From Multifractional Brownian Motion to Generalized Multifractional Brownian Motion

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Main parts of the seminar

1. Introduction and motivation
2. Construction of GMBM
3. Some important properties of GMBM

1-Introduction and motivation

In the first seminar we have seen that it is important to construct stochastic processes satisfying the following 3 properties:

(i) They extend Fractional Brownian Motion, which is denoted by $\{B_H(t)\}_{t\in\mathbb{R}}$ where $H \in (0,1)$ is the Hurst parameter. Recall that

$$B_H(t) = \int_{\mathbb{R}} \frac{e^{it\xi} - 1}{|\xi|^{H+1/2}}\, d\widehat{W}(\xi). \tag{1}$$

(ii) Their paths are with probability 1 continuous functions.

(iii) Their pointwise Hölder exponents are allowed to change from one place to another. $\{\alpha_X(t)\}_{t\in\mathbb{R}}$, the pointwise Hölder exponent of a process $\{X(t)\}_{t\in\mathbb{R}}$, with continuous and nowhere differentiable paths, is defined as,

$$\alpha_X(t) = \sup\Big\{\alpha \in [0,1] \,:\, \limsup_{s\to 0} \frac{|X(t+s) - X(t)|}{|s|^{\alpha}} = 0\Big\}.$$

The paradigmatic example of such processes is Multifractional Brownian Motion (MBM) denoted by $\{X(t)\}_{t\in\mathbb{R}}$. It is obtained by substituting to the Hurst parameter a deterministic function $h:\mathbb{R}\to(0,1)$; therefore $\{X(t)\}_{t\in\mathbb{R}}$ can be represented as,

$$X(t)=\int_{\mathbb{R}}\frac{e^{it\xi}-1}{|\xi|^{h(t)+1/2}}\,d\widehat{W}(\xi). \tag{2}$$

We have seen that:

(R_1) A necessary and sufficient condition for having the continuity, with probability 1, of the paths of $\{X(t)\}_{t\in\mathbb{R}}$, is that the function h be continuous.

(R_2) When h is on any compact subset K, a $\beta(K)$-Hölder continuous function such that $\max_{t\in K} h(t)<\beta(K)$, then,

$$\mathbb{P}\big\{\forall t\in\mathbb{R}:\ \alpha_X(t)=h(t)\big\}=1.$$

The result (R_2) means that the pointwise Hölder exponent of the MBM $\{X(t)\}_{t\in\mathbb{R}}$ **is allowed to change from one place to another only in a smooth way.** Let us show that **this is a strong limitation both from an applied point of view and a theoretical one.**

It is a limitation from an applied point of view since, large classes of real life signals exhibit **a very irregular behavior**: their Hölder exponents may change widely from point to point.

Examples:

- turbulence data (turbulent flows are not spatially homogenous, so the irregularity of the velocity of fully developed turbulence seems to differ widely from point to point);
- speech signals (they have Hölder exponents with sharp fluctuations, especially within consonants);
- computer traffic (empirical studies have shown that some traces of data traffic exhibit multifractal behaviors)

$\Rightarrow$ For modeling such signals it is important to construct functions or processes with **the most general pointwise Hölder exponent**.

The fact the pointwise Hölder exponent of MBM is allowed to change from one place to another only in a smooth way, is also a limitation from a theoretical point of view. It is possible to construct continuous functions with a very fluctuating local Hölder regularity. A famous example is **the Riemann function**,

$$R(x) = \sum_{k=1}^{+\infty} \frac{1}{n^2} \sin(\pi n^2 x).$$

Riemann would have proposed it as an example of continuous and nowhere differentiable function; it turned out that the regularity of R varies strongly from one point to another. R is even differentiable at some points (1967, J.L. Gerver a student of S. Lang).

The most general form of the pointwise Hölder exponent of an arbitrary deterministic continuous function f defined on $\mathbb{R}$ (or on $\mathbb{R}^N$) is given by the following theorem.

Theorem 1 (Anderson, Daoudi, Jaffard and Lévy Véhel)

A nonnegative function h is the pointwise Hölder exponent of a continuous function f, if and only if there exists $(h_n)_{n\in\mathbb{N}}$ **a sequence of continuous functions** *such that for all $t \in \mathbb{R}$, one has*

$$h(t) = \liminf_{n\to+\infty} h_n(t). \qquad (3)$$

Note that, very recently, using a "wavelet-leader" reformulation of a nice characterization of pointwise Hölder regularity due to Anderson, Jaffard and Ayache have shown that **Theorem 1 remains valid when f is a discontinuous and locally integrable function.**

Remarks:

(R_1) α_f the pointwise Hölder exponent of an arbitrary continuous function f, can be very fluctuating, for example α_f can be the indicator of the irrationnal numbers, but it cannot be the indicator of the rational numbers.

