Fractional and Multifractional fields from a wavelet point of view

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1-Introduction

The goal of our talk: To present the main ideas of the solutions of 3 connected problems in the setting of **multivariate Fractional Brownian Motion (FBM)** and **multivariate Multifractional Brownian Motion (MBM)**.

Though it is **rich enough** (FBM is non-Markovian nor a semimartingale), this setting remains **simple enough** (thus we can avoid some technical complications).

Wavelet methods will play a crucial role in the solutions of these problems.

FBM is a quite classical example of a fractal field. It is denoted by $\{B_H(t)\}_{t\in\mathbb{R}^N}$ since it depends on a unique parameter $H \in (0,1)$, called **the Hurst parameter.** The covariance kernel of this centered Gaussian field with stationary increments is given, for all $s \in \mathbb{R}^N$ and $t \in \mathbb{R}^N$, by

$$\mathbb{E}\left(B_H(s)B_H(t)\right) = \frac{c_H}{2}\left\{|s|^{2H} + |t|^{2H} - |s-t|^{2H}\right\}. \tag{1}$$

$\{B_{1/2}(t)\}_{t\in\mathbb{R}^N}$ is the usual Brownian motion.
Up to a multiplicative constant, FBM can be represented, for all $t \in \mathbb{R}^N$, as

$$B_H(t) = \int_{\mathbb{R}^N} \frac{e^{it\cdot\xi} - 1}{|\xi|^{H+N/2}}\, dW(\xi). \tag{2}$$

It is a fractal objet since, for all $a > 0$,

$$\{B_H(at)\}_{t\in\mathbb{R}^N} \overset{\text{f.d.d.}}{=} \{a^H B_H(t)\}_{t\in\mathbb{R}^N}.$$

The first problem: As FBM is a continuous Gaussian field, it can be represented on any compact $K \subset \mathbb{R}^N$ as,

$$B_H(t) = \sum_{n=1}^{+\infty} \epsilon_n f_n(t), \tag{3}$$

where:

- The ϵ_n's are independent $\mathcal{N}(0,1)$ real-valued Gaussian random variables.
- The f_n's are deterministic real-valued continuous functions over K.
- The series in (3) is, with probability 1, uniformly convergent in $t \in K$.

The representation (3) is far from being unique and it seems natural to look for **optimal** representations i.e.

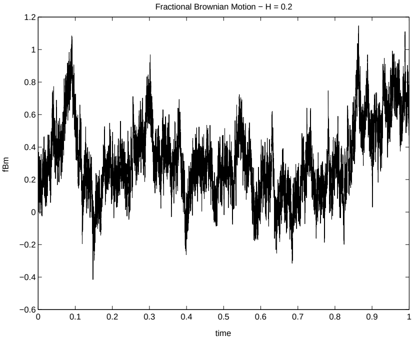
$$\mathbb{E}\left(\sup_{t\in K}\Big|\sum_{n=m}^{+\infty} \epsilon_n f_n(t)\Big|\right) \longrightarrow 0,$$

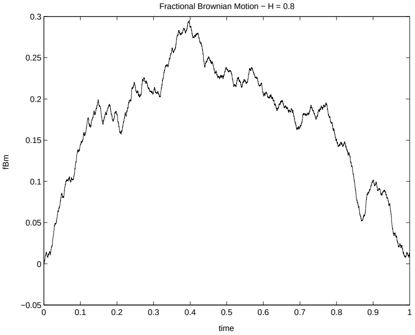
as fast as possible, when $m \to +\infty$.

$\rightarrow$ In section 2 we will introduce a radom wavelet series represention of FBM and show that it is optimal.

Let us now present the main motivations behind the second problem.

Tough, FBM turned out to be very useful in many areas (signal and image processing, telecommunication, ...) it has some drawbacks; an important one is that **the local Hölder regularity of FBM remains the same all along its trajectory**:





The local Hölder regularity of a stochastic field $\{X(t)\}_{t\in\mathbb{R}^N}$ can be measured through its **pointwise Hölder exponent** $\{\alpha_X(t)\}_{t\in\mathbb{R}^N}$. When $\{X(t)\}_{t\in\mathbb{R}^N}$ is continuous and nowhere differentiable (this is the case of FBM) then, for all $t \in \mathbb{R}^N$,

$$\alpha_X(t) = \sup\Big\{\alpha \in [0,1] \,:\, \limsup_{s\to 0} \frac{|X(t+s)-X(t)|}{|s|^\alpha} = 0\Big\}.$$

The local Hölder regularity of FBM remains the same all along its trajectory since:

$$\mathbb{P}\Big\{\forall t \in \mathbb{R}^N : \alpha_{B_H}(t) = H\Big\} = 1. \tag{4}$$

Basically, (4) is due to the fact that the increments of B_H are stationary.

The constancy of the pointwise Hölder exponent of FBM is a strong limitation in many situations (detection of cancer tumors in medical images, description of traces of Internet network traffic,...). This is why a more flexible model called **Multifractional Brownian Motion (MBM)** was introduced by Benassi, Jaffard and Roux and also by Lévy Vehel and Peltier.

We denote MBM by $\{X(t)\}_{t\in\mathbb{R}^N}$. It can be represented, for all $t\in\mathbb{R}^N$ as,

$$X(t)=\int_{\mathbb{R}^N}\frac{e^{it\cdot\xi}-1}{|\xi|^{h(t)+N/2}}\,dW(\xi), \tag{5}$$

where $h(\cdot)$ is a continuous functional parameter with values in $(0,1)$.

Theorem 1 (Lévy Vehel et al. and Benassi et al.)

When $h(\cdot)$ *satisfies, on each compact cube* $K \subset \mathbb{R}^N$*, a uniform Hölder condition of order* $\beta = \beta(K)$*, i.e. for all* $s, t \in K$*,*

$$|h(s) - h(t)| \leq c|s - t|^{\beta}$$

and β *is such that* $\sup_{t \in K} h(t) < \beta$*. Then for all* $t \in \mathbb{R}^N$*,*

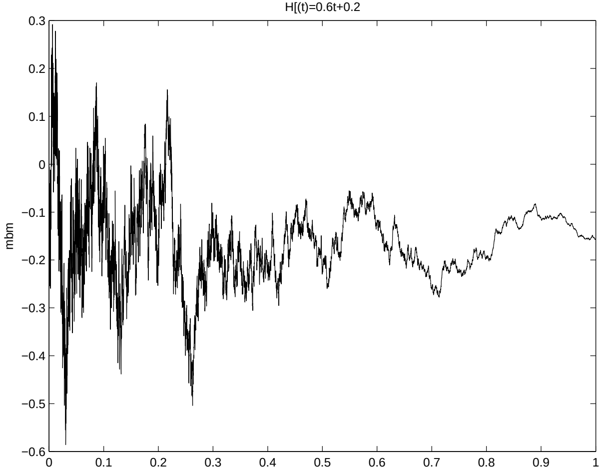
$$\mathbb{P}\{\alpha_X(t) = h(t)\} = 1.$$

The second problem we intend to solve in this talk (see section 3) is to show that:

$$\mathbb{P}\{\forall t \in \mathbb{R}^N : \alpha_X(t) = h(t)\} = 1.$$

Simulation of a trajectory of MBM

In this simulation the functional parameter of MBM has been chosen such that $h(t) = 0,6t + 0,2$ for all $t \in [0,1]$



The third problem we intend to solve in this talk (see section 4) is to show that **the local time of MBM is jointly continuous**. To this end we need to prove that MBM satisfies the so-called property of **local nondeterminism (LND)**.

