A mini-course on Wavelets and Fractional Processes

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Abstract

In this mini-course we explain how one can construct an orthonormal wavelet basis starting from a multiresolusition analysis. Also, we present one of the most standard random wavelet series representation of Fractional Brownian Motion and use it in order to solve a problem concerning the pointwise Hölder regularity of the trajectories of the latter process

1 Introduction

Fractional Brownian Motion is a quite natural generalization of Brownian Motion (BM) and its wavelet series representations introduced in [25], can be viewed as a natural extension to the fractional setting of the representation of BM in the Schauder system due to Lévy [18]. In this introduction we will make a brief recall concerning the latter representation of BM.

Let us first recall that one says that a sequence $\{e_n\}_n$ in a Hilbert space H (note that all the Hilbert spaces we consider in this course are over $\mathbb{C}$ the field of complex numbers) equipped with the inner product $\langle \cdot, \cdot \rangle$, is an an orthonormal basis of H if and only if (i) and (ii) are satisfied:

(i) For all n', n'' one has $\langle e_{n'}, e_{n''} \rangle = 1$ if $n' = n''$ and $\langle e_{n'}, e_{n''} \rangle = 0$ else.

(ii) the (finite) linear combinations of the e_n's are dense in H.

For example:

- The trigonometric system $\{(2\pi)^{-1/2} e^{il\xi} : l \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R}/2\pi\mathbb{Z})$, the space of the 2π-periodic functions defined on the real line and whose squares of the modulus are Lebesgue integrable on an (or equivalently on each) interval of lenght 2π. Note in passing that one says that a function f defined on the real line is 2π-periodic if one has for almost all $\xi \in \mathbb{R}$, $f(\xi + 2\pi) = f(\xi)$.

- The Haar system [13] $\{\mathbb{1}_{[0,1]}(s)\} \cup \{2^{j/2}h(2^j s - k) \,:\, j \in \mathbb{N},\, k \in \mathbb{N} \text{ and } 0 \le k \le 2^j - 1\}$ is an orthonormal basis $L^2[0,1]$, here $h(x) = \mathbb{1}_{[0,1/2)}(s) - \mathbb{1}_{[1/2,1]}(s)$.

Now let $\{W(t) \,:\, t \in [0,1]\}$ be a BM over $[0,1]$. It can be expressed as the Wiener integral

$$W(t) = \int_0^1 \mathbb{1}_{[0,t]}(s)\, dW(s).$$

By expanding the function $s \mapsto \mathbb{1}_{[0,t]}(s)$ in the Haar system, it follows that

$$\mathbb{1}_{[0,t]}(s) = t\mathbb{1}_{[0,1]}(s) + \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} 2^{-j/2}\tau(2^j t - k) 2^{j/2} h(2^j s - k),$$

where τ is the triangle function based on $[0,1]$ such that $\tau(1/2) = 1/2$ and where the series converges in $L^2[0,1]$. Next, using the isometry property of Wiener integral one gets that

$$W(t) = t\varepsilon_0 + \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} 2^{-j/2}\tau(2^j t - k)\varepsilon_{j,k}, \tag{1.1}$$

where $\varepsilon_0 = \int_0^1 \mathbb{1}_{[0,1]}(s)\, dW(s)$ and $\varepsilon_{j,k} = \int_0^1 2^{j/2}h(2^j s - k)\, dW(s)$. Observe that $\{\varepsilon_0\} \cup \{\varepsilon_{j,k} \,:\, j \in \mathbb{N},\, k \in \mathbb{N} \text{ and } 0 \le k \le 2^j - 1\}$ is a sequence of $\mathcal{N}(0,1)$ Gaussian random variables. Also observe that not only the series in (1.1) is convergent for every fixed t in $L^2(\Omega)$ (Ω being the underlying probability space) but also it is, with probability 1, uniformly convergent in t.

To conclude this introduction, let us emphasize that the representation of BM in the Schauder system has turned out to be quite useful in the study of fine properties of its trajectories.

2 Multiresolution Analyses and Wavelet bases

The goal of this section is to present some important results related to the construction of orthonormal wavelet bases of $L^2(\mathbb{R})$. We refer to the books [10, 14, 20, 23, 24, 29] for detailed presentations of the wavelet theory and some of its applications.

An orthonormal wavelet basis of $L^2(\mathbb{R})$ is an orthonormal basis of $L^2(\mathbb{R})$ obtained by dilations and translations of a function usually denoted by ψ and called a mother wavelet; more precisely such a basis is of the form

$$\{2^{j/2}\psi(2^j x - k) \,:\, j \in \mathbb{Z} \text{ and } k \in \mathbb{Z}\};$$

usually one sets $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$.

The first proofs which have been given in the literature to show that by dilating and translating a well chosen function one can obtain an orthonormal basis of $L^2(\mathbb{R})$ rely on many tricky computations [22, 28] and therefore seem to be a bit miraculous. However when the concept of Multiresolution Analysis (MRA) was introduced the construction of orthonormal wavelet bases became quite natural.

Definition 2.1 *A MRA of $L^2(\mathbb{R})$ is a sequence $(V_j)_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ satisfying the following properties:*

(a) For every $j \in \mathbb{Z}$, $V_j \subset V_{j+1}$.

(b) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$.

(c) $\cup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ i.e. $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$.

(d) For every $j \in \mathbb{Z}$, $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$.

(e) There exists a function $g \in V_0$ such that the sequence $\{g(x-k) : k \in \mathbb{Z}\}$ forms a Riesz basis of V_0.

Definition 2.2 *Let H be a separable Hilbert space, one says that a sequence $\{e_k : k \in \mathbb{Z}\}$ forms a Riesz basis of H if it satisfies the following properties:*

(i) The (finite) linear combinations of the e_k's are dense in H i.e. $H = \overline{span\{e_k : k \in \mathbb{Z}\}}$.

(ii) $\{e_k : k \in \mathbb{Z}\}$ is a Riesz sequence i.e. there are two constants $0 < c < c'$ such that for each complex-valued sequence $(a_k)_{k\in\mathbb{Z}}$ with a finite number of non vanishing terms, one has

$$c \sum_{k\in\mathbb{Z}} |a_k|^2 \leq \Big\| \sum_{k\in\mathbb{Z}} a_k e_k \Big\|^2 \leq c' \sum_{k\in\mathbb{Z}} |a_k|^2. \tag{2.1}$$

Remarks 2.3

(i) Any Riesz basis of H is the image of an orthonormal basis of H by an isomorphism of H.

(ii) Observe that there are many sequences of subspaces of $L^2(\mathbb{R})$ which satisfies properties (a), (b) and (c); whereas properties (d) and (e) are specific to the the concept of MRA.

$\square$

Now we are going to explain more precisely properties (d) and (e).

Proposition 2.4 *Property (d) means that each space V_j is a dilated version of the reference space V_0, namely one has that*

$$V_j = \{f(2^j x) \,:\, f(x) \in V_0\}. \tag{2.2}$$

The proof of Proposition 2.4 is obvious.

Proposition 2.5 *Property (e) implies that V_0 is the subspace of $L^2(\mathbb{R})$ of the functions f whose Fourier transforms can be expressed for almost all $\xi \in \mathbb{R}$ as*

$$\widehat{f}(\xi) = \lambda_f(\xi)\widehat{g}(\xi), \tag{2.3}$$

where $\lambda_f \in L^2(\mathbb{R}/2\pi\mathbb{Z})$.

In order to be able to prove Proposition 2.5 we need the following lemma.

Lemma 2.6 *Let $h \in L^2(\mathbb{R})$ and let $0 < c \leq c'$ be two constants, then the following two assertions are equivalent.*

(i) For each complex-valued sequence $(a_k)_{k\in\mathbb{Z}}$ with a finite number of non vanishing terms, one has

$$c\sum_{k\in\mathbb{Z}} |a_k|^2 \leq \int_{\mathbb{R}} \Big|\sum_{k\in\mathbb{Z}} a_k h(x-k)\Big|^2 \, dx \leq c'\sum_{k\in\mathbb{Z}} |a_k|^2. \tag{2.4}$$

(ii) One has for almost all $\xi \in \mathbb{R}$,

$$c \leq \sum_{k\in\mathbb{Z}} |\widehat{h}(\xi + 2k\pi)|^2 \leq c'. \tag{2.5}$$

Observe that (i) means that $\{h(x-k) \,:\, k \in \mathbb{Z}\}$ is a Riesz sequence.