(R_2) The fact that "α_f **the pointwise Hölder exponent of an arbitrary continuous and nowhere differentiable function** f**, can always be expressed as a lower limit of a sequence of continuous functions denoted by** $(h_n)_{n\in\mathbb{N}}$", can be proved thanks to the following tricky idea due to Y.Meyer: for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$, one sets,

$$h_n(t) = \inf_{2^{-n-2}\le|s|<2^{-n-1}} \left\{ \frac{\log\left(|f(t+s)-f(t)| + 2^{-n^2}\right)}{\log|s|} \right\}. \qquad (4)$$

(R_3) The reciprocal result: **"being given an arbitrary sequence $(h_n)_{n\in\mathbb{N}}$ of nonnegative continuous functions there always exists a continuous function f, such that $\alpha_f(t) = \liminf_{n\to+\infty} h_n(t)$ for all t"**, requires much more effort to be obtained. Different constructive proofs have been proposed:

- Daoudi, Lévy Véhel and Meyer gave 3 different constuctions; respectively based on Weierstrass function, Schauder basis and IFS's.
- On the other hand Jaffard gave a construction based on Wavelets.

All of these deterministic constructions are **extremely peculiar** (we will soon present one of them); therefore they could not be used in any realistic simulation.

$\Rightarrow$ Thus it would be useful to have **a natural random construction of continuous functions whose pointwise Hölder exponents can be of the most general form.**

Question: (Jaffard and Lévy Véhel) Is it possible to construct a stochastic process $\{X(t)\}_{t\in\mathbb{R}}$ which has the following two properties ?

- $\{X(t)\}_{t\in\mathbb{R}}$ is an entension of FBM;
- with probability 1, $\{X(t)\}_{t\in\mathbb{R}}$ paths are continuous functions whose pointwise Hölder exponents are equal to the lower limit of an arbitrary sequence $(h_n)_{n\in\mathbb{N}}$ of nonnegative continuous functions.

The GMBM, we will present in this seminar, **provides a positive answer to this question; in the case where the functions h_n are with values in $[0,1]$.**

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Since we are interested in a local problem, we can restrict to a compact interval, for example the interval $[0, 1]$.
Let $(\tilde{h}_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions; for simplicity we assume that these functions are defined on the interval $[0, 1]$ and with values in a fixed interval $[a, b] \subset (0, 1)$. For all $t \in [0, 1]$, we set

$$h(t) = \liminf_{n\to+\infty} \tilde{h}_n(t).$$

The proof of the following lemma mainly relies on Stone-Weierstrass Theorem.

Lemma 1 (Daoudi, Meyer and Lévy Véhel)

One can always construct a sequence $(h_n)_{n\in\mathbb{N}}$ *of Lipschitz functions defined on* $[0,1]$ *and with values in* $[a,b]$ *such that:*

$(\mathcal{A}_1)$ $\|h_n\|_{Lip} = O(n)$;

$(\mathcal{A}_2)$ $h(t) = \liminf_{n\to+\infty} h_n(t)$ *for all* $t\in[0,1]$.

Recall that one says that g is a Lipschitz function on $[0,1]$ if,

$$\|g\|_{\mathrm{Lip}} = \sup_{s,t\in[0,1]} \frac{|g(s)-g(t)|}{|s-t|} < \infty.$$

So, from now on we assume that $(h_n)_{n\in\mathbb{N}}$ is a sequence of Lipschitz functions satisfying $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$.

The construction of GMBM we will give, is to a certain extent inspired by that of the **Generalized Weierstrass function** due to Daoudi, Meyer and Lévy Véhel; so let us first recall the latter construction.
$\rightarrow$ The usual **Weierstrass function** is the continuous and nowhere differentiable function f defined as,

$$f(t) = \sum_{n=1}^{+\infty} \lambda^{-nH} \sin(\lambda^n t),$$

where $H \in (0,1)$ and $\lambda \in (1,+\infty)$ are two constant parameters.
$\rightarrow$ When one replaces H by a β-Hölder function h ranging in $(0,1)$ and such that $\max_{t\in[0,1]} h(t) < \beta$, then one obtains a continuous Weierstrass function whose pointwise Hölder exponent equals h.
$\rightarrow$ When one replaces "abruptly" H by a discontinous function, then one generally obtains a discontinuous Weierstrass function.