The concept of LND was first introduced by Berman. Roughly speaking, it means that **the increments are asymptotically independent.** Thanks to it, many of the results on local times of Brownian motion, were extended to more general stochastic fields.

l-numbers and optimal representations:

A sharp lower bound of the rates of convergence of all the series representations over a compact $K \subset \mathbb{R}^N$ of the FBM $\{B_H(t)\}_{t \in K}$ is given by $(l_m(B_H))_{m \geq 1}$ **the sequence of l-numbers of** B_H, defined as

$$l_m(B_H) = \inf \left\{ \mathbb{E} \sup_{t \in K} \left| \sum_{n=m}^{+\infty} \epsilon_n f_n(t) \right| \, : \, X(t) = \sum_{n=1}^{+\infty} \epsilon_n f_n(t) \right\}. \qquad (6)$$

A representation $B_H(t) = \sum_{n=1}^{\infty} \epsilon_n f_n(t)$, is said to be **optimal**, if and only if, one has when $m \to +\infty$,

$$\mathbb{E} \sup_{t \in K} \left| \sum_{n=m}^{+\infty} \epsilon_n f_n(t) \right| = O(l_m(B_H)).$$

In this section, from now on we assume that $K = [0, 1]^N$.

Recently, by using operator theory, Linde and Ayache **have obtained the following sharp estimations of $l_m(B_H)$:** For all $m \geq 2$

$$c_2 m^{-H/N}\sqrt{\log m} \leq l_m(B_H) \leq c_1 m^{-H/N}\sqrt{\log m},$$

where $0 < c_2 \leq c_1$ are two constants.

A natural question one can address is that whether it is possible to construct explicitly optimal series representation of B_H.

We will show that the wavelet series representation of B_H, which will soon be introduced, provides a positive answer to this question.

Note that by using spherical harmonics Malyarenko has recently given another optimal series representation of B_H.

An orthonormal wavelet basis of $L^2(\mathbb{R}^N)$ is a basis of the form:

$$\Big\{2^{jN/2}\psi_l(2^j x-k)\ :\ 1\le l\le 2^N-1,\, j\in\mathbb{Z},\, k\in\mathbb{Z}^N\Big\},$$

where the functions ψ_l are called **the mother wavelets.** We always assume that:

(i) The ψ_l's belong to **the Schwartz class** $S(\mathbb{R}^N)$ i.e. they are infinitely differentiable and satisfy,

$$\sup\Big\{\big(1+|x|\big)^p\big|(\partial^\gamma\psi_l)(x)\big|\ :\ x\in\mathbb{R}^N\Big\}<+\infty.$$

(ii) For all l, $\widehat{\psi}_l(\xi)=(2\pi)^{-N/2}\int_{\mathbb{R}^N} e^{-i\xi\cdot x}\psi_l(x)\,dx$, the Fourier transform of ψ_l, is **compactly supported and vanishes in a neighborhood of the origin.**

The first construction of mother wavelets satisfying (i) and (ii) was given given by Meyer and Lemarié.

Wavelet series representation of FBM:

The isometry property of Fourier transform implies that:

$$\left\{2^{-jN/2}e^{i2^{-j}k\cdot\xi}\overline{\widehat{\psi}_l(2^{-j}\xi)}\ :\ 1\le l\le 2^N-1,\, j\in\mathbb{Z},\, k\in\mathbb{Z}^N\right\},$$

forms **an orthonormal basis of** $L^2(\mathbb{R}^N)$. By expanding for every fixed $t\in\mathbb{R}^N$ and $H\in(0,1)$, the kernel of FBM $\xi\mapsto\frac{e^{it\cdot\xi}-1}{|\xi|^{H+N/2}}$ in the latter basis, it follows that

$$\frac{e^{it\cdot\xi}-1}{|\xi|^{H+N/2}}=\sum_{l=1}^{2^N-1}\sum_{j=-\infty}^{+\infty}\sum_{k\in\mathbb{Z}^N}c_{l,j,k}(t,H)2^{-jN/2}e^{i2^{-j}k\cdot\xi}\overline{\widehat{\psi}_l(2^{-j}\xi)}, \qquad (7)$$

where the series converges in $L^2(\mathbb{R}^N)$ and each of its coefficients is given by

$$c_{l,j,k}(t,H)=2^{-jN/2}\int_{\mathbb{R}^N}\frac{(e^{it\cdot\xi}-1)}{|\xi|^{H+N/2}}\times e^{-i2^{-j}k\cdot\xi}\widehat{\psi}_l(2^{-j}\xi)\,d\xi. \qquad (8)$$

Setting in (8), $\eta = 2^{-j}\xi$ one gets that

$$c_{l,j,k}(t,H) = 2^{-jH}\big(\Psi_l(2^j t - k, H) - \Psi_l(-k,H)\big), \qquad (9)$$

where

$$\Psi_l(x,H) = \int_{\mathbb{R}^N} e^{ix\cdot\eta} \frac{\widehat{\psi}_l(\eta)}{|\eta|^{H+N/2}}\, d\eta. \qquad (10)$$

Putting together (7), (9) and the isometry property of Wiener integral one obtains **the wavelet series representation of FBM:**

$$B_H(t) = \sum_{l=1}^{2^N-1} \sum_{j=-\infty}^{+\infty} \sum_{k\in\mathbb{Z}^N} 2^{-jH}\epsilon_{l,j,k}\big(\Psi_l(2^j t - k, H) - \Psi_l(-k,H)\big), \qquad (11)$$

where the $\epsilon_{l,j,k}$'s are the **independent** $\mathcal{N}(0,1)$ real-valued Gaussian random variables defined as

$$\epsilon_{l,j,k} = 2^{-jN/2} \int_{\mathbb{R}^N} e^{i2^{-j}k\cdot\xi}\, \overline{\widehat{\psi}_l(2^{-j}\xi)}\, dW(\xi).$$

A priori, the wavelet series representation of FBM is, for every fixed (t, H), convergent in $L^2(\Omega)$, Ω being the underlying probability space.

