Optimal series representations of continuous Gaussian random fields

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Organization of the talk

Definition 1.1

A random variable X *defined on a probability space* $(\Omega, \mathcal{A}, \mathbb{P})$ *and with values in a separable Banach space* $(E, \mathcal{B}_E)$, *is said to be* Gaussian centered, *when* $\langle X, x^* \rangle$ *is* a real-valued centered Gaussian random variable *for any* $x^* \in E^*$, *the topological dual of* E.

Remark 1.1

Let K *be a compact metric space and let* $\mathcal{C}(K)$ *be the Banach space of the continuous real-valued functions defined on* K. *Assume that* $\{X(t)\}_{t \in K}$ *is a real-valued centered Gaussian process, defined on* $(\Omega, \mathcal{A}, \mathbb{P})$ *and with continuous paths (for all fixed* $\omega \in \Omega$, *the function* $X(\cdot, \omega) : K \to \mathbb{R}$, $t \mapsto X(t, \omega)$ *is continuous). Then, the map* $(\Omega, \mathcal{A}, \mathbb{P}) \to (\mathcal{C}(K), \mathcal{B}_{\mathcal{C}(K)})$, $\omega \mapsto X(\cdot, \omega)$ *is a centered Gaussian random variable with values in* $(\mathcal{C}(K), \mathcal{B}_{\mathcal{C}(K)})$.

Theorem 1.1

X can be represented as an almost surely convergent random series of the form:

$$X = \sum_{k=1}^{+\infty} \epsilon_k \alpha_k, \tag{1.1}$$

where the ϵ_k's are independent $\mathcal{N}(0,1)$ real-valued Gaussian random variables *and the α_k's are* some deterministic elements of E*.*

→ See the book of Lifshits (1995) or that of Ledoux and Talagrand (1991) for a proof Theorem 1.1.
→ In the case where X is a Brownian motion $B = \{B(t)\}_{t\in[0,1]}$, two classical examples of such series representations are:

- the representation in the trigonometric system,

$$B(t) = \epsilon_0 t + \sqrt{2} \sum_{k=1}^{+\infty} \epsilon_k \frac{\sin(\pi k t)}{\pi k};$$

- the representation in the Faber-Schauder system (it will be presented later).

It is natural to look for **optimal series representations of** X i.e. those where the tail of the series:

$$\sum_{k=n}^{+\infty} \epsilon_k \alpha_k,$$

tends to zero as fast as possible. This leads to the study of the quantity

$$l_n(X) = \inf \left\{ \left(E \left\| \sum_{k=n}^{+\infty} \epsilon_k \alpha_k \right\|^2 \right)^{1/2} ;\ X = \sum_{k=1}^{+\infty} \epsilon_k \alpha_k \right\}. \tag{1.2}$$

$l_n(X)$ is called the **nth** l**-number of** X.

Remark 1.2

The value of $l_n(X)$ remains the same even if the random variables ϵ_k are allowed to be dependent.

Clearly, $\lim_{n\to+\infty} l_n(X) = 0$. The rate of convergence is closely connected with the small ball behaviour of X.

Theorem 1.2 (Li and Linde 1999)

Let $a > 0$ and $b \in \mathbb{R}$ be fixed.

1. *If for some constant $c_1 > 0$ and every integer $n \geq 1$,*

$$l_n(X) \leq c_1 n^{-a}(1 + \log n)^b. \tag{1.3}$$

Then, there is a constant $c_2 > 0$ such that for any $\epsilon > 0$ small enough,

$$\log(\mathbb{P}(\|X\| \leq \epsilon) \geq -c_2 \epsilon^{-1/a}(\log 1/\epsilon)^{b/a}. \tag{1.4}$$

2. *Conversely, if (1.4) holds, then there is a constant $c_3 > 0$ such that for each integer $n \geq 1$,*

$$l_n(X) \leq c_3 n^{-a}(1 + \log n)^{b+1}. \tag{1.5}$$

Remark 1.3 (Li and Linde 1999)

When E is K-convex (e.g. L^p, $1 < p < \infty$) then (1.3) and (1.4) are equivalent.

The main three goals of our talk are the following:

(a) To connect the Hölder regularity (in the mean-square sense) of a centered continuous real-valued Gaussian process with the rate of convergence of its l-numbers.

(b) To present the Meyer, Sellan and Taqqu wavelet series representations of fractional Brownian motion (fBm), and to show that they are optimal; notice that fBm is the most natural real-valued continuous Gaussian process which extends Brownian motion.

(c) To investigate, for the Riemann-Liouville process (that is the high-frequency part of fBm), the optimality of the series representations obtained via the Haar and the trigonometric bases of $L^2[0,1]$.

Organization of the talk

Throughout this section, $X = \{X(t)\}_{t\in[0,1]^N}$ denotes a centered real-valued Gaussian process with continuous paths.

Definition 2.1

Let $H \in (0,1]$. One says that X is H-Hölder (in the mean-square sense), if there exists a constant $c > 0$ such that for all $t, s \in [0,1]^N$,

$$\mathbb{E}\left(|X(t) - X(s)|^2\right) \le c|t-s|^{2H}. \tag{2.1}$$

Remark 2.1

1. *It follows from (2.1) that the paths of X are almost surely γ-Hölder functions for any $\gamma \in (0,H)$; that is, for all $t, s \in [0,1]^N$,*

$$\big|X(t,\omega) - X(s,\omega)\big| \le C'(\omega)|t-s|^{\gamma}. \tag{2.2}$$

2. *Conversely, if the paths of X are almost surely H-Hölder functions then (2.1) is verified.*

A classical example of a real-valued, centered, continuous and H-Hölder Gaussian process, is the multivariate **fractional Brownian motion** $B_H = \{B_H(t)\}_{t\in[0,1]^N}$ whose covariance is given by:

$$\mathbb{E}B_H(t)B_H(s) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right). \tag{2.3}$$

The following theorem, is the main result of this section.

Theorem 2.1 (A. and Linde 2008)

Assume that $X = \{X(t)\}_{t\in[0,1]^N}$ is a real-valued, centered, continuous and H-Hölder Gaussian process. Then there exists a constant $c > 0$ such that for every $n \geq 2$,

$$l_n(X) \leq cn^{-H/N}\sqrt{\log n}. \tag{2.4}$$

For proving Theorem 2.1 we will use the operators theory.

Gaussian processes and operators

Let $X = \{X(t)\}_{t\in[0,1]^N}$ be a centered continuous real-valued Gaussian process.