Let a' and b' be two reals satisfying $0 < a' < a < b < b' < 1$. Let $L = (l_p)_{p \geq 1}$ the sequence of integers such that $l_1 = 1$ and for all $p \geq 1$,

$$l_{p+1} = \left[\frac{1 - a'}{1 - b'} l_p \right] + 1.$$

Then g, the **Generalized Weierstrass function** is defined as

$$g(t) = \sum_{n \in L} \lambda^{-n h_n(t)} \sin(\lambda^n t).$$

This function is continuous and satisfies for every t, $\alpha_g(t) = h(t) = \liminf_{n \to +\infty} h_n(t)$.

First definition of GMBM:

We set $D_0 = \{\xi \in \mathbb{R} : |\xi| < 1\}$ and for all $n \geq 1$, $D_n = \{\xi \in \mathbb{R} : 2^{n-1} \leq |\xi| < 2^n\}$. The GMBM $\{\widetilde{X}(t)\}_{t\in[0,1]}$ is defined as,

$$\widetilde{X}(t) = \int_{\mathbb{R}} \Big(\sum_{n=0}^{+\infty} \frac{e^{it\xi} - 1}{|\xi|^{h_n(t)+1/2}} \mathbb{1}_{D_n}(\xi) \Big) \, d\widehat{W}(\xi). \qquad (5)$$

$\rightarrow$ In contrast with FBM and MBM, in the case of GMBM the low frequencies and frequencies are governed by different functional parameters, namely the first terms and the tail of the sequence $(h_n)_{n\in\mathbb{N}}$.
$\rightarrow$ In fact, it is more covenient to replace the indicator functions in (8) by smooth functions. In order to precisely explain this point, let us introduce some notations.

Let $\widehat{f_0}$ be a $\mathcal{C}^d(\mathbb{R})$ function with values in $[0,1]$ which satisfies

$$\widehat{f_0}(\xi) = \begin{cases} 1 \text{ if } |\xi| \leq 2\pi/3 \\ 0 \text{ if } |\xi| \geq \pi \end{cases} \tag{6}$$

For every integer $n \geq 1$ and all $\xi \in \mathbb{R}$, we set

$$\widehat{f_n}(\xi) = \widehat{f}(2^{-n}\xi) - \widehat{f}(2^{-(n-1)}\xi) = \widehat{f_1}(2^{-(n-1)}\xi). \tag{7}$$

Then, for all $n \geq 1$,

- $\mathrm{Supp}\,\widehat{f_n} \subseteq \{\xi \in \mathbb{R} \,:\; 2^n(\pi/3) \leq |\xi| \leq 2^n\pi\}$;
- when $2^{n-1}\pi \leq |\xi| \leq 2^{n+1}(\pi/3)$, one has $\widehat{f_n}(\xi) = 1$,
- for all $\xi \in \mathbb{R}$, one has $\sum_{n=0}^{+\infty} \widehat{f_n}(\xi) = 1$.

Such a sequence $(\widehat{f_n})_{n\in\mathbb{N}}$ is called **a Littelwood-Paley Analysis.**

Second definition of GMBM:
The GMBM $\{X(t)\}_{t\in[0,1]}$ is defined as,

$$X(t) = \int_{\mathbb{R}} \Big(\sum_{n=0}^{+\infty} \frac{e^{it\xi}-1}{|\xi|^{h_n(t)+1/2}} \widehat{f}_n(\xi) \Big)\, d\widehat{W}(\xi). \tag{8}$$

The field $\{Y(u,v)\}_{(u,v)\in[0,1]^2}$ generating the GMBM $\{X(t)\}_{t\in[0,1]}$ is defined as,

$$Y(u,v) = \int_{\mathbb{R}} \Big(\sum_{n=0}^{+\infty} \frac{e^{iu\xi}-1}{|\xi|^{h_n(v)+1/2}} \widehat{f}_n(\xi) \Big)\, d\widehat{W}(\xi). \tag{9}$$

Observe that for all $t \in [0,1]$,

$$X(t) = Y(t,t). \tag{10}$$

The second definition of GMBM is more convenient than the first one since it allows to obtain the following quite useful lemma.