We are going to show that **it is also almost surely convergent in the Banach space of continuous functions $\mathcal{C}(K)$ (equipped with the uniform norm). Here K denotes an arbitrary compact subset of** $\mathbb{R}^N \times (0,1)$; for simplicity we assume that $K = [0,1]^N \times [a,b]$.

In view of **Itô-Nisio Theorem**, it is sufficient to prove that $\big(\{B_H^n(t)\}_{(t,H)\in K}\big)_{n\geq 1}$, the sequence of the partial sums of the series, is weakly relatively compact in $\mathcal{C}(K)$, which can be obtained (see e.g. Billingsley) by showing that, for all $(t_1, H_1) \in K$ and $(t_2, H_2) \in K$,

$$\mathbb{E}\left|B_{H_1}^n(t_1) - B_{H_2}^n(t_2)\right|^2 \leq c\big(|t_1 - t_2|^2 + |H_1 - H_2|^2\big)^a. \qquad (12)$$

where $c > 0$ is a constant **non depending on** n.

$B_H^n(t)$ can be expressed as

$$B_H^n(t) = \sum_{(l,j,k)\in\mathcal{D}_n} 2^{-jH}\epsilon_{l,j,k}\big(\Psi_l(2^j t - k, H) - \Psi_l(-k, H)\big),$$

where $\mathcal{D}_n$ is a finite set whose cardinality depends on n. By using the fact that the $\epsilon_{l,j,k}$'s are independent $\mathcal{N}(0,1)$ Gaussian variables, one obtains that

$$\begin{aligned}
&\mathbb{E}\left|B_{H_1}^n(t_1) - B_{H_2}^n(t_2)\right|^2 \\
&= \sum_{(l,j,k)\in\mathcal{D}_n} \Big|2^{-jH_1}\big(\Psi_l(2^j t_1 - k, H_1) - \Psi_l(-k, H_1)\big) \\
&\qquad - 2^{-jH_2}\big(\Psi_l(2^j t_2 - k, H_2) - \Psi_l(-k, H_2)\big)\Big|^2 \\
&\leq \sum_{l=1}^{2^N-1}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |2^{-jH_1}\big(\Psi_l(2^j t_1 - k, H_1) - \Psi_l(-k, H_1)\big) \\
&\qquad - 2^{-jH_2}\big(\Psi_l(2^j t_2 - k, H_2) - \Psi_l(-k, H_2)\big)\Big|^2 \\
&= \mathbb{E}\left|B_{H_1}(t_1) - B_{H_2}(t_2)\right|^2.
\end{aligned}$$

Then standard computations allow to show that

$$\mathbb{E}\left|B_{H_1}(t_1) - B_{H_2}(t_2)\right|^2 \leq c\left(|t_1 - t_2|^2 + |H_1 - H_2|^2\right)^a.$$

Let us now show that the wavelet series representation of FBM is optimal.

Lemma 1

For every $p \in \mathbb{N}$ and $\gamma \in \mathbb{N}^N$ one has

$$\sup\left\{\left(1+|x|\right)^p\left|(\partial_x^\gamma \Psi_l)(x,H)\right| \ :\ x \in \mathbb{R}^N \text{ and } H \in (0,1)\right\} < +\infty.$$

In view of Lemma 1, from now on we will make **the heuristical assumption** that for all $H \in (0,1)$, one has

$$\text{supp } \Psi_l(\cdot, H) \subset [0,1)^N.$$

Proof of Lemma 1: One has for all $(x, H) \in \mathbb{R}^N \times (0,1)$,

$$\Psi_l(x,H) = \int_{\mathbb{R}^N} e^{ix\cdot\eta} \frac{\widehat{\psi}_l(\eta)}{|\eta|^{H+N/2}}\, d\eta,$$

where $\widehat{\psi}_l$ is a C^∞ compactly supported function vanishing in a neighborhood of the origin. Thus,

$$(\partial_x^\gamma \Psi_l)(x,H) = i^{|\gamma|} \int_{\mathbb{R}^N} e^{ix\cdot\eta} \frac{\eta^\gamma \widehat{\psi}_l(\eta)}{|\eta|^{H+N/2}}\, d\eta,$$

Finally, several integrations by parts allow to obtain the lemma. □

Lemma 2

There is a random variable $C > 0$ of finite moment of any order such that one has almost surely for all $l \in \{1, \dots, 2^N - 1\}$, $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^N$,

$$|\epsilon_{l,j,k}| \le C\sqrt{\log\left(2 + |j| + |k|\right)}.$$

Theorem 2 (Linde and Ayache)

The wavelet series representation of FBM is optimal on each compact cube of $\mathbb{R}^N$.

Heuristic proof of Theorem 2: We will show the optimality on the cube $[0,1]^N$. Therefore, in view of our heuristical assumption: supp $\Psi_l(\cdot, H) \subset [0,1)^N$, many terms in the wavelet series representation of FBM:

$$B_H(t) = \sum_{l=1}^{2^N-1} \sum_{j=-\infty}^{+\infty} \sum_{k \in \mathbb{Z}^N} 2^{-jH} \epsilon_{l,j,k} \big(\Psi_l(2^j t - k, H) - \Psi_l(-k, H) \big)$$

vanish and it reduces to

$$B_H(t) = \sum_{l=1}^{2^N-1} \sum_{j=-\infty}^{-1} 2^{-jH} \epsilon_{l,j,k} \Psi_l(2^j t, H) + \sum_{l=1}^{2^N-1} \sum_{j=0}^{+\infty} \sum_{k \in \{0,\dots,2^j-1\}^N} 2^{-jH} \epsilon_{l,j,k} \Psi_l(2^j t - k, H).$$

We approximate the latter series by the finite sum

$$B_H^J(t) = \sum_{l=1}^{2^N-1} \sum_{j=-[2^{J/2}]}^{-1} 2^{-jH} \epsilon_{l,j,k} \Psi_l(2^j t, H)$$
$$+ \sum_{l=1}^{2^N-1} \sum_{j=0}^{J} \sum_{k\in\{0,\dots,2^j-1\}^N} 2^{-jH} \epsilon_{l,j,k} \Psi_l(2^j t - k, H),$$

where J is a big enough integer. Up to a multicative constant (only depending on N) **there are** 2^{JN} **terms in the latter sum**. Let us derive a sharp upper bound of

$$\left\| B_H - B_H^J \right\|_\infty = \sup_{t\in[0,1]^N} \left| B_H(t) - B_H^J(t) \right|.$$

By using the smoothness of $\Psi_l(\cdot, H)$ and the fact $\Psi_l(0, H) = 0$, it follows that for all $j < 0$ and $t \in [0,1]^N$,

$$\left|\Psi_l(2^j t, H)\right| \leq c_1 2^j,$$

where $c_1 > 0$ is a constant. Then, in view of Lemma 2, one has

$$\begin{aligned}
&\Big\| \sum_{l=1}^{2^N-1} \sum_{j=-\infty}^{-[2^{J/2}]-1} 2^{-jH} \epsilon_{l,j,k} \Psi_l(2^j \cdot, H) \Big\|_\infty \\
&\leq C_2 \sum_{j=-\infty}^{-[2^{J/2}]-1} 2^{j(1-H)} \sqrt{\log(2+|j|)} \\
&\leq C_3 2^{-(1-H)2^{J/2}} \sqrt{J},
\end{aligned} \tag{13}$$

where C_2 and C_3 are two random variables (non depending on J) of finite moment of any order.