1. There exists $u : L^2 = L^2[0,1] \to \mathcal{C} = \mathcal{C}[0,1]^N$ a bounded operator such that for any $t, s \in [0,1]^N$,

$$\mathbb{E}X(t)X(s) = \langle u^*\delta_t, u^*\delta_s\rangle_{L^2}, \tag{2.5}$$

where $u^* : \mathcal{C}^* \to L^2$ is the dual operator of u, and δ_t is the Dirac measure in $t \in [0,1]^N$; one calls u: **an operator generating X.**

2. In fact, one can prove that u belongs to the **the class $\mathcal{G} = \mathcal{G}(L^2[0,1], \mathcal{C}[0,1]^N)$, that is for each orthonormal basis $(f_k)_{k\geq1}$ of L^2 and any sequence $\{\epsilon_k\}_{k\geq1}$ of independent real-valued $\mathcal{N}(0,1)$ Gaussian random variables the sum**

$$\sum_{k=1}^{+\infty} \epsilon_k u(f_k),$$

converges almost surely in $\mathcal{C}$.

Some useful properties of $\mathcal{G} = \mathcal{G}(L^2[0,1], \mathcal{C}[0,1]^N)$

- Any $v \in \mathcal{G}$ is a compact operator (see for example the book of Pisier (1989)).
- The l-norm of any $v \in \mathcal{G}$ is defined as

$$l(v) = \left(\mathbb{E} \left\| \sum_{k=1}^{+\infty} \epsilon_k v(f_k) \right\|_\infty^2 \right)^{1/2} ; \tag{2.6}$$

  this norm is known to be independent of the special choice of the orthonormal basis $(f_k)_{k \geq 1}$.
- $(\mathcal{G}, l)$ is a Banach space (Linde and Peitsch 1973).
- If v has finite rank and X_0 is an arbitrary standard Gaussian random variable with values in $(\ker v)^\perp$. Then

$$l(v) = \left(\mathbb{E} \| v(X_0) \|_\infty^2 \right)^{1/2}.$$

Another useful property of $\mathcal{G} = \mathcal{G}(L^2[0,1], \mathcal{C}[0,1]^N)$

Let $v \in \mathcal{G}$ and define the kernel $K : [0,1]^N \times [0,1] \to \mathbb{R}$ by

$$K(t,x) = (v^* \delta_t)(x). \tag{2.7}$$

Then $t \mapsto \Phi(t) = K(t,\cdot)$ is a continuous function from $[0,1]^N$ into $L^2[0,1]$. Moreover, for any $h \in L^2[0,1]$ and $t \in [0,1]^N$ one has

$$(vh)(t) = \int_0^1 K(t,x)h(x)\,dx. \tag{2.8}$$

Proof of (2.8): Using (2.7), one obtains that for all $t \in [0,1]^N$,

$$\int_0^1 K(t,x)h(x)\,dx = \langle v^*\delta_t, h\rangle_{L^2,L^2} = \langle vh, \delta_t\rangle_{C,C^*} = (vh)(t).$$

□

Proof of the continuity of $\Phi : t \mapsto K(t,\cdot)$: Using (2.7) and

$$\|f\|_{L^2} = \sup_{\|h\|_{L^2}\leq 1} |\langle f, h\rangle|$$

for all $f \in L^2$, one gets that for any $t_1, t_2 \in [0,1]^N$,

$$\begin{aligned} &\|K(t_1,\cdot) - K(t_2,\cdot)\|_{L^2} \\ &= \|v^*(\delta_{t_1}) - v^*(\delta_{t_2})\|_{L^2} \\ &= \sup_{\|h\|_{L^2}\leq 1} |\langle v^*(\delta_{t_1}) - v^*(\delta_{t_2}), h\rangle| \\ &= \sup_{\|h\|_{L^2}\leq 1} |(vh)(t_1) - (vh)(t_2)|. \end{aligned} \tag{2.9}$$

Since v is compact, the set $\{v(h) : \|h\|_{L^2} \leq 1\}$ is equicontinuous (Arzela-Ascoli Theorem).

Thus (2.9) implies that Φ is a continuous function from $[0,1]^N$ into $L^2[0,1]$. □

Proposition 2.1

Let $X = \{X(t)\}_{t\in[0,1]^N}$ be a centered, continuous and real-valued Gaussian process. Let $u \in \mathcal{G}(L^2, \mathcal{C})$ be an operator generating X i.e. for all $t, s \in [0,1]^N$,

$$\mathbb{E}X(t)X(s) = \langle u^*\delta_t, u^*\delta_s\rangle_{L^2}, \qquad (2.10)$$

and

$$K(t,x) = (u^*\delta_t)(x), \qquad (2.11)$$

the kernel corresponding to u. Then

$$law\left\{X(t)\,:\,t\in[0,1]^N\right\} = law\left\{\int_0^1 K(t,x)\,dB(x)\,:\,t\in[0,1]^N\right\},$$

where dB is a Brownian measure.

Proof: It follows from (2.10) and (2.11) that for all $t, s \in [0,1]^N$,

$$\mathbb{E}X(t)X(s) = \int_0^1 K(t,x)K(s,x)\,dx.$$

On the other hand, using the isometry property of Wiener integral one has for all $t, s \in [0,1]^N$,

$$\mathbb{E}\left(\int_0^1 K(t,x)\,dB(x)\int_0^1 K(s,x)\,dB(x)\right) = \int_0^1 K(t,x)K(s,x)\,dx.$$

□

Corollary 2.1

Let $X = \{X(t)\}_{t \in [0,1]^N}$ be a centered, continuous and real-valued Gaussian process. Recall that X can be represented as,

$$X(t) = \sum_{k=1}^{+\infty} \epsilon_k \alpha_k(t).$$

In fact, a possible (but not unique) choice of the continuous functions α_k is

$$\alpha_k = u(f_k),$$

where u is an operator generating X and $(f_k)_{k \geq 1}$ is an arbitrary orthonormal basis of $L^2[0,1]$.

Proof: The process X can be represented for every $t \in [0,1]^N$ as

$$X(t) = \int_0^1 K(t,x)\, dB(x), \tag{2.12}$$

where K is the kernel corresponding to u i.e. for every $h \in L^2$ and $t \in [0,1]^N$,

$$(uh)(t) = \int_0^1 K(t,x) h(x)\, dx. \tag{2.13}$$

Expanding, for all fixed $t \in [0,1]^N$, the function $K(t,\cdot)$ in the basis $(f_k)_{k\geq 1}$ and then using (2.13), one obtains that

$$K(t,\cdot) = \sum_{k=1}^{+\infty} (uf_k)(t) f_k, \tag{2.14}$$

where the series converges in $L^2([0,1], dx)$. Finally, it follows from (2.14), (2.12) and the isometry property of Wiener integral that

$$X(t) = \sum_{k=1}^{+\infty} \epsilon_k (uf_k)(t), \tag{2.15}$$

where

$$\epsilon_k = \int_0^1 f_k(x)\, dB(x), \quad \text{for all } k.$$

□

Remark 2.2

A priori, the series in (2.15) converges in $L^2(\Omega, \mathbb{R})$ for every fixed t. However, since $u \in \mathcal{G}(L^2[0,1], \mathcal{C}[0,1]^N)$, one can show that it also converges almost surely uniformly in $t \in [0,1]^N$ or equivalently that it converges in $L^2(\Omega, \mathcal{C}[0,1]^N)$.

