Jean-Pierre Ramis
Jacques Sauloy
Changgui Zhang

LOCAL ANALYTIC CLASSIFICATION OF q-DIFFERENCE EQUATIONS

Jean-Pierre Ramis

Institut de France (Académie des Sciences) and Laboratoire Émile Picard, Institut de Mathématiques, CNRS UMR 5219, U.F.R. M.I.G., Université Paul Sabatier (Toulouse 3), 31062 Toulouse CEDEX 9.

E-mail : `ramis.jean-pierre@wanadoo.fr`

Jacques Sauloy

Laboratoire Émile Picard, Institut de Mathématiques, CNRS UMR 5219, U.F.R. M.I.G., Université Paul Sabatier (Toulouse 3), 31062 Toulouse CEDEX 9.

E-mail : `sauloy@math.univ-toulouse.fr`

Url : `www.math.univ-toulouse.fr/~sauloy/`

Changgui Zhang

Laboratoire Paul Painlevé, UMR-CNRS 8524, UFR Mathématiques Pures et Appliquées, Université des Sciences et Technologies de Lille (Lille 1), Cité Scientique, 59655 Villeneuve d'Ascq Cedex.

E-mail : `Changgui.Zhang@math.univ-lille1.fr`

Url : `www-gat.univ-lille1.fr/~zhang/`

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Abstract. — We essentially achieve Birkhoff's program for q-difference equations by giving three different descriptions of the moduli space of isoformal analytic classes. This involves an extension of Birkhoff-Guenther normal forms, q-analogues of the so-called Birkhoff-Malgrange-Sibuya theorems and a new theory of summation. The results were announced in [**37, 38**] and in various seminars and conferences between 2004 and 2006.

Résumé **(Classification analytique locale des équations aux q-différences)**

Nous achevons pour l'essentiel le programme de Birkhoff pour la classification des équations aux q-différences en donnant trois descriptions distinctes de l'espace des modules des classes analytiques isoformelles. Cela passe par une extension des formes normales de Birkhoff-Guenther, des q-analogues des théorèmes dits de Birkhoff-Malgrange-Sibuya et une nouvelle théorie de la sommation. Ces résultats ont été annoncés dans [**37, 38**] ainsi que dans divers sminaires et conférences de 2004 à 2006.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1. The problem

1.1.1. The generalized Riemann problem and the allied problems. — This paper is a contribution to a large program stated and begun by G.D. Birkhoff in the beginning of XXth century [**8**]: *the generalized Riemann problem for linear differential equations and the allied problems for linear difference and q-difference equations*. Such problems are now called *Riemann-Hilbert-Birkhoff problems*. Today the state of achievement of the program *as it was formulated by Birkhoff* in [**8**]: is the following.

- For linear *differential* equations the problem is completely closed (both in the regular-singular case and in the irregular case).
- For linear *q-difference* equations ($|q| \neq 1$), taking account of preceding results due to Birkhoff [**8**], the second author [**44**] and van der Put-Reversat [**51**] in the regular-singular case, and to Birkhoff-Guenther [**9**] in the *irregular case*, the present work *essentially closes the problem* and moreover answers related questions formulated later by Birkhoff in a joint work with his PhD student P.E. Guenther [**9**] (*cf.* below 1.1.2).
- For linear *difference* equations the problem is closed for global regular-singular equations (Birkhoff, J. Roques [**40**]) and there are some important results in the irregular case [**22**], [**11**].

1.1.2. The Birkhoff-Guenther program. — We quote the conclusion of [**9**], it contains a program which is one of our central motivations for the present work. We shall call it *Birkhoff-Guenther program*.

Up to the present time, the theory of q-difference equations has lagged noticeabely behind the sister theories of linear difference and q-difference equations. In the opinion of the authors, the use of the canonical system, as formulated above in a special case, is destined to carry the theory of q-difference equations to a comparable degree of completeness. This program includes in particular the complete theory of convergence and divergence of formal series, the explicit determination of the essential invariants (constants in the canonical form), the inverse Riemann theory both for the neighborhood of $x = \infty$ *and in the complete plane (case of rational coefficients), explicit integral representations of the solutions, and finally the definition of q-sigma periodic matrices, so far defined essentially only in the case* $n = 1$*. Because of its extensiveness this material cannot be presented here.*

The paper [**9**] appeared in 1941 and Birkhoff died in 1944; as far as we know "this material" never appeared and the corresponding questions remained opened and forgotten for a long time.

1.1.3. What this paper could contain but does not. — Before describing the contents of the paper in the following paragraph, we shall first say briefly what it could contain but does not.

The kernel of the present work is the detailed proofs of some results announced in [**37**], [**38**] and in various seminars and conferences between 2004 and 2006, but there are also some new results in the same spirit and some examples.

In this paper, we limit ourselves to the case $|q| \neq 1$. The problem of classification in the case $|q| = 1$ involves diophantine conditions, it remains open but for the only exception [**15**]. Likewise, we do not study problems of confluence of our invariants towards invariants of differential equations, that is of q-Stokes invariants towards classical Stokes invariants (*cf.* in this direction [**17**], [**55**]).

In this work, following Birkhoff, we classify analytically equations admitting a fixed normal form, a moduli problem. There is another way to classify equations: in terms of representations of a *"fundamental group"*, in Riemann's spirit. This is related to the Galois theory of q-difference equations, we will not develop this topic here, limiting ourselves to the following remarks, even if the two types of classification are strongly related.

The initial work of Riemann was a "description" of hypergeometric differential equations as two-dimensional *representations* of the free non-abelian *group* generated by two elements. Later Hilbert asked for a classification of meromorphic linear differential equations in terms of *finite dimensional representations of free groups*. Apparently the idea of Birkhoff was to get classifications of meromorphic linear differential, difference and q-difference equations in terms of elementary linear algebra and combinatorics but *not* in terms of group representations. For many reasons it is interesting to work in the line of Riemann and Hilbert and to translate Birkhoff style invariants in terms of group representations. The corresponding groups will be *fundamental groups* of the Riemann sphere minus a finite set or *generalized fundamental groups*. It is possible to define categories of linear differential, difference and q-difference equations, these categories are tannakian categories; then applying the fundamental theorem of Tannakian categories we can interpret them in terms of finite dimensional representations of pro-algebraic groups, the Tannakian groups. The fundamental groups and the (hypothetic) generalized fundamental groups will be Zariski dense subgroups of the Tannakian groups.

In the case of differential equations the work was achieved by the first author, the corresponding group is the *wild fundamental group* which is Zariski dense in the Tannakian group. In the case of difference equations almost nothing is known. In the case of q-difference equations, the situation is the following.

1. In the *local regular singular case* the work was achieved by the second author in [**45**], the generalized fundamental group is abelian, its semi-simple part is abelian free on two generators and its unipotent part is isomorphic to the additive group $\mathbf{C}$.
2. In the *global regular singular case* only the abelian case is understood [**45**], using the geometric class field theory. In the general case some non abelian class field theory is needed.
3. In the *local irregular case*, using the results of the present paper the first and second authors recently got a generalized fundamental group [**32, 36, 35**].
4. Using case 3, the solution of the *global general case* should follows easily from the solution of case 2.

1.1.3.1. About the asumption that the slopes are integral. — The "abstract" part of our paper does not require any assumption on the slopes: that means general structure theorems, *e.g.* theorem 3.1.4 and proposition 3.4.2. The same is true of the decomposition of the local Galois group of an irregular equation as a semi-direct product of the formal group by a unipotent group in [**32, 36**].

However, all our *explicit* constructions (Birkhoff-Guenther normal form, privileged cocycles, discrete summation) rest on the knowledge of a normal form for pure modules, that we have found only in the case of integral slopes.

In [**51**] van der Put and Reversat classified pure modules with non integral slopes. The adaptation of [**51**] to our results does not seem to have been done - and would be very useful.

1.2. Contents of the paper

We shall classify analytically isoformal q-difference equations. The isoformal analytic classes form an affine space, we shall give *three descriptions* of this space respectively in chapters 3, 4 and 6 (this third case is a particular case of the second and is based upon chapter 5) using different constructions of the analytic invariants. A *direct and explicit* comparison between the second and the third description is straightforward but such a comparison between the first and the second or the third construction is quite subtle; in the present paper it will be clear for the "abelian case", of two integral slopes; for the general case we refer the reader to [**32, 36**] and also to work in progress [**48**].

Chapter 2 deals with the general setting of the problem (section 2.1) and introduces two fundamental tools: the Newton polygon and the slope filtration (section 2.2). From this, the problem of analytic isoformal classification admits a purely algebraic formulation: the isograded classification of filtered difference modules. An algebraic theory is developped in section 2.3. Since the space under study looks like some generalized extension module, it relies on homological algebra and index computations. Last, in view of dealing with specific examples, some practical aspects are discussed in section 2.4.