Lemma 2

With probability 1, *the paths of the field* $\{Y(u,v)\}_{(u,v)\in[0,1]^2}$ *are Lipschitz functions in* v *uniformly in* u. *More precisely, there is a random variable* C *of finite moment of any order, such that one has, a.s., for all* $v_1, v_2 \in [0,1]$,

$$\sup_{u\in[0,1]} \left| Y(u,v_1) - Y(u,v_2) \right| \leq C|v_1 - v_2|. \tag{11}$$

Main parts of the seminar

Theorem 2 (Jaffard, Taqqu and Ayache)

$\{\alpha_X(t)\}_{t\in[0,1]}$ the pointwise Hölder exponent of the GMBM $\{X(t)\}_{t\in[0,1]}$ satisfies,

$$\mathbb{P}\Big\{\forall t\in[0,1]\,:\,\alpha_X(t)=h(t)=\liminf_{n\to+\infty} h_n(t)\Big\}=1. \tag{12}$$

In this seminar we will only show that for all fixed $t\in[0,1]$,

$$\mathbb{P}\Big\{\alpha_X(t)=h(t)\Big\}=1. \tag{13}$$

Let $\{Z(u)\}_{u\in[0,1]}$ be the process defined for all $u\in[0,1]$ as

$$Z(u)=Y(u,t), \tag{14}$$

where Y is the field generating GMBM. Notice that for proving (13), it is sufficient to show that

$$\mathbb{P}\Big\{\alpha_Z(t)=h(t)\Big\}=1.$$

Indeed, one has that

$$
\begin{aligned}
&\big|X(t+s) - X(t)\big| = \big|Y(t+s, t+s) - Y(t,t)\big| \\
&\leq \big|Y(t+s, t+s) - Y(t+s, t)\big| + \big|Y(t+s, t) - Y(t,t)\big| \\
&\leq \sup_{u\in[0,1]} \big|Y(u, t+s) - Y(u,t)\big| + \big|Z(t+s) - Z(t)\big| \\
&= O(|s|) + \big|Z(t+s) - Z(t)\big|.
\end{aligned}
$$

The following lemma allows to show that with probability 1, the paths $\{Z(u)\}_{u\in[0,1]}$ are Hölder functions of any order $\gamma < h(t)$; which implies that

$$\mathbb{P}\{\alpha_Z(t) \geq h(t)\} = 1.$$

Lemma 3

For all $\varepsilon > 0$ there is a constant c such that one has for $u_1\, u_2 \in [0,1]$,

$$\mathbb{E}\left|Z(u_1) - Z(u_2)\right|^2 \leq c|u_1 - u_2|^{2h(t)-2\varepsilon}. \tag{15}$$

Proof: Since $h(t) = \liminf_{n\to+\infty} h_n(t)$, there is an integer n_0, such that for every $n \geq n_0 + 1$,

$$h_n(t) \geq h(t) - \varepsilon. \tag{16}$$

Then using the definition $\{Z(u)\}_{u\in[0,1]}$, the latter inequality and the equality $\sum_{n=0}^{+\infty} \widehat{f}_n(\xi) = 1$, one has,

$$
\begin{aligned}
&\mathbb{E}\left|Z(u_1)-Z(u_2)\right|^2\\
&=\int_{\mathbb{R}}\left|e^{i(u_1-u_2)\xi}-1\right|^2\Big(\sum_{n=0}^{+\infty}\frac{\widehat{f}_n(\xi)}{|\xi|^{h_n(t)+1/2}}\Big)^2\,d\xi\\
&\leq 2\int_{\mathbb{R}}\left|e^{i(u_1-u_2)\xi}-1\right|^2\Big(\sum_{n=0}^{n_0}\frac{\widehat{f}_n(\xi)}{|\xi|^{h_n(t)+1/2}}\Big)^2\,d\xi\\
&\qquad +2\int_{\mathbb{R}}\frac{\left|e^{i(u_1-u_2)\xi}-1\right|^2}{|\xi|^{2h(t)+1-2\varepsilon}}\Big(\sum_{n=n_0+1}^{+\infty}\widehat{f}_n(\xi)\Big)^2\,d\xi\\
&\leq 2|u_1-u_2|^2\int_{\mathbb{R}}|\xi|^2\Big(\sum_{n=0}^{n_0}\frac{\widehat{f}_n(\xi)}{|\xi|^{h_n(t)+1/2}}\Big)^2\,d\xi\\
&\qquad +2|u_1-u_2|^{2(h(t_0)-\varepsilon)}\int_{\mathbb{R}}\frac{\left|e^{i\eta}-1\right|^2}{|\eta|^{2h(t)+1-2\varepsilon}}\,d\eta.
\end{aligned}
$$