Remark 2.7 In this course the Fourier transform of a function $f \in L^1(\mathbb{R})$ is defined for all $\xi \in \mathbb{R}$ as

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx.$$

□

Proof of Lemma 2.6: We will only show that $(i) \Rightarrow (ii)$, the proof of $(ii) \Rightarrow (i)$ is not difficult. Let us assume that (2.4) holds, then it follows from the isometry property of Fourier transform that

$$\int_{\mathbb{R}} \Big| \sum_{k\in\mathbb{Z}} a_k h(x-k) \Big|^2 \, dx = \int_{\mathbb{R}} |\lambda_a(\xi)|^2 |\widehat{h}(\xi)|^2 \, d\xi, \tag{2.6}$$

where λ_a is the trigonometric polynomial

$$\lambda_a(\xi) = \sum_{k\in\mathbb{Z}} a_k e^{-ik\xi}. \tag{2.7}$$

Next, using the 2π-periodicity of λ_a one obtains that

$$\int_{\mathbb{R}} |\lambda_a(\xi)|^2 |\widehat{h}(\xi)|^2 \, d\xi = \int_{-\pi}^{\pi} |\lambda_a(\xi)|^2 \Big(\sum_{l\in\mathbb{Z}} |\widehat{h}(\xi + 2l\pi)|^2 \Big) \, d\xi. \tag{2.8}$$

Let now assume that that $\xi_0 \in \mathbb{R}$ is arbitrary and fixed and the sequence $(a_k)_{k\in\mathbb{Z}}$ has been chosen in such a way that for all $\xi \in \mathbb{R}$,

$$\lambda_a(\xi) = \frac{1}{\sqrt{2\pi N}} \sum_{k=0}^{N-1} e^{ik(\xi_0 - \xi)},$$

where the integer $N \geq 1$ is arbitrary and fixed. One has therefore

$$\int_{-\pi}^{\pi} |\lambda_a(\xi)|^2 \, d\xi = 1 \text{ and } |\lambda_a(\xi)|^2 = \frac{1}{2\pi N} \frac{\sin^2(N(\xi_0 - \xi)/2)}{\sin^2((\xi_0 - \xi)/2)}.$$

Then using (2.4), (2.6), (2.7) and (2.8) one gets that

$$c \leq \int_{-\pi}^{\pi} \frac{1}{2\pi N} \frac{\sin^2(N(\xi_0 - \xi)/2)}{\sin^2((\xi_0 - \xi)/2)} \Big(\sum_{l\in\mathbb{Z}} |\widehat{h}(\xi + 2l\pi)|^2 \Big) \, d\xi \leq c'.$$

Finally the following lemma allows us to finish our proof. □

Lemma 2.8 *For every integer $N \geq 1$, let $K_N \in L^1(\mathbb{R}/2\pi\mathbb{Z})$ be a nonnegative function satisfying $\int_{-\pi}^{\pi} K_N(\xi)\, d\xi = 1$ and*

$$K_N(\xi) \leq c \frac{N}{1 + N^2 \xi^2},$$

for almost all $|\xi| \leq \pi$, where $c > 0$ is a constant non depending on N and ξ. For all $F \in L^1(\mathbb{R}/2\pi\mathbb{Z})$ and $\xi_0 \in \mathbb{R}$ we set

$$K_N * F(\xi_0) = \int_{-\pi}^{\pi} K_N(\xi_0 - \xi) F(\xi) \, d\xi.$$

Then $K_N * F$ converges to F in $L^1(\mathbb{R}/2\pi\mathbb{Z})$ when $N \to +\infty$. As a consequence, there is a subsequence $m \mapsto N_m$ such that $\lim_{m\to+\infty} K_{N_m} * F(\xi_0) = F(\xi_0)$ for almost all $\xi_0 \in \mathbb{R}$.

The proof of Lemma 2.8 only requires classical techniques on convolution product, this is why it is left to the reader.

We are now in position to prove Proposition 2.5.

PROOF OF PROPOSITION 2.5: In view of the fact that $\{g(x-k)\,:\,k\in\mathbb{Z}\}$ is a Riesz basis of V_0, any function $f\in V_0$, can be written as

$$f(x)=\sum_{k\in\mathbb{Z}} b_k g(x-k), \tag{2.9}$$

where the series converges in $L^2(\mathbb{R})$ and where the sequence $(b_k)_{k\in\mathbb{Z}}$ belongs to $l^2(\mathbb{Z})$. Then it follows from (2.9) and the isometry property of Fourier tranform that

$$\widehat{f}(\xi)=\sum_{k\in\mathbb{Z}} b_k e^{-ik\xi}\widehat{g}(\xi), \tag{2.10}$$

where the series converges in $L^2(\mathbb{R})$. Finally, let us show that for almost all ξ,

$$\sum_{k\in\mathbb{Z}} b_k e^{-ik\xi}\widehat{g}(\xi)=\lambda_f(\xi)\widehat{g}(\xi),$$

where $\lambda_f\in L^2(\mathbb{R}/2\pi\mathbb{Z})$ is defined as $\lambda_f(\xi)=\sum_{k\in\mathbb{Z}} b_k e^{-ik\xi}$. To this end, it is sufficient to prove that

$$\lim_{N\to+\infty}\int_{\mathbb{R}}\Big|\lambda_f(\xi)-\sum_{k=-N}^{N} b_k e^{-ik\xi}\Big|^2|\widehat{g}(\xi)|^2\,d\xi=0. \tag{2.11}$$

By using the 2π-periodicity of the function $\Big|\lambda_f(\xi)-\sum_{k=-N}^{N} b_k e^{-ik\xi}\Big|^2$ as well as (2.5), one obtains that

$$\begin{aligned}\int_{\mathbb{R}}\Big|\lambda_f(\xi)-\sum_{k=-N}^{N} b_k e^{-ik\xi}\Big|^2|\widehat{g}(\xi)|^2\,d\xi &= \int_0^{2\pi}\Big|\lambda_f(\xi)-\sum_{k=-N}^{N} b_k e^{-ik\xi}\Big|^2\Big(\sum_{k\in\mathbb{Z}}|\widehat{g}(\xi+2k\pi)|^2\Big)\,d\xi\\ &\le c'\int_0^{2\pi}\Big|\lambda_f(\xi)-\sum_{k=-N}^{N} b_k e^{-ik\xi}\Big|^2\,d\xi\end{aligned}$$

and the latter inequality clearly implies that (2.11) holds. □

Proposition 2.9 *For every $j\in\mathbb{Z}$ denote by W_j the subspace of V_{j+1} defined by the condition*

$$V_{j+1}=V_j\oplus W_j; \tag{2.12}$$

note that we impose to the spaces V_j and W_j to be orthogonal. Then it follows from (2.2) that

$$W_j=\{f(2^jx)\,:\,f\in W_0\}. \tag{2.13}$$

Moreover, it follows from properties (a), (b) and (c) that for every $J \in \mathbb{Z}$,

$$L^2(\mathbb{R}) = V_J \oplus \Big(\bigoplus_{j=J}^{+\infty} W_j \Big) \text{ and } L^2(\mathbb{R}) = \bigoplus_{j=-\infty}^{+\infty} W_j.$$

Let us now give two examples of MRA's.

Example 2.10 Assume that for every $j \in \mathbb{Z}$,

$$V_j = \Big\{ f \in L^2(\mathbb{R}) \ : \ \forall\, k \in \mathbb{Z},\ f_{/_{[\frac{k}{2^j}, \frac{k+1}{2^j})}} = \text{constant} \Big\},$$

where $f_{/_{[\frac{k}{2^j}, \frac{k+1}{2^j})}}$ denotes the restriction of f to the dyadic interval $[\frac{k}{2^j}, \frac{k+1}{2^j})$, also assume that $g = \mathbb{1}_{[0,1)}$. Then $(V_j)_{j \in \mathbb{Z}}$ forms a MRA of $L^2(\mathbb{R})$ usually called Haar MRA; observe that in this case not only $\{g(x-k) : k \in \mathbb{Z}\}$ is a Riesz basis of V_0 but it is also an orthonormal basis of this space. $\square$

The main advantage of the Haar MRA is its simplicity. However, in this case the function g which generates V_0 is discontinuous and this might be a drawback. So, let us now introduce "a regularised version" of the Haar MRA.

Example 2.11 (a regularised version of Example 2.10) Let $m \geq 1$ be a fixed integer and for every $j \in \mathbb{Z}$ set

$$V_j = \Big\{ f \in L^2(\mathbb{R}) \cap C^{m-1}(\mathbb{R}) \ : \ \forall\, k \in \mathbb{Z},\ f_{/_{[\frac{k}{2^j}, \frac{k+1}{2^j})}} = \text{a polynomial of degree } m \Big\},$$

where $C^{m-1}(\mathbb{R})$ denotes the space of $m-1$ times continuously differentiable functions on the real line. Also assume that g is the B-spline of order m i.e. $g = \mathbb{1}_{[0,1)}^{*m}$ (g is the convolution product of the indicator of $[0,1)$, m times by itself. Then $(V_j)_{j \in \mathbb{Z}}$ forms a MRA of $L^2(\mathbb{R})$. Observe that $\operatorname{supp} g = [0, m+1]$ and also $\{g(x-k) : k \in \mathbb{Z}\}$ is not an orthonormal sequence but only a Riesz sequence. $\square$

Now we are going to give some general results concerning MRA's. The first result shows that starting from g, one can always construct a function φ such that $\{\varphi(x-l) : l \in \mathbb{Z}\}$ forms an orthonormal basis of V_0.