The first attack at the classification problem for q-difference equations comes in chapter 3. Section 3.1 specializes the results of section 2.3 to q-differences, and section 3.2 provides the proof of some related index computations. One finds that the space of classes is an affine scheme (theorem 3.1.4) and computes its dimension. This is a rather abstract result. In order to provide explicit descriptions (normal forms, coordinates, invariants ...) from section 3.3 up to the end of the paper, we assume that *the slopes of the Newton polygon are integers*. This allows for the more precise theorem 3.3.5 and the existence of Birkhoff-Guenther normal forms originating in [**9**]. This also makes easier the explicit computations of the following chapters. Then some precisions about q-Gevrey classification are given in section 3.4.

Analytic isoformal classification by normal forms is a special feature of the q-difference case, such a thing does not exist for differential equations. To tackle this case Birkhoff introduced functional 1-cochains using Poincaré asymptotics [**7, 8**]. Later, in the seventies, Malgrange interpreted Birkhoff cochains using sheaves on a circle S^1 (the real blow up of the origin of the complex plane). Here, we modify these constructions in order to deal with the q-difference case, introducing a new asymptotic theory and replacing the circle S^1 by the elliptic curve $\mathbf{E}_q = \mathbf{C}^*/q^{\mathbf{Z}}$.

In this spirit, chapter 4 tackles the extension to q-difference equations of the so called Birkhoff-Malgrange-Sibuya theorems. In section 4.1 is outlined an asymptotic theory adapted to q-difference equations but weaker than that of section 5.2 of the next chapter: the difference is the same as between classical Gevrey versus Poincaré asymptotics. The counterpart of the Poincaré version of Borel-Ritt is theorem 4.1.4; also, comparison with Whitney conditions is described in lemma 4.1.3. Indeed, the geometric methods of section 4.3 rest on the integrability theorem of Newlander-Niremberg. They allow the proof of the first main theorem, the q-analogue of the abstract Birkhoff-Malgrange-Sibuya theorem (theorem 4.3.10). Then, in section 4.4, it is applied to the classification problem for q-difference equations and one obtains the q-analogue of the concrete Birkhoff-Malgrange-Sibuya theorem (theorem 4.4.1): there is a natural bijection from the space $\mathcal{F}(M_0)$ of analytic isoformal classes in the formal class of M_0 with the first cohomology set $H^1(\mathbf{E}_q, \Lambda_I(M_0))$ of the "Stokes sheaf". The latter is the sheaf of automorphisms of M_0 infinitely tangent to identity, a sheaf of unipotent groups over the elliptic curve $\mathbf{E}_q = \mathbf{C}^*/q^{\mathbf{Z}}$. The proof of theorem 4.4.1appeals to the fundamental theorem of existence of asymptotic solutions, previously proved in section 4.2.

Chapter 5 aims at developping a summation process for q-Gevrey divergent series. After some preparatory material in section 5.1, an asymptotic theory "with estimates" well suited for q-difference equations is expounded in section 5.2. Here, the sectors of the classical theory are replaced by preimages in $\mathbf{C}^*$ of the Zariski open sets of the elliptic curve $\mathbf{C}^*/q^{\mathbf{Z}}$, that is, complements of finite unions of discrete q-spirals in $\mathbf{C}^*$; and the growth conditions at the boundary of the sectors are replaced by polarity conditions along the discrete spirals. The q-Gevrey analogue of the classical theorems are stated and proved in sections 5.3 and 5.4: the counterpart of the Gevrey version of Borel-Ritt is theorem 5.3.3 and multisummability conditions appear in theorems 5.4.3 and 5.4.7. The theory is then applied to q-difference equations and to their classification in section 5.5, where is proved the summability of solutions (theorem 5.5.3), the existence of asymptotic solutions coming as a consequence (theorem 5.5.5) and the description of Stokes phenomenon as an application (theorem 5.5.7).

Chapter 6 deals with some complementary information on the geometry of the space $\mathcal{F}(M_0)$ of analytic isoformal classes, through its identification with the cohomology set $H^1(\mathbf{E}_q, \Lambda_I(M_0))$ obtained in chapter 4. Theorem 4.4.1 of chapter 4 implicitly attaches cocycles to analytic isoformal classes and theorem 5.5.7 of chapter 5 shows how to obtain them by a summation process. In section 6.1, we give yet another construction of "privileged cocycles" (from [**46**]) and study their properties. In section 6.2, we show how the devissage of the sheaf $\Lambda_I(M_0)$ by holomorphic vector bundles over $\mathbf{E}_q$ allows to identify $H^1(\mathbf{E}_q, \Lambda_I(M_0))$ with an affine space, and relate it to the corresponding result of theorem 3.3.5. In section 6.3, we recall how holomorphic vector bundles over $\mathbf{E}_q$ appear naturally in the theory of q-difference equations and we apply them to an interpretation of the formula for the dimension of $\mathcal{F}(M_0)$.

Chapter 7 provides some elementary examples motivated by their relation to q-special functions, either linked to modular functions or to confluent basic hypergeometric series.

It is important to notice that the construction of q-analogs of classical objects (special functions. . .) is not canonical, there are in general several "good"

q-analogs. So there are several q-analogs of the Borel-Laplace summation (and multisummation): there are several choices for Borel and Laplace kernels (depending on a choice of q-analog of the exponential function) and several choices of the integration contours (continuous or discrete in Jackson style) [**34**], [**39**], [**53, 56, 54**]. Our choice of summation here seems quite "optimal": the entries of our Stokes matrices are *elliptic functions* (*cf.* the "q-sigma periodic matrices" of Birkhoff-Guenther program), moreover Stokes matrices are meromorphic in the parameter of "q-direction of summation"; this is essential for applications to q-difference Galois theory (*cf.* [**32, 36, 35**]). Unfortunately we did not obtain explicit integral formulae for this summation (except for some particular cases), in contrast with what happens for other summations introduced before by the third author.

1.3. General notations

Generally speaking, in the text, the sentence $A := B$ means that the term A is defined by formula B. But for some changes, the notations are the same as in [**37**], [**38**], etc. Here are the most useful ones.

We write $\mathbf{C}\{z\}$ the ring of convergent power series (holomorphic germs at $0 \in \mathbf{C}$) and $\mathbf{C}(\{z\})$ its field of fractions (meromorphic germs). Likewise, we write $\mathbf{C}[[z]]$ the ring of formal power series and $\mathbf{C}((z))$ its field of fractions.

We fix once for all a complex number $q \in \mathbf{C}$ such that $|q| > 1$. Then, the dilatation operator σ_q is an automorphism of any of the above rings and fields, well defined by the formula:

$$\sigma_q f(z) := f(qz).$$

Other rings and fields of functions on which σ_q operates will be introduced in the course of the paper. This operator also acts coefficientwise on vectors, matrices... over any of these rings and fields.

We write $\mathbf{E}_q$ the complex torus (or elliptic curve) $\mathbf{E}_q := \mathbf{C}^*/q^{\mathbf{Z}}$ and $p : \mathbf{C}^* \to \mathbf{E}_q$ the natural projection. For all $\lambda \in \mathbf{C}^*$, we write $[\lambda; q] := \lambda q^{\mathbf{Z}} \subset \mathbf{C}^*$ the *discrete logarithmic q-spiral* through the point $\lambda \in \mathbf{C}^*$. All the points of $[\lambda; q]$ have the same image $\overline{\lambda} := p(\lambda) \in \mathbf{E}_q$ and we may identify $[\lambda; q] = p^{-1}\left(\overline{\lambda}\right)$ with $\overline{\lambda}$.

A linear analytic (resp. formal) q-difference equation (implicitly: at $0 \in \mathbf{C}$) is an equation:

$$\sigma_q X = AX, \tag{1}$$

where $A \in \mathrm{GL}_n(\mathbf{C}(\{z\}))$ (resp. $A \in \mathrm{GL}_n(\mathbf{C}((z)))$).

1.3.1. Theta Functions. — Jacobi theta functions pervade the theory of q-difference equations. We shall mostly have use for two slightly different forms of them. In section 5, we shall use:

$$\theta(z;q) := \sum_{n\in\mathbf{Z}} q^{-n(n-1)/2} z^n = \prod_{n\in\mathbf{N}} (1-q^{-n-1})(1+q^{-n}z)(1+q^{-n-1}/z). \tag{2}$$

The second equality is Jacobi's celebrated triple product formula. When obvious, the dependency in q will be omitted and we shall write $\theta(z)$ instead of $\theta(z;q)$. One has:

$$\theta(qz) = q\ z\ \theta(z) \text{ and } \theta(z) = \theta(1/qz).$$

In section 7, we shall rather use:

$$\theta_q(z) := \sum_{n\in\mathbf{Z}} q^{-n(n+1)/2} z^n = \prod_{n\in\mathbf{N}} (1-q^{-n-1})(1+q^{-n-1}z)(1+q^{-n}/z). \tag{3}$$

One has of course $\theta_q(z) = \theta(q^{-1}z;q)$ and:

$$\theta_q(qz) = z\theta_q(z) = \theta_q(1/z).$$

Both functions are analytic over the whole of $\mathbf{C}^*$ and vanish on the discrete q-spiral $-q^{\mathbf{Z}}$ with simple zeroes.

