Harmonizable Fractional Stable Fields: Local Nondeterminism and Joint Continuity of the Local Times

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Organization of the talk

Harmonizable Fractional Stable Field (HFSF)

The real-valued HFSF, denoted by $X = \{X(t), t \in \mathbb{R}^N\}$, is one of the most classical extensions of the well-known Fractional Brownian Field (FBF), to the setting of heavy-tailed stable distributions. It is parameterized by $\alpha \in (0,2)$ and $H \in (0,1)$, the *stability* and *Hurst* parameters; moreover it is defined as:

$$X(t) := c\,\mathrm{Re} \int_{\mathbb{R}^N} \frac{e^{it\cdot\xi} - 1}{|\xi|^{H+N/\alpha}}\, \widetilde{M}_\alpha(d\xi), \qquad \forall\ t \in \mathbb{R}^N; \tag{1.1}$$

- $\widetilde{M}_\alpha$ denotes a complex-valued rotationally invariant α-stable random measure with Lebesgue control measure, thus,

$$\left\| \mathrm{Re} \int_{\mathbb{R}^N} f(\xi)\, \widetilde{M}_\alpha(d\xi) \right\|_\alpha^\alpha = \int_{\mathbb{R}^N} \left| f(\xi) \right|^\alpha d\xi, \qquad \forall f \in L^\alpha(\mathbb{R}^N); \tag{1.2}$$

- X is H-self-similar with stationary and isotropic increments;
- $c = c(\alpha, H, N) > 0$ denotes the normalizing constant chosen such that

$$\|X(t)\|_\alpha = |t|^H, \qquad \forall\ t \in \mathbb{R}^N. \tag{1.3}$$

Several authors have studied various path properties of HFSF and its local times, for example:

- using LePage series, Kôno and Maejima (1991), showed that, on any compact rectangle of $\mathbb{R}^N$, the paths are, almost surely, Hölder continuous functions of an arbitrary order $\gamma < H$; later, Xiao (2010) obtained a more precise uniform modulus of continuity for them;
- In the case where $\alpha \in [1, 2)$, Nolan (1989) investigated the regularity of the local times of an $\mathbb{R}^d$-valued HFSF (its components are i.i.d. real-valued HFSF); Kôno and Shieh (1993) as well as Shieh (1993), studied existence and joint continuity of the intersection local times.

The concept of *local nondeterminism (LND)* is the keystone of most of the works on regularity of local times of Gaussian and stable fields/processes. Roughly speaking, *it is a kind of generalization of the notion of independent increments.*

The concept of LND was first introduced by Berman (1973) in the framework of Gaussian processes. Pitt (1978) extended it to the framework of Gaussian fields, and Nolan (1989) to that of stable fields.

For HFSF, the property of LND was proved by Nolan (1989) for $1 \leq \alpha < 2$. However, the problem for the case $0 < \alpha < 1$ had remained open. The main objective of this talk is to resolve this problem.

Organization of the talk

Let $Y=\{Y(t), t\in\mathbb{R}^N\}$ be an arbitrary real-valued stable field such that,

$$Y(t)=\mathrm{Re}\int_{\mathbb{R}^N} h(t,\xi)\,\widetilde{S}_\alpha(d\xi), \qquad \forall\ t\in\mathbb{R}^N; \tag{2.1}$$

- $\widetilde{S}_\alpha$ is a complex-valued rotationally invariant α-stable random measure on $\mathbb{R}^N$ with an arbitrary control measure Δ;
- $h(t,\cdot):\mathbb{R}^N\to\mathbb{C}$ $(t\in\mathbb{R}^N)$ is a family of functions belonging to $L^\alpha(\Delta)$ i.e.

$$\big\|h(t,\cdot)\big\|_{L^\alpha(\Delta)}^\alpha := \int_{\mathbb{R}^N} |h(t,\xi)|^\alpha\,\Delta(d\xi)<\infty, \qquad \forall\ t\in\mathbb{R}^N. \tag{2.2}$$

Let the random vector $\mathbb{Y}_{t^1,\ldots,t^m} := \big(Y(t^1),\ldots,Y(t^m)\big)$ where $t^1,\ldots,t^m\in\mathbb{R}^N$ are arbitrary; its characteristic function $\Upsilon_{\mathbb{Y}_{t^1,\ldots,t^m}}$ defined as,

$$\Upsilon_{\mathbb{Y}_{t^1,\ldots,t^m}}(d_1,\ldots,d_m) := \mathbb{E}\exp\left(i\sum_{n=1}^m d_n Y(t^n)\right), \qquad \forall\ (d_1,\ldots,d_m)\in\mathbb{R}^m, \tag{2.3}$$

can be expressed as,

$$\Upsilon_{\mathbb{Y}_{t^1,\ldots,t^m}}(d_1,\ldots,d_m) = \exp\left(-\Big\|\sum_{n=1}^m d_n h(t^n,\cdot)\Big\|_{L^\alpha(\Delta)}^\alpha\right). \tag{2.4}$$

