$$

Proof. It follows from [2, Lemma 7.3] that

$$\sum_k h\left(\frac{4k}{K}\right)ET_1 \ll \sum_k h\left(\frac{4k}{K}\right)\sum_{l<k^{1+\epsilon}}\frac{1}{\sqrt{l}}\frac{l^{-1/24}}{\sqrt{k}}\exp\left(-c\frac{k}{l^{1/4}}\right) \ll K^{-A}.$$

□

6.2. **Error term of the second type.** In this section we prove the following estimate

$$\sum_k h\left(\frac{4k}{K}\right) ET_2 \ll K^{-A} \tag{6.5}$$

for any fixed $A > 0$. The idea is to apply the Liouville-Green method for the function Φ_{2k}. In order to obtain the estimate K^{-A} it is required to take N terms in the Liouville-Green approximation [2, Theorems 6.5, 6.10] for sufficiently large $N = N(A)$. In fact, each of these terms can be treated similarly. Therefore, for the sake of simplicity we consider only the first term corresponding to $N = 1$ in the lemma below. This yields the estimate $K^{1/4+\theta}$. Remark following the lemma provides further explainations how to modify the arguments in order to sharpen this to K^{-A}.

Lemma 6.2. *We have*

$$\sum_k h\left(\frac{4k}{K}\right) ET_2 \ll K^{1/4+\theta}. \tag{6.6}$$

Proof. We decompose the sum over l in (6.2) into two parts:

$$\sum_{l=1}^{\infty} = \sum_{l\neq\square} + \sum_{l=\square}.$$

Suppose that l is not a full square. To approximate the function Φ_{2k} we apply [2, Theorem 6.10] with

$$\cos^2\sqrt{\xi} := n^2/(4l), \tag{6.7}$$

namely

$$\Phi_{2k}(\cos^2\sqrt{\xi}) = \frac{-\pi}{\xi^{1/4}(\sin\sqrt{\xi})^{1/2}}\left[(1+C_J)Z_J(\xi) + C_Y Z_Y(\xi)\right].$$

Using [2, Theorem 6.5] to evaluate $Z_J(\xi)$, $Z_Y(\xi)$ and [2, Corollary 6.9] to approximate the constants C_J, C_Y, we have

$$\Phi_{2k}(\cos^2\sqrt{\xi}) = \frac{-\pi}{\xi^{1/4}(\sin\sqrt{\xi})^{1/2}}\Bigg[\sqrt{\xi}Y_0((4k-1)\sqrt{\xi}) + \sqrt{\xi}J_0((4k-1)\sqrt{\xi}) + O\left(\frac{1}{k}\left|\sqrt{\xi}Y_0((4k-1)\sqrt{\xi})\right|\right)\Bigg]. \tag{6.8}$$

In order to apply [20, Eq. 10.7.8] to approximate the Bessel functions above we show that the argument is large, namely

$$((4k-1)\sqrt{\xi}) \gg k^{1/2-\epsilon}.$$

Indeed, since l is not a full square there exist u and m such that $4l = m^2 + u$, $1 \le u \le 2m$. Therefore,

$$2\sqrt{l} - m = \frac{u}{\sqrt{m^2+u}+m} < 1$$

and

$$[2\sqrt{l}] = m, \quad \{2\sqrt{l}\} = \frac{u}{\sqrt{m^2+u}+m}.$$

Consequently,

$$\frac{\{2\sqrt{l}\}}{4\sqrt{l}} = \frac{u}{m+\sqrt{m^2+u}}\frac{1}{2\sqrt{m^2+u}} \ge \frac{1}{4m^2}.$$

This inequality can be applied to estimate

$$(4k-1)\sqrt{\xi} = (4k-1)\arccos\frac{n}{2\sqrt{l}}, \quad 1 \le n \le [2\sqrt{l}].$$

Changing the variable n to $[2\sqrt{l}] - n$ we have

$$(4k-1)\sqrt{\xi} = (4k-1)\arccos\frac{[2\sqrt{l}]-n}{2\sqrt{l}}, \quad 0 \le n \le [2\sqrt{l}] - 1.$$

Further, trigonometric identities give

$$(4k-1)\sqrt{\xi} = 2(4k-1)\arcsin\sqrt{\frac{n+\{2\sqrt{l}\}}{4\sqrt{l}}}$$
$$\ge 2(4k-1)\arcsin\sqrt{\frac{\{2\sqrt{l}\}}{4\sqrt{l}}} \gg \frac{k}{m} \gg \frac{k}{\sqrt{l}} \gg k^{1/2-\epsilon},$$

as required.

Next, we insert the asymptotic expansion (6.8) into ET_2. The contribution of the error term in (6.8) is majorized by

$$ET_{2,1} := \sum_k h\left(\frac{4k}{K}\right)\frac{1}{k}\sum_{\substack{l\le k^{1+\epsilon}\\ l\neq\square}}\frac{1}{l^{3/4}}\sum_{n<2\sqrt{l}}(4l-n^2)^{\theta}\frac{\xi^{1/4}}{(\sin\sqrt{\xi})^{1/2}}\times$$
$$\left|Y_0((4k-1)\sqrt{\xi})\right| \ll \sum_k h\left(\frac{4k}{K}\right)\frac{1}{k}\sum_{l\le k^{1+\epsilon}}\frac{1}{l^{3/4}}\sum_{n<2\sqrt{l}}\frac{(4l-n^2)^{\theta}}{k^{1/2}(\sin\sqrt{\xi})^{1/2}}.$$