1.3.2. q-Gevrey levels. — As in [**6**],[**33**], we introduce the space of formal series of *q-Gevrey order s*:

$$\mathbf{C}[[z]]_{q;s} := \left\{ \sum a_n z^n \in \mathbf{C}[[z]] \mid \exists A > 0\ :\ f_n = O\left(A^n q^{sn^2/2}\right) \right\}.$$

We also say that $f \in \mathbf{C}[[z]]_{q;s}$ is of *q-Gevrey level* $1/s$. It understood that $\mathbf{C}[[z]]_0 = \mathbf{C}\{z\}$ and $\mathbf{C}[[z]]_\infty = \mathbf{C}[[z]]$ (but it is *not* true that $\bigcap_{s>0} \mathbf{C}[[z]]_s = \mathbf{C}\{z\}$, nor that $\bigcup_{s>0} \mathbf{C}[[z]]_s = \mathbf{C}[[z]]$).
Similar considerations apply to spaces of Laurent formal series:

$$\mathbf{C}((z))_{q;s} := \mathbf{C}[[z]]_{q;s}[1/z].$$

More generally, one can speak of q-Gevrey sequences of complex numbers. Let $k \in \mathbf{R}^* \cup \{\infty\}$ and $s := \dfrac{1}{k}$. A sequence $(a_n) \in \mathbf{C}^{\mathbf{N}}$ is *q-Gevrey of order s* if it is dominated by a sequence of the form $\left(CA^n\,|q|^{n^2/(2k)}\right)$, for some constants

$C, A > 0$.
Note that this terminology is all about sequences, or coefficients of series. Extension to q-Gevrey asymptotics is explained in definition 5.2.1, while the q-Gevrey interpolation by growth of decay of functions is dealt with in definitions 5.2.7 and 5.4.1.

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CHAPTER 2

SOME GENERAL NONSENSE

2.1. The category of q-difference modules

General references for this section are [**52**], [**47**]

2.1.1. Some general facts about difference modules. — Here, we also refer to the classical litterature about difference fields, like [**13**] and [**18**] (see also [**31**]). We shall also use specific results proved in section 2.3.

We call *difference field*[1] a pair (K, σ), where K is a (commutative) field and σ a field automorphism of K. We write indifferently $\sigma(x)$ or σx the action of σ on $x \in K$. One can then form the Ore ring of difference operators:

$$\mathcal{D}_{K,\sigma} := K\langle T, T^{-1}\rangle$$

characterized by the twisted commutation relation:

$$\forall k \in \mathbf{Z}\ ,\ x \in K\ ,\ T^k x = \sigma^k(x) T^k.$$

We shall rather write somewhat improperly $\mathcal{D}_{K,\sigma} := K\langle \sigma, \sigma^{-1}\rangle$ and, for short, $\mathcal{D} := \mathcal{D}_{K,\sigma}$ in this section. The center of $\mathcal{D}$ is the "field of constants":

$$K^\sigma := \{x \in K \mid \sigma(x) = x\}.$$

The ring $\mathcal{D}$ is left euclidean and any ideal is generated by an entire unitary polynomial $P = \sigma^n + a_1\sigma^{n-1} + \cdots + a_n$, but such a generator is by no ways unique. For instance, it is an easy exercice to show that $\sigma - a$ and $\sigma - a'$ (with $a, a' \in K^*$) generate the same ideal if, and only if, a'/a belongs to the

(1) Much of what follows will be extended to the case of difference *rings* in section 2.3, where the basic linear constructions will be described in great detail.

subgroup $\{\frac{\sigma b}{b} \mid b \in K^*\}$ of K^*.

Any (left) $\mathcal{D}$-module $M \in Mod_{\mathcal{D}}$ can be seen as a K-vector space E and the left multiplication $x \mapsto \sigma.x$ as a *semi-linear* automorphism $\Phi : E \to E$, which means that $\Phi(\lambda x) = \sigma(\lambda)\Phi(x)$; and any pair (E, Φ) of a K-vector space E and a semi-linear automorphism Φ of E defines a $\mathcal{D}$-module; we just write $M = (E, \Phi)$. Morphisms from (E, Φ) to (E', Φ') in $Mod_{\mathcal{D}}$ are linear maps $u \in \mathcal{L}_K(E, E')$ such that $\Phi' \circ u = u \circ \Phi$.

The $\mathcal{D}$-module M has finite length if, and only if, E is a finite dimensional K-vector space. A finite length $\mathcal{D}$-module is called a *difference module* over (K, σ), or over K for short. The full subcategory of $Mod_{\mathcal{D}}$ whose objects are of difference modules is written $DiffMod(K, \sigma)$. The categories $Mod_{\mathcal{D}}$ and $DiffMod(K, \sigma)$ are abelian and K^σ-linear.

By choosing a basis of E, we can identify any difference module with some (K^n, Φ_A), where $A \in \mathrm{GL}_n(K)$ and $\Phi_A(X) := A^{-1}\sigma X$ (with the natural operation of σ on K^n); the reason for using A^{-1} will become clear soon. If $B \in \mathrm{GL}_p(K)$, then morphisms from (K^n, Φ_A) to (K^p, Φ_B) can be identified with matrices $F \in \mathrm{Mat}_{p,n}(K)$ such that $(\sigma F)A = BF$ (and composition amounts to the product of matrices).

2.1.1.1. Unity. — An important particular object is the *unity* $\underline{1}$, which may be described either as (K, σ) or as $\mathcal{D}/\mathcal{D}P$ with $P = \sigma - 1$. For any difference module $M = (E, \Phi)$ the K^σ-vector space $\mathrm{Hom}(\underline{1}, M)$ can be identified with the kernel of the K^σ-linear map $\Phi - Id : E \to E$; in case $M = (K^n, \Phi_A)$, this boils down to the space $\{X \in K^n \mid \sigma X = AX\}$ of solutions of a "σ-difference system"; whence our definition of Φ_A. The *functor of solutions* $M \rightsquigarrow \Gamma(M) := \mathrm{Hom}(\underline{1}, M)$ is left exact and K^σ-linear. We shall have use of its right derived functors $\Gamma^i(M) = \mathrm{Ext}^i(\underline{1}, M)$ (*see* [**10**]).

2.1.1.2. Internal Hom. — Let $M = (E, \Phi)$ and $N = (E, \Psi)$. The map $T_{\Phi,\Psi} : f \mapsto \Psi \circ f \circ \Phi^{-1}$ is a semi-linear automorphism of the K-vector space $\mathcal{L}_K(E, F)$, whence a difference module $\underline{\mathrm{Hom}}(M, N) := \big(\mathcal{L}_K(E, F), T_{\Phi,\Psi}\big)$. Then one has $\underline{\mathrm{Hom}}(\underline{1}, M) = M$ and $\Gamma\big(\underline{\mathrm{Hom}}(M, N)\big) = \mathrm{Hom}(M, N)$. The *dual* of M is $M^\vee := \underline{\mathrm{Hom}}(M, \underline{1})$, so that $\mathrm{Hom}(M, \underline{1}) = \Gamma(M^\vee)$. For instance, the dual of $M = (K^n, \Phi_A)$ is $M^\vee = (K^n, \Phi_{A^\vee})$, where $A^\vee := {}^tA^{-1}$.

2.1.1.3. Tensor product. — Let $M = (E, \Phi)$ and $N = (E, \Psi)$. The map $\Phi \otimes \Psi : x \otimes y \mapsto \Phi(x) \otimes \Psi(y)$ from $E \otimes_K F$ to itself is well defined and it is a semi-linear automorphism, whence a difference module $M \otimes N = (E \otimes_K F, \Phi \otimes \Psi)$. The obvious morphism yields the adjunction relation:

$$\mathrm{Hom}\big(M, \underline{\mathrm{Hom}}(N, P)\big) = \mathrm{Hom}(M \otimes N, P).$$

We also have functorial isomorphisms $\underline{1} \otimes M = M$ and $\underline{\mathrm{Hom}}(M, N) = M^\vee \otimes N$. The classical computation of the rank through $\underline{1} \to M^\vee \otimes M \to \underline{1}$ yields $\dim_K E$ as it should. We write $\mathrm{rk}\, M$ this number.