Proposition 2.12 *Let φ be the function defined as*

$$\widehat{\varphi}(\xi) = \frac{\widehat{g}(\xi)}{\Big(\sum_{k \in \mathbb{Z}} |\widehat{g}(\xi + 2k\pi)|^2 \Big)^{1/2}}. \tag{2.14}$$

Then there are two sequences $(a_l)_{l\in\mathbb{Z}}$ and $(b_l)_{l\in\mathbb{Z}}$ of $l^2(\mathbb{Z})$ such that

$$\varphi(x) = \sum_{l\in\mathbb{Z}} a_l g(x-l) \text{ and } g(x) = \sum_{l\in\mathbb{Z}} b_l \varphi(x-l), \tag{2.15}$$

where the series converge in $L^2(\mathbb{R})$; moreover $\{\varphi(x-l) : l \in \mathbb{Z}\}$ is an orthonormal basis of V_0. As a consequence, for all $J \in \mathbb{Z}$, $\{2^{J/2}\varphi(2^J x - l) : l \in \mathbb{Z}\}$ is an orthonormal basis of V_J.

In order to be able to show that Proposition 2.12 holds, we need the following lemma.

Lemma 2.13 *Let $h \in L^2(\mathbb{R})$. The following assertions are equivalent.*

(i) $\{h(x-k) : k \in \mathbb{Z}\}$ is an orthonormal sequence.

(ii) For almost all $\xi \in \mathbb{R}$,

$$\sum_{k\in\mathbb{Z}} |\widehat{h}(\xi + 2k\pi)|^2 = 1. \tag{2.16}$$

Proof of Lemma 2.13: Observe that the fact that $\{h(x-k) : k \in \mathbb{Z}\}$ is an orthonormal sequence is equivalent to: for all sequence $(d_k)_{k\in\mathbb{Z}}$ with a finite number of non vanishing terms, one has

$$\int_{\mathbb{R}} \Big| \sum_{k\in\mathbb{Z}} d_k h(x-k) \Big|^2 dx = \sum_{k\in\mathbb{Z}} |d_k|^2.$$

Thus, taking in Lemma 2.6 $c = c' = 1$, one obtains Lemma 2.13. □

We are now in position to prove Proposition 2.12.

Proof of Proposition 2.12: It is worth to notice that Lemma 2.6 entails that the 2π-periodic functions $\Big(\sum_{k\in\mathbb{Z}} \widehat{g}(\xi+2k\pi)|^2\Big)^{-1/2}$ and $\Big(\sum_{k\in\mathbb{Z}} \widehat{g}(\xi+2k\pi)|^2\Big)^{1/2}$ belong to $L^2(\mathbb{R}/2\pi\mathbb{Z})$. Therefore, there are two sequences $(a_l)_{l\in\mathbb{Z}}$ and $(b_l)_{l\in\mathbb{Z}}$ of $l^2(\mathbb{Z})$ such that

$$\lim_{N\to+\infty} \int_0^{2\pi} \Big| \Big(\sum_{k\in\mathbb{Z}} \widehat{g}(\xi + 2k\pi)|^2 \Big)^{-1/2} - \sum_{l=-N}^{N} a_l e^{-il\xi} \Big|^2 d\xi = 0 \tag{2.17}$$

and

$$\lim_{N\to+\infty} \int_0^{2\pi} \Big| \Big(\sum_{k\in\mathbb{Z}} \widehat{g}(\xi + 2k\pi)|^2 \Big)^{1/2} - \sum_{l=-N}^{N} b_l e^{-il\xi} \Big|^2 d\xi = 0. \tag{2.18}$$

Let us show that (2.17) and (2.18) imply that

$$\lim_{N\to+\infty} \int_{\mathbb{R}} \Big| \varphi(x) - \sum_{l=-N}^{N} a_l g(x-l) \Big|^2 dx = 0 \tag{2.19}$$

and

$$\lim_{N\to+\infty}\int_{\mathbb{R}}\Big|g(x)-\sum_{l=-N}^{N} b_l\varphi(x-l)\Big|^2\,dx=0. \tag{2.20}$$

It follows from the isometry property of Fourier transform, from (2.14) and from Lemma 2.6 that

$$\begin{aligned}
&\int_{\mathbb{R}}\Big|\varphi(x)-\sum_{l=-N}^{N} a_l g(x-l)\Big|^2\,dx\\
&=\int_{\mathbb{R}}\Big|\widehat{\varphi}(\xi)-\sum_{l=-N}^{N} a_l e^{-il\xi}\widehat{g}(\xi)\Big|^2\,d\xi\\
&=\int_{\mathbb{R}}\Big|\Big(\sum_{k\in\mathbb{Z}}\widehat{g}(\xi+2k\pi)|^2\Big)^{-1/2}-\sum_{l=-N}^{N} a_l e^{-il\xi}\Big|^2|\widehat{g}(\xi)|^2,d\xi\\
&=\int_0^{2\pi}\Big|\Big(\sum_{k\in\mathbb{Z}}|\widehat{g}(\xi+2k\pi)|^2\Big)^{-1/2}-\sum_{l=-N}^{N} a_l e^{-il\xi}\Big|^2\Big(\sum_{m\in\mathbb{Z}}|\widehat{g}(\xi+2m\pi)|^2\Big)\,d\xi\\
&\le c'\int_0^{2\pi}\Big|\Big(\sum_{k\in\mathbb{Z}}|\widehat{g}(\xi+2k\pi)|^2\Big)^{-1/2}-\sum_{l=-N}^{N} a_l e^{-il\xi}\Big|^2\,d\xi.
\end{aligned}$$

Finally combining the latter inequality with (2.17) one gets (2.19). (2.20) can be proved in the same way.

Let us now show that $\{\varphi(x-k)\,:\,k\in\mathbb{Z}\}$ is an orthonormal basis of V_0. It follows (2.14) and Lemma 2.13 that $\{\varphi(x-k)\,:\,k\in\mathbb{Z}\}$ is an orthonormal sequence. Moreover, (2.15) and the fact that $V_0=\overline{\mathrm{span}\{g(x-k)\,:\,k\in\mathbb{Z}\}}$ imply that $V_0=\overline{\mathrm{span}\{\varphi(x-k)\,:\,k\in\mathbb{Z}\}}$. $\square$

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Proposition 2.14 *Let φ be a function such that $\{\varphi(x-k)\,:\,k\in\mathbb{Z}\}$ is an orthonormal basis of V_0. Then there is a function $m_0\in L^2(\mathbb{R}/2\pi\mathbb{Z})$ such that for almost all $\xi\in\mathbb{R}$ one has*

$$\widehat{\varphi}(2\xi)=m_0(\xi)\widehat{\varphi}(\xi) \tag{2.21}$$

and

$$|m_0(\xi)|^2+|m_0(\xi+\pi)|^2=1. \tag{2.22}$$

Note that one usually calls φ a scaling function since it satisfies (2.21).

Proof of Proposition 2.14: It follows from (d) and (a) that $2^{-1}\varphi(2^{-1}x)\in V_{-1}\subset V_0$, moreover $\widehat{\varphi}(2\xi)$ is the Fourier transform of $2^{-1}\varphi(2^{-1}x)$. Thus using Proposition 2.5, (2.14) and Lemma 2.6 one can show that (2.21) is satisfied. Let us show that (2.22) also holds.

Lemma 2.13 combined with (2.21) and the 2π-periodicity of m_0, implies that for almost all $\xi \in \mathbb{R}$,

$$\begin{aligned} 1 = \sum_{k\in\mathbb{Z}} |\widehat{\varphi}(2\xi + 2k\pi)|^2 &= |m_0(\xi)|^2 \sum_{k\in\mathbb{Z}} |\widehat{\varphi}(\xi + 2l\pi)|^2 + |m_0(\xi+\pi)|^2 \sum_{k\in\mathbb{Z}} |\widehat{\varphi}(\xi + \pi + 2l\pi)|^2 \\ &= |m_0(\xi)|^2 + |m_0(\xi+\pi)|^2. \end{aligned}$$

□

Now we are going to see how on can construct an orthormal wavelet basis of $L^2(\mathbb{R})$ starting from a scaling function. Before stating the main result, it is useful to make the following remark.

Remark 2.15 We use use the same notations as in Proposition 2.14. Let $m_1 \in L^2(\mathbb{R}/2\pi\mathbb{Z})$ be the function defined for almost all $\xi \in \mathbb{R}$ as

$$m_1(\xi) = e^{-i\xi}\overline{m_0(\xi+\pi)}. \tag{2.23}$$

Then, one has for almost all $\xi \in \mathbb{R}$

$$|m_1(\xi)|^2 + |m_1(\xi+\pi)|^2 = 1 \tag{2.24}$$

and

$$m_0(\xi)\overline{m_1(\xi)} + m_0(\xi+\pi)\overline{m_1(\xi+\pi)} = 0 \tag{2.25}$$

□

Let us now state the main result of this section.

Theorem 2.16 *(Mallat and Meyer 1986) We use the same notations as in Proposition 2.14 and Remark 2.15. For almost all ξ we set*

$$\widehat{\psi}(2\xi) = m_1(\xi)\widehat{\varphi}(\xi). \tag{2.26}$$

Then:

(i) $\{\psi(x-k) \,:\, k \in \mathbb{Z}\}$ *is an orthonormal basis of* W_0.

(ii) For every $J \in \mathbb{Z}$, $\{2^{J/2}\varphi(2^J x - l) \,:\, l \in \mathbb{Z}\} \cup \{2^{j/2}\psi(2^j x - k) \,:\, j \in \mathbb{Z},\, j \geq J$ *and* $k \in \mathbb{Z}\}$ *is an orthonormal basis of* $L^2(\mathbb{R})$.