□

In order to show that

$$\mathbb{P}\{\alpha_Z(t) \le h(t)\} = 1,$$

we will use the following lemma, which generally speaking allows to obtain a.s. an upper bound of the pointwise Hölder exponent of a Gaussian process at a given point.

Lemma 4

Let $\{X(u)\}_{u\in[0,1]}$ be a Gaussian process and let $t \in [0,1]$ be a fixed point. Assume that there is $\mu \in (0,1)$ satisfying the following property: for all $\varepsilon > 0$ and $n \in \mathbb{N}$ one has

$$\mathbb{E}\left|X(t+s_n) - X(t)\right|^2 \ge c|s_n|^{2(\mu+\varepsilon)}, \tag{17}$$

where $c > 0$ is a constant and $(s_n)_{n\in\mathbb{N}}$ is a sequence of non vanishing real numbers which converges to 0. Then

$$\mathbb{P}\{\alpha_X(t) \le h(t)\} = 1.$$

Lemma 5

For all $\varepsilon > 0$, there is a constant $c > 0$ and a subsequence $l \mapsto n_l$ such that

$$\mathbb{E}\left|Z(t+2^{-n_l}) - Z(t)\right|^2 \geq c2^{-2n_l(h(t)+\varepsilon)}. \tag{18}$$

Proof: Since $h(t) = \liminf_{n\to+\infty} h_n(t)$, for all $\varepsilon > 0$, there is a subsequence $l \mapsto n_l$ such that,

$$h_{n_l}(t) \leq h(t) + \varepsilon.$$

Then using the definition of $\{Z(u)\}_{u\in[0,1]}$, the fact that $\widehat{f}_{n_l}(\xi)=1$ for all $\xi\in[2^{n_l-1}\pi,2^{n_l+1}\pi/3]$, as well as the latter inequality one obtains that

$$\begin{aligned}\mathbb{E}\left|Z(t+2^{-n_l})-Z(t)\right|^2 &\geq \int_{2^{n_l-1}\pi}^{2^{n_l+1}\pi/3}\frac{\left|e^{i2^{-n_l}\xi}-1\right|^2}{|\xi|^{2h_{n_l}(t)+1}}\,d\xi\\ &\geq \int_{2^{n_l-1}\pi}^{2^{n_l+1}\pi/3}\frac{\left|e^{i2^{-n_l}\xi}-1\right|^2}{|\xi|^{2h(t)+1+2\varepsilon}}\,d\xi\\ &\geq 2^{-2n_l(h(t)+\varepsilon)}\int_{\pi/2}^{2\pi/3}\frac{\left|e^{i\eta}-1\right|^2}{|\eta|^{2h(t)+1+2\varepsilon}}\,d\xi.\end{aligned}$$

□

Theorem 3 (Lévy Véhel and Ayache)

Recall that the functional parameters h_n *of the GMBM* $\{X(t)\}_{t\in\mathbb{R}}$ *are assumed to be with values in* $[a,b]\subset(0,1)$. *Let* $s\in\mathbb{R}$ *be a point satisfying the following property: there exist two constants* c *and* δ *(a priori depending on* s*) such that for all* $n\in\mathbb{N}$

$$\big|h_n(s)-h(s)\big|\leq c2^{-2\delta n}. \tag{19}$$

Then $\{X(t)\}_{t\in[0,1]}$ *is* **locally asymptotically self-similar of order** $h(s)$ **at** s. *More precisely,*

$$\lim_{\rho\to 0^+} law\left\{\frac{X(s+\rho u)-X(s)}{\rho^{h(s)}}\right\}_{u\in\mathbb{R}} = law\left\{B_{h(s)}(u)\right\}_{u\in\mathbb{R}}, \tag{20}$$

where $\left\{B_{h(s)}(u)\right\}_u$ *is the FBM of Hurst parameter* $h(s)$ *and where the convergence in distribution holds for the topology of uniform convergence on compact sets.*

Remark: The condition $\big|h_n(s) - h(s)\big| \le c2^{-2\delta n}$ under which Theorem 3 is valid is not, in fact, very restrictive. Actually, at least in each of the following two cases (i) and (ii), it is possible to construct a sequence of functions $(h_n)_{n\in\mathbb{N}}$ which satisfies this condition at any point $s \in \mathbb{R}$.