According to (6.7) we have

$$\sin\sqrt{\xi} = \frac{\sqrt{4l-n^2}}{2\sqrt{l}},$$

and therefore,

$$ET_{2,1} \ll \sum_k h\left(\frac{4k}{K}\right)\frac{1}{k^{3/2}}\sum_{l\le k^{1+\epsilon}}\frac{1}{l^{3/4}}\sum_{n<2\sqrt{l}}(4l-n^2)^{\theta-1/4}l^{1/4} \ll K^{\theta+1/4}.$$

It remains to estimate the contribution of the main term in (6.8) given by

$$ET_{2,2} := \sum_k h\left(\frac{4k}{K}\right)\sum_l \frac{V_k(l,0)}{l^{3/4}}\sum_{n<2\sqrt{l}}\mathscr{L}_{n^2-4l}(1/2)\times$$
$$\frac{\sin((4k-1)\sqrt{\xi}-\pi/4)}{\sqrt{4k-1}(\sin\sqrt{\xi})^{1/2}} = \sum_{l\ll K^{1+\epsilon}}\frac{1}{l^{1/2}}\sum_{n<2\sqrt{l}}\frac{\mathscr{L}_{n^2-4l}(1/2)}{(4l-n^2)^{1/4}}\times$$
$$\sum_k h\left(\frac{4k}{K}\right)V_k(l,0)\frac{\sin((4k-1)\arccos\frac{n}{2\sqrt{l}}-\pi/4)}{\sqrt{4k-1}}.$$

The inner sum over k can be evaluated similarly to [1, Lemma 7.3], as we show now. Using the Poisson summation formula

$$\sum_k h\left(\frac{4k}{K}\right)V_k(l,0)\frac{\sin((4k-1)\arccos\frac{n}{2\sqrt{l}}-\pi/4)}{\sqrt{4k-1}} = \sum_m I(m),$$

where

$$I(m) = \int_{-\infty}^{\infty} h\left(\frac{4y}{K}\right)V_y(l,0)\frac{\sin((4y-1)\arccos\frac{n}{2\sqrt{l}}-\pi/4)}{\sqrt{4y-1}}\times$$
$$\exp(-my)dy \ll K\left|\int_{-\infty}^{\infty} h(y)\exp(ig(y))V_{yK/4}(l,0)\frac{dy}{\sqrt{yK-1}}\right|$$

with

$$g(y) = \frac{K}{4}y\left(-2\pi m \pm 4\arccos\frac{n}{2\sqrt{l}}\right).$$

Integrating a times by parts we obtain

$$I(m) \ll \begin{cases}\sqrt{K}/(Km)^a & m\neq 0\\ \sqrt{K}/(K\arccos\frac{n}{2\sqrt{l}})^a & m=0.\end{cases}$$

Finally, since

$$\arccos\frac{n}{2\sqrt{l}} \gg \frac{1}{\sqrt{l}}$$

we have

$$\sum_m I(m) \ll \frac{\sqrt{K}}{(K\arccos\frac{n}{2\sqrt{l}})^a} \ll \frac{\sqrt{K}}{(K/\sqrt{l})^a}$$

and

$$ET_{2,2} \ll K^{-a}.$$

The last step is to evaluate the sum over $l = \square$. Making the change of variables from l to l^2 we have to estimate

$$E_{2,3} := \sum_k h\left(\frac{4k}{K}\right) \sum_l \frac{V_k(l^2,0)}{l^{3/2}} \sum_{n<2l} \mathscr{L}_{n^2-4l^2}(1/2)\Phi_{2k}\left(\frac{n^2}{4l^2}\right).$$

Using the subconvexity estimate (2.11) we have

$$ET_{2,3} \ll \sum_{l \ll K^{1/2+\epsilon}} \frac{1}{l^{3/2}} \sum_{n<2l} (4l^2-n^2)^{\theta} \left| \sum_k h\left(\frac{4k}{K}\right) V_k(l^2,0)\Phi_{2k}\left(\frac{n^2}{4l^2}\right) \right|.$$

Changing the variable n to $2l-n$ gives

$$ET_{2,3} \ll \sum_{l \ll K^{1/2+\epsilon}} \sum_{n<2l} \frac{(nl)^{\theta}}{l^{3/2}} \left| \sum_k h\left(\frac{4k}{K}\right) V_k(l^2,0)\Phi_{2k}\left(\left(1-\frac{n^2}{4l^2}\right)^2\right) \right|.$$

Let

$$\xi := 4(\arcsin\sqrt{\frac{n}{4l}})^2.$$

Consequently,

$$\left(1-\frac{n}{2l}\right)^2 = \cos^2\sqrt{\xi}.$$

Applying [2, Theorem 6.10], we have

$$\Phi_{2k}(\cos^2\sqrt{\xi}) = \frac{-\pi}{\xi^{1/4}(\sin\sqrt{\xi})^{1/2}}\left[(1+C_J)Z_J(\xi) + C_Y Z_Y(\xi)\right].$$

Note that now

$$(4k-1)\sqrt{\xi} = 2(4k-1)\arcsin\sqrt{\frac{n}{4l}} \gg \frac{k}{\sqrt{l}} \gg k^{3/4}.$$

In the same way as in the case $l \neq \square$ we prove

$$ET_{2,3} \ll K^{-1/4+\theta}.$$

Finally,

$$ET_2 \ll ET_{2,1} + ET_{2,2} + ET_{2,3} \ll K^{1/4+\theta}.$$

$\square$

Remark. Using [2, Thm. 6.5] with sufficiently large N depending on A and taking more terms in the asymptotics for C_Y, C_J we obtain equation (6.8) with additional series of main terms plus the error term

$$O\left(\frac{1}{k^{A+2}}\left|\sqrt{\xi}Y_0((4k-1)\sqrt{\xi})\right|\right).$$

As a result, following the proof of Lemma 6.2 we obtain the estimate (6.5).

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