Hence, the scale parameter of the symmetric α-stable random variable $\sum_{n=1}^{m} d_n Y(t^n)$ is given by

$$\Big\| \sum_{n=1}^{m} d_n Y(t^n) \Big\|_\alpha = \Big\| \sum_{n=1}^{m} d_n h(t^n, \cdot) \Big\|_{L^\alpha(\Delta)}. \tag{2.5}$$

When $m \geq 2$, the $L^\alpha(\Delta)$-distance from $h(t^m, \cdot)$ to the subspace of $L^\alpha(\Delta)$ spanned by $\big\{ h(t^1, \cdot), \ldots, h(t^{m-1}, \cdot) \big\}$ is defined as,

$$\inf \Big\{ \Big\| h(t^m, \cdot) - \sum_{n=1}^{m-1} b_n h(t^n, \cdot) \Big\|_{L^\alpha(\Delta)} : \ \forall \ b_1, \ \ldots, \ b_{m-1} \in \mathbb{R} \Big\}; \tag{2.6}$$

notice that the infimum is attained. Because of the analogy with the Gaussian case ($\alpha = 2$), this infimum can be viewed as the L^α-error of predicting $Y(t^m)$, given $Y(t^1), \ldots, Y(t^{m-1})$; therefore we denote it by,

$$\big\| Y(t^m) \big| Y(t^1), \ldots, Y(t^{m-1}) \big\|_\alpha.$$

To state the property of LND in the sense of Nolan (1989), we will make use of the following partial order in $\mathbb{R}^N$: for any $m \geq 2$ distinct points $t^1, \ldots, t^m \in \mathbb{R}^N$,

$$t^1 \preccurlyeq t^2 \preccurlyeq \cdots \preccurlyeq t^m \Longleftrightarrow \forall\ 1 \leq i < j \leq m,\ \ |t^j - t^{j-1}| \leq |t^j - t^i|; \tag{2.7}$$

note that the partial order defined by (2.7) is not unique (including the case $N = 1$); one reason is that the choice of the point to be labeled t^m can be done in m different ways.

Definition 2.1 (Nolan (1989), local nondeterminism (LND))

Let $I \subset \mathbb{R}^N$ be a compact rectangle. The stable field Y is said to be LND on I, if:

(a) $\big\|Y(t)\big\|_\alpha > 0$, *for every* $t \in I$;

(b) $\big\|Y(s) - Y(t)\big\|_\alpha > 0$ *for all distinct* $s, t \in I$ *sufficiently close together;*

(c) *for each fixed integer* $m \geq 2$,

$$\liminf_{t^1 \preccurlyeq t^2 \preccurlyeq \cdots \preccurlyeq t^m,\ |t^m - t^1| \to 0} \frac{\big\|Y(t^m)\big|Y(t^1), \ldots, Y(t^{m-1})\big\|_\alpha}{\|Y(t^m) - Y(t^{m-1})\|_\alpha} > 0. \tag{2.8}$$

Conditions (a) and (b) rule out degeneracy; condition (c) is the important one. The ratio in (2.8) is a relative linear prediction error and is always between 0 and 1. If the ratio is bounded away from 0 as $|t^m - t^1| \to 0$, then we can approximate $Y(t^m)$ in $\|\cdot\|_\alpha$ (quasi)-norm, by the "most recent" value $Y(t^{m-1})$ alone, with the same order of error as with all the "past" $Y(t^1), \ldots, Y(t^{m-1})$.

Definition 2.2 (Nolan (1989), locally appoximately independent increments)

Y is said to have locally appoximately independent increments on I, if:

(a) $\|Y(t)\|_\alpha > 0$, *for every* $t \in I$;

(b) $\|Y(s) - Y(t)\|_\alpha > 0$ *for all distinct* $s, t \in I$ *sufficiently close together;*

(c′) *for each fixed integer* $m \geq 2$, *there is a constant* $c_m > 0$, *such that the inequalities,*

$$\begin{aligned} c_m^{-1}\Big(\big\|b_1 Y(t^1)\big\|_\alpha \;&+\; \sum_{j=2}^m \big\|b_j\big(Y(t^j) - Y(t^{j-1})\big)\big\|_\alpha\Big) \\ &\leq \Big\|b_1 Y(t^1) + \sum_{j=2}^m b_j\big(Y(t^j) - Y(t^{j-1})\big)\Big\|_\alpha \qquad (2.9) \\ &\leq c_m\Big(\big\|b_1 Y(t^1)\big\|_\alpha + \sum_{j=2}^m \big\|b_j\big(Y(t^j) - Y(t^{j-1})\big)\big\|_\alpha\Big), \end{aligned}$$

hold, for every $b_1, \ldots, b_m \in \mathbb{R}$ *and all* $t^1 \preccurlyeq t^2 \preccurlyeq \cdots \preccurlyeq t^m$ *sufficiently close.*

Theorem 2.3 (Nolan (1989))

Y is LND on I if and only if Y has locally appoximately independent increments on I.