The sequence of l-**numbers of** u, is defined as,

$$l_n(u) = \inf \left\{ \left(\mathbb{E} \left\| \sum_{k=n}^{\infty} \epsilon_k u(f_k) \right\|_{\infty}^2 \right)^{1/2} : (f_k)_{k\geq 1} \text{ ONB in } L^2[0,1] \right\}; \qquad (2.16)$$

recall that, the sequence of l-**numbers of** X is defined as:

$$l_n(X) = \inf \left\{ \left(\mathbb{E} \left\| \sum_{k=n}^{\infty} \epsilon_k \alpha_k \right\|_{\infty}^2 \right)^{1/2} : X = \sum_{k=0}^{\infty} \epsilon_k \alpha_k \text{ a.s.} \right\}. \qquad (2.17)$$

According to Corollary 2.1, a possible choice of the α_k's is $\alpha_k = u(f_k)$ for all k; hence

$$l_n(X) \leq l_n(u). \qquad (2.18)$$

Observe that, the reverse inequality is also true, thus,

$$l_n(X) = l_n(u). \qquad (2.19)$$

Proof of the main result of the section

Recall that the goal of this section is to prove that:

Theorem 2.1

Assume that $X = \{X(t)\}_{t\in[0,1]^N}$ is a real-valued, centered, continuous and H-Hölder Gaussian process. Then,

$$l_n(X) \le \mathcal{O}\left(n^{-H/N}\sqrt{\log n}\right). \tag{2.20}$$

Reformulation of Theorem 2.1 in terms of operators theory:

Theorem 2.1 (reformulated)

Let $H \in (0,1]$ and **let** $u \in \mathcal{G}(L^2[0,1], \mathcal{C}[0,1]^N)$ **be a** H-**Hölder operator**, which means that, there exists a constant $c_2 > 0$ such that for any $h \in L^2[0,1]$ and $t_1, t_2 \in [0,1]^N$,

$$|(uh)(t_1) - (uh)(t_2)| \le c_2 \|h\|_{L^2} |t_1 - t_2|^H. \tag{2.21}$$

Then $l_n(u) = \mathcal{O}\left(n^{-H/N}\sqrt{\log n}\right)$.

To prove Theorem 2.1 (reformulated), we will use the following proposition:

Proposition 2.2

Let $u \in \mathcal{G}(L^2[0,1], \mathcal{C}[0,1]^N)$. For each $a > 0$ and $b \in \mathbb{R}$, (i) is equivalent to (ii).

(i) *The l-numbers of u, satisfies,*

$$l_n(u) = \mathcal{O}\left(n^{-a}(\log n)^b\right). \tag{2.22}$$

(ii) *There exists a sequence $(u_j)_{j\geq 1}$ of* **finite rank operators**, *such that:*

1. *One has, in the sense of the l-norm,*

$$u = \sum_{j=1}^{\infty} u_j;$$

2. *the ranks of the u_j's satisfy*

$$rk(u_j) = \mathcal{O}\left(2^j\right) \tag{2.23}$$

and, their l-norms satisfy

$$l(u_j) = \mathcal{O}\left(2^{-aj} j^b\right). \tag{2.24}$$

Sketch of the proof of Theorem 2.1 (reformulated): We suppose that $N=1$. For any integers $j\geq 0$ and $0\leq k\leq 2^j$ set

$$t_k^j = k/2^j \text{ and } \lambda_k^j = [t_k^j, t_{k+1}^j),$$

with the convention that $\lambda_{2^j}^j = \{1\}$. We denote by $B([0,1])$, the Banach space of the real-valued bounded function on $[0,1]$, equipped with the uniform norm. Let $v_j : L^2[0,1] \to B([0,1])$ be the finite rank operator defined for all $h \in L^2$ as,

$$v_j h = \sum_{k=0}^{2^j} (uh)(t_k^j) \mathbf{1}_{\lambda_k^j}. \tag{2.25}$$

It is clear that,

$$\mathrm{rk}\,(v_j) \leq 2^{j+1}. \tag{2.26}$$

moreover, since u is a H-Hölder operator, one has

$$\|u - v_j\| = \mathcal{O}\left(2^{-jH}\right). \tag{2.27}$$

Let $u_0 = v_0$ and for all $j\geq 1$, $u_j = v_j - v_{j-1}$. (2.26) implies that $\mathrm{rk}\,(u_j) \leq 2^{j+2}$ and (2.27) that $u = \sum_{j=0}^{+\infty} u_j$. It remains to show that $l(u_j) = \mathcal{O}\left(2^{-jH}\sqrt{j+1}\right)$.

This can be obtained by using the fact that $l(u_j) = \left(\mathbb{E}\|u_j(X_j)\|_\infty^2\right)^{1/2}$, where X_j is a standard Gaussian random variable with values in $(\ker u_j)^\perp$.

□

Organization of the talk

→ $\{B_H(t)\}_{t\in[0,1]}$, the **fractional Brownian motion** (fBm) of Hurst parameter $H \in (0,1)$, is the continuous centered real-valued Gaussian process with covariance,

$$\mathbb{E}\left(B_H(s)B_H(t)\right) = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |s-t|^{2H}\right); \tag{3.1}$$

when $H = 1/2$, fBm reduces to Brownian motion, otherwise its increments are correlated and even display long-range dependence when $H \in (1/2, 1)$.
→ Kühn and Linde (2002), have obtained sharp estimates of the asymptotic behaviour of $(l_n(B_H))_{n\geq 1}$, the sequence of the l-numbers of fBm; namely, there exist two constants $0 < c_1 \leq c_2$, such that for all $n \geq 2$,

$$c_1 n^{-H}\sqrt{\log n} \leq l_n(B_H) \leq c_2 n^{-H}\sqrt{\log n} \tag{3.2}$$

Observe that Theorem 2.1 allows to recover the second inequality in (3.2).
→ It seems natural to look for explicit optimal random series representations of fBm; in view of (3.2) their rate of convergence must be, $n^{-H}\sqrt{\log n}$.