2.1.1.4. Extension of scalars. — An extension difference field (K', σ') of (K, σ) consists in an extension K' of K and an automorphism σ' of K' which restricts to σ on K. Any difference module $M = (E, \Phi)$ over (K, σ) then gives rise to a difference module $M' = (E', \Phi')$ over $(K', \sigma)'$, where $E' := K' \otimes_L E$ and $\Phi' := \sigma' \otimes \Phi$ is defined the same way as above. We then write $\Gamma_{K'}(M)$ the $K'^{\sigma'}$-vector space $\Gamma(M')$. The *functor of solutions (with values) in* K' $M \rightsquigarrow \Gamma(M') = \mathrm{Hom}(\underline{1}, M')$ is left exact and K^σ-linear. The functor $M \rightsquigarrow M'$ from $DiffMod(K, \sigma)$ to $DiffMod(K', \sigma')$ is compatible with unity, internal Hom, tensor product and dual. The image of $M = (K^n, \Phi_A)$ is $M' = (K'^n, \Phi_A)$.

2.1.2. q-difference modules. — We now restrict our attention to the difference fields $\big(\mathbf{C}(\{z\}), \sigma_q\big)$ and $\big(\mathbf{C}((z)), \sigma_q\big)$, and to the corresponding categories of analytic, resp. formal, q-difference modules: they are $\mathbf{C}$-linear abelian categories since $\mathbf{C}(\{z\})^{\sigma_q} = \mathbf{C}((z))^{\sigma_q} = \mathbf{C}$. In both settings, we consider the q-difference module $M = (K^n, \Phi_A)$ as an abstract model for the linear q-difference system $\sigma_q X = AX$. Isomorphisms from (the system with matrix) A to (the system with matrix) B in either category correspond to analytic, resp. formal, gauge transformations, *i.e.* matrices $F \in \mathrm{GL}_n(\mathbf{C}(\{z\}))$, resp. $F \in \mathrm{GL}_n(\mathbf{C}((z)))$, such that $B = F[A] := (\sigma_q F) A F^{-1}$. We write $\mathcal{D}_q$ indifferently for $\mathcal{D}_{\mathbf{C}(\{z\}), \sigma_q}$ or $\mathcal{D}_{\mathbf{C}((z)), \sigma_q}$ when the distinction is irrelevant.

***Lemma 2.1.1* (Cyclic vector lemma).** — *Any (analytic or formal) q-difference module is isomorphic to a module $\mathcal{D}_q / \mathcal{D}_q P$ for some unitary entire q-difference operator P.*

Proof. — See [**14**], [**44**]. □

Theorem 2.1.2. — *The categories $DiffMod(\mathbf{C}(\{z\}), \sigma_q)$ and $DiffMod(\mathbf{C}((z)), \sigma_q)$ are abelian $\mathbf{C}$-linear rigid tensor categories. (They actually are neutral tannakian categories, but we won't use this fact.)*

Proof. — See [**52**], [**45**]. □

2.1.2.1. The functor of solutions. — It is customary in $\mathcal{D}$-module theory to call *solution* of M a morphism $M \to \underline{1}$. Indeed, a solution of $\mathcal{D}/\mathcal{D}P$ is then an element of Ker P. We took the dual convention, yielding a *covariant* $\mathbf{C}$-linear left exact functor Γ for the following reason. To any analytic q-difference module $M = (\mathbf{C}(\{z\})^n, \Phi_A)$, one can associate a holomorphic vector bundle F_A over the elliptic curve (or complex torus) $\mathbf{E}_q := \mathbf{C}^*/q^{\mathbf{Z}}$ in such a way that the space of global sections of F_A can be identified with $\Gamma(M)$. We thus think of Γ as a functor of global sections, and, the functor $M \rightsquigarrow F_A$ being exact, the Γ^i can be defined through sheaf cohomology.

There is an interesting relationship between the space $\Gamma(M) = \mathrm{Hom}(\underline{1}, M)$ of solutions of M and the space $\Gamma(M^\vee) = \mathrm{Hom}(M, \underline{1})$ of "cosolutions" of M. Starting from a unitary entire (analytic or formal) q-difference operator $P = \sigma_q^n + a_1\sigma_q^{n-1} + \cdots + a_n \in \mathcal{D}_q$, one studies the solutions of the q-difference equation:

$$P.f := \sigma_q^n f + a_1\sigma_q^{n-1} f + \cdots + a_n f = 0$$

by vectorializing it into the form:

$$\sigma_q X = AX, \text{ where } X = \begin{pmatrix} f \\ \sigma_q f \\ \vdots \\ \sigma_q^{n-2} f \\ \sigma_q^{n-1} f \end{pmatrix} \text{ and } A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & - \ldots & -a_1 \end{pmatrix}.$$

Solutions of P then correspond to solutions of $M = (\mathbf{C}(\{z\})^n, \Phi_A)$ or $(\mathbf{C}((z))^n, \Phi_A)$. Now, by lemma 2.1.1, one has $M = \mathcal{D}_q/\mathcal{D}_q Q$ for some unitary entire q-difference operator Q. Any such polynomial Q is *dual* to P. An explicit formula for a particular dual polynomial is given in [**47**].

From the derived functors $\Gamma^i(M) = \mathrm{Ext}^i(\underline{1}, M)$, one can recover general Ext-modules:

Proposition 2.1.3. — *There are functorial isomorphisms:*

$$\mathit{Ext}^i(M, N) \simeq \Gamma^i(M^\vee \otimes N).$$

Proof. — The covariant functor $N \rightsquigarrow \mathrm{Hom}(M, N)$ is obtained by composing the exact covariant functor $N \rightsquigarrow M^\vee \otimes N$ with the left exact covariant functor Γ. One can then derive it using [**20**], chap. III, §7, th. 1, or directly by writing down a right resolution of Γ and applying the exact functor, then the cohomology sequence of Γ. $\square$

Remark 2.1.4. — For any difference field extension (K, σ) of $(\mathbf{C}(\{z\}), \sigma_q)$, the K^σ-vector space of solutions of M in K has dimension $\dim_{K^\sigma} \Gamma_K(M) \leq \operatorname{rk} M$: this follows from the q-wronskian lemma (*see* [**14**]). One can show that the functor Γ_K is a fibre functor if, and only if, (K, σ) is a *universal field of solutions*, *i.e.* such that one always have the equality $\dim_{K^\sigma} \Gamma_K(M) = \operatorname{rk} M$. The field $\mathcal{M}(\mathbf{C}^*)$ of meromorphic functions over $\mathbf{C}^*$, with the automorphism σ_q, is a universal field of solutions, thus providing a fibre functor over $\mathcal{M}(\mathbf{C}^*)^{\sigma_q} = \mathcal{M}(\mathbf{E}_q)$ (field of elliptic functions). Actually, no subfield of $\mathcal{M}(\mathbf{C}^*)$ gives a fibre functor over $\mathbf{C}$ (*see* [**45**]). In [**52**], van der Put and Singer use an algebra of symbolic solutions which is reduced but not integral. A transcendental construction of a fibre functor over $\mathbf{C}$ is described in [**32**].

2.2. The Newton polygon and the slope filtration

We summarize results from [**47**] (2).

2.2.1. The Newton polygon. — The contents of this section are valid as well in the analytic as in the formal setting. To the (analytic or formal) q-difference operator $P = \sum a_i \sigma_q^i \in \mathcal{D}_q$, we associate the *Newton polygon* $N(P)$, defined as the convex hull of $\{(i,j) \in \mathbf{Z}^2 \mid j \geq v_0(a_i)\}$, where v_0 denotes the z-adic valuation in $\mathbf{C}(\{z\}), \mathbf{C}((z))$. Multiplying P by a unit $a\sigma_q^k$ of $\mathcal{D}_q$ just translates the Newton polygon by a vector of $\mathbf{Z}^2$, and we shall actually consider $N(P)$ as defined up to such a translation (or, which amounts to the same, choose a unitary entire P). The relevant information therefore consists in the lower part of the boundary of $N(P)$, made up of vectors $(r_1, d_1), \ldots, (r_k, d_k)$, $r_i \in \mathbf{N}^*$, $d_i \in \mathbf{Z}$. Going from left to right, the *slopes* $\mu_i := \dfrac{d_i}{r_i}$ are rational numbers such that $\mu_1 < \cdots < \mu_k$. Their set is written $S(P)$ and r_i is the *multiplicity* of $\mu_i \in S(P)$. The most convenient object is however the *Newton function* $r_p : \mathbf{Q} \to \mathbf{N}$, such that $\mu_i \mapsto r_i$ and null out of $S(P)$.

(2) Note however that, from [**32**], we changed our terminology: the slopes of a q-difference module are the *opposites* of what they used to be; and what we call a pure isoclinic (resp. pure) module was previously called pure (resp. tamely irregular).