From the other hand, by using the heuristical assumtion supp $\Psi_l(\cdot, H) \subset [0,1)^N$, and Lemma 2, it follows that

$$\Big\| \sum_{l=1}^{2^N-1} \sum_{j=J+1}^{+\infty} \sum_{k\in\{0,\dots,2^j-1\}^N} 2^{-jH} \epsilon_{l,j,k} \Psi_l(2^j \cdot -k, H) \Big\|_\infty$$
$$\leq C_4 \Big\| \sum_{j=J+1}^{+\infty} \sum_{k\in\{0,\dots,2^j-1\}^N} 2^{-jH} \mathbb{1}_{\prod_{q=1}^N [2^{-j}k_q, 2^{-j}(k_q+1))}(\cdot) \sqrt{\log(2+j+|k|)} \Big\|_\infty$$
$$\leq C_5 \sum_{j=J+1}^{+\infty} 2^{-jH} \sqrt{j} \leq C_6 2^{-JH} \sqrt{J}, \tag{14}$$

where C_4, C_5 and C_6 are random variables (non depending J) of finite moment of any order. Finally combining (13) with (14) one obtains the theorem. □

3-Local Hölder regularity of MBM

The MBM $\{X(t)\}_{t\in\mathbb{R}^N}$ is defined for every $t \in \mathbb{R}^N$ as

$$X(t) = B_{h(t)}(t) = \int_{\mathbb{R}^N} \frac{e^{it\cdot\xi} - 1}{|\xi|^{h(t)+N/2}}\, dW(\xi),$$

where $h(\cdot)$ is a deterministic funtion with values in $(0,1)$ which satisfies on each compact cube $K \subset \mathbb{R}^N$ a uniform Hölder condition of order $\beta > \max_{t\in K} h(t)$.

The local Hölder regularity of $\{X(t)\}_{t\in\mathbb{R}^N}$ can be measured through its pointwise Hölder exponent $\{\alpha_X(t)\}_{t\in\mathbb{R}^N}$ which is defined for all $t \in \mathbb{R}^N$ as,

$$\alpha_X(t) = \sup\Big\{\alpha \in [0,1] \,:\, \limsup_{s\to 0} \frac{|X(t+s) - X(t)|}{|s|^\alpha} = 0\Big\}.$$

The goal of this section is to show that

Theorem 3 (Jaffard, Taqqu and Ayache)

$$\mathbb{P}\big\{\forall t \in \mathbb{R}^N :\, \alpha_X(t) = h(t)\big\} = 1.$$

It not very difficult to prove that

$$\mathbb{P}\Big\{\forall t\in\mathbb{R}^N :\ \alpha_X(t)\ge h(t)\Big\}=1. \tag{15}$$

Indeed, one has for all $\omega\in\Omega$ and $t\in\mathbb{R}^N$,

$$\alpha_X(t,\omega)\ge \sup\Big\{\beta_X(K,\omega)\ :\ K \text{ is a cube s.t. } t\in \mathring{K}\Big\},$$

where

$$\beta_X(K)=\sup\Big\{\beta\in[0,1]\ :\ \sup_{t,t+s\in K}\frac{|X(t+s)-X(t)|}{|s|^{\beta}}<+\infty\Big\},$$

is the **uniform Hölder exponent of** $\{X(t)\}_{t\in\mathbb{R}^N}$ **over** K. Therefore to derive (15), it is sufficient to obtain a convenient lower bound of $\beta_X(K)$; to this end we will use a strong version of Kolmogorov criterion.

Proposition 1 (Taqqu and Ayache)

The uniform Hölder exponent of MBM satisfies

$$\mathbb{P}\left\{\beta_X(K)=\min_{x\in K} h(x)\ :\ \textit{for all cube } K\right\}=1.$$

In view of the continuity of $h(\cdot)$, Proposition 1 implies that

$$\mathbb{P}\left\{\forall t\in\mathbb{R}^N:\ h(t)=\sup\left\{\beta_X(K)\ :\ K \text{ is a cube s.t. } t\in\mathring{K}\right\}\right\}=1.$$

Proof of Proposition 1: Let K be an arbitrary cube. Standard computations allow to show that for all $t, t+s\in K$

$$\mathbb{E}\left|X(t+s)-X(t)\right|^2=\int_{\mathbb{R}^N}\left|\frac{e^{i(t+s)\cdot\xi}-1}{|\xi|^{h(t+s)+N/2}}-\frac{e^{it\cdot\xi}-1}{|\xi|^{h(t)+N/2}}\right|^2 d\xi\le |s|^{2\min_{x\in K}h(x)},$$

where $c>0$ is a constant only depending on K. Then it follows from a strong version of Kolmogorov criterion (see for example Karatzas and Shreve) that

$$\mathbb{P}\left\{\beta_X(K)=\min_{x\in K} h(x)\right\}=1.$$

A straightforward consequence is that

$$\mathbb{P}\left\{\beta_X(K) = \min_{x\in K} h(x)\ :\ \text{for all cube } K \text{ with rational vertices}\right\} = 1.$$

Finally by the continuity of $h(\cdot)$ and the decreaseness of $\beta_X(\cdot)$, one obtains the proposition. □

The fact that

$$\mathbb{P}\{\forall t \in \mathbb{R}^N : \alpha_X(t) \leq h(t)\} = 1. \tag{16}$$

is more difficult to prove. This will result from the following two lemmas.