The main goal of this section is to present the Meyer, Sellan and Taqqu wavelet representations of fBm, and to show that they are optimal. Roughly speaking, they consist in expressing fBm as a series of approximations with successive scale refinements; they are reminiscent of the representation of Brownian motion in the Faber-Schauder system.

Representation of Brownian motion in the Faber-Schauder system

$$B(t) = \epsilon_0 t + \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} \epsilon_{j,k} 2^{-j/2} \tau(2^j t - k), \tag{3.3}$$

where:

- ϵ_0 and $\epsilon_{j,k}$ are independent real-valued $\mathcal{N}(0,1)$ Gaussian random variables,
- τ is the triangle function based on $[0,1]$,
- the series is almost surely uniformly convergent in $t \in [0,1]$.

The expansion (3.3) was introduced by Paul Lévy in 1948. It has turned out to be very useful in the (fine) study of Brownian motion; for instance it allows to prove that Brownian paths do not satisfy a uniform or a pointwise Hölder condition of order $1/2$.

Proposition 3.1

The series expansion of Brownian motion in the Faber-Schauder system is optimal.

For proving Proposition 3.1, we need the following two lemmas.

Lemma 3.1

There is a constant $c > 0$ such that for any integer $N \geq 1$ and for each centered Gaussian real-valued sequence $Z_1, \ldots, Z_N$ one has,

$$\mathbb{E}\left(\sup_{1\leq k\leq N} |Z_k|\right) \leq c(1+\log N)^{1/2} \sup_{1\leq k\leq N} \left(\mathbb{E}|Z_k|^2\right)^{1/2}. \qquad (3.4)$$

Lemma 3.2

For any $t \in [0,1]$ and for each integer $j \geq 0$, there is at most one integer k, such that $0 \leq k < 2^j$ and $\tau(2^j t - k) \neq 0$.

Proof of Proposition 3.1: Lemma 3.2 implies that for all $j \geq 0$ and $t \in [0,1]$,

$$\sum_{k=0}^{2^j-1} |\epsilon_{j,k}||\tau(2^j t-k)| \leq \left(\sup_{0\leq k<2^j} |\epsilon_{j,k}| \right) \|\tau\|_\infty. \tag{3.5}$$

Using (3.5), the gaussianity of the $\epsilon_{j,k}$'s and Lemma 3.1, one has for all $m \in \mathbb{N}$,

$$\begin{aligned} Q_m &= \mathbb{E}\left(\sup_{t\in[0,1]} \left| B(t) - \epsilon_0 t - \sum_{j=0}^{m} \sum_{k=0}^{2^j-1} 2^{-j/2} \epsilon_{j,k} \tau(2^j t-k) \right| \right) \\ &\leq c_1 \sum_{j=m+1}^{+\infty} 2^{-j/2} \mathbb{E}\left(\sup_{0\leq k<2^j} |\epsilon_{j,k}| \right) \\ &\leq c_2 \sum_{j=m+1}^{+\infty} 2^{-j/2} (1+j)^{1/2} \left(\sup_{0\leq k<2^j} \mathbb{E}|\epsilon_{j,k}|^2 \right)^{1/2} \\ &= c_2 \sum_{j=m+1}^{+\infty} 2^{-j/2} (1+j)^{1/2} \leq c_3 2^{-m/2} (1+m)^{1/2}. \end{aligned}$$

Thus, the expansion of $\{B(t)\}_{t\in[0,1]}$ in the Faber-Schauder system is optimal.
□

Wavelet representation of fBm without a scaling function

We suppose that:

- ψ is a smooth and well-localized real-valued mother wavelet defined on $\mathbb{R}$,
- $\{2^{j/2}\psi(2^j t - k);\, j \in \mathbb{Z} \text{ and } k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$,
- $\widehat{\psi}$, the Fourier transform of ψ, is a smooth and well-localized function.

Under these assumptions, for any $H \in (0,1)$,

$$\Psi_H(t) = \int_{\mathbb{R}} e^{it\xi} \frac{\widehat{\psi}(\xi)}{(i\xi)^{H+1/2}}\, d\xi \text{ and } \Psi_{-H}(t) = \int_{\mathbb{R}} e^{it\xi} (i\xi)^{H+1/2} \widehat{\psi}(\xi)\, d\xi,$$

the **fractional primitive** of ψ of order $H + 1/2$, and its **fractional derivative** of order $H + 1/2$, are well-defined, smooth and well-localized real-valued functions.

Theorem 3.1 (Meyer, Sellan and Taqqu 1999)

The sequences of functions $\{2^{j/2}\Psi_H(2^j t - k);\, j \in \mathbb{Z} \text{ and } k \in \mathbb{Z}\}$ and $\{2^{j/2}\Psi_{-H}(2^j t - k);\, j \in \mathbb{Z} \text{ and } k \in \mathbb{Z}\}$ are two biorthogonal bases of $L^2(\mathbb{R})$.

These bases are well-adapted to the analysis of fBm:

Theorem 3.2 (Meyer, Sellan and Taqqu 1999)

The fBm $\{B_H(t)\}_{t\in[0,1]}$ can be expressed as:

$$B_H(t) = \sum_{j=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} 2^{-jH} \left(\Psi_H(2^j t - k) - \Psi_H(-k)\right) \epsilon_{j,k}, \tag{3.6}$$

where the $\epsilon_{j,k}$'s are independent $\mathcal{N}(0,1)$ Gaussian random variables and the series is almost surely uniformly convergent in t. Moreover,

$$\epsilon_{j,k} = 2^{j(H+1)} \int_{\mathbb{R}} B_H(t)\Psi_{-H}(2^j t - k)\, dt. \tag{3.7}$$

$\rightarrow$ (3.6) is almost a Karhunen-Loeve expansion of B_H.
$\rightarrow$ It allows to obtain some local and asymptotic properties of B_H (nowhere differentiability, moduli of continuity, behaviour at infinity, and so on).