For proving this theorem Nolan made use of arguments from Linear Algebra relying on generalized Grammian:

$$\begin{aligned} g(V^1,\dots,V^m) \quad := \quad & \big\|V^1\big\| \times \big\|V^2 - \mathrm{span}\{V^1\}\big\| \times \\ & \big\|V^3 - \mathrm{span}\{V^1,V^2\}\big\| \times \dots \times \big\|V^m - \mathrm{span}\{V^1,\dots,V^{m-1}\}\big\|, \end{aligned}$$

which, generally speaking, provides a measure of dependence between arbitrary vectors $V^1,\dots,V^m$ in an (quasi)-normed linear space.

Organization of the talk

Theorem 3.1 (Ayache and Xiao (2013))

Let I be a compact rectangle of $\mathbb{R}^N$ such that $0 \notin I$. Then, for all the possible values of its parameters α and H, the HFSF $X = \{X(t),\, t \in \mathbb{R}^N\}$ satisfies the following property: for any fixed integer $m \geq 2$, there exists a constant $c_1 = c_1(m) > 0$, depending on m, α, H, N and I only, such that for all $t^1, \ldots, t^m \in I$, one has,

$$\left\| X(t^m) \middle| X(t^1), \ldots, X(t^{m-1}) \right\|_\alpha \geq c_1 \min \left\{ |t^m - t^n|^H : 1 \leq n < m \right\}. \qquad (3.1)$$

Theorem 3.1 entails that X is LND on I, in the sense of Nolan (1989). Indeed, the equality

$$\|X(s) - X(t)\|_\alpha = |s - t|^H, \qquad \forall\, s, t \in \mathbb{R}^N, \qquad (3.2)$$

implies that $\|X(s) - X(t)\|_\alpha > 0$ when $s \neq t$; moreover, combining (3.2) with $X(0) = 0$ and $0 \notin I$, one gets that $\inf_{t \in I} \|X(t)\|_\alpha > 0$. On the other hand, assuming that $t^1 \preccurlyeq t^2 \preccurlyeq \cdots \preccurlyeq t^m$ and using (3.2), the minimum in (3.1), reduces to $\|X(t^m) - X(t^{m-1})\|_\alpha$; thus it results from (3.1) that,

$$\inf_{t^1 \preccurlyeq t^2 \preccurlyeq \cdots \preccurlyeq t^m} \frac{\left\| X(t^m) \middle| X(t^1), \ldots, X(t^{m-1}) \right\|_\alpha}{\|X(t^m) - X(t^{m-1})\|_\alpha} \geq c_1 > 0. \qquad (3.3)$$

From now on, we suppose that $I := [\varepsilon, 1]^N$, $\varepsilon \in (0,1)$ being arbitrary. Recall that

$$X(t) := c\,\mathrm{Re} \int_{\mathbb{R}^N} \mathcal{K}(t,\xi)\, \widetilde{M}_\alpha(d\xi), \quad \text{where} \quad \mathcal{K}(t,\xi) := \frac{e^{it\cdot\xi} - 1}{|\xi|^{H+N/\alpha}}; \tag{3.4}$$

thus, for proving the previous theorem, one has to show that, the following inequality holds, for all $t^1, \ldots, t^m \in I$ and $b_1, \ldots, b_m \in \mathbb{R}$,

$$\begin{aligned} &\Big\| X(t^m) - \sum_{n=1}^{m-1} b_n X(t^n) \Big\|_\alpha^\alpha \qquad (3.5)\\ &= \int_{\mathbb{R}^N} \Big| \mathcal{K}(t^m,\xi) - \sum_{n=1}^{m-1} b_n\, \mathcal{K}(t^n,\xi) \Big|^\alpha d\xi \geq c_1^\alpha \min\left\{ |t^m - t^n|^{\alpha H} : 1 \leq n < m \right\}. \end{aligned}$$

Lemma 3.2

It is sufficient to establish the inequality in (3.5), only when $\max_{1\leq n<m} |b_n| \leq 1$.