Lemma 3

For all compact cube K and all reals $0 < a < b < 1$, there is a random variable $C > 0$ of finite moment of any order, such that one has almost surely for all $H_1, H_2 \in [a, b]$,

$$\sup_{t\in K} |B_{H_1}(t) - B_{H_2}(t)| \leq C|H_1 - H_2|. \tag{17}$$

Lemma 4

$\{\alpha_{B_H}(t)\}_{t\in\mathbb{R}^N}$ the pointwise Hölder exponent of the FBM $\{B_H(t)\}_{t\in\mathbb{R}^N}$ satisfies

$$\mathbb{P}\big\{\alpha_{B_H}(t)\le H\ :\ \forall\, t\in\mathbb{R}^N \text{ and } H\in(0,1)\big\}=1.$$

Proof of (16): Let K be a compact cube. The increments of the MBM $\{X(t)\}_{t\in K}=\{B_{h(t)}(t)\}_{t\in K}$ satisfy

$$|X(t+s)-X(t)|\ge \big|B_{h(t)}(t+s)-B_{h(t)}(t)\big|-\sup_{x\in K}\big|B_{h(t+s)}(x)-B_{h(t)}(x)\big|. \tag{18}$$

Moreover, Lemma 3 and the fact that $h(\cdot)$ is a β-Hölder function, imply that

$$\sup_{x\in K}\big|B_{h(t+s)}(x)-B_{h(t)}(x)\big|\le C\big|h(t+s)-h(t)\big|\le C'|s|^{\beta}. \tag{19}$$

where $\beta>h(t)$. Putting together (18), (19) and Lemma 4, it follows that almost surely, for all $\theta\in\big(h(t),\beta\big)$,

$$\limsup_{s\to 0}\frac{|X(t+s)-X(t)|}{|s|^{\theta}}=+\infty.$$

Proof of Lemma 3: We have to show that a.s. for all $H_1, H_2 \in [a,b] \subset (0,1)$ one has

$$\sup_{t \in K} |B_{H_1}(t) - B_{H_2}(t)| \leq C|H_1 - H_2|. \tag{20}$$

For simplicity, we assume that $K = [0,1]^N$. By using the wavelet representation of FBM, one has

$$B_{H_1}(t) - B_{H_2}(t) = \sum_{l=1}^{2^N-1} \sum_{j=-\infty}^{-1} \epsilon_{l,j,0} \big(2^{-jH_1}\Psi_l(2^j t, H_1) - 2^{-jH_2}\Psi_l(2^j t, H_2)\big)$$
$$+ \sum_{l=1}^{2^N-1} \sum_{j=0}^{+\infty} \sum_{k \in \{0,\dots,2^j-1\}^N} \epsilon_{l,j,k} \big(2^{-jH_1}\Psi_l(2^j t - k, H_1) - 2^{-jH_2}\Psi_l(2^j t - k, H_2)\big). \tag{21}$$

It follows from the Mean Value Theorem That there is a constant $c_1 > 0$ such that for all $1 \leq l \leq 2^N - 1$, $j < 0$, H_1, H_2 and $t \in [0,1]^N$,

$$\left|2^{-jH_1}\Psi_l(2^j t, H_1) - 2^{-jH_2}\Psi_l(2^j t, H_2)\right| \leq c_1 |j| 2^{j(1-b)} |H_1 - H_2|. \tag{22}$$

By using again this theorem and our heuristical assumption that for all $1 \leq l \leq 2^N - 1$ and $H \in (0,1)$, $\operatorname{supp} \Psi_l(\cdot, H) \subset [0,1)^N$, one can show that there is a constant $c_2 > 0$ such that for all $1 \leq l \leq 2^N - 1$, $j \geq 0$ $k \in \{0, \dots, 2^j - 1\}^N$, H_1, H_2 and $t \in [0,1]^N$,

$$\begin{aligned} &\left|2^{-jH_1}\Psi_l(2^j t - k, H_1) - 2^{-jH_2}\Psi_l(2^j t - k, H_2)\right| \\ &\qquad \leq c_2 |j| 2^{-ja} |H_1 - H_2| \mathbb{1}_{\prod_{q=1}^N [2^{-j}k_q, 2^{-j}(k_q+1))}(t). \end{aligned} \tag{23}$$

Putting together (21), (22), (23) and the fact that $|\epsilon_{l,j,k}| \leq C_3 \sqrt{\log(2 + |j| + |k|)}$ one obtains (20). □

Now, our aim will be to prove Lemma 4. Let us set for each $x \in \mathbb{R}^N$ and $H \in (0,1)$,

$$\widetilde{\Psi}_1(x,H) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{ix\cdot\xi} |\xi|^{H+N/2} \widehat{\psi}_1(\xi)\, d\xi. \tag{24}$$

The function $\widetilde{\Psi}_1$ is C^∞ over $\mathbb{R}^N \times (0,1)$ and satisfies:

(i)

$$\sup\Big\{ \big(1+|x|\big)^p \big|(\partial_x^\gamma \widetilde{\Psi}_1)(x,H)\big| \,:\, x \in \mathbb{R}^N \text{ and } H \in (0,1) \Big\} < +\infty.$$

(ii) For every $H \in (0,1)$, the first moment of $\widetilde{\Psi}_1(\cdot, H)$ vanishes i.e.

$$\int_{\mathbb{R}^N} \widetilde{\Psi}_1(x,H)\, dx = 0.$$

The following proposition allows us to understand the motivation behind the introduction of the function $\widetilde{\Psi}_1$

Proposition 2 (Taqqu and Ayache)

Recall that the FBM $\{B_H(t)\}_{t\in\mathbb{R}^N}$ *can be expressed as*

$$B_H(t) = \sum_{l=1}^{2^N-1} \sum_{j=-\infty}^{+\infty} \sum_{k\in\mathbb{Z}^N} 2^{-jH} \epsilon_{l,j,k} \big(\Psi_l(2^j t - k, H) - \Psi_l(-k, H)\big).$$

One has, almost surely for all $H \in (0,1)$ *and* $(j,k) \in \mathbb{Z} \times \mathbb{Z}^N$,

$$2^{(N+H)j} \int_{\mathbb{R}^N} B_H(t) \widetilde{\Psi}_1(2^j t - k, H)\, dt = \epsilon_{1,j,k}.$$

Proposition 2 is a straightforward consequence of the following lemma.

Lemma 5

For every $H \in (0,1)$ *the sequences of functions*

$$\left\{2^{jN/2}\widetilde{\Psi}_1(2^j \cdot -k, H) \,:\, j \in \mathbb{Z},\, k \in \mathbb{Z}^N\right\},$$

and

$$\left\{2^{jN/2}\Psi_l(2^j \cdot -k, H) \,:\, 1 \leq l \leq 2^N - 1,\, j \in \mathbb{Z},\, k \in \mathbb{Z}^N\right\},$$

are biorthogonal i.e.

$$2^{(j+j')N/2} \int_{\mathbb{R}^N} \Psi_l(2^j t - k, H)\widetilde{\Psi}_1(2^{j'} t - k', H)\, dt = \begin{cases} 1 \text{ if } (l,j,k) = (1,j',k') \\ 0 \text{ else.} \end{cases}$$