Proof of Theorem 3.2: Let us start from the **harmonizable representation** of fBm:

$$B_H(t) = \int_{\mathbb{R}} \frac{e^{it\xi} - 1}{(i\xi)^{H+1/2}}\, d\widehat{B}(\xi), \tag{3.8}$$

where $d\widehat{B}$ is the "Fourier transform" of the Brownian measure dB. Expanding the function $\xi \mapsto \frac{e^{it\xi}-1}{(i\xi)^{N+1/2}}$ in the orthonormal basis

$$\left\{ 2^{-j/2} e^{ik\xi/2^j} \overline{\widehat{\psi}(2^{-j}\xi)};\ j \in \mathbb{Z} \text{ and } k \in \mathbb{Z} \right\}$$

and then, using the isometry property of the integral $\displaystyle\int_{\mathbb{R}} (\cdot) d\widehat{B}$, it follows that

$$B_H(t) = \int_{\mathbb{R}} \frac{e^{it\xi}-1}{(i\xi)^{H+1/2}}\, d\widehat{B}(\xi) = \sum_{j,k\in\mathbb{Z}} \alpha_{j,k}(t)\epsilon_{j,k}, \tag{3.9}$$

where the

$$\epsilon_{j,k} = 2^{-j/2}\int_{\mathbb{R}} e^{ik\xi/2^j}\,\overline{\widehat{\psi}(2^{-j}\xi)}\, d\widehat{B}(\xi)$$

are independent $\mathcal{N}(0,1)$ real-valued Gaussian random variables and

$$\alpha_{j,k}(t) = 2^{-j/2}\int_{\mathbb{R}} \frac{e^{it\xi}-1}{(i\xi)^{H+1/2}}\, e^{-ik\xi/2^j}\,\widehat{\psi}(2^{-j}\xi)\, d\xi. \tag{3.10}$$

The series (3.9) is almost surely uniformly convergent in t (Itô-Nisio Theorem).

Let us now prove that

$$\alpha_{j,k}(t) = 2^{-jH}\left(\Psi_H(2^j t - k) - \Psi_H(-k)\right). \tag{3.11}$$

As the first two moments of the wavelet ψ vanish, one has,

$$\widehat{\psi}(\xi) = \mathcal{O}(\xi^2), \tag{3.12}$$

which implies that

$$\alpha_{j,k}(t) = 2^{-j/2}\int_{\mathbb{R}} e^{i(t-k/2^j)\xi}\,\frac{\widehat{\psi}(2^{-j}\xi)}{(i\xi)^{H+1/2}}\,d\xi - 2^{-j/2}\int_{\mathbb{R}} e^{-ik\xi/2^j}\,\frac{\widehat{\psi}(2^{-j}\xi)}{(i\xi)^{H+1/2}}\,d\xi. \tag{3.13}$$

Finally, setting $\eta = 2^{-j}\xi$ in the latter two integrals, one obtains (3.11).
□

Wavelet representation with a well-localized scaling function

The mother wavelet ψ satisfies the same conditions as previously and ϕ is a corresponding scaling function.

As $\widehat{\phi}(0) = 1 \neq 0$, the fractional primitive of ϕ of order $H + 1/2$, can only be defined when $H \in (0, 1/2)$; moreover, it is irregular and bad localized.

$\Rightarrow$ The problem of finding a wavelet expansion of fBm with a well-localized scaling function is tricky. However, we need to have this well-localization for the expansion to be optimal.

To overcome this difficulty Meyer, Sellan and Taqqu have replaced the fractional primitive of ϕ, by the function Φ_H defined as,

$$\widehat{\Phi}_H(\xi) = \left(\frac{1 + e^{-i\xi}}{i\xi} \right)^{H+1/2} \widehat{\Phi}(\xi). \tag{3.14}$$

Theorem 3.3 (Meyer, Sellan and Taqqu 1999)

Let Ψ_H and Ψ_{-H} be the fractional primitive of order $H+1/2$ and the fractional derivative of order $H+1/2$, of the mother wavelet ψ. Let Φ_H be as in (3.14) and let Φ_{-H} be defined as,

$$\widehat{\Phi}_{-H}(\xi) = \left(\frac{1+e^{-i\xi}}{i\xi}\right)^{-H-1/2} \widehat{\Phi}(\xi). \tag{3.15}$$

Then for any $m \in \mathbb{Z}$,

$$\left\{2^{m/2}\Phi_H(2^m t - l) : \, l \in \mathbb{Z}\right\} \bigcup \left\{2^{j/2}\Psi_H(2^j t - k) : \, j \in \mathbb{Z}, \, j \geq m, \, k \in \mathbb{Z}\right\}$$

and

$$\left\{2^{m/2}\Phi_{-H}(2^m t - l) : \, l \in \mathbb{Z}\right\} \bigcup \left\{2^{j/2}\Psi_{-H}(2^j t - k) : \, j \in \mathbb{Z}, \, j \geq m, \, k \in \mathbb{Z}\right\}$$

are two biorthogonal bases of $L^2(\mathbb{R})$.

Theorem 3.4 (Meyer, Sellan and Taqqu 1999)

For any $m \in \mathbb{Z}$, the fBm $\{B_H(t)\}_{t\in[0,1]}$ can be expressed as:

$$\begin{aligned} B_H(t) &= 2^{-mH} \sum_{l=-\infty}^{+\infty} \Phi_H(2^m t - l) S_{m,l}^{(H)} \\ &\quad + \sum_{j=m}^{+\infty} \sum_{k=-\infty}^{+\infty} 2^{-jH} \Psi_H(2^j t - k) - b_0, \end{aligned} \tag{3.16}$$

where $\{S_{m,l}^{(H)}\}_{l\in\mathbb{Z}}$ is a FARIMA(0,H-1/2,0) random walk and $\{\epsilon_{j,k}\}_{j\geq m,k\in\mathbb{Z}}$ is a sequence of independent real-valued $\mathcal{N}(0,1)$ Gaussian random variables. The series in (3.16), are almost surely uniformly convergent in t; moreover,

$$S_{m,l}^{(H)} = 2^{m(H+1)} \int_{\mathbb{R}} B_H(t) \Phi_{-H}(2^m t - l)\, dt$$

and

$$\epsilon_{j,k} = 2^{j(H+1)} \int_{\mathbb{R}} B_H(t) \Psi_{-H}(2^j t - k)\, dt.$$

The main two advantages of the latter representation with respect to the representation without scaling function are the following.

1. The first term isolates the low frequencies and gives the tendency while the second term involves fluctuations around it.
2. FBm can be approximated by the first term and Mallat pyramidal algorithm allows to compute by induction the coefficients $S_{m,l}$, $l \in \mathbb{Z}$ for any $m \geq 1$.