Theorem 2.2.1. — *(i) For a given q-difference module M, all unitary entire P such that $M \simeq \mathcal{D}_q/\mathcal{D}_q P$ have the same Newton polygon; the* Newton polygon *of M is defined as $N(M) := N(P)$, and we put $S(M) := S(P)$, $r_M := r_P$.*
(ii) The Newton polygon is additive: for any exact sequence,

$$0 \to M' \to M \to M'' \to 0 \Longrightarrow r_M = r_{M'} + r_{M''}.$$

(iii) The Newton polygon is multiplicative:

$$\forall \mu \in \mathbf{Q}\ ,\ r_{M_1 \otimes M_1}(\mu) = \sum_{\mu_1+\mu_2=\mu} r_{M_1}(\mu_1) r_{M_2}(\mu_2). \textit{Also: } r_{M^\vee}(\mu) = r_M(-\mu).$$

(iv) Let $\ell \in \mathbf{N}^*$, introduce a new variable $z' := z^{1/\ell}$ (ramification) and make $K' := \mathbf{C}((z))[z'] = \mathbf{C}((z'))$ or $\mathbf{C}(\{z\})[z'] = \mathbf{C}(\{z'\})$ a difference field extension by putting $\sigma'(z') = q'z'$, where q' is any ℓ^{th} root of q. The Newton polygon of the q'-difference module $M' := K' \otimes M$ (computed w.r.t. variable z') is given by the formula:

$$r_M(\mu) = r_{M'}(\ell\mu).$$

Example 2.2.2. — Vectorializing an analytic equation of order two yields:

$$\sigma_q^2 f + a_1 \sigma_q f + a_2 f = 0 \Longleftrightarrow \sigma_q X = AX, \text{ with } X = \begin{pmatrix} f \\ \sigma_q f \end{pmatrix} \text{ and } A = \begin{pmatrix} 0 & 1 \\ -a_2 & -a_1 \end{pmatrix}.$$

The associated module is $(\mathbf{C}(\{z\})^2, \Phi_A)$. Putting $e := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, one has $\Phi_A(e) = \begin{pmatrix} -a_1/a_2 \\ 1 \end{pmatrix}$ and:

$$\Phi_A^2(e) = \frac{-1}{a_2} e + \frac{-\sigma_q a_1}{\sigma_q a_2} \Phi_A(e) \Longrightarrow M \simeq \mathcal{D}_q/\mathcal{D}_q \hat{L}, \text{ where } \hat{L} := \sigma_q^2 + \frac{\sigma_q a_1}{\sigma_q a_2} \sigma_q + \frac{1}{a_2},$$

$\hat{L}$ thus being a dual of L. Note that if we started from vector $e' := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we would compute likewise:

$$\Phi_A^2(e') = \frac{-1}{\sigma_q a_2} e' + \frac{-a_1}{\sigma_q a_2} \Phi_A(e') \Longrightarrow M \simeq \mathcal{D}_q/\mathcal{D}_q \hat{L}', \text{ where } \hat{L}' := \sigma_q^2 + \frac{a_1}{\sigma_q a_2} \sigma_q + \frac{1}{\sigma_q a_2},$$

another dual of L. However, they give the same Newton polygon, since $v_0(a_1/\sigma_q a_2) = v_0(\sigma_q a_1/\sigma_q a_2)$ and $v_0(1/\sigma_q a_2) = v_0(1/a_2)$.
We now specialize to the equation satisfied by a q-analogue of the Euler series, the so-called *Tshakaloff series*:

$$\mathrm{Ch}(z) := \sum_{n \geq 0} q^{n(n-1)/2} z^n. \tag{4}$$

Then $\hat{\phi} := \mathrm{Ch}$ satisfies:

$$\hat{\phi} = 1 + z\sigma_q \hat{\phi} \Longrightarrow L.\hat{\phi} = 0,$$

where:

$$qzL := (\sigma_q - 1)(z\sigma_q - 1) = qz\sigma_q^2 - (1+z)\sigma_q + 1.$$

By our previous definitions, $S(L) = \{0, 1\}$ (both multiplicities equal 1). The second computation of a dual above implies that:

$$M \simeq \mathcal{D}_q/\mathcal{D}_q P, \text{ where } P := \sigma_q^2 - q(1+z)\sigma_q + q^2 z = (\sigma_q - qz)(\sigma_q - q),$$

whence $S(M) = S(P) = \{-1, 0\}$ (both multiplicities equal 1).
This computation relies on the obvious vectorialisation with matrix $A = \begin{pmatrix} 0 & 1 \\ -1/qz & (1+z)/qz \end{pmatrix}$. However, we also have:

$$L.f = 0 \Longleftrightarrow \sigma_q Y = BY, \text{ with } X = \begin{pmatrix} f \\ z\sigma_q f - f \end{pmatrix} \text{ and } B = \begin{pmatrix} z^{-1} & z^{-1} \\ 0 & 1 \end{pmatrix}.$$

The fact that matrix A is analytically equivalent to an upper triangular matrix comes from the analytic factorisation of $\hat{L}$; the exponents of z on the diagonal are the slopes: this, as we shall see, is a general fact when the slopes are integral.

2.2.2. Pure modules. — We call *pure isoclinic (of slope μ)* a module M such that $S(M) = \{\mu\}$ and *pure* a direct sum of pure isoclinic modules. We call *fuchsian* a pure isoclinic module of slope 0. The following description is valid whether $K = \mathbf{C}((z))$ or $\mathbf{C}(\{z\})$.

Lemma 2.2.3. — *(i) A pure isoclinic module of slope μ over K can be written:*

$$M \simeq \mathcal{D}_q/\mathcal{D}_q P, \quad \textit{where} \quad P = a_n\sigma_q^n + a_{n-1}\sigma_q^{n-1} + \cdots + a_0 \in \mathcal{D}_q \quad \textit{with}:$$

$$a_0 a_n \neq 0 \;;\; \forall i \in \{1, n-1\} \;,\; v_0(a_i) \geq v_0(a_0) + \mu i \;\textit{and}\; v_0(a_n) = v_0(a_0) + \mu n.$$

(ii) If $\mu \in \mathbf{Z}$, it further admits the following description:

$$M = (K^n, \Phi_{z^\mu A}) \; \textit{with} \; A \in GL_n(\mathbf{C}).$$

Proof. — The first description is immediate from the definitions. The second is proved in [**47**]. □

2.2.2.1. Pure modules with integral slopes. — In particular, any fuchsian module is equivalent to some module (K^n, Φ_A) with $A \in \mathrm{GL}_n(\mathbf{C})$. One may moreover require that A has all its eigenvalues in the fundamental annulus:

$$\forall \lambda \in \mathrm{Sp} A \; , \; 1 \leq |\lambda| < |q| \, .$$

Last, two fuchsian modules (K^n, Φ_A), $(K^{n'}, \Phi_{A'})$ with $A \in \mathrm{GL}_n(\mathbf{C})$, $A' \in \mathrm{GL}_{n'}(\mathbf{C})$ having all their eigenvalues in the fundamental annulus are isomorphic if and only if the matrices A, A' are similar (so that $n = n'$).

It follows that any pure isoclinic module of slope μ is equivalent to some module $(K^n, \Phi_{z^\mu A})$ with $A \in \mathrm{GL}_n(\mathbf{C})$, the matrix A having all its eigenvalues in the fundamental annulus; and that two such modules $(K^n, \Phi_{z^\mu A})$, $(K^{n'}, \Phi_{z^\mu A'})$ are isomorphic if and only if the matrices A, A' are similar.

2.2.2.2. Pure modules of arbitrary slopes. — The classification of pure modules of arbitrary (not necessarily integral) slopes was obtained by van der Put and Reversat in [**51**]. It is cousin to the classification by Atiyah of vector bundles over an elliptic curve, which it allows to recover in a simple and elegant way. Although we shall not need it, we briefly recall that result.

The first step is the classification of irreducible (that is, simple) modules. An irreducible q-difference module M is automaticaly pure isoclinic of slope say μ. We write $\mu = d/r$ with d, r coprime and may assume that $r \geq 2$ (the case $r = 1$ is already known). Let $K' := K[z^{1/r}] = K[z']$ and q' an arbitrary rth root of q. Then M is isomorphic to the restriction of some q'-difference module M' of rank 1 over K'; and M' is isomorphic to $(K', \Phi_{cz'^d})$ for a unique $c \in \mathbf{C}^*$ such that $1 \leq |c| < |q'|$. Actually:

$$M \simeq E(r, d, c^r) := \mathcal{D}_q / \mathcal{D}_q \left(\sigma_q^r - {q'}^{-dr(r-1)/2} c^{-r} z^{-d} \right) .$$

Moreover, for $E(r, d, a)$ and $E(r', d', a')$ to be isomorphic, it is necessary (and sufficient) that $(r', d', a') = (r, d, a)$.