Proof of Lemma 5: It follows from Plancherel formula and from the definitions of Ψ_l and $\widetilde{\Psi}_1$ that

$$2^{(j+j')N/2} \int_{\mathbb{R}^N} \Psi_l(2^j t - k, H)\widetilde{\Psi}_1(2^{j'} t - k', H)\, dt =$$

$$2^{-(j+j')N/2} \int_{\mathbb{R}^N} \left(e^{-i2^{-j}k\cdot\xi} \frac{\widehat{\psi}_l(2^{-j}\xi)}{|2^{-j}\xi|^{H+N/2}}\right)\left(e^{i2^{-j'}k'\cdot\xi}|2^{-j'}\xi|^{H+N/2}\overline{\widehat{\psi}_l(2^{-j'}\xi)}\right) d\xi$$

$$= 2^{(H+N/2)(j-j')-(j+j')N/2} \int_{\mathbb{R}^N} \left(e^{-i2^{-j}k\cdot\xi}\widehat{\psi}_l(2^{-j}\xi)\right) \left(e^{i2^{-j'}k'\cdot\xi}\overline{\widehat{\psi}_l(2^{-j}\xi)}\right) d\xi$$

$$= 2^{(H+N/2)(j-j')-(j+j')N/2} \int_{\mathbb{R}^N} \psi_l(2^j t - k)\psi_l(2^{j'} t - k')\, dt.$$

Then using the fact that

$$\left\{2^{jN/2}\psi_l(2^j \cdot - k, H) \,:\, 1 \le l \le 2^N - 1,\, j \in \mathbb{Z},\, k \in \mathbb{Z}^N\right\},$$

is an orthonormal sequence, we obtain the lemma. □

Proof of Lemma 4: We have to show that

$$\mathbb{P}\{\alpha_{B_H}(t) \le H \,:\, \forall\, t \in \mathbb{R}^N \text{ and } H \in (0,1)\} = 1. \tag{25}$$

Suppose ad absurdum that $\mathbb{P}(A) > 0$ where

$$A = \Big\{\omega \in \Omega \,:\, \text{there is } (t_0, H_0) \in \mathbb{R}^N \times (0,1) \text{ s.t. } \alpha_{B_{H_0}}(t_0,\omega) > H_0\Big\}.$$

Let $\omega \in A$, it follows from the definition of $\alpha_{B_{H_0}}(t_0)$, that there are two constants $\theta_0 > H_0$ and $c_0 > 0$ such that for all t close to t_0 one has

$$\big|B_{H_0}(t,\omega) - B_{H_0}(t_0,\omega)\big| \le c_0|t - t_0|^{\theta_0}. \tag{26}$$

Note that (26) remains valid for all $t \in \mathbb{R}^N$ because of the continuity of FBM and the slowness of its increase at infinity.

By using Proposition 2, the fact that $\int_{\mathbb{R}^N} \widetilde{\Psi}_1(x, H_0)\,dx = 0$, (26), the change of variable $s = 2^j t - k$ and the triangle inequality, one has for all j, k,

$$
\begin{aligned}
|\epsilon_{1,j,k}(\omega)| &= 2^{(N+H_0)j}\Big|\int_{\mathbb{R}^N} B_{H_0}(t,\omega)\widetilde{\Psi}_1(2^j t - k, H_0)\,dt\Big| \\
&= 2^{(N+H_0)j}\Big|\int_{\mathbb{R}^N} \Big(B_{H_0}(t,\omega) - B_{H_0}(t_0,\omega)\Big)\widetilde{\Psi}_1(2^j t - k, H_0)\,dt\Big| \\
&\le c_0 2^{(N+H_0)j}\int_{\mathbb{R}^N} |t - t_0|^{\theta_0}\big|\widetilde{\Psi}_1(2^j t - k, H_0)\big|dt \\
&= c_0 2^{jH_0}\int_{\mathbb{R}^N} |2^{-j}s + 2^{-j}k - t_0|^{\theta_0}\big|\widetilde{\Psi}_1(s, H_0)\big|\,ds \\
&\le c_1 2^{-j(\theta_0 - H_0)}\big(1 + |2^j t_0 - k|\big)^{\theta_0}.
\end{aligned}
\tag{27}
$$

where $c_1 > 0$ is a constant non depending on j, k.

Finally (27) implies that

$$\lim_{j\to+\infty} \sup\Big\{|\epsilon_{1,j,k}(\omega)| \,:\; k\in\mathbb{Z}^N \text{ and } \big|2^j t_0 - k\big| \le j\Big\} = 0,$$

but this is impossible (Borell-Cantelli Lemma) since $\{\epsilon_{1,j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}^N}$ is a sequence of independent standard Gaussian random variables. $\square$

The easy part in the problem we solved in the previous section, was to show that the MBM $\{X(t)\}_{t\in\mathbb{R}^N}$ has everywhere, locally, *a certain degree of regularity* i.e. $\alpha_X(t) \geq h(t)$; whereas the difficult part in this problem was to to show that it has everywhere, locally, *a certain degree of irregularity* i.e. $\alpha_X(t) \leq h(t)$.

Generally speaking, it is often a tricky problem to prove that, with probability 1, **the trajectories of a Gaussian field are everywhere locally irregular** as for example they are nowhere differentiable.

$\rightarrow$ **The notion of local time is very useful in such a problem: "When the local time of a field is regular then the field is irregular" (Berman).** For example, when a field has a jointly continuous local time then, with probability 1, the trajectories of the field are nowhere differentiable functions.

It is worth noticing that another method to solve the difficult part of the problem we study in the previous section consists in showing that **the local time of MBM satisfies appropriate Hölder conditions**.

In this section, we will content ourself with presenting the main ideas to show that the local time of MBM is jointly continuous.

Let $\{X(t)\}_{t\in\mathbb{R}^N}$ be a Gaussian field and $I \subseteq \mathbb{R}^N$ a fixed Borel set.

μ_I **the occupation measure** of X on I, is defined for each borel set $B \subseteq \mathbb{R}$ as

$$\mu_I(B) = \lambda_N\big\{t \in I : X(t) \in B\big\},$$

where λ_N denotes the Lebesgue measure on $\mathbb{R}^N$.

If μ_I is absolutely continuous with respect to λ, then X is said to have **a local time** on I and its local time $L(\cdot, I)$ is defined as the Radon–Nikodým derivative of μ_I with respect to λ, i.e. for all $x \in \mathbb{R}$,

$$L(x, I) = \frac{d\mu_I}{d\lambda}(x).$$

x is the *space variable*, and I is the *time variable*. Sometimes, we write $L(x, t)$ in place of $L\big(x, \prod_{l=1}^{N}[0, t_l]\big)$.

If $L(x, I)$ exists, then for all Borel set $J \subset I$, $L(x, J)$ also exists.