Optimality of both representations

Theorem 3.5 (A. and Taqqu 2003)

For each integer $J \geq 0$, *let*
$F_J = \{(j,k) \in \mathbb{Z}^2;\ 0 \leq j \leq J \text{ and } |k| \leq (J-j+1)^{-2} 2^{J+4}\}$, *and*
$P_J = \{(j,k) \in \mathbb{Z}^2;\ -J \leq j \leq -1 \text{ and } |k| \leq 2^{J/2}\}$. *For every integer* $n \geq \beta$ *with* $\beta = 128 \sum_{p=1}^{+\infty} p^{-2}$, *let* $\mathcal{I}_n \subset \mathbb{Z}^2$ *be a set satisfying the following properties:*

- $\mathcal{I}_n$ *contains, at most,* n *indices* (j,k),
- *for all* $n \geq \beta$, $F_{J(n)} \cup P_{J(n)} \subset \mathcal{I}_n$, *where* $J(n)$ *is the unique integer such that* $\beta 2^{J(n)} \leq n < \beta 2^{J(n)+1}$.

At last let

$$B_{H,n}(t) = \sum_{(j,k) \in \mathcal{I}_n} 2^{-jH} \left(\Psi_H(2^j t - k) - \Psi_H(-k)\right) \epsilon_{j,k}. \tag{3.17}$$

Then there is a r. v. $C > 0$ *of finite moments such that a.s. for all* $n \geq \beta$,

$$\sup_{t \in [0,1]} |B_H(t) - B_{H,n}(t)| \leq C n^{-H} (1 + \log n)^{1/2}. \tag{3.18}$$

Theorem 3.6 (A. and Taqqu 2003)

We suppose that β, F_J *and* $J(n)$ *are the same as in Theorem 3.5. For every integer* $J \geq 0$ *let* $Q_J = \{l \in \mathbb{Z};\, |l| \leq 2^J\}$. *For every integer* $n \geq \beta$, *let* $\mathcal{I}'_n \subset \mathbb{Z}$ *and* $\mathcal{I}''_n \subset \mathbb{Z}^2$ *be two sets satisfying the following properties:*

- $\mathcal{I}'_n$ *contains at most* $n/2$ *indices* l *and* $\mathcal{I}''_n$ *contains at most* $n/2$ *indices* (j,k),
- *for every* $n \geq \beta$, $Q_{J(n)} \subset \mathcal{I}'_n$ *and* $F_{J(n)} \subset \mathcal{I}''_n$.

At last let

$$B_{H,n}(t) = \sum_{l \in \mathcal{I}'_n} \Phi_H(t-l) S_{0,l}^{(H)} + \sum_{(j,k) \in \mathcal{I}''_n} 2^{-jH} \left(\Psi_H(2^j t - k) - \Psi_H(-k) \right) \epsilon_{j,k}. \tag{3.19}$$

Then there is a r. v. $C > 0$ *of finite moments such that a.s. for all* $n \geq \beta$,

$$\sup_{t \in [0,1]} |B_H(t) - B_{H,n}(t)| \leq C n^{-H} (1 + \log n)^{1/2}. \tag{3.20}$$

Remark 3.1

Another optimal series representation of fBm has been introduced in 2003 by Dzhaparidze and Van Zanten. It has some similarities with the expansion of Brownian motion in the trigonometric system.

Organization of the talk

Some generalities on Riemann-Liouville process

$\{R^\alpha(t)\}_{t\in[0,1]}$, the Reimann-Liouville process (RLp) of parameter $\alpha > 1/2$, is defined as the Wiener integral,

$$R^\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^1 (t-s)_+^{\alpha-1}\, dB(s), \tag{4.1}$$

with the convention that for all $(x,\gamma) \in \mathbb{R}^2$, $(x)_+^\gamma = x^\gamma$ if $x > 0$ and $(x)_+^\gamma = 0$ else.

Remark 4.1

- $\{R^1(t)\}_{t\in[0,1]}$ *is the usual Brownian motion.*
- *If* $1/2 < \alpha < 3/2$, *then* $\{R^\alpha(t)\}_{t\in[0,1]}$ *differs from* $\{B_{\alpha-1/2}(t)\}_{t\in[0,1]}$, *the fBm of Hurst parameter* $H = \alpha - 1/2$, *only by a very smooth process, namely,*

$$Q^\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^1 \Big\{ (t-s)_+^{\alpha-1} - (-s)_+^{\alpha-1} \Big\}\, dB(s);$$

sometime $\{R^\alpha(t)\}_{t\in[0,1]}$ *is called the high-frequency part of fBm and* $\{Q^\alpha(t)\}_{t\in[0,1]}$ *the low-frequency part.*

- An advantage of RLp with respect to fBm is that **its parameter α can take any value strictly greater than** $1/2$, while the Hurst parameter of fBm is necessarily obliged to belong to $(0,1)$.
- **The semigroup property of RLp:** for each real number $\beta > 1/2$, we denote by $\mathcal{C}^{\beta-1/2}[0,1]$, the Hölder space on $[0,1]$ of order $\beta - 1/2$, and we denote by $I_\beta : L^2[0,1] \to \mathcal{C}^{\beta-1/2}[0,1]$ the fractional primitive operator of order β, defined, for all $h \in L^2[0,1]$ and $t \in [0,1]$ as,

$$(I_\beta h)(t) = \frac{1}{\Gamma(\beta)} \int_0^1 (t-s)_+^{\beta-1} h(s)\, ds; \qquad (4.2)$$

  then one has $(I_\beta R^\alpha)(t) = R^{\alpha+\beta}(t)$.

Series representation of RLP by an orthonormal basis:

Let $\mathbf{F} = (f_k)_{k\geq 1}$ be an arbitrary orthonormal basis of $L^2[0,1]$. Expanding, for every fixed $t \in [0,1]$, the function $s \mapsto (t-s)_+^{\alpha-1}$ in this basis, one gets that

$$(t-\cdot)_+^{\alpha-1} = \sum_{k=1}^{\infty} (I_\alpha f_k)(t)\varphi_k(\cdot). \tag{4.3}$$

Next, using the isometry property of Wiener integral, it follows that

$$R^\alpha(t) = \sum_{k=1}^{\infty} (I_\alpha f_k)(t)\epsilon_k, \tag{4.4}$$

where

$$(\epsilon_k)_{k\geq 1} = \Big(\int_0^1 f_k(s)\, dB(s) \Big)_{k\geq 1}, \tag{4.5}$$

is a sequence of independent real-valued $\mathcal{N}(0,1)$ Gaussian random variables.

Remark 4.2

- *The previous series is called the series representation of RLp by the basis* $\mathbf{F}$.
- *A priori, this series is, for every fixed* t*, convergent in* $L^2(\Omega)$*, where* Ω *denotes the underlying probability space. However, Itô-Nisio Theorem allows to show that it is also convergent, with probability* 1*, uniformly in* t.