Sketch of the Proof of Lemma 3.2: the proof can be done by induction on m, here we only give it at the initial step $m = 2$. By the assumption of the lemma, one knows that, for some constant $c_1 > 0$, the inequality,

$$\left\|X(t^2) - b_1 X(t^1)\right\|_\alpha^\alpha \geq c_1^\alpha |t^2 - t^1|^{\alpha H}, \tag{3.6}$$

holds, for all $t^1, t^2 \in I$ and $b_1 \in \mathbb{R}$ satisfying $|b_1| \leq 1$. Let us show that it remains true when b_1 is replaced by a real number a_1 such that $|a_1| > 1$. One clearly has,

$$\begin{aligned} \left\|X(t^2) - a_1 X(t^1)\right\|_\alpha^\alpha &= |a_1|^\alpha \left\|X(t^1) - a_1^{-1} X(t^2)\right\|_\alpha^\alpha \\ &\geq \left\|X(t^1) - a_1^{-1} X(t^2)\right\|_\alpha^\alpha; \end{aligned} \tag{3.7}$$

then using (3.6), in which, t^1 and t^2 are interchanged, and $b_1 = a_1^{-1}$, one gets that

$$\left\|X(t^2) - a_1 X(t^1)\right\|_\alpha^\alpha \geq c_1^\alpha |t^2 - t^1|^{\alpha H}. \tag{3.8}$$

$\square$

From now on, we assume that $\max_{1\le n<m}|b_n|\le 1$. We set

$$\rho := 2^{-1}\varepsilon\, N^{-1/2}\min\left\{|t^m - t^n| : 1\le n<m\right\}; \tag{3.9}$$

thus, $\rho\in[0,2^{-1}\varepsilon]$, moreover, there is no restriction to suppose that $\rho\neq 0$. Let $c_*\in\big(0,(16N)^{-1/2}\big]$ be an arbitrary constant, later we will impose to it to be very small. Denote by $j_0\ge 1$, the unique integer such that

$$2^{-j_0-1} < c_*\rho \le 2^{-j_0}. \tag{3.10}$$

Thus, the fact that $\mathcal{K}(t,\xi) := \frac{e^{it\cdot\xi}-1}{|\xi|^{H+N/\alpha}}$, and the change of variable $\eta = 2^{-j_0}\xi$, show that, the inequality

$$\int_{\mathbb{R}^N}\Big|\mathcal{K}(t^m,\xi)-\sum_{n=1}^{m-1} b_n\,\mathcal{K}(t^n,\xi)\Big|^{\alpha}d\xi \ge c_1^{\alpha}\rho^{\alpha H}, \tag{3.11}$$

we want to prove, is equivalent to the inequality

$$\int_{\mathbb{R}^N}\Big|\mathcal{K}(2^{j_0}t^m,\eta)-\sum_{n=1}^{m-1} b_n\,\mathcal{K}(2^{j_0}t^n,\eta)\Big|^{\alpha}d\eta \ge c_2>0, \tag{3.12}$$

where $c_2>0$ is some constant non depending on the t^n's and the b_n's.

In order to establish (3.12), we will make use of the "Fourier transform" of a Lemarié-Meyer orthonormal wavelet basis of $L^2(\mathbb{R}^N)$. We denote such a basis, by

$$\Big\{2^{jN/2}\psi_l(2^j\cdot -k):\, 1\le l\le 2^N-1; j\in\mathbb{Z},\, k\in\mathbb{Z}^N\Big\};$$

a specificity of it, is that the mother wavelets ψ_l, belong to the Schwartz class $\mathcal{S}(\mathbb{R}^N)$, and have compactly supported C^∞ Fourier transforms, satisfying,

$$\operatorname{supp}\,\widehat{\psi}_l\subseteq\Delta:=\Big[-\frac{8\pi}{3},\frac{8\pi}{3}\Big]^N\setminus\Big(-\frac{2\pi}{3},\frac{2\pi}{3}\Big)^N. \tag{3.13}$$

In view of (3.13), let us consider $\widehat{g}:\mathbb{R}^N\to[0,1]$, an even compactly supported C^∞ function, such that,

$$\operatorname{supp}\,\widehat{g}\subseteq\Delta \quad \text{and} \quad \widehat{g}(4\pi/3,0,\ldots,0)=1. \tag{3.14}$$

Then, for each fixed $t\in\mathbb{R}^N$, $\widetilde{\mathcal{K}}(t,\cdot):=\mathcal{K}(t,\cdot)\widehat{g}(\cdot)$, is a C^∞ function with a compact support included in Δ.

In view of the fact that $\widehat{g}$ is with values in $[0,1]$, for proving (3.12), it is sufficient to show that

$$A_\alpha := \int_{\mathbb{R}^N} \Big| \widetilde{\mathcal{K}}(2^{j_0} t^m, \eta) - \sum_{n=1}^{m-1} b_n \, \widetilde{\mathcal{K}}(2^{j_0} t^n, \eta) \Big|^\alpha d\eta \geq c_2 > 0. \tag{3.15}$$