(iii) $\{2^{j/2}\psi(2^j x - k) \,:\, j \in \mathbb{Z}$ *and* $k \in \mathbb{Z}\}$ *is an orthonormal basis of* $L^2(\mathbb{R})$.

In order to be able to prove Theorem 2.16 we need the following two lemmas.

Lemma 2.17 *Let $f \in L^1(\mathbb{R}/2\pi\mathbb{Z})$ be a function whose all Fourier coefficients vanish i.e. $\int_0^{2\pi} e^{-ik\xi} f(\xi)\, d\xi = 0$ for all $k \in \mathbb{Z}$. Then for almost all $\xi \in \mathbb{R}$, $f(\xi) = 0$.*

Proof of Lemma 2.17: For each trigonometric polynomial $T(\xi)$ one has $\int_0^{2\pi} T(\xi) f(\xi)\, d\xi = 0$. Next assuming that

$$T(\xi) = \Big| \frac{1}{\sqrt{2\pi N}} \sum_{k=0}^{N-1} e^{ik(\xi_0 - \xi)} \Big|^2 = \frac{1}{2\pi N} \frac{\sin^2(N(\xi_0 - \xi)/2)}{\sin^2((\xi_0 - \xi)/2)},$$

where the real ξ_0 and the integer $N \geq 1$ are arbitrary and fixed, it follows that

$$\int_0^{2\pi} \frac{1}{2\pi N} \frac{\sin^2(N(\xi_0 - \xi)/2)}{\sin^2((\xi_0 - \xi)/2)} f(\xi)\, d\xi = 0.$$

From the other hand, Lemma 2.8 entails that there is a subsequence $m \mapsto N_m$ such that for almost all $\xi_0 \in \mathbb{R}$,

$$\lim_{m \to +\infty} \int_0^{2\pi} \frac{1}{2\pi N_m} \frac{\sin^2(N_m(\xi_0 - \xi)/2)}{\sin^2((\xi_0 - \xi)/2)} f(\xi)\, d\xi = f(\xi_0).$$

□

Lemma 2.18 *Let $F, G \in L^2(\mathbb{R})$, the following two assertions are equivalent.*

(i) One has for all $l \in \mathbb{Z}$, $\int_{\mathbb{R}} F(x) \overline{G(x-l)}\, dx = 0$.

(ii) One has for almost all $\xi \in \mathbb{R}$, $\sum_{k \in \mathbb{Z}} \widehat{F}(\xi + 2k\pi) \overline{\widehat{G}(\xi + 2k\pi)} = 0$.

Proof of Lemma 2.18: it follows from Plancherel formula and from the 2π periodicity of $e^{il\xi}$ that

$$\int_{\mathbb{R}} F(x) \overline{G(x-l)}\, dx = \int_{\mathbb{R}} e^{il\xi} \widehat{F}(\xi) \overline{\widehat{G}(\xi)}\, d\xi = \int_0^{2\pi} e^{il\xi} \Big(\sum_{k \in \mathbb{Z}} \widehat{F}(\xi + 2k\pi) \overline{\widehat{G}(\xi + 2k\pi)} \Big)\, d\xi,$$

which means that the sequence $\Big(\int_{\mathbb{R}} F(x) \overline{G(x-l)}\, dx \Big)_{l \in \mathbb{Z}}$ is the sequence of the Fourier coefficients of the function $\sum_{k \in \mathbb{Z}} \widehat{F}(\xi + 2k\pi) \overline{\widehat{G}(\xi + 2k\pi)}$. Then Lemma 2.17 allows us to finish our proof. □

Proof of Theorem 2.16: First notice that (2.26) implies that for every $k \in \mathbb{Z}$, $\frac{1}{2}\psi(\frac{x}{2} - k) \in V_0$ and consequently that $\psi(x-k) \in V_1$.

Let us show that $\psi(x-k) \in W_0$. It is sufficient to show that $\int_{\mathbb{R}} \psi(x-k)\overline{\varphi(x-l)}\, dx = 0$ for all $l \in \mathbb{Z}$, which is equivalent to $\int_{\mathbb{R}} \psi(x)\overline{\varphi(x-l)}\, dx = 0$ for all $l \in \mathbb{Z}$. In view of Lemma 2.18, this remains to show that for almost all $\xi \in \mathbb{R}$,

$$\sum_{k\in\mathbb{Z}} \widehat{\varphi}(\xi + 2k\pi)\overline{\widehat{\psi}(\xi + 2k\pi)} = 0. \tag{2.27}$$

By using (2.21), (2.26), the equality $\sum_{m\in\mathbb{Z}} |\widehat{\varphi}(\eta+2m\pi)|^2 = 1$ for almost all $\eta \in \mathbb{R}$ and (2.25) one obtains that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} \widehat{\varphi}(\xi + 2k\pi)\overline{\widehat{\psi}(\xi + 2k\pi)} \\
&= m_0(\frac{\xi}{2})\overline{m_1(\frac{\xi}{2})} \sum_{m\in\mathbb{Z}} |\widehat{\varphi}(\frac{\xi}{2} + 2m\pi)|^2 + m_0(\frac{\xi}{2} + \pi)\overline{m_1(\frac{\xi}{2} + \pi)} \sum_{m\in\mathbb{Z}} |\widehat{\varphi}(\frac{\xi}{2} + \pi + 2m\pi)|^2 \\
&= m_0(\frac{\xi}{2})\overline{m_1(\frac{\xi}{2})} + m_0(\frac{\xi}{2} + \pi)\overline{m_1(\frac{\xi}{2} + \pi)} = 0
\end{aligned}$$

and thus one gets (2.27).

Let us now show that $\{\psi(x-k) \,:\, k \in \mathbb{Z}\}$ *is an orthonormal sequence.* In view of Lemma 2.13, it is sufficient to prove that for almost all $\xi \in \mathbb{R}$,

$$\sum_{k\in\mathbb{Z}} |\widehat{\psi}(\xi + 2k\pi)|^2 = 1.$$

This is a straightforward consequence of (2.26), the equality $\sum_{m\in\mathbb{Z}} |\widehat{\varphi}(\eta + 2m\pi)|^2 = 1$ for almost all $\eta \in \mathbb{R}$ and (2.24).

Let us now show that $W_0 = \overline{\mathit{span}\{\psi(x-k) \,:\, k \in \mathbb{Z}\}}$. In view of the equalities $V_0 = \overline{\mathrm{span}\{\varphi(x-k) \,:\, k \in \mathbb{Z}\}}$ and $V_1 = V_0 \oplus W_0$, this is equivalent to show that

$$V_1 = \overline{\mathrm{span}\{\varphi(x-k),\, \psi(x-k) \,:\, k \in \mathbb{Z}\}}$$

i.e. for any $f \in V_1$ satisfying for all $k \in \mathbb{Z}$,

$$\int_{\mathbb{R}} \varphi(x-k)\overline{f(x)}\, dx = \int_{\mathbb{R}} \psi(x-k)\overline{f(x)}\, dx = 0, \tag{2.28}$$

one has $f(x) = 0$ for almost all $x \in \mathbb{R}$. Observe that the fact that $f \in V_1$, Proposition 2.4 and Proposition 2.5 imply that there is $\lambda_f \in L^2(\mathbb{R}/2\pi\mathbb{Z})$ such that for almost all $\xi \in \mathbb{R}$ one has

$$\widehat{f}(\xi) = \lambda_f(\xi/2)\widehat{\varphi}(\xi/2). \tag{2.29}$$

Using (2.28), Plancherel formula, (2.29), (2.21) and (2.26) one gets that

$$\int_{\mathbb{R}} e^{-i2k\xi} m_0(\xi)\lambda_f(\xi)|\widehat{\varphi}(\xi)|^2\, d\xi = 0 \tag{2.30}$$

and

$$\int_{\mathbb{R}} e^{-i2k\xi} m_1(\xi)\lambda_f(\xi)|\widehat{\varphi}(\xi)|^2 \, d\xi = 0. \tag{2.31}$$

(2.30), (2.31), the equality $\sum_{k\in\mathbb{Z}} |\widehat{\varphi}(\xi+2k\pi)|^2 = 1$ for almost all $\xi \in \mathbb{R}$ and the 2π-periodicity of $m_0(\xi)\lambda_f(\xi)$ and $m_1(\xi)\lambda_f(\xi)$ imply that

$$\int_0^{2\pi} e^{-i2k\xi} m_0(\xi)\lambda_f(\xi) \, d\xi = \int_0^{2\pi} e^{-i2k\xi} m_1(\xi)\lambda_f(\xi) \, d\xi = 0.$$

Thus we obtain that for all $p \in \mathbb{Z}$

$$\int_0^{2\pi} e^{-ip\xi} \Big(m_0(\xi)\lambda_f(\xi) + m_0(\xi+\pi)\lambda_f(\xi+\pi) \Big) \, d\xi = 0$$

and

$$\int_0^{2\pi} e^{-ip\xi} \Big(m_1(\xi)\lambda_f(\xi) + m_1(\xi+\pi)\lambda_f(\xi+\pi) \Big) \, d\xi = 0.$$

Therefore using Lemma 2.17 we get that for almost all $\xi \in \mathbb{R}$,

$$\begin{cases} m_0(\xi)\lambda_f(\xi) + m_0(\xi+\pi)\lambda_f(\xi+\pi) = 0 \\ m_1(\xi)\lambda_f(\xi) + m_1(\xi+\pi)\lambda_f(\xi+\pi) = 0. \end{cases} \tag{2.32}$$

Moreover, it follows (2.23) and (2.22) that for almost all $\xi \in \mathbb{R}$,

$$\det \begin{pmatrix} m_0(\xi) & m_0(\xi+\pi) \\ m_1(\xi) & m_1(\xi+\pi) \end{pmatrix} = -e^{-i\xi} \neq 0.$$

Thus (2.32) entails that $\lambda_f(\xi) = 0$ for almost all $\xi \in \mathbb{R}$.