Theorem 4.1 (Kühn and Linde 2002)

For all $\alpha > 1/2$*, there are two constants* $0 < c_1 \leq c_2$ *such that for every* $n \geq 2$*,*

$$c_1 n^{-\alpha+1/2}\sqrt{\log n} \leq l_n(R^\alpha) \leq c_2 n^{-\alpha+1/2}\sqrt{\log n},$$

where $l_n(R^\alpha)$ *denotes the nth l-number of RLp.*

Theorem 4.2 (Schack 2009)

For all $\alpha > 1/2$*, any Daubechies wavelet basis of* $L^2[0,+\infty)$ *with at least* $\max\{2[\alpha+1/2]+3, 7\}$ *vanishing moments, generates an optimal random series representation of* R^α.

Optimality of classical series representations

The main two results of this section:

Theorem 4.3 (A. and Linde 2009)

For all $1/2 < \alpha < 3/2$, *the Haar basis of* $L^2[0,1]$,

$$\mathbf{H} := \{\mathbf{1}\} \cup \{h_{j,k} \ : \ j \in \mathbb{N}_0 \text{ and } 0 \le k < 2^j\},$$

where

$$h_{j,k} = 2^{j/2}\Big(\mathbf{1}_{[\frac{2k}{2^{j+1}},\frac{2k+1}{2^{j+1}}]} - \mathbf{1}_{[\frac{2k+1}{2^{j+1}},\frac{2k+2}{2^{j+1}}]}\Big),$$

generates an optimal series representation of R^α.

Theorem 4.4 (A. and Linde 2009)

For all $1 < \alpha \le 2$, *the trigonometric basis of* $L^2[0,1]$,

$$\mathbf{T} := \{\mathbf{1}\} \cup \{\sqrt{2}\cos(k\pi\cdot) \ : \ k \in \mathbb{N}\},$$

generates an optimal series representation of R^α.

Remark 4.3

- *When* $\alpha > 3/2$*, the random series representation of* R^α *generated by the Haar basis is rearrangement non-optimal i.e. the optimality does not hold even if the terms of the series are renumbered (A. and Linde 2009); the latter result remains true when* $\alpha = 1$ *(Lifshits 2009).*
- *The random series representation of* R^α *generated by the trigonometric basis is rearrangement non-optimal if* $\alpha > 2$ *(A. and Linde 2009).*
- *Kühn and Linde (2002) have shown that the random series representation of* R^α *generated by the trigonometric basis is optimal if* $\alpha \in (1/2, 1]$*.*

From now on, we will mainly focus on the proof of the theorem which concerns the Haar basis.
Let us set

$$h_{-1} = \mathbf{1} \text{ and } \epsilon_{-1} = \int_0^1 h_{-1}(s)\, dB(s).$$

We denote by $\{\epsilon_{j,k} : j \in \mathbb{N}_0,\, 0 \le k \le 2^j - 1\}$, the sequence of the real-valued $\mathcal{N}(0,1)$ Gaussian random variables, defined as,

$$\epsilon_{j,k} = \int_0^1 h_{j,k}(s)\, dB(s).$$

At last, recall that $I_\alpha : L^2[0,1] \to C^{\alpha-1/2}[0,1]$, is the fractional primitive operator of order α.

The series representation of R^α by the Haar basis can be expressed as:

$$R^\alpha(t) = \epsilon_{-1}(I_\alpha h_{-1})(t) + \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} \epsilon_{j,k}(I_\alpha h_{j,k})(t). \tag{4.6}$$

The latter infinite sum will be approximated by the finite sum,

$$R_J^\alpha(t) = \epsilon_{-1}(I_\alpha h_{-1})(t) + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} \epsilon_{j,k}(I_\alpha h_{j,k})(t), \tag{4.7}$$

in which there are exactly 2^J terms.

Theorem 4.5 (A. and Linde 2009)

Suppose $1/2 < \alpha < 1$. *Then there is a random variable* $C > 0$ *of finite moments of any order, such that one has almost surely, for every* $J \in \mathbb{N}$,

$$\sup_{t \in [0,1]} \left| R^\alpha(t) - R_J^\alpha(t) \right| \leq C 2^{-(\alpha - 1/2)J} \sqrt{1+J}.$$

Theorem 4.6 (A. and Linde 2009)

Suppose $1 < \alpha < 3/2$. *Then there is a constant* $c > 0$, *such that one has almost surely, for every* $J \in \mathbb{N}$,

$$\mathbb{E} \sup_{t \in [0,1]} \left| R^\alpha(t) - R_J^\alpha(t) \right| \leq c 2^{-(\alpha - 1/2)J} \sqrt{1+J}.$$

In order to be able to prove the latter two theorems, we need some preliminary results.

Remark 4.4

One has

$$
\begin{aligned}
&(I_\alpha h_{j,k})(t)\\
&= \frac{2^{j/2}}{\Gamma(\alpha+1)} \left\{ \int_{2k/2^{j+1}}^{(2k+1)/2^{j+1}} (t-s)_+^{\alpha-1}\, ds - \int_{(2k+1)/2^{j+1}}^{(2k+2)/2^{j+1}} (t-s)_+^{\alpha-1}\, ds \right\}\\
&= \frac{2^{j/2}}{\Gamma(\alpha+1)} \left\{ \Big(t - \frac{2k+2}{2^{j+1}}\Big)_+^{\alpha} - 2\Big(t - \frac{2k+1}{2^{j+1}}\Big)_+^{\alpha} + \Big(t - \frac{2k}{2^{j+1}}\Big)_+^{\alpha} \right\}.
\end{aligned}
$$

The following lemma allows to estimate $(I_\alpha h_{j,k})(t)$.

Lemma 4.1

For each reals $\gamma > 0$ *and* $t \in [0,1]$, *and for all integers* $j \in \mathbb{N}_0$ *and* $0 \le k \le 2^j - 1$, *we set*

$$A_{\gamma,j,k}(t) := \Big(t - \frac{2k+2}{2^{j+1}}\Big)_+^{\gamma} - 2\Big(t - \frac{2k+1}{2^{j+1}}\Big)_+^{\gamma} + \Big(t - \frac{2k}{2^{j+1}}\Big)_+^{\gamma}. \tag{4.8}$$

Furthermore, let $\widetilde{k}_j(t)$ *the unique integer satisfying:*

$$\frac{\widetilde{k}_j(t)}{2^j} \le t < \frac{\widetilde{k}_j(t)+1}{2^j}. \tag{4.9}$$

with the convention that $\widetilde{k}_j(1) = 2^j - 1$. *Then,*

(i) *for all* $k \ge \widetilde{k}_j(t) + 1$, *one has* $A_{\gamma,j,k}(t) = 0$;

(ii) *there is a constant* $c > 0$, *only depending on* γ, *such that for all* $j \in \mathbb{N}_0$ *and* $0 \le k \le \widetilde{k}_j(t)$, *one has*

$$|A_{\gamma,j,k}(t)| \le c2^{-j\gamma}\Big(1 + \widetilde{k}_j(t) - k\Big)^{\gamma-2}. \tag{4.10}$$

The following lemma, results from Borel-Cantelli Lemma; it provides sharp estimates of the asymptotic behaviour of the sequence of the real-valued $\mathcal{N}(0,1)$ Gaussian random variables, $\{\epsilon_{j,k} \,:\, j \in \mathbb{N}_0,\, 0 \le k \le 2^j - 1\}$.