Then van der Put and Reversat prove that an indecomposable module M (that is, M is not a non trivial direct sum) comes from successive extensions of isomorphic irreducible modules; indeed, it has the form $M \simeq E(r, d, a) \otimes (K^m, \Phi_U)$ for some indecomposable unipotent constant matrix U. Last, any pure isoclinic module is a direct sum of indecomposable modules in an essentially unique way.

2.2.3. The slope filtration. — Submodules, quotient modules, sums... of pure isoclinic modules of a given slope keep the same property. It follows that each module M admits a biggest pure submodule M' of slope μ; one then has *a priori* rk $M' \leq r_M(\mu)$. Like in the case of differential equations, one wants to "break the Newton polygon" and find pure submodules of maximal possible rank. In the formal case, the results are similar, but in the analytic case, we get a bonus.

2.2.3.1. Formal case. — Any q-difference operator P over $\mathbf{C}((z))$ admits, for all $\mu \in S(P)$, a factorisation $P = QR$ with $S(Q) = \{\mu\}$ and $S(R) = S(P)\backslash\{\mu\}$. As a consequence, writing $M_{(\mu)}$ the biggest pure submodule of slope μ of M, one has rk $M_{(\mu)} = r_M(\mu)$ and $S(M/M_{(\mu)}) = S(M) \setminus \{\mu\}$. Moreover, all modules are pure:

$$M = \bigoplus_{\mu \in S(M)} M_{(\mu)}.$$

The above splitting is canonical, functorial (preserved by morphisms) and compatible with all tensor operations (tensor product, internal Hom, dual).

2.2.3.2. Analytic case. — Here, from old results due (independantly) to Adams and to Birkhoff and Guenther, one draws that, if $\mu := \min S(P)$ is the smallest slope, then P admits, a factorisation $P = QR$ with $S(Q) = \{\mu\}$ and $S(R) = S(P)\backslash\{\mu\}$. As a consequence, the biggest pure submodule of slope μ of M, call it M', again satisfies rk $M' \leq r_M(\mu)$, so that $S(M/M') = S(M) \setminus \{\mu\}$.

Theorem 2.2.4. — *(i) Each q-difference module M over $\mathbf{C}(\{z\})$ admits a unique filtration with pure isoclinic quotients. It is an ascending filtration $(M_{\leq\mu})$ characterized by the following properties:*

$$S(M_{\leq\mu}) = S(M) \cap]-\infty, \mu] \text{ and } S(M/M_{\leq\mu}) = S(M) \cap]\mu, +\infty[\,.$$

(ii) This filtration is strictly functorial, i.e. *all morphisms are strict.*
(iii) Writing $M_{<\mu} := \bigcup_{\mu'<\mu} M_{\leq\mu'}$ and $M_{(\mu)} := M_{\leq\mu}/M_{<\mu}$ (which is pure isoclinic of slope μ and rank $r_M(\mu)$), the functor:

$$M \rightsquigarrow grM := \bigoplus_{\mu \in S(M)} M_{(\mu)}$$

is exact, $\mathbf{C}$-linear, faithful and $\otimes$-compatible.

Proof. — See [**47**] □

2.2.4. Application to the analytic isoformal classification. — The formal classification of analytic q-difference modules is equivalent to the classification of pure modules: the formal class of M is equal to the formal class of $\mathrm{gr}M$, and the latter is essentially the same thing as the analytic class of $\mathrm{gr}M$. The classification of pure modules has been described in paragraph 2.2.2.

Let us fix a formal class, that is, a pure module:

$$M_0 := P_1 \oplus \cdots \oplus P_k.$$

Here, each P_i is pure isoclinic of slope $\mu_i \in \mathbf{Q}$ and rank $r_i \in \mathbf{N}^*$ and we assume that $\mu_1 < \cdots < \mu_k$. All modules M such that $\mathrm{gr}M \simeq M_0$ have the same Newton polygon $N(M) = N(M_0)$. They constitute a formal class and we want to classify them analytically. The following definition is inspired by [**2**].

Definition 2.2.5. — We write $\mathcal{F}(M_0)$ or $\mathcal{F}(P_1, \ldots, P_k)$ for the set of equivalence classes of pairs (M, g) of an analytic q-difference module M and an isomorphism $g : \mathrm{gr}(M) \to M_0$, where (M, g) is said to be equivalent to (M', g') if there exists a morphism $f : M \to M'$ such that $g = g' \circ \mathrm{gr}(f)$.

Note that f is automatically an isomorphism. The goal of this paper is to describe precisely $\mathcal{F}(P_1, \ldots, P_k)$.

Remark 2.2.6. — The group $\prod\limits_{1 \leq i \leq k} \mathrm{Aut}(P_i)$ naturally operates on $\mathcal{F}(P_1, \ldots, P_k)$ in the following way: if (ϕ_i) is an element of the group, thus defining $\phi \in \mathrm{Aut}(M_0)$, then send the class of the isomorphism $g : \mathrm{gr}(M) \to M_0$ to the class of $\phi \circ g$. The quotient of $\mathcal{F}(P_1, \ldots, P_k)$ by this action is the set of analytic classes within the formal class of M_0, but it is not naturally made into a space of moduli.
For instance, taking $P_1 := (K, \sigma_q)$ and $P_2 := (K, z^{-1}\sigma_q)$, it will follow of our study that $\mathcal{F}(P_1, P_2)$ is the affine space $\mathbf{C}$. Also, $\mathrm{Aut}(P_1) = \mathrm{Aut}(P_2) = \mathbf{C}^*$ with the obvious action on $\mathcal{F}(P_1, P_2) = \mathbf{C}$. But the quotient $\mathbf{C}/\mathbf{C}^*$ is rather badly behaved: it consists of two points $0, \overline{1}$, the second being dense.

2.2.4.1. The case of two slopes. — The case $k = 1$ is of course trivial. The case where $k = 2$ is "linear" or "abelian": the set of classes is naturally a finite dimensional vector space over $\mathbf{C}$. We call it "one level case", because the *q-Gevrey level* $\mu_2 - \mu_1$ (see the paragraph 1.3.2 of the general notations in the introduction for its definition) is the fundamental parameter. This will be illustrated in section 3.4, in chapter 5

. Let P, P' be pure analytic q-difference modules with ranks r, r' and slopes $\mu < \mu'$.

Proposition 2.2.7. — *There is a natural one-to-one correspondance:*

$$\mathcal{F}(P, P') \to \mathit{Ext}(P', P).$$

Proof. — Here, Ext denotes the space of extension classes in the category of left $\mathcal{D}_q$-modules; the homological interpretation is discussed in the remark below.
Note that an extension of $\mathcal{D}_q$-modules of finite length has finite length, so that an extension of q-difference modules is a q-difference module. To give $g : \mathrm{gr}M \simeq P \oplus P'$ amounts to give an isomorphism $M_{\leq \mu} \simeq P$ and an isomorphism $M/M_{\leq \mu} \simeq P'$, *i.e.* a monomorphism $i : P \to M$ and an epimorphism $p : M \to P'$ with kernel $i(P)$, *i.e.* an extension of P' by P. Reciprocally, for any such extension, one automatically has $M_{\leq \mu} = i(P)$, thus an isomorphism $g : \mathrm{gr}M \simeq P \oplus P'$. The condition of equivalence of pairs (M, g) is then exactly the condition of equivalence of extensions. □

Remark 2.2.8. — By the classical identification of Ext spaces with Ext^1 modules, we thus get a description of $\mathcal{F}(P, P')$ as a $\mathbf{C}$-vector space. However, we have only shown the map is one-to-one: linearity will be proved later, in section 2.3, thereby giving a way to compute the natural structure on $\mathcal{F}(P, P')$.

2.2.4.2. Matricial description. — We now go for a preliminary matricial description, valid without any restriction on the slopes (number or integrity). Details can be found in section 2.3. From the theorem 2.2.4, we deduce that each analytic q-difference module M with $S(M) = \{\mu_1, \ldots, \mu_k\}$, these slopes being indexed in increasing order: $\mu_1 < \cdots < \mu_k$ and having multiplicities $r_1, \ldots, r_k \in \mathbf{N}^*$, can be written $M = (\mathbf{C}(\{z\})^n, \Phi_A)$ with:
(5)

$$A_0 := \begin{pmatrix} B_1 & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & 0 & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots \\ \ldots & 0 & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \ldots & B_k \end{pmatrix} \text{ and } A = A_U := \begin{pmatrix} B_1 & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & U_{i,j} & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots \\ \ldots & 0 & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \ldots & B_k \end{pmatrix},$$

where, for $1 \leq i \leq k$, $B_i \in \mathrm{GL}_{r_i}(\mathbf{C}(\{z\}))$ encodes a pure isoclinic module $P_i = (K^{r_i}, \Phi_{B_i})$ of slope μ_i and rank r_i and where, for $1 \leq i <$

$j \leq k$, $U_{i,j} \in \mathrm{Mat}_{r_i,r_j}(\mathbf{C}(\{z\}))$. Here, U stands short for $(U_{i,j})_{1\leq i<j\leq k} \in \prod_{1\leq i<j\leq k} \mathrm{Mat}_{r_i,r_j}(\mathbf{C}(\{z\}))$. Call $M_U = M$ the module thus defined: it is implicitly endowed with an isomorphism from $\mathrm{gr}M_U$ to $M_0 := P_1 \oplus \cdots \oplus P_k$, here identified with (K^n, Φ_{A_0}). If moreover the slopes are integral: $S(M) \subset \mathbf{Z}$, then one may take, for $1 \leq i \leq k$, $B_i = z^{\mu_i} A_i$, where $A_i \in \mathrm{GL}_{r_i}(\mathbf{C})$.