Observe that,

$$\sup_{(t,\xi)\in\mathbb{R}^N\times\mathbb{R}^N} \big|\widetilde{\mathcal{K}}(t,\xi)\big| \leq \sup_{(t,\xi)\in\mathbb{R}^N\times\Delta} \frac{|e^{it\cdot\xi}-1|}{|\xi|^{H+N/\alpha}} \leq 2\Big(\frac{3}{2\pi}\Big)^{H+N/\alpha} := c_3; \tag{3.16}$$

therefore, using $\max_{1\leq n<m} |b_n| \leq 1$ and $\alpha \in (0,2)$, one gets,

$$\begin{aligned} A_\alpha &= m^\alpha c_3^\alpha \int_{\mathbb{R}^N} m^{-\alpha} c_3^{-\alpha} \Big| \widetilde{\mathcal{K}}(2^{j_0} t^m, \eta) - \sum_{n=1}^{m-1} b_n \, \widetilde{\mathcal{K}}(2^{j_0} t^n, \eta) \Big|^\alpha d\eta \\ &\geq m^\alpha c_3^\alpha \int_{\mathbb{R}^N} m^{-2} c_3^{-2} \Big| \widetilde{\mathcal{K}}(2^{j_0} t^m, \eta) - \sum_{n=1}^{m-1} b_n \, \widetilde{\mathcal{K}}(2^{j_0} t^n, \eta) \Big|^2 d\eta \\ &\geq m^{\alpha-2} c_3^{\alpha-2} A_2. \end{aligned} \tag{3.17}$$

A_2 is in fact the square of the $L^2(\mathbb{R}^N)$-norm of the function

$$F(\cdot) := \widetilde{\mathcal{K}}(2^{j_0} t^m, \cdot) - \sum_{n=1}^{m-1} b_n \widetilde{\mathcal{K}}(2^{j_0} t^n, \cdot).$$

Thus, denoting by $\left\{ w_{l,j,k} : 1 \le l \le 2^N - 1; j \in \mathbb{Z},\ k \in \mathbb{Z}^N \right\}$ the sequence of the coefficients of $F(\cdot)$, in the orthonormal basis of $L^2(\mathbb{R}^N)$,

$$\left\{ 2^{jN/2} \overline{\mathcal{F}\big(\psi_l(2^j \cdot - k)\big)} : 1 \le l \le 2^N - 1; j \in \mathbb{Z},\ k \in \mathbb{Z}^N \right\};$$

it follows from Parseval identity that

$$A_2 = \sum_{l=1}^{2^N - 1} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^N} \left| w_{l,j,k} \right|^2 \ge \sum_{k \in \mathbb{Z}^N} \left| w_{1,0,k} \right|^2. \tag{3.18}$$

From now on, we set $w_k := w_{1,0,k}$ and $\psi := \psi_1$.

$$\begin{aligned}
w_k &:= \int_{\mathbb{R}^N} F(\eta) e^{-ik\cdot\eta} \widehat{\psi}(\eta)\, d\eta \\
&= \int_{\mathbb{R}^N} \widetilde{\mathcal{K}}(2^{j_0} t^m, \eta) e^{-ik\cdot\eta} \widehat{\psi}(\eta)\, d\eta - \sum_{n=1}^{m-1} b_n \int_{\mathbb{R}^N} \widetilde{\mathcal{K}}(2^{j_0} t^n, \eta) e^{-ik\cdot\eta} \widehat{\psi}(\eta)\, d\eta \\
&= \int_{\mathbb{R}^N} \left(e^{i 2^{j_0} t^m \cdot \eta} - 1\right) e^{-ik\cdot\eta} \frac{\widehat{g}(\eta) \widehat{\psi}(\eta)}{|\eta|^{H+N/\alpha}}\, d\eta \\
&\qquad - \sum_{n=1}^{m-1} b_n \int_{\mathbb{R}^N} \left(e^{i 2^{j_0} t^n \cdot \eta} - 1\right) e^{-ik\cdot\eta} \frac{\widehat{g}(\eta) \widehat{\psi}(\eta)}{|\eta|^{H+N/\alpha}}\, d\eta \\
&= \Theta\left(2^{j_0} t^m - k\right) - \sum_{n=0}^{m-1} b_n \Theta\left(2^{j_0} t^n - k\right),
\end{aligned} \tag{3.19}$$

where, $t^0 := 0$, $b_0 := 1 - \sum_{n=1}^{m-1} b_n$ and Θ is the function of $\mathcal{S}(\mathbb{R}^N)$ defined as,

$$\Theta(x) := \int_{\mathbb{R}^N} e^{ix\cdot\eta} \frac{\widehat{g}(\eta)\, \widehat{\psi}(\eta)}{|\eta|^{H+N/\alpha}}\, d\eta, \qquad \forall\, x \in \mathbb{R}^N. \tag{3.20}$$