Suppose we fix a rectangle $I = \prod_{l=1}^{N}[d_l, d_l + h_l]$. If there is a version of the local time, still denoted by $L\big(x, \prod_{l=1}^{N}[d_l, d_l + t_l]\big)$ such that it is, almost surely, a continuous function of $(x, t_1, \dots, t_N) \in \mathbb{R} \times \prod_{l=1}^{N}[0, h_l]$, X is said to have **a jointly continuous local time on I**.

The joint continuity of the local time is quite useful when one studies the path behavior of X, for example when this property is satisfied then (see Adler) $L(x, \cdot)$ can be extended to be **a finite measure supported on the level set**

$$X^{-1}(x) \cap I = \{t \in I : X(t) = x\},$$

which allows to obtain a sharp lower bound of the Hausdorff dimension of this set (Frostman Lemma).

From now on we assume that $I = [\epsilon, 1]^N$, where $\epsilon > 0$ is fixed.

Theorem 4 (Shieh, Xiao and Ayache)

The MBM X has a local time $L(\cdot, I)$ on I. Moreover, one has almost surely $\int_{\mathbb{R}} \big(L(x, I)\big)^2\, dx < +\infty$.

Proof of Theorem 4: In view of Plancherel Theorem, it is sufficient to show that

$$\Phi_{\mu_I}(\xi) = \int_{\mathbb{R}} e^{i\xi x}\, d\mu_I(x) = \int_I e^{i\xi X(t)}\, dt, \tag{28}$$

the characteristic function of the occupation measure μ_I of the MBM, satisfies

$$\mathbb{E}\left(\int_{\mathbb{R}} \left|\Phi_{\mu_I}(\xi)\right|^2 d\xi \right) < +\infty. \tag{29}$$

It follows from Tonelli Theorem and (28) that

$$\mathbb{E}\left(\int_{\mathbb{R}} \left|\Phi_{\mu_I}(\xi)\right|^2 d\xi\right) = \int_I \int_I \int_{\mathbb{R}} \mathbb{E}\left(e^{i\xi(X(t)-X(s))}\right) d\xi\, ds\, dt.$$

Observe that $\xi \mapsto E\big(e^{i\xi(X(t)-X(s))}\big)$ is the characteristic function of the centered Gaussian random variable $X(t)-X(s)$. One has therefore

$$E\big(e^{i\xi(X(t)-X(s))}\big) = \exp\left\{-\frac{1}{2}\Big(\sigma\big(X(t)-X(s)\big)\xi\Big)^2\right\}.$$

where $\sigma\big(X(t)-X(s)\big) = \Big(\mathbb{E}\left|X(t)-X(s)\right|^2\Big)^{1/2}$. Setting $\eta = \sigma\big(X(t)-X(s)\big)\xi$, it follows that

$$\int_{\mathbb{R}} \mathbb{E}\left(e^{i\xi(X(t)-X(s))}\right) d\xi = \sigma\big(X(t)-X(s)\big)^{-1} \int_{\mathbb{R}} e^{-\eta^2/2}\, d\eta.$$

Thus, in order to show that

$$\mathbb{E}\Big(\int_{\mathbb{R}} \big|\Phi_{\mu_I}(\xi)\big|^2\, d\xi\Big) < +\infty,$$

it is sufficient to prove that

$$\int_I \int_I \sigma\big(X(t) - X(s)\big)^{-1}\, ds\, dt < +\infty.$$

The finiteness of the latter integral is a straightforward consequence of the fact that

$$\sigma\big(X(t) - X(s)\big) \geq c|t - s|^b,$$

where $b = \max_{t\in I} h(t) < 1 < N$. $\square$

The proof of the joint continuity of the local time of MBM mainly relies on the following two lemmas.

Lemma 6

For every integer $m \geq 1$, there exists a constant $c > 0$, only depending on m, N, $b = \max_{t \in I} h(t)$ and I, such that for all $x \in \mathbb{R}$ and all cube $T \subseteq I$, one has

$$\mathbb{E}\left[L(x,T)^m\right] \leq c\,(Diam\,T)^{m(N-b)}.$$

Lemma 7

For every even integer $m \geq 1$, there exists a constant $c > 0$, only depending on m, N, b and I, such that for all cube $T \subseteq I$, all $x, y \in \mathbb{R}$ satisfying $|x - y| \leq 1$ and all $\gamma \in (0,1)$ small enough, one has

$$\mathbb{E}\left[(L(x,T) - L(y,T))^m\right] \leq c|x-y|^{m\gamma} \times (Diam\,T)^{m(N-b-\gamma)}.$$

Theorem 5 (Shieh, Xiao and Ayache)

MBM has a jointly continuous local time over $I = [\epsilon, 1]^N$.

Proof of Theorem 5: We will use Kolmogorov criterion. For simplicity, we suppose that $N = 1$. Denote by $m \geq 2$ an arbitrary even integer. One has for all reals x, y, s, t satisfying $|x - y| \leq 1$ and $\epsilon \leq s < t \leq 1$,

$$\begin{aligned} &\mathbb{E}\left[\left(L(x,[\epsilon,t]) - L(y,[\epsilon,s])\right)^m\right] \\ &\leq c_1 \mathbb{E}\left[\left(L(x,[s,t])\right)^m\right] + c_1 \mathbb{E}\left[\left(L(x,[\epsilon,s]) - L(y,[\epsilon,s])\right)^m\right] \\ &\leq c_2 |s-t|^{m(N-b)} + c_2 |x-y|^{m\gamma} \times |s-t|^{m(N-b-\gamma)}. \end{aligned}$$

Thus, assuming that m is big enough, it follows that

$$\mathbb{E}\left[\left(L(x,[\epsilon,t]) - L(y,[\epsilon,s])\right)^m\right] \leq c_4 \left(|s-t| + |x-y|\right)^3.$$

Finally, applying Kolmogorov criterion, one gets the joint continuity of the local time. □

We will only give the proof of Lemma 6 since that of Lemma 7 is in the same spirit. The notion of local nondeterminism, which will be soon defined, will play a crucial role in this proof.

Proof of Lemma 6: By using a classical result on local times (see for instance Geman and Horowitz)

$$\mathbb{E}\,[L(x,T)^m] = (2\pi)^{-m}\int_{T^m}\int_{\mathbb{R}^m} \exp\Big(-ix\sum_{j=1}^m u^j\Big) \times \mathbb{E}\exp\Big(i\sum_{j=1}^m u^j X(t^j)\Big)\,d\overline{u}\,d\overline{t} \tag{30}$$

where $\overline{u} = (u^1,\dots,u^m) \in \mathbb{R}^m$ and $\overline{t} = (t^1,\dots,t^m) \in T^m$.