Now, a morphism from M_U to M_V compatible with the graduation (as in definition 2.2.5) is a matrix:
(6)

$$F := \begin{pmatrix} I_{r_1} & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & F_{i,j} & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & I_{r_k} \end{pmatrix}, \text{ with } (F_{i,j})_{1\leq i<j\leq k} \in \prod_{1\leq i<j\leq k} \mathrm{Mat}_{r_i,r_j}(\mathbf{C}(\{z\})),$$

such that $(\sigma_q F) A_U = B_U F$. The corresponding relations for the $F_{i,j}, U_{i,j}, V_{i,j}$ will be detailed later. Here, we just note that the above form of F characterizes a unipotent algebraic subgroup $\mathfrak{G}$ of GL_n, which is completely determined by the Newton polygon of M_0. The condition of equivalence of M_U and M_V reads:

$$M_U \sim M_V \Longleftrightarrow \exists F \in \mathfrak{G}(\mathbf{C}(\{z\})) \ : \ F[A_U] = A_V.$$

The set of classes $\mathcal{F}(P_1, \dots, P_k)$ may therefore be identified with the quotient of $\prod_{1\leq i<j\leq k} \mathrm{Mat}_{r_i,r_j}(\mathbf{C}(\{z\}))$ by that equivalence relation, *i.e.* by the action of $\mathfrak{G}(\mathbf{C}(\{z\}))$. For a more formal description, we introduce the following set:

$$\mathfrak{G}^{A_0}(\mathbf{C}((z))) := \{\hat{F} \in \mathfrak{G}(\mathbf{C}((z))) \mid \hat{F}[A_0] \in \mathrm{GL}_n(\mathbf{C}(\{z\}))\}.$$

If $\hat{F} \in \mathfrak{G}^{A_0}(\mathbf{C}((z)))$ and $F \in \mathfrak{G}(\mathbf{C}(\{z\}))$, then $(F\hat{F})[A_0] = F\big[\hat{F}[A_0]\big] \in \mathrm{GL}_n(\mathbf{C}(\{z\}))$, so that $F\hat{F} \in \mathfrak{G}^{A_0}(\mathbf{C}((z)))$: therefore, the group $\mathfrak{G}(\mathbf{C}(\{z\}))$ operates on the set $\mathfrak{G}^{A_0}(\mathbf{C}((z)))$.

Proposition 2.2.9. — *The map $\hat{F} \mapsto \hat{F}[A_0]$ induces a one-to-one correspondance:*

$$\mathfrak{G}^{A_0}(\mathbf{C}((z)))/\mathfrak{G}(\mathbf{C}(\{z\})) \to \mathcal{F}(P_1, \dots, P_k).$$

Proof. — From the formal case in 2.2.3, it follows that, for any $U \in \prod_{1\leq i<j\leq k} \mathrm{Mat}_{r_i,r_j}(\mathbf{C}(\{z\}))$ there exists a unique $\hat{F} \in \mathfrak{G}(\mathbf{C}((z)))$ such that $\hat{F}[A_0] = A_U$. The equivalence of $\hat{F}[A_0]$ with $\hat{F}'[A_0]$ is then just the relation $\hat{F}'\hat{F}^{-1} \in \mathfrak{G}(\mathbf{C}(\{z\}))$. $\square$

Remark 2.2.10. — Write $\hat{F}_U$ for the $\hat{F}$ in the above proof. More generally, for any $U, V \in \prod_{1 \leq i < j \leq k} \mathrm{Mat}_{r_i,r_j}(\mathbf{C}(\{z\}))$ there exists a unique $\hat{F} \in \mathfrak{G}(\mathbf{C}((z)))$ such that $\hat{F}[A_U] = A_V$; write it $\hat{F}_{U,V}$. Then $\hat{F}_{U,V} = \hat{F}_V \hat{F}_U^{-1}$ and the condition of equivalence of M_U and M_V reads:

$$M_U \sim M_V \Longleftrightarrow \hat{F}_{U,V} \in \mathfrak{G}(\mathbf{C}(\{z\})).$$

Giving analyticity conditions for a formal object strongly hints towards a resummation problem! (See chapters 4 and 5.)

2.3. Classification of isograded filtered difference modules

2.3.1. General setting of the problem. — Definition 2.2.5 obviously admits a purely algebraic generalization, which is related to some interesting problems in homological algebra. We shall now describe in some detail results in that direction, from [**23**]; this will give us an adequate frame to formulate and prove our first structure theorem for the space of isoformal analytic classes.

Let C a commutative ring and $\mathcal{C}$ an abelian C-linear category. We fix a finitely graded object:

$$P = P_1 \oplus \cdots \oplus P_k,$$

and intend to classify pairs $(\underline{M}, \underline{u})$ made up of a finitely filtered object:

$$\underline{M} = (0 = M_0 \subset M_1 \subset \cdots \subset M_k = M),$$

and of an isomorphism from $\mathrm{gr}M$ to P:

$$\underline{u} = (u_i : M_i/M_{i-1} \simeq P_i)_{1 \leq i \leq k}.$$

As easily checked, it amounts to the same as giving k exact sequences:

$$0 \to M_{i-1} \overset{w_i}{\to} M_i \overset{v_i}{\to} P_i \to 0.$$

The pairs $(\underline{M}, \underline{u})$ and $(\underline{M'}, \underline{u'})$ are said to be equivalent if (with obvious notations) there exists a morphism from M to M' which is compatible with the filtrations and with the structural isomorphisms, that is, making the following diagram commutative:

$$\begin{tikzcd} \mathrm{gr}M \arrow[rr, "\mathrm{gr}f"] \arrow[dr, "u"'] & & \mathrm{gr}M' \arrow[dl, "u'"] \\ & P & \end{tikzcd}$$

In the description by exact sequences, the equivalence relation translates as follows: there should exist morphisms $f_i : M_i \to M'_i$ making the following diagrams commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_{i-1} & \xrightarrow{w_i} & M_i & \xrightarrow{v_i} & P_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_{i-1}} & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle \mathrm{Id}_{P_i}} & & \\
0 & \longrightarrow & M'_{i-1} & \xrightarrow{w'_i} & M'_i & \xrightarrow{v'_i} & P_i & \longrightarrow & 0
\end{array}$$

Note that such a morphism (if it exists) is automatically strict and an isomorphism.

We write $\mathcal{F}(P_1, \ldots, P_k)$ the set[(3)] of equivalence classes of pairs $(\underline{M}, \underline{u})$.

2.3.1.1. Small values of k. — For $k = 1$, the set $\mathcal{F}(P_1)$ is a singleton. For $k = 2$, the set $\mathcal{F}(P_1, P_2)$ has a natural identification with the set $\mathrm{Ext}(P_2, P_1)$ of classes of extensions of P_2 by P_1, which carries a structure of C-module[(4)]. The identification generalizes the one that was described in proposition 2.2.7 and can be obtained as follows. To give a filtered module $\underline{M} = (0 = M_0 \subset M_1 \subset M_2 = M)$ endowed with an isomorphism from $\mathrm{gr}M$ to $P_1 \oplus P_2$ amounts to give an isomorphism from M_1 to P_1 and an isomorphism from M/M_1 to P_2, that is, a monomorphism i from P_1 to M and an epimorphism p from M to P_2 with kernel $i(P_1)$, that is, an exact sequence:

$$0 \to P_1 \xrightarrow{i} M \xrightarrow{p} P_2 \to 0,$$

that is, an extension of P_2 by P_1. One then checks easily that our equivalence relation thereby corresponds with the usual isomorphism of extensions.