Recall that

$$\rho := 2^{-1}\varepsilon N^{-1/2}\min\{|t^m - t^n| : 1 \le n < m\}; \tag{3.21}$$

and that $j_0 \ge 1$ is defined as the unique positive integer such that

$$2^{-j_0-1} < c_*\rho \le 2^{-j_0}, \tag{3.22}$$

where, till now, $c_* \in \big(0, (16N)^{-1/2}\big]$ is an arbitrary constant. Let $\nu = \nu(t^m, \rho, c_*)$ be the set of indices k located "near t^m", defined as,

$$\nu := \left\{k \in \mathbb{Z}^N : |t^m - 2^{-j_0}k| \le 2^{-1}\rho\right\}. \tag{3.23}$$

One clearly has,

$$A_2^{1/2} \ge \Big(\sum_{k\in\mathbb{Z}^N}|w_k|^2\Big)^{1/2} \ge \Big(\sum_{k\in\nu}|w_k|^2\Big)^{1/2}; \tag{3.24}$$

thus, using the expression of w_k in terms of Θ, and the triangle inequality, one gets,

$$A_2^{1/2} \geq \Big(\sum_{k\in\nu} \big|\Theta\big(2^{j_0} t^m - k\big)\big|^2\Big)^{1/2} - \sum_{n=0}^{m-1} |b_n| \Big(\sum_{k\in\nu} \big|\Theta\big(2^{j_0} t^n - k\big)\big|^2\Big)^{1/2}. \qquad (3.25)$$

From now on, our goal is to show that the second term in the right-hand side of the inequality (3.25), is negligible with respect to the first one; to this end, we will use the fact that Θ is a well localized function, namely:

$$|\Theta(x)| \leq c_4 \big(1 + |x|\big)^{-1}, \qquad \forall\ x \in \mathbb{R}^N. \qquad (3.26)$$

It follows from (3.26), $|t^n - t^m| \geq \rho$, $|t^m - 2^{-j_0} k| \leq 2^{-1}\rho$, and the triangle inequality, that $|2^{j_0} t^n - k| \geq 4^{-1} c_*^{-1}$ and consequently that

$$\sum_{k\in\nu} \big|\Theta\big(2^{j_0} t^n - k\big)\big|^2 \leq c_5 c_*, \qquad (3.27)$$

where $c_5 > 0$ is some constant non depending on j_0, the t^n's and c_*.

On the other hand, using the fact that, for all $x \in \mathbb{R}^N$ and $k \in \mathbb{Z}^N$,

$$\begin{aligned} \Theta(x-k) &:= \int_{\mathbb{R}^N} e^{i(x-k)\cdot\eta} \, \frac{\widehat{g}(\eta)\,\widehat{\psi}(\eta)}{|\eta|^{H+N/\alpha}} \, d\eta \qquad (3.28) \\ &= \int_{[0,2\pi]^N} e^{-ik\cdot\eta} \left(\sum_{q\in\mathbb{Z}^N} e^{ix\cdot(\eta+2\pi q)} \, \frac{\widehat{g}(\eta+2\pi q)\,\widehat{\psi}(\eta+2\pi q)}{|\eta+2\pi q|^{H+N/\alpha}} \right) d\eta, \end{aligned}$$

one can show that

$$c_6 := \inf \Big\{ \sum_{k\in\mathbb{Z}^N} \big|\Theta(x-k)\big|^2 : x \in \mathbb{R}^N \Big\} > 0. \qquad (3.29)$$

Then, the localization property of Θ and the inequality $|2^{j_0} t^m - k| \geq 4^{-1} c_*^{-1}$ for all $k \notin \nu$, imply that

$$\sum_{k\in\nu} \big|\Theta\big(2^{j_0} t^m - k\big)\big|^2 \geq c_6 - c_7 c_*, \qquad (3.30)$$

where $c_7 > 0$ is some constant non depending on j_0, the t^n's and c_*.

Finally, using (3.25), (3.27), (3.30), and the inequality $\max_{0 \le n \le m} |b_n| \le m+1$, one gets that

$$\begin{aligned} A_2^{1/2} &\ge \Big(\sum_{k \in \nu} \big| \Theta\big(2^{j_0} t^m - k\big) \big|^2 \Big)^{1/2} - \sum_{n=0}^{m-1} |b_n| \Big(\sum_{k \in \nu} \big| \Theta\big(2^{j_0} t^n - k\big) \big|^2 \Big)^{1/2} \\ &\ge \sqrt{c_6 - c_7 c_*} - m(m+1)\sqrt{c_5 c_*}, \end{aligned} \tag{3.31}$$

which in turn entails that

$$A_2^{1/2} \ge 2^{-1} \sqrt{c_6} > 0, \tag{3.32}$$

provided that one imposes to the constant c_* to be small enough.
□

Organization of the talk

Now, we consider the $\mathbb{R}^d$-valued HFSF $\vec{X} = \{\vec{X}(t), t \in \mathbb{R}^N\}$ defined by

$$\vec{X}(t) = \big(X_1(t), \ldots, X_d(t)\big), \qquad \forall t \in \mathbb{R}^N, \tag{4.1}$$

where $X_1, \ldots, X_d$ are independent copies of the real-valued HFSF X. Thanks to the LND property of X we have just derived, we will establish the joint continuity for the local times of $\vec{X}$.