We have proved that $\{\psi(x-k) \,:\, k \in \mathbb{Z}\}$ is an orthonormal basis of W_0. Then (2.13) implies that for all $j \in \mathbb{Z}$, $\{2^{j/2}\psi(2^j x - k) \,:\, k \in \mathbb{Z}\}$ is an orthonormal basis of W_j. From the other hand, for all $J \in \mathbb{Z}$, $\{2^{J/2}\varphi(2^J x - l) \,:\, l \in \mathbb{Z}\}$ is an orthonormal basis of V_J (see Proposition 2.12). Thus it follows from the last part of Proposition 2.9, that $\{2^{J/2}\varphi(2^J x - l) \,:\, l \in \mathbb{Z}\} \cup \{2^{j/2}\psi(2^j x - k) \,:\, j \in \mathbb{Z},\, j \geq J \text{ and } k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$ and $\{2^{j/2}\psi(2^j x - k) \,:\, j \in \mathbb{Z} \text{ and } k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$ as well. $\square$

To end this section, let us state (whithout proof) a result which will be quite useful in the sequel.

First recall that $S(\mathbb{R})$ the Schwartz class is the space of infinitely differentiable functions f which satisfy for all $p \in \mathbb{N}$ and $N \in \mathbb{N}$,

$$\sup_{x\in\mathbb{R}} \Big(1+|x|\Big)^N |f^{(p)}(x)| < +\infty, \tag{2.33}$$

where $f^{(p)}$ denotes the derivative of f of order p (with the covention that $f^{(0)} = f$).

Theorem 2.19 *(Meyer 1985) There exist a scaling function φ and a mother wavelet ψ which generate an orthonormal basis of $L^2(\mathbb{R})$ and satisfy the following properties*

(i) $\varphi,\, \psi \in S(\mathbb{R})$.

(ii) *supp* $\widehat{\varphi} \subset [-\frac{4\pi}{3}, \frac{4\pi}{3}]$ *and for all* $\xi \in [-\frac{2\pi}{3}, \frac{2\pi}{3}]$, $\widehat{\varphi}(\xi) = 1$.

(iii) *supp* $\widehat{\psi} \subset \{\xi \in \mathbb{R} \,:\, \frac{2\pi}{3} \le |\xi| \le \frac{8\pi}{3}\}$.

A wavelet basis generated by such φ and ψ is called a Meyer wavelet basis.

3 Fractional Brownian Motion

The goal of this section is to present some results concerning the uniform and the pointwise Hölder regularity of fractional Brownian motion (FBM). We refer to the books [11, 12, 27] for detailed presentations of the properties of FBM and related processes.

Let $\{X(t) \,:\, t \in \mathbb{R}\}$ be a centered real-valued Gaussian process, recall that its law (i.e. its distribution) is completely determined by its covariance function defined for all $t_1, t_2 \in \mathbb{R}$ as $\mathbb{E}X(t_1)X(t_2)$.

One says that $\{X(t) \,:\, t \in \mathbb{R}\}$ is self-similar of parameter H (where $H \in (0,1)$) if for all $a > 0$

$$\text{law}\{x(at) \,:\, t \in \mathbb{R}\} = \text{law}\{a^H X(t) \,:\, t \in \mathbb{R}\}; \tag{3.1}$$

Observe that (3.1) implies that $X(0) = 0$ almost surely. Indeed, one has for all $a > 0$, $\mathbb{E}|X(0)|^2 = \mathbb{E}|X(a0)|^2 = a^{2H}\mathbb{E}|X(0)|^2$, then letting a goes to 0 one obtains that $\mathbb{E}|X(0)|^2 = 0$.

One says that $\{X(t) \,:\, t \in \mathbb{R}\}$ is with stationary increments if for all $t_1, t_2 \in \mathbb{R}$,

$$\text{law}(X(t_2) - X(t_1)) = \text{law}(X(t_2 - t_1) - X(0)). \tag{3.2}$$

Proposition 3.1 *For every $H \in (0,1)$, up to a multiplicative constant c_H, there is a unique (in distribution) H-self-similar and stationary increments Gaussian process. This process is called FBM of Hurst parameter H and denoted $\{B_H(t) \,:\, t \in \mathbb{R}\}$. Moreover its covariance function satisfies for all $t_1, t_2 \in \mathbb{R}$,*

$$\mathbb{E}B_H(t_1)B_H(t_2) = \frac{c_H}{2}\Big\{|t_1|^{2H} + |t_2|^{2H} - |t_1 - t_2|^{2H}\Big\}. \tag{3.3}$$

Proof of Proposition 3.1: Let $\{X(t) \,:\, t \in \mathbb{R}\}$ be a H-self-similar and stationary increments Gaussian process. It follows from (3.2) and (3.1) that for every reals t_1 and t_2 one has

$$\mathbb{E}|X(t_2) - X(t_1)|^2 = E|X(|t_2 - t_1|)|^2 = c_H|t_2 - t_1|^{2H},$$

where $c_H = \mathbb{E}|X(1)|^2$. From the other hand one has

$$\mathbb{E}X(t_1)X(t_2) = \frac{1}{2}\Big\{\mathbb{E}|X(t_1)|^2 + \mathbb{E}|X(t_2)|^2 - \mathbb{E}|X(t_1) - X(t_2)|^2\Big\}.$$

$\square$

Remark 3.2 FBM reduces to a Brownian Motion (BM) when $H = 1/2$. FBM was introduced in 1940 by Kolmogorov as a way to generate Gaussian "spirals" in Hilbert space [16] and it was made popular by Mandelbrot and Van Ness [21] in 1968. It is quite useful in many areas: biology, hydrology, geology, telecommunications and so on. One of its main advantages with respect to BM is that its increments are correlated and they even display long-range dependence when $H > 1/2$:

$$\sum_{l=0}^{+\infty} |\mathbb{E}\{(B_H(l+1) - B_H(l))(B_H(1) - B_H(0))| = +\infty.$$

Indeed, it follows from (3.3) that for all $|l|$ big enough,

$$\begin{aligned}
|\mathbb{E}\{(B_H(l+1) - B_H(l))(B_H(1) - B_H(0))| &= \frac{c_H}{2}\Big||l+1|^{2H} - 2|l|^{2H} + |l-1|^{2H}\Big| \\
&= \frac{c_H}{2}|l|^{2H}\Big||1+l^{-1}|^{2H} + |1-l^{-1}|^{2H} - 2\Big| \\
&\sim \frac{c_H 2H|2H-1|}{4}|l|^{2H-2}.
\end{aligned}$$

$\square$

Proposition 3.3 *(stochastic integral representations of FBM)*

(i) Non anticipative moving average representation: up to a multiplicative constant

$$law\{B_H(t) \,:\, t \in \mathbb{R}\} = law\Big\{\int_{\mathbb{R}} \Big((t-s)_+^{H-1/2} - (-s)_+^{H-1/2}\Big)\, dW(s) \,:\, t \in \mathbb{R}\Big\}, \tag{3.4}$$

where $(x)_+^{H-1/2} = x^{H-1/2}$ if $x > 0$ and $(x)_+^{H-1/2} = 0$ else, and where dW is the real-valued white noise i.e. the derivative in the sense of tempered distribution of the Wiener process.

(ii) Harmonizable representation: up to a multiplicative constant

$$law\{B_H(t) \,:\, t \in \mathbb{R}\} = law\Big\{\int_{\mathbb{R}} \frac{e^{it\xi} - 1}{(i\xi)^{H+1/2}}\, d\widehat{W}(\xi) \,:\, t \in \mathbb{R}\Big\}, \tag{3.5}$$

where $(i\xi)^{H+1/2} = |\xi|^{H+1/2}e^{i(H+1/2)\pi/2}$ *if* $\xi > 0$ *and* $(i\xi)^{H+1/2} = |\xi|^{H+1/2}e^{-i(H+1/2)\pi/2}$ *if* $\xi < 0$ *and where* $d\widehat{W}$ *is the complex-valued white-noise obtained by "Fourier transformation" of the real-valued white noise i.e. for every* $f \in L^2(\mathbb{R})$,

$$\int_{\mathbb{R}} f(s)\, dW(s) = \int_{\mathbb{R}} \widehat{f}(\xi)\, d\widehat{W}(\xi). \tag{3.6}$$

Let us now determine the uniform Hölder regularity of the trajectories of FBM.