(i) h is both a lower-semi-continuous and a piecewise continuous function;

(ii) h is a function of the $b + (a-b)\mathbb{1}_F$ where F is an arbitrary closed subset of $\mathbb{R}$.

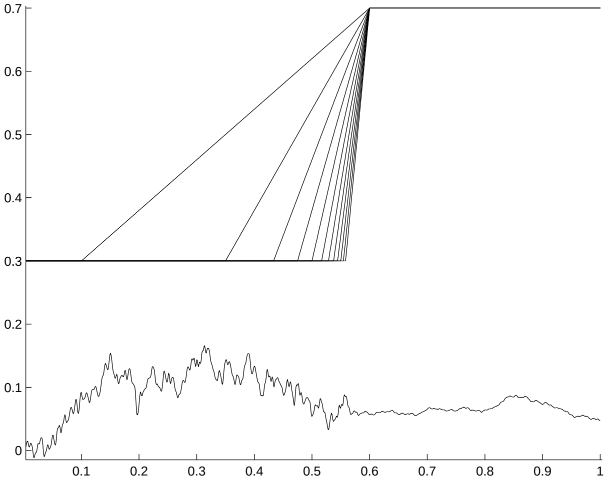

Figure: Simulation of a path of a GMBM such that $h = 0.3\mathbb{1}_{[0,0.6)} + 0.7\mathbb{1}_{[0.6,1]}$ and $(h_n)_n$ is a sequence of piecewise affine functions converging to h at each point

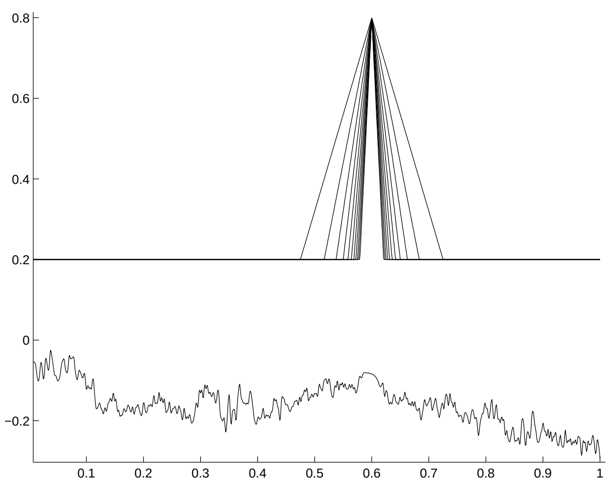

Figure: Simulation of a path of a GMBM such that $h = 0.2\mathbb{1}_{[0,1]\setminus\{0.6\}} + 0.8\mathbb{1}_{\{0.6\}}$ and $(h_n)_n$ is a sequence of piecewise affine functions converging to h at each point

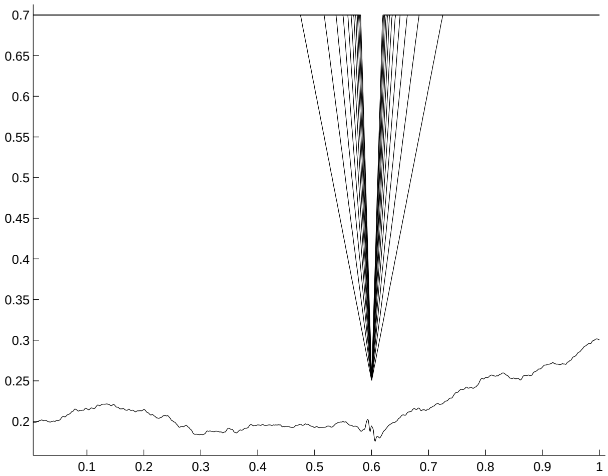

Figure: Simulation of a path of a GMBM such that $h = 0.7\mathbb{1}_{[0,1]\setminus\{0.6\}} + 0.25\mathbb{1}_{\{0.6\}}$ and $(h_n)_n$ is a sequence of piecewise affine functions converging to h at each point