Let us first make a brief recall on local times and their joint continuity. For any fixed compact rectangle $T \subset \mathbb{R}^N$, the pathwise occupation measure of $\vec{X}$ on T, denoted by μ_T, is defined as the following Borel measure on $\mathbb{R}^d$:

$$\mu_T(\bullet) = \lambda_N\big\{t \in T : \vec{X}(t) \in \bullet\big\}. \tag{4.2}$$

If μ_T is almost surely absolutely continuous with respect to the Lebesgue measure λ_d on $\mathbb{R}^d$, then $\vec{X}$ is said to have local times on T, and its local time $L(\cdot, T)$ is defined as the Radon-Nikodým derivative of μ_T with respect to λ_d, i.e.,

$$L(x, T) = \frac{d\mu_T}{d\lambda_d}(x), \qquad \forall x \in \mathbb{R}^d.$$

In the above, x is the *space variable*, and T is the *time variable*.

Notice that when $\vec{X}$ has local times on T, then it also has local times on any rectangle $S \subseteq T$ (consequence of Radon-Nikodým-Lebesgue Theorem).
Express the rectangle T as $T = \prod_{l=1}^{N}[a_l, a_l + h_l]$ where $a_l \in \mathbb{R}$ and $h_l \in \mathbb{R}_+$, for all $l = 1, \ldots, N$. Then $\vec{X}$ is said to have jointly continuous local times on T, whenever we can choose a version of the local times, still denoted by $L\big(\cdot, \bullet\big)$, such that, almost surely, the function

$$(x, t_1, \cdots, t_N) \longmapsto L\Big(x, \prod_{l=1}^{N}[a_l, a_l + t_l]\Big),$$

is continuous on $\mathbb{R}^d \times \prod_{i=1}^{N}[0, h_i]$.
We refer to Geman and Horowitz (1980) and Dozzi (2002) for further information on local times of random fields. At last, it is worth mentioning that it is shown in Adler (1981) that, when local times are jointly continuous, then, for each fixed $x \in \mathbb{R}^d$, $L(x, \bullet)$ can be extended to be a finite Borel measure supported on the level set

$$\vec{X}_T^{-1}(x) = \big\{t \in T : \vec{X}(t) = x\big\}; \tag{4.3}$$

thus, local times are useful in studying various fractal properties of level sets and inverse images of the random field $\vec{X}$.

Theorem 4.1 (Ayache and Xiao (2013))

$\vec{X} = \{\vec{X}(t), t \in \mathbb{R}^N\}$, the $\mathbb{R}^d$-valued HFSF with Hurst parameter H, has a jointly continuous local times on T when $N > Hd$.

The strategy of the proof of Theorem 4.1 is rather classical. It is based on Kolmogorov continuity Theorem, which can be used thanks to the following two estimates:

(*i*) For all integer $m \geq 1$, there exists a finite constant $c_1 > 0$, which depends on m, such that for any cube $B \subseteq T$ and $x \in \mathbb{R}^d$,

$$\mathbb{E}\big[L(x, B)^m\big] \leq c_1 \big(\mathrm{diam}(B)\big)^{m(N-Hd)}. \tag{4.4}$$

(*ii*) For all even integer $m \geq 2$ and $\gamma \in \big(0,\, 1 \wedge \frac{1}{2}(\frac{N}{Hd} - 1)\big)$, there exists a finite constant $c_2 > 0$, which depends on m and γ, such that for any cube $B \subseteq T$ and $x, y \in \mathbb{R}^d$ with $|x - y| \leq 1$,

$$\mathbb{E}\Big[\big(L(x, B) - L(y, B)\big)^m\Big] \leq c_2\, |x - y|^{m\gamma} \big(\mathrm{diam}(B)\big)^{m(N-H(d+\gamma))}. \tag{4.5}$$

For proving (i) and (ii), let us recall the following two identities from Geman and Horowitz (1980):

(i') For all $x \in \mathbb{R}^d$ and integer $m \geq 1$,

$$\mathbb{E}\Big[L(x,B)^m\Big] \;=\; (2\pi)^{-md}\int_{B^m}\int_{\mathbb{R}^{md}} \exp\Big(-i\sum_{j=1}^{m} u^j\cdot x\Big)\Phi(\overline{u},\overline{t})\,d\overline{u}\,d\overline{t} \tag{4.6}$$

.