Proposition 3.4 $\{B_H(t) \,:\, t \in \mathbb{R}\}$ *has a modification* $\{\widetilde{B}_H(t) \,:\, t \in \mathbb{R}\}$ *whose trajectories satisfy, with probability* 1, *a uniform Hölder condition of any order* $\gamma < H$ *on each compact subset* $K \subset \mathbb{R}$ *i.e. there is an event* Ω^* *of probability* 1 *such that for all* $\omega \in \Omega^*$, *all* $\gamma \in (0, H)$ *and all* $t_1, t_2 \in K$,

$$|\widetilde{B}_H(t_1, \omega) - \widetilde{B}_H(t_2, \omega)| \leq c(\omega)|t_1 - t_2|^{\gamma}, \tag{3.7}$$

where $C > 0$ *is a random variable of finite moment of any order only depending on* K *and* γ. *From now on* $\{B_H(t) \,:\, t \in \mathbb{R}\}$ *will be identified with* $\{\widetilde{B}_H(t) \,:\, t \in \mathbb{R}\}$.

Proposition 3.4 is a straightforward consequence of the following lemma.

Lemma 3.5 *Let* T *be an arbitrary positive real number and let* $\{X(t) \,:\, t \in [-T, T]\}$ *be a Gaussian process. Assume that for all* $t_1, t_2 \in [-T, T]$,

$$\mathbb{E}|X(t_1) - X(t_2)|^2 \leq c|t_1 - t_2|^{2H}, \tag{3.8}$$

where $H \in (0, 1)$ *and* $c > 0$ *are two constants. Then there exists a modification of* $\{X(t) \,:\, t \in [-T, T]\}$ *whose trajectories satisfy, with probability* 1, *a uniform Hölder condition of any order* $\gamma \in (0, H)$.

Lemma 3.5 can be obtained (see for instance [3, 12]) by using the equivalence of Gaussian moments (i.e. if Z is a centered Gaussian random variable then for all real $p > 0$ there is a constant $c(p) > 0$ only depending on p such that $\mathbb{E}|Z|^p = (\mathbb{E}|Z|^2)^{p/2}$) and the following generalized version of Kolmogorov criterion (see for instance [15]).

Lemma 3.6 *(a generalized version of Kolmogorov criterion) If a stochastic process* $\{X(t) \,:\, t \in [-T, T]\}$ *satisfies for all* $t_1, t_2 \in [-T, T]$,

$$\mathbb{E}|X(t_1) - X(t_2)|^{\delta} \leq c|t_1 - t_2|^{1+\varepsilon},$$

for some constants $\delta > 0$, $\varepsilon > 0$ *and* $c > 0$. *Then* $\{X(t) \,:\, t \in [-T, T]\}$ *has a modification whose trajectories satisfy with probability* 1 *a uniform Hölder condition of any order* $\gamma \in [0, \varepsilon/\delta)$.

Proposition 3.7 *With probability* 1*, the trajectories of* $\{B_H(t) \,:\, t \in \mathbb{R}\}$ *fail to satisfy a uniform Hölder condition of any order* $\gamma > H$ *on any interval* I *with non empty interior.*

PROOF OF PROPOSITION 3.7: Assume ad absurdum that there are I a compact interval with non empty interior and $\varepsilon > 0$ such that

$$\mathbb{P}\Big(\sup_{t_1,t_2\in I} \frac{|B_H(t_1) - B_H(t_2)|}{|t_1 - t_2|^{H+\varepsilon}} < \infty\Big) > 0,$$

with the convention that $\frac{B_H(t_1)-B_H(t_2)}{|t_1-t_2|^{H+\varepsilon}} = 0$ if $t_1 = t_2$. Then it follows from the zero-one law for Gaussian processes (see for instance [19]) that, almost surely,

$$\sup_{t_1,t_2\in I} \frac{|B_H(t_1) - B_H(t_2)|}{|t_1 - t_2|^{H+\varepsilon}} < \infty.$$

Finally using a Borell type inequality (see for instance [1]) one gets that

$$\mathbb{E}\Big(\sup_{t_1,t_2\in I} \frac{|B_H(t_1) - B_H(t_2)|^2}{|t_1 - t_2|^{2H+2\varepsilon}}\Big) < \infty.$$

But this is impossible since

$$\lim_{|t_1-t_2|\to 0} \frac{\mathbb{E}|B_H(t_1) - B_H(t_2)|^2}{|t_1 - t_2|^{2H+2\varepsilon}} = +\infty.$$

□

Propositions 3.4 and 3.7 show that the critical uniform Hölder exponent of a typical trajectory of FBM is H. Let us now determine the critical pointwise Hölder regularity of a typical trajectory of FBM. To this end, we need to define the notion of pointwise Hölder exponent, for sake of simplicity we will restrict to the setting of locally bounded and nowhere differentiable functions, note in passing that the notion of pointwise Hölder exponent can be defined in a much more general setting (see for instance [2, 7]).

Definition 3.8 *Let* $\{X(t) \,:\, t \in \mathbb{R}\}$ *be a stochastic process whose trajectories are with probability* 1 *locally bounded and nowhere differentiable functions. Let* $t_0 \in \mathbb{R}$ *be fixed.* $\alpha_X(t_0)$ *the pointwise Hölder exponent of* X *at* t_0 *is defined as*

$$\alpha_X(t_0) = \sup\Big\{\alpha \in \mathbb{R}_+ \,:\, \limsup_{h\to 0} \frac{|X(t_0+h) - X(t_0)|}{|h|^\alpha} = 0\Big\}.$$

$\alpha_X(t_0)$ *is with values in* $[0,1]$*, the smaller it is the more oscillating is the process* X *in a neighbourhood of* t_0*.*

Let us now state the main result of this section.

Theorem 3.9 *The pointwise Hölder exponent of FBM satisfies the following property: There is Ω^* an event with probability 1 such that for all $t_0 \in \mathbb{R}$ and $\omega \in \Omega^*$ one has*

$$\alpha_{B_H}(t_0, \omega) = H.$$

Remarks 3.10

(i) The difficult part in the proof of Theorem 3.9 is the inequality $\alpha_{B_H}(t_0, \omega) \leq H$, the other inequality, namely $\alpha_{B_H}(t_0, \omega) \geq H$, is a straightforward consequence of the fact that a typical trajectory of FBM satisfies a uniform Hölder condition of any order $\gamma \in (0, H)$.

(ii) The event Ω^* in Theorem 3.9 does not depend on t_0; if one allows this event to depend on t_0 then the theorem easily results from the following lemma which has been obtained in [3].

□

Lemma 3.11 *Let $\{X(t) : t \in \mathbb{R}\}$ be a centered Gaussian process whose trajectories are continuous with probability 1 and let $t_0 \in \mathbb{R}$ be a fixed point. Assume that there is a nonnegative real δ which satisfies for all arbitrarily small $\varepsilon > 0$*

$$\limsup_{h \to 0} \frac{\mathbb{E}|X(t_0 + h) - X(t_0)|^2}{|h|^{2\delta + \varepsilon}} = +\infty. \tag{3.9}$$

Then there exists $\Omega^*_{t_0}$, an event of probability 1 which a priori depends on t_0, such that one has for all $\omega \in \Omega^*_{t_0}$, $\alpha_X(t_0, \omega) \leq \delta$.

Remark 3.12 In Lemma 3.11, for sake of simplicity, we have assumed that that the trajectories of $\{X(t) : t \in \mathbb{R}\}$ are continuous with probability 1, however this hypothesis can be weakned. □

Proof of Lemma 3.11: Assume ad absurdum that there is $\varepsilon_0 > 0$ such that

$$\mathbb{P}(\alpha_X(t_0) > \delta + \varepsilon_0/2) > 0.$$

This implies that

$$\mathbb{P}\Big(\sup_{h \in [-1,1]} \frac{|X(t_0 + h) - X(t_0)|}{|h|^{\delta + \varepsilon_0/2}} < \infty\Big) > 0,$$

with the convention that $\frac{X(t_0+h)-X(t_0)}{|h|^{\delta+\varepsilon_0/2}} = 0$ if $h = 0$. Then, it follows from the zero-one law for Gaussian processes (see for instance [19]) that, almost surely,

$$\sup_{h\in[-1,1]} \frac{|X(t_0+h)-X(t_0)|}{|h|^{\delta+\varepsilon_0/2}} < \infty.$$

Next, using a Borell type inequality (see for instance [1]) one gets that

$$\mathbb{E}\Big(\sup_{h\in[-1,1]} \frac{|X(t_0+h)-X(t_0)|^2}{|h|^{2\delta+\varepsilon_0}}\Big) < \infty,$$

which contradicts (3.9). $\square$

We will prove Theorem 3.9 by means of a wavelet method which uses the same ideas as in [6, 5]. To this end, now we are going to present one of the most standard wavelet series representation of FBM which has been in introduced in [25] (and in another form in [8]). First, it is convenient to define the left-sided fractional primitive of order $H + 1/2$ and the right-sided fractional derivative of order $H+1/2$ of a Meyer mother wavelet. We refer to the book [26] for a detailed presentation of the notions of fractional integrals and derivatives.