By using the fact that $\overline{u} \mapsto \mathbb{E} \exp\left(i \sum_{j=1}^m u^j X(t^j)\right)$ is the characteristic function of the centered Gaussian vector $\left(X(t^1), \ldots, X(t^m)\right)$, it follows that

$$\mathbb{E} \exp\left(i \sum_{j=1}^m u^j X(t^j)\right) = \exp\left(-\frac{1}{2}\overline{u}' \Gamma_X(\overline{t})\overline{u}\right),$$

where $\Gamma_X(\overline{t})$ denotes the covariance matrix of this Gaussian vector. Therefore one has that

$$\int_{\mathbb{R}^m} \mathbb{E} \exp\left(i \sum_{j=1}^m u^j X(t^j)\right) d\overline{u} = (2\pi)^{m/2} \left[\det(\Gamma_X(t))\right]^{-1/2}$$

and as a consequence

$$\mathbb{E}\left[L(x,T)^m\right] \leq (2\pi)^{-m/2} \int_{T^m} \left[\det(\Gamma_X(\overline{t}))\right]^{-1/2} d\overline{t}.$$

From the other hand, the determinant of Γ_Z the covariance matrix of an arbitrary Gaussian vector $Z = (Z_1, \ldots, Z_m)$ can be expressed as,

$$\det(\Gamma_Z) = \mathrm{Var}(Z_1) \prod_{n=2}^{m} \mathrm{Var}(Z_n | Z_n, \ldots, Z_{n-1}).$$

Thus one needs to find a convenient lower bound of

$$\mathrm{Var}\big(X(t^n) | X(t^1), \ldots, X(t^{n-1})\big).$$

The concept of local nondeterminism (LND) was introduced to this end.

A Gaussian field $\{X(t)\}_{t\in\mathbb{R}^N}$ is said to be LND over a cube I, if for all integer $n\geq 2$, there exist two constants $c>0$ and $\delta>0$, only depending on n, such that one has,

$$\mathrm{Var}\big(X(t^n)|X(t^1),\ldots,X(t^{n-1})\big)\geq c\min_{1\leq p\leq n-1}\left\{\mathrm{Var}\big(X(t^n)-X(t^p)\big)\right\}, \tag{31}$$

for all $t^1,\ldots,t^n\in I$ satisfying $|t^i-t^j|\leq\delta$ for every $i,j\in\{1,\ldots,n\}$.

(31) suggests that the increments of X over I are **asymptotically independent.**

Theorem 6 (Shieh, Xiao and Ayache)

The MBM $\{X(t)\}_{t\in\mathbb{R}^N}$ is LND over the cube $I=[\epsilon,1]^N$.

Heuristic proof of Theorem 6: Standard computations allow to show that

$$c_1|s-t|^{2\max\{h(s),h(t)\}} \le \mathrm{Var}\big(X(s)-X(t)\big) \le c_2|s-t|^{2\max\{h(s),h(t)\}}, \tag{32}$$

where $0<c_1\le c_2$ are two constants. Thus it is sufficient to prove that

$$\mathrm{Var}\big(X(t^n)|X(t^1),\ldots,X(t^{n-1})\big) \ge c_3\min\Big\{|t^n-t^p|^{2h(t^n)}\,:\,1\le p\le n-1\Big\}, \tag{33}$$

for all $t^1,\ldots,t^n\in I$,

or equivalently that

$$\mathbb{E}\left|X(t^n)-\sum_{p=1}^{n-1}\lambda_p X(t^p)\right|^2 \geq c_3 \min\left\{\left|t^n-t^p\right|^{2h(t^n)} : 1\leq p\leq n-1\right\}, \quad (34)$$

for all $\lambda_1,\ldots,\lambda_{n-1}\in\mathbb{R}$. We will use the wavelet series representation of MBM derived from that of FBM:

$$X(t)=B_{h(t)}(t) = \sum_{l=1}^{2^N-1}\sum_{j=-\infty}^{+\infty}\sum_{k\in\mathbb{Z}^N} 2^{-jh(t)}\epsilon_{l,j,k}\big(\Psi_l(2^j t-k,h(t))-\Psi_l(-k,h(t))\big).$$

As the functions $\Psi_l(x,\theta)$ are well-localized in x uniformly in θ, we suppose heuristically that, for all $(x,\theta) \in \mathbb{R}^N \times (0,1)$,

$$\Psi_l(x,\theta) = \mathbb{1}_{[0,1)^N}(x). \tag{35}$$

Let j_0 be the unique integer such that

$$2^{-j_0-1} < 2^{-1}\epsilon N^{-1/2} \min\left\{|t^n - t^p| \,:\, 1 \le p \le n-1\right\} \le 2^{-j_0}. \tag{36}$$

It follows from (35) and (36) that there is a unique $k^n \in \mathbb{Z}^N$ which satisfies

$$\Psi_l(2^{j_0}t^n - k^n, h(t^n)) = 1, \tag{37}$$

$$\Psi_l(-k^n, h(t^p)) = 0, \quad \text{for every} \quad p = 0,1,\dots,n \tag{38}$$

and

$$\Psi_l(2^{j_0}t^p - k^n, h(t^p)) = 0, \quad \text{for every} \quad p = 1,\dots,m. \tag{39}$$

From the other hand, by using the wavelet series representation of MBM and the fact that the random variables $\epsilon_{l,j,k}$ in it, are independent $\mathcal{N}(0,1)$ Gaussian random variables, one obtains that

$$
\begin{aligned}
&\mathbb{E}\left|X(t^n)-\sum_{p=1}^{n-1}\lambda_p X(t^p)\right|^2 \\
&=\sum_{l=1}^{2^N-1}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^N}\Bigg|2^{-jh(t^n)}\Big(\Psi_l(2^j t^n-k,h(t^n))-\Psi_l(-k,h(t^n))\Big) \\
&\qquad -\sum_{p=1}^{n-1}2^{-jh(t^p)}\lambda_p\Big(\Psi_l(2^j t^p-k,h(t^p))-\Psi_l(-k,h(t^p))\Big)\Bigg|^2 \\
&\geq\Bigg|2^{-j_0h(t^n)}\Big(\Psi_1(2^{j_0}t^n-k^n,h(t^n))-\Psi_1(-k^n,h(t^n))\Big) \\
&\qquad -\sum_{p=1}^{n-1}2^{-j_0h(t^p)}\lambda_p\Big(\Psi_1(2^{j_0}t^n-k_0,h(t^p))-\Psi_1(-k_0,h(t^p))\Big)\Bigg|^2
\end{aligned}
\tag{40}
$$

Finally, it follows from (36), (37), (38), (39) and (40) that

$$\begin{aligned}\mathbb{E}\left|X(t^n) - \sum_{p=1}^{n-1} \lambda_p X(t^p)\right|^2 \\ &= 2^{-2j_0 h(t^n)} \\ &\geq c_3 \min\left\{|t^n - t^p|^{2h(t^n)} \,:\, 1 \leq p \leq n-1\right\}.\end{aligned}$$