(ii') For all $x, y \in \mathbb{R}^d$ and even integer $m \geq 2$,

$$\mathbb{E}\Big[\big(L(x,B)-L(y,B)\big)^m\Big] = (2\pi)^{-md}\int_{B^m}\int_{\mathbb{R}^{md}}\prod_{j=1}^{m}\Big[e^{-iu^j\cdot x}-e^{-iu^j\cdot y}\Big] \times \Phi(\overline{u},\overline{t})\,d\overline{u}\,d\overline{t}, \tag{4.7}$$

where $\overline{u} = (u^1,\ldots,u^m)$, $\overline{t} = (t^1,\ldots,t^m)$ and $\Phi(\cdot,\overline{t})$ is the characteristic function of $\big(\vec{X}(t^1),\ldots,\vec{X}(t^m)\big)$.

From now on, our goal is to explain the main ideas for bounding the integrals in (4.6) and (4.7), by expressing, in a convenient way

$$\Phi(\overline{u},\overline{t}) := \mathbb{E}\exp\left(i\sum_{j=1}^{m} u^j \cdot \vec{X}(t^j)\right), \qquad \forall\ (\overline{u},\overline{t}) \in \mathbb{R}^{md} \times B^m. \tag{4.8}$$

Observe that for any $c \in (0,+\infty)$ and $t^0 \in (0,+\infty)^N$,

$$\Phi(\overline{u},\overline{t}) = \mathbb{E}\exp\left(i\sum_{j=1}^{m} c^H u^j \cdot \left(\vec{X}\big(c^{-1}(t^j+t^0)\big) - \vec{X}\big(c^{-1}t^0\big)\right)\right), \tag{4.9}$$

thanks to the stationarity of increments and self-similarity of $\vec{X}$. Moreover, it can be seen that, given any compact rectangle $T \subset \mathbb{R}^N$, we can choose $\varepsilon \in (0,1)$, $c \in (0,+\infty)$ and $t^0 \in (0,\infty)^N$, such that $s^0 := c^{-1}t^0 \in I := [\varepsilon,1]^N$ and $c^{-1}(t+t^0) \in I$ for each $t \in T$. Thus, the change of variables

$$v^j = c^H u^j, \quad s^j = c^{-1}(t^j+t^0), \quad j = 1,\dots,m, \tag{4.10}$$

allows to replace in the integrals (4.6) and (4.7), $\Phi(\overline{u},\overline{t})$, by

$$\widetilde{\Phi}(\overline{v},\overline{s}) := \mathbb{E}\exp\left(\sum_{j=1}^{m} v^j \cdot (\vec{X}(s^j) - \vec{X}(s^0))\right). \tag{4.11}$$

Using an Abel transform one has

$$\widetilde{\Phi}(\overline{v},\overline{s}) = \check{\Phi}(\overline{w},\overline{s}), \tag{4.12}$$

where

$$\begin{cases} w^j := \sum_{k=j}^m v^k, \qquad j = 1,\dots,m, \\ \check{\Phi}(\overline{w},\overline{s}) := \mathbb{E}\exp\left(i\sum_{j=1}^m w^j\cdot(\vec{X}(s^j)-\vec{X}(s^{j-1}))\right). \end{cases} \tag{4.13}$$

Next, denoting by w_l^j, $l = 1,\dots,N$ the coordinates w^j, one gets

$$\check{\Phi}(\overline{w},\overline{s}) = \prod_{l=1}^N \mathbb{E}\exp\left(i\sum_{j=1}^m w_l^j(X(s^j)-X(s^{j-1}))\right), \tag{4.14}$$

since the N coordinates of $\vec{X}$ are independent copies of the HFSF X. Next, noticing that $\mathbb{E}\exp\left(i\sum_{j=1}^m w_l^j(X(s^j)-X(s^{j-1}))\right)$ is the value at 1 of the characteristic function of the symmetric α-stable random variable $\sum_{j=1}^m w_l^j(X(s^j)-X(s^{j-1}))$, one obtains

$$\mathbb{E}\exp\left(i\sum_{j=1}^{m} w_l^j\cdot\left(X(s^j)-X(s^{j-1})\right)\right)=\exp\left(-\Big\|\sum_{j=1}^{m} w_l^j\left(X(s^j)-X(s^{j-1})\right)\Big\|_\alpha^\alpha\right). \tag{4.15}$$

It is at this stage that the LND property of X plays a crucial role. Indeed, thanks to it, we know that the increments $X(s^j)-X(s^{j-1})$, $j=1,\ldots m$, are approximately independent; therefore there exists a constant $c>0$, non depending on the w_l^j's and the s^j's, such that,

$$\begin{aligned}\Big\|\sum_{j=1}^{m} w_l^j\left(X(s^j)-X(s^{j-1})\right)\Big\|_\alpha^\alpha &\geq c\sum_{j=1}^{m}|w_l^j|^\alpha\Big\|X(s^j)-X(s^{j-1})\Big\|_\alpha^\alpha\\ &= c\sum_{j=1}^{m}|w_l^j|^\alpha|s^j-s^{j-1}|^{\alpha H}.\end{aligned} \tag{4.16}$$