Definition 3.13 *Let ψ be a Meyer mother wavelet.*

(i) The left-sided fractional primitive of order $H + 1/2$ of ψ is denoted by Ψ_H and defined for all $x \in \mathbb{R}$ as

$$\Psi_H(x) = \int_{\mathbb{R}} e^{ix\xi} \frac{\widehat{\psi}(\xi)}{(i\xi)^{H+1/2}}\, d\xi. \tag{3.10}$$

(ii) The right-sided fractional derivative of order $H + 1/2$ of ψ is denoted by Ψ_{-H} and defined for all $x \in \mathbb{R}$ as

$$\Psi_{-H}(x) = \int_{\mathbb{R}} e^{ix\xi}\widehat{\psi}(\xi)(-i\xi)^{H+1/2}\, d\xi. \tag{3.11}$$

Remarks 3.14

- $\Psi_H, \Psi_{-H} \in S(\mathbb{R})$ since $\psi \in S(\mathbb{R})$ and $\widehat{\psi}$ is compactly supported and vanishes in neighbourhood of zero.
- By using the fact that ψ is real-valued one can show that Ψ_H and Ψ_{-H} are real-valued.

$\square$

Theorem 3.15 *Let* $\{B_H(t) \,:\, t \in \mathbb{R}\} = \left\{ \int_{\mathbb{R}} \frac{e^{it\xi}-1}{(i\xi)^{H+1/2}}\, d\widehat{W}(\xi) \,:\, t \in \mathbb{R} \right\}$ *be the FBM. Let* $\{\varepsilon_{j,k} \,:\, (j,k) \in \mathbb{Z}\times\mathbb{Z}\}$ *be the sequence of i.i.d. real-valued* $\mathcal{N}(0,1)$ *Gaussian random variables defined as*

$$\varepsilon_{j,k} = \int_{\mathbb{R}} 2^{-j/2} e^{ik2^{-j}\xi} \widehat{\psi}(-2^{-j}\xi)\, d\widehat{W}(\xi) = \int_{\mathbb{R}} 2^{j/2} \psi(-2^{j}t - k)\, dW(\xi). \tag{3.12}$$

Then $\{B_H(t) \,:\, t \in \mathbb{R}\}$ *can be represented as*

$$B_H(t) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} 2^{-jH} \varepsilon_{j,k} (\Psi_H(2^j t - k) - \Psi_H(-k)), \tag{3.13}$$

where the series is with probability 1 *uniformly convergent in* t *on each compact subset of* $\mathbb{R}$.

Proof of Theorem 3.15: First observe that for every $(j,k) \in \mathbb{Z}\times\mathbb{Z}$, the Fourier transform of the function $2^{j/2}\psi(2^j x - k)$ is $2^{-j/2}e^{-ik2^{-j}\xi}\widehat{\psi}(2^{-j}\xi)$. Also observe that

$$\overline{2^{-j/2}e^{-ik2^{-j}\xi}\widehat{\psi}(2^{-j}\xi)} = 2^{-j/2}e^{ik2^{-j}\xi}\widehat{\psi}(-2^{-j}\xi),$$

since the Meyer wavelet ψ is real-valued. By using the fact that

$$\{2^{j/2}\psi(2^j x - k) \,:\, j \in \mathbb{Z},\, k \in \mathbb{Z}\}$$

is an orhtonormal basis of $L^2(\mathbb{R})$ and the isometry property of Fourier transform, it follows that

$$\{2^{-j/2}e^{ik2^{-j}\xi}\widehat{\psi}(-2^{-j}\xi) \,:\, j \in \mathbb{Z},\, k \in \mathbb{Z}\}$$

is an orthonormal basis of $L^2(\mathbb{R})$. By expanding for every fixed $t \in \mathbb{R}$, the function $\xi \mapsto \frac{e^{it\xi}-1}{(i\xi)^{H+1/2}}$ in the latter basis, it follows that

$$\frac{e^{it\xi}-1}{(i\xi)^{H+1/2}} = \sum_{(j,k)\in\mathbb{Z}\times\mathbb{Z}} c_{j,k}(t) 2^{-j/2} e^{ik2^{-j}\xi} \widehat{\psi}(-2^{-j}\xi), \tag{3.14}$$

where the series is convergent in $L^2(\mathbb{R})$ and

$$c_{j,k}(t) = 2^{-j/2} \int_{\mathbb{R}} \frac{e^{it\xi}-1}{(i\xi)^{H+1/2}} e^{-ik2^{-j}\xi} \widehat{\psi}(2^{-j}\xi)\, d\xi;$$

observe that by setting in the latter integral $\eta = 2^{-j}\xi$, one has that

$$c_{j,k}(t) = 2^{-jH}(\Psi_H(2^j t - k) - \Psi_H(-k)).$$

Next, using (3.14) and the isometry property of Wiener integral, one obtains that

$$\int_{\mathbb{R}} \frac{e^{it\xi}-1}{(i\xi)^{H+1/2}}\, d\widehat{W}(\xi) = \sum_{(j,k)\in\mathbb{Z}\times\mathbb{Z}} c_{j,k}(t)\varepsilon_{j,k}, \tag{3.15}$$

where $\varepsilon_{j,k} = \int_{\mathbb{R}} 2^{-j/2} e^{ik2^{-j}\xi} \widehat{\psi}(-2^{-j}\xi)\, d\widehat{W}(\xi)$. A priori, the series in (3.15) is, for every fixed t, convergent in $L^2(\Omega)$, Ω being the underlying probability space. Let us show that it is also, with probability 1, uniformly convergent in $t \in K$, K being an arbitrary compact subset of $\mathbb{R}$. We denote by $(D_n)_{n\in\mathbb{N}}$ a sequence of finite subsets of $Z \times \mathbb{Z}$ satisfying $D_n \subset D_{n+1}$ for every n and $Z \times \mathbb{Z} = \cup_{n\in\mathbb{N}} D_n$. It follows from Theorem 12.3 in [9] that the functional sequence $(\sum_{(j,k)\in D_n} c_{j,k}(\cdot)\varepsilon_{j,k})_{n\in\mathbb{N}}$ is weakly relatively compact in $C(K)$ the space of continuous function over K equipped with the uniform norm. Indeed, one has for every $t_1, t_2 \in K$

$$\begin{aligned}\mathbb{E}\Big| \sum_{(j,k)\in D_n} (c_{j,k}(t_1) - c_{j,k}(t_2))\varepsilon_{j,k} \Big|^2 \leq \sum_{(j,k)\in\mathbb{Z}\times\mathbb{Z}} |c_{j,k}(t_1) - c_{j,k}(t_2)|^2 &= \int_{\mathbb{R}} \frac{|e^{i(t_1-t_2)\xi} - 1|^2}{|\xi|^{2H+1}}\, d\xi \\ &= c|t_1 - t_2|^{2H},\end{aligned}$$

where $c > 0$ is a constant non depending on n. Finally, it follows from Itô-Nisio Theorem (see Theorem 2.1.1 in [17]) that, with probability 1, $\sum_{(j,k)\in D_n} c_{j,k}(t)\varepsilon_{j,k}$ converges to $\int_{\mathbb{R}} \frac{e^{it\xi}-1}{(i\xi)^{H+1/2}}\, d\widehat{W}(\xi)$ in $C(K)$ as $n \to +\infty$. □

Let us now state a very useful lemma, whose proof mainly relies on Borel-Cantelli Lemma (see for instance [25, 4]).

Lemma 3.16 *There are an event Ω_1^* of probability 1 and a random variable C_1 of finite moment of any order such that one has for all $\omega \in \Omega_1^*$, $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$*

$$|\varepsilon_{j,k}(\omega)| \leq C_1(\omega)\sqrt{\log(2 + |j| + |k|)}.$$

We also need the following proposition which provides a sharp estimation of the asymtotic behaviour of FBM at infinity.

Proposition 3.17 *There is a random variable C_2 of finite moment of any order such that for all $\omega \in \Omega_1^*$ and $t \in \mathbb{R}$ one has*

$$|B_H(t,\omega)| \leq \sum_{(j,k)\in\mathbb{Z}^2} 2^{-jH}|\varepsilon_{j,k}(\omega)||\Psi_H(2^j t - k) - \Psi_H(-k)| \leq C_2(\omega)(1+|t|)^H \sqrt{|\log|\log(2+|t|)|}. \tag{3.16}$$

The proof of Proposition 3.17 is a bit technical, it relies on Lemma 3.16 as well as on the fact that $\Psi_H \in S(\mathbb{R})$. We refer to [5] for a proof of a more general result in the setting of Fractional Brownian Sheets.

Proposition 3.18 *For all $\omega \in \Omega_1^*$ and $(j,k) \in \mathbb{Z}\times\mathbb{Z}$ one has*

$$\varepsilon_{j,k}(\omega) = 2^{j(1+H)} \int_{\mathbb{R}} B_H(t,\omega)\Psi_{-H}(2^j t - k)\, dt. \tag{3.17}$$

In order to be able to prove Proposition 3.18 we need the following lemma.