Lemma 4.2

There exists a random variable $C > 0$ of finite moment of any order such that one has almost surely, for every $j \in \mathbb{N}_0$ and $0 \le k < 2^j - 1$,

$$|\epsilon_{j,k}| \le C\sqrt{1+j}.$$

Proof of the optimality when $1/2 < \alpha < 1$:

It follows from the latter two lemmas that almost surely for every $t \in [0,1]$ and $J \in \mathbb{N}$,

$$
\begin{aligned}
|R^{\alpha}(t) - R^{\alpha}_J(t)| &\leq \sum_{j=J}^{+\infty} \sum_{k=0}^{2^j-1} |\epsilon_{j,k}| |(I_\alpha h_{j,k})(t)| \\
&\leq C_1 \sum_{j=J}^{+\infty} 2^{-j(\alpha-1/2)} \sqrt{j+1} \sum_{k=0}^{\widetilde{k}_j(t)} \left(1 + \widetilde{k}_j(t) - k\right)^{\alpha-2} \\
&\leq C_2 2^{-J(\alpha-1/2)} \sqrt{J+1}.
\end{aligned}
$$

Observe that the condition $1/2 \leq \alpha < 1$ plays a crucial role in the proof. Indeed, one has

$$
\sum_{k=0}^{\widetilde{k}_j(t)} \left(1 + \widetilde{k}_j(t) - k\right)^{\alpha-2} \leq \sum_{p=1}^{+\infty} p^{\alpha-2} < \infty, \tag{4.11}
$$

only when it is satisfied.

□

Main ideas of the proof of the optimality when $1 < \alpha < 3/2$:

We want to show that there is a constant $c_1 > 0$, such that one has for every $J \in \mathbb{N}$,

$$\mathbb{E}(A_J) \leq c_1 2^{-(\alpha-1/2)J}\sqrt{1+J}, \tag{4.12}$$

where

$$A_J = \sup_{t\in[0,1]} \left|R^\alpha(t) - R_J^\alpha(t)\right|. \tag{4.13}$$

First step: We prove that there is a constant $c_2 > 0$, such that one has for every $J \in \mathbb{N}$,

$$\mathbb{E}(B_J) \leq c_2 2^{-(\alpha-1/2)J}\sqrt{1+J}, \tag{4.14}$$

where

$$B_J = \sup_{0\leq K<2^J, K\in\mathbb{N}_0} \left|R^\alpha(K2^{-J})) - R_J^\alpha(K2^{-J}))\right|. \tag{4.15}$$

Second step: We prove that there is a constant $c_3 > 0$, such that one has for every $J \in \mathbb{N}$,

$$0 \leq \mathbb{E}(A_J - B_J) \leq c_3 2^{-(\alpha-1/2)J}\sqrt{1+J}. \tag{4.16}$$

Proof of the first step:

Lemma 4.3

There exists a constant $c > 0$ such that for any $N \in \mathbb{N}$ and any real-valued centered Gaussian sequence $Z_1, \ldots, Z_N$ one has,

$$\mathbb{E}\Big\{ \sup_{1\le k\le N} |Z_k| \Big\} \le c\,(1+\log N)^{1/2} \sup_{1\le k\le N} \big(E|Z_k|^2\big)^{1/2}. \tag{4.17}$$

Therefore, there is a constant $c_4 > 0$ such that for all $J \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{E} \sup_{0\le K<2^J, K\in\mathbb{N}_0} \Big| R^\alpha(K2^{-J})) - R^\alpha_J(K2^{-J}))\Big| \\
&\le c_4(1+J)^{1/2} \sup_{0\le K<2^J, K\in\mathbb{N}_0} \Big(\mathbb{E}\Big| R^\alpha(K2^{-J})) - R^\alpha_J(K2^{-J}))\Big|^2\Big)^{1/2}.
\end{aligned}$$

Then the following lemma allows to finish the proof.

Lemma 4.4

There is a constant $c > 0$ such that one has for all $t \in [0,1]$ and $J \in \mathbb{N}$,

$$\mathbb{E}\Big|R^{\alpha}(t) - R^{\alpha}_J(t))\Big|^2 \leq c2^{-J(2\alpha-1)}.$$

Proof Lemma 4.4: Using the fact that $\{\epsilon_{j,k} : j \in \mathbb{N}_0,\, 0 \leq k \leq 2^j - 1\}$ is a sequence of independent real-valued $\mathcal{N}(0,1)$ Gaussian random variables and using the previous estimations of the $(I_\alpha h_{j,k})(t)$'s, one gets that for all $J \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{E}\Big|R^{\alpha}(t) - R^{\alpha}_J(t))\Big|^2 \\
&= \mathbb{E}\Big(\sum_{j=J}^{+\infty}\sum_{k=0}^{2^j-1}\epsilon_{j,k}(I_\alpha h_{j,k})(t)\Big)^2 = \sum_{j=J}^{+\infty}\sum_{k=0}^{2^j-1}|(I_\alpha h_{j,k})(t)|^2 \\
&\leq c_1\sum_{j=J}^{+\infty}2^{-j(2\alpha-1)}\sum_{k=0}^{\widetilde{k}_j(t)}\Big(1+\widetilde{k}_j(t)-k\Big)^{-2(2-\alpha)} \leq c_2 2^{-J(2\alpha-1)}.
\end{aligned}$$

□

Optimality of the representention by the trigonometric basis:

Roughly speaking, when $\alpha \in (1/2, 2]$, the optimality of the representention by the trigonometric basis, can be shown, by following the main lines as in the proof of the latter theorem, and by using the fact there is a constant $c > 0$ such that for all integer $k \geq 1$,

$$\sup_{t\in[0,1]} |(I_\alpha\{\cos(k\pi\cdot)\})(t)| \leq ck^{-\alpha} \tag{4.18}$$

and

$$\sup_{t\in[0,1]} |(I_\alpha\{\sin(k\pi\cdot)\})(t)| \leq ck^{-\alpha} \tag{4.19}$$