Lemma 3.19 *(i)* $\int_{\mathbb{R}} \Psi_{-H}(t)\,dt = \int_{\mathbb{R}} \Psi_H(t)\,dt = 0.$

(ii) For all $(j,k) \in \mathbb{Z}\times\mathbb{Z}$ *and* $(j',k') \in \mathbb{Z}\times\mathbb{Z}$ *one has*

$$2^{(j'+j)/2}\int_{\mathbb{R}} \Psi_H(2^{j'}t-k')\Psi_{-H}(2^jt-k)\,dt = \begin{cases} 1 \ \textit{if}\ (j,k)=(j',k') \\ 0 \ \textit{else}. \end{cases}$$

Proof of Lemma 3.19: Let us first give the proof of part (i). One has $\int_{\mathbb{R}} \Psi_{-H}(t)\,dt = \sqrt{2\pi}\widehat{\Psi}_{-H}(0) = 0$, since $\widehat{\Psi}_{-H}(\xi) = (-i\xi)^{H+1/2}\widehat{\psi}(\xi)$ and $\widehat{\psi}$ vanishes in a neighbourhood of 0. Similarly, one can show that $\int_{\mathbb{R}} \Psi_H(t)\,dt = 0$.

Let us now give the proof of part (ii). It follows from Plancherel formula that

$$\begin{aligned}
&2^{(j'+j)/2}\int_{\mathbb{R}} \Psi_H(2^{j'}t-k')\Psi_{-H}(2^jt-k)\,dt \\
&= 2^{-(j'+j)/2}\int_{\mathbb{R}} e^{-ik'2^{-j'}\xi}(i2^{-j'}\xi)^{-H-1/2}\widehat{\psi}(2^{-j'}\xi)\overline{e^{-ik2^{-j}\xi}(-i2^{-j}\xi)^{H+1/2}\widehat{\psi}(2^{-j}\xi)}\,d\xi \\
&= 2^{(H+1/2)(j'-j)-(j'+j)/2}\int_{\mathbb{R}} e^{-ik'2^{-j'}\xi}\widehat{\psi}(2^{-j'}\xi)\overline{e^{-ik2^{-j}\xi}\widehat{\psi}(2^{-j}\xi)}\,d\xi \\
&= 2^{(H+1/2)(j'-j)+(j'+j)/2}\int_{\mathbb{R}} \psi(2^{j'}t-k')\psi(2^jt-k)\,dt.
\end{aligned}$$

Finally, by using the orthonormality of the functions $2^{j/2}\psi(2^jt-k)$, $j\in\mathbb{Z}$ and $k\in\mathbb{Z}$ we can finish our proof. □

Proof of Proposition 3.18: It follows from the dominated convergence Theorem (we can use this theorem thanks to Proposition 3.17 and to the fact that $\Psi_{-H} \in S(\mathbb{R})$) and from Lemma 3.19 that

$$\begin{aligned}
&2^{j(1+H)}\int_{\mathbb{R}} B_H(t,\omega)\Psi_{-H}(2^jt-k)\,dt \\
&= 2^{j(1+H)}\int_{\mathbb{R}}\left(\sum_{(j',k')\in\mathbb{Z}^2} 2^{-j'H}\varepsilon_{j',k'}(\omega)\Big(\Psi_H(2^{j'}t-k')-\Psi_H(-k')\Big)\right)\Psi_{-H}(2^jt-k)\,dt \\
&= \sum_{(j',k')\in\mathbb{Z}^2} 2^{j+(j-j')H}\varepsilon_{j',k'}(\omega)\int_{\mathbb{R}}\Big(\Psi_H(2^{j'}t-k')-\Psi_H(-k)\Big)\Psi_{-H}(2^jt-k)\,dt \\
&= \sum_{(j',k')\in\mathbb{Z}^2} 2^{j+(j-j')H}\varepsilon_{j',k'}(\omega)\int_{\mathbb{R}}\Psi_H(2^{j'}t-k')\Psi_{-H}(2^jt-k)\,dt \\
&= \varepsilon_{j,k}(\omega).
\end{aligned}$$

□

For every $j \geq 1$ and $l \in \mathbb{Z}$ let ν_j^l be the random variable defined as

$$\nu_j^l = \max\{|\varepsilon_{j,jl+m}| \,:\, 0 \leq m \leq j-1\}. \tag{3.18}$$

Lemma 3.20 *Assume that one has for some $\omega_0 \in \Omega_1^*$ and some $t_0 \in \mathbb{R}$, $\alpha_{B_H}(t_0,\omega_0) > H$. Then*

$$\limsup_{j\to+\infty} \nu_j^{l_j(t_0)}(\omega_0) = 0,$$

where $l_j(t_0) = \max\{l \in \mathbb{Z} \,:\, jl \leq 2^j t_0\}$.

Proof of Lemma 3.20: The assumption $\alpha_{B_H}(t_0,\omega_0) > H$ and Proposition 3.17 imply that there exist $\varepsilon_0 > 0$ and $c_0 > 0$ such that one has for all $t \in \mathbb{R}$,

$$|B_H(t,\omega_0) - B_H(t_0,\omega_0)| \leq c_0 |t-t_0|^{H+\varepsilon_0}. \tag{3.19}$$

Next by using the fact that $2^{j(1+H)}\int_{\mathbb{R}} \Psi_{-H}(2^j t - k)\,dt = 0$ and (3.19), it follows that

$$\begin{aligned}
|\varepsilon_{j,k}| &= 2^{j(1+H)}\Big|\int_{\mathbb{R}} \Big(B_H(t,\omega_0) - B_H(t_0,\omega_0)\Big)\Psi_{-H}(2^j t - k)\,dt\Big| \\
&\leq 2^{j(1+H)}\int_{\mathbb{R}} |B_H(t,\omega_0) - B_H(t_0,\omega_0)||\Psi_{-H}(2^j t - k)|\,dt \\
&\leq c_0 2^{j(1+H)}\int_{\mathbb{R}} |t-t_0|^{H+\varepsilon_0}|\Psi_{-H}(2^j t - k)|\,dt \\
&\leq c_0 2^{jH}\int_{\mathbb{R}} |2^{-j}s - (t_0 - 2^{-j}k)|^{H+\varepsilon_0}|\Psi_{-H}(s)|\,ds \text{ (setting } s = 2^j t - k) \\
&\leq c_1 2^{-j\varepsilon_0}\Big(1 + |2^j t_0 - k|\Big)^{H_0+\varepsilon_0},
\end{aligned} \tag{3.20}$$

where $c_1 > 0$ is a constant which does not depend on j and k. Let us now assume that k is of the form

$$k = jl_j(t_0) + m$$

where $m \in \{0,\ldots,j-1\}$. Observe that it then follows from the definition $l_j(t_0)$ that

$$0 \leq 2^j t_0 - jl_j(t_0) < j.$$

One has therefore

$$\Big(1 + |2^j t_0 - jl_j(t_0) - m|\Big)^{H_0+\varepsilon_0} \leq (2j)^{H_0+\varepsilon_0}. \tag{3.21}$$

Finally, (3.20) and (3.21) imply that for all $j \geq 1$,

$$\nu_j^{l_j(t_0)}(\omega_0) = \max\{|\varepsilon_{j,jl_j(t_0)+m}(\omega_0)| \,:\, 0 \leq m \leq j-1\} \leq c_1 2^{-j\varepsilon_0}(2j)^{H+\varepsilon_0}.$$

Thus we obtain that $\limsup_{j\to+\infty} \nu_j^{l_j(t_0)}(\omega_0) = 0$. $\square$

Lemma 3.21 *There is Ω_2^* an event of probability 1 included in Ω_1^* such that for all $p \in \mathbb{Z}$ and all $\omega \in \Omega_2^*$ one has*

$$\liminf_{j\to+\infty} \min\{\nu_j^l(\omega) \,:\, (p-1)2^j \leq jl \leq (p+1)2^j\} \geq 1/2.$$

Proof of Lemma 3.21: Let $p \in \mathbb{Z}$ be fixed. by using (3.18) and the fact that $\{\varepsilon_{j,jl+m} : 0 \leq m \leq j-1\}$ is a sequence of $\mathcal{N}(0,1)$ independent random variables, one obtains that for every $j \geq 1$,

$$\begin{aligned}
&\mathbb{P}\Big(\min\{\nu_j^l(\omega) \,:\, (p-1)2^j \leq jl \leq (p+1)2^j\} < 1/2\Big) \\
&\leq \sum_{(p-1)2^j \leq jl \leq (p+1)2^j} \mathbb{P}\Big(\cap_{m=0}^{j-1} \{|\varepsilon_{j,jl+m}| < 1/2\}\Big) \\
&\leq \Big(\frac{(p+1)2^j}{j} - \frac{(p-1)2^j}{j} + 1\Big)\Big(\frac{1}{\sqrt{2\pi}} \int_{-1/2}^{1/2} e^{-x^2/2}\,dx\Big)^j \\
&\leq 3j^{-1}\Big(\sqrt{\frac{2}{\pi}}\Big)^j.
\end{aligned}$$

Therefore, one has that

$$\sum_{j=1}^{+\infty} \mathbb{P}\Big(\min\{\nu_j^l(\omega) \,:\, (p-1)2^j \leq jl \leq (p+1)2^j\} < 1/2\Big) < \infty.$$

Finally by using Borel-Cantelli Lemma we can finish our proof □

Proof of Theorem 3.9: The fact that for all $\omega \in \Omega_2^*$ and $t \in \mathbb{R}$ one has $\alpha_{B_H}(t,\omega) \leq H$ is a straightforward consequence of Lemmas 3.20 and 3.21. □

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