When $k = 3$, the description of $\mathcal{F}(P_1, P_2, P_3)$ amounts to the classification of *blended extensions* ("extensions panachées"). These were introduced by Grothendieck in [**21**]. We refer to the studies [**4, 5**] by Daniel Bertrand, whose conventions we use. Start from a representative of a class in $\mathcal{F}(P_1, P_2, P_3)$, in the form of three exact sequences:

$$0 \to M_0 \xrightarrow{w_1} M_1 \xrightarrow{v_1} P_1 \to 0, \quad 0 \to M_1 \xrightarrow{w_2} M_2 \xrightarrow{v_2} P_2 \to 0, \quad 0 \to M_2 \xrightarrow{w_3} M_3 \xrightarrow{v_3} P_3 \to 0.$$

---

(3) We admit that $\mathcal{F}(P_1, \ldots, P_k)$ is indeed a set; from the devissage arguments that follow, (see 2.3.1.2), it easily seen to be true if all the Ext spaces are sets, *e.g.* in the category of left modules over a ring.

(4) Note however that the latter comes by identifying Ext with Ext^1, and that there are two opposite such identifications, see for instance [**12**], exercice 1, p. 308. We shall systematically use the conventions of [**10**] and, from now on, make no difference between Ext and Ext^1.

Also recall that $M_0 = 0$ and $M_3 = M$. These give rise to two further exact sequences; first:

$$0 \longrightarrow P_1 \xrightarrow{w_2 \circ v_1^{-1}} M_2 \xrightarrow{v_2} P_2 \longrightarrow 0$$

Indeed, $v_1 : M_1 \to P_1$ is an isomorphism, so that $v_1^{-1} : P_1 \to M_1$ is well defined; and the exactness is easy to check. To describe the second exact sequence, note that $v_2 : M_2 \to P_2$ induces an isomorphism $\overline{v_2} : M_2/w_2(M_1) \to P_2$, whence $(\overline{v_2})^{-1} : P_2 \to M_2/w_2(M_1)$; then $w_3 : M_2 \to M_3 = M$ induces a morphism $\overline{w_3} : M_2/w_2(M_1) \to M'$, where we put $M' := M/w_3 \circ w_2(M_1)$, and, by composition, a morphism $\overline{w_3} \circ (\overline{v_2})^{-1} : P_2 \to M'$; last, $v_3 : M_3 \to P_3$ is trivial on $w_3(M_2)$, therefore on $w_3 \circ w_2(M_1)$, so that it induces a morphism $\overline{v_3} : M' \to P_3$. Now we get the sequence:

$$0 \longrightarrow P_2 \xrightarrow{\overline{w_3} \circ (\overline{v_2})^{-1}} M' \xrightarrow{\overline{v_3}} P_3 \longrightarrow 0$$

We leave for the reader to verify the exactness of this sequence. These two sequences can be *blended* ("panachées") to give the following commutative diagram of exact sequences:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & P_1 \arrow[r, "w_2 \circ v_1^{-1}"] \arrow[d, equal] & M_2 \arrow[r, "v_2"] \arrow[d, "w_3"] & P_2 \arrow[r] \arrow[d, "\overline{w_3} \circ (\overline{v_2})^{-1}"] & 0 \\
0 \arrow[r] & P_1 \arrow[r, "w_3 \circ w_2 \circ v_1^{-1}"] & M \arrow[r, "can"] \arrow[d, "v_3"] & M' \arrow[r] \arrow[d, "\overline{v_3}"] & 0 \\
 & & P_3 \arrow[r, equal] \arrow[d] & P_3 \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

(We call *can* the canonical projection from M to M'.) Equivalence of such diagrams, with fixed P_1, P_2, P_3, is easy to define, and one can prove that one thus gets a bijective mapping from $\mathcal{F}(P_1, P_2, P_3)$ to the set of equivalence classes of such diagrams.

2.3.1.2. The devissage. — Our goal is to give conditions ensuring that $\mathcal{F}(P_1, \ldots, P_k)$ carries the structure of an affine space over C, and to compute its dimension. The case $k = 2$ suggests that we should assume the C-modules

$\mathrm{Ext}(P_j, P_i)$ to be free of finite rank. (As we shall see, the only pairs that matter are those with $i < j$.) Then, aiming at an induction argument, one invokes a natural onto mapping:

$$\mathcal{F}(P_1, \ldots, P_k) \longrightarrow \mathcal{F}(P_1, \ldots, P_{k-1}),$$

sending the class of $(\underline{M}, \underline{u})$ defined as above to the class of $(\underline{M'}, \underline{u'})$ defined by:

$$\underline{M'} = (0 = M_0 \subset M_1 \subset \cdots \subset M_{k-1} = M') \text{ et } \underline{u'} = (u_i)_{1 \leq i \leq k-1}.$$

The preimage of the class of $(\underline{M'}, \underline{u'})$ described above is identified with $\mathrm{Ext}(P_k, M')$; and note that $\mathrm{Ext}(P_k, M')$ indeed only depends (up to a canonical isomorphism) on the class of $(\underline{M'}, \underline{u'})$. Under the assumptions we shall choose, we shall see that $\mathrm{Ext}(P_k, M')$ can in turn be unscrewed (dévissé) in the $\mathrm{Ext}(P_k, P_i)$ for $i < k$, and we expect to get a space with dimension $\sum\limits_{1 \leq i < j \leq k} \dim \mathrm{Ext}(P_j, P_i)$.

Remark 2.3.1. — Once described the space $\mathcal{F}(P_1, \ldots, P_k)$, one can ask for the seemingly more natural problem of the classification of those objects M such that $\mathrm{gr} M \simeq P$ (without prescrit of the "polarization" u). One checks that the group $\prod \mathrm{Aut}(P_i)$ operates on the space $\mathcal{F}(P_1, \ldots, P_k)$: actually, $(\phi_i) \in \prod \mathrm{Aut}(P_i)$ acts on the "class" of all pairs $(\underline{M}, \underline{u})$ through left compositions $\phi_i \circ u_i$. Then, our new classification comes by quotienting $\mathcal{F}(P_1, \ldots, P_k)$ by this action. We shall not deal with that problem.

The use of homological algebra in classification problems for functional equations is ancient, but it seems that the first step, the algebraic modelisation, has sometimes been tackled rather casually: for instance, when identifying a module of extensions with a cokernel, the explicit description of a map is almost always given; the proof of its bijectivity comes sometimes; the proof of its additivity seldom; the proof of its linearity (seemingly) never. For that reason, very great care has been given here to detailed algebraic constructions and proofs of "obvious" isomorphisms.

2.3.2. Difference modules over difference rings. — In order to study a relative situation from 2.3.4 on, we now generalize to difference rings our basic constructions. Let K be a commutative ring and σ a ring automorphism of K. As noticed in 2.1.1, most constructions and statements about difference fields and modules remain valid over the *difference ring* (K, σ). Now let C be a commutative ring (we may think of the the field $\mathbf{C}$ of complex numbers, or else some arbitrary field of "constants"). Assume that K is a commutative

C-algebra and σ a C-algebra automorphism, that is, a C-linear ring automorphism. Then the ring of constants $K^\sigma := \{x \in K \mid \sigma x = x\}$ is actually a sub C-algebra of K, and the Öre ring of difference operators $\mathcal{D}_{K,\sigma} := K < \sigma, \sigma^{-1} >$ is a C-algebra with center K^σ. (As before, its elements are non-commutative Laurent polynomials with the twisted commutation relations $\sigma^k.\lambda = \sigma^k(\lambda)\sigma^k$).

A *difference module* over the *difference C-algebra* (K, σ) (more shortly, over K) will be defined to be a left $\mathcal{D}_{K,\sigma}$-module which, by restriction of scalars to K, yields a finite rank projective K-module. We write $DiffMod(K, \sigma)$ the full subcategory of $\mathcal{D}_{K,\sigma} - Mod$ with objects the difference modules over K. Both categories are abelian and C-linear. Left $\mathcal{D}_{K,\sigma}$-modules (resp. difference modules over K) can be realized as pairs (E, Φ), where E is a K-module (resp. projective of finite rank) and Φ is a semi-linear automorphism of E, that is, a group automorphism such that $\forall \lambda \in K$, $\forall x \in E$, $\Phi(\lambda x) = \sigma(\lambda)\Phi(x)$. In this description, a morphism of $\mathcal{D}_{K,\sigma}$-modules from (E, Φ) to (F, Ψ) is a map $u \in \mathcal{L}_K(E, F)$ such that $\Psi \circ u = u \circ \Phi$.

2.3.2.1. Matricial description of difference modules. — Here, we assume that the K-module E is *free* of finite rank n. This assumption will be maintained in 2.3.2.2 and 2.3.3.2, where we pursue the matricial description. Choosing a basis $\mathcal{B}$ allows one to identify E with K^n. It is then clear that $\Phi(\mathcal{B})$ is also a basis of E, whence the existence of $A \in GL_n(K)$ such that $\Phi(\mathcal{B}) = \mathcal{B}A^{-1}$. If $x \in E$ has the coordinate column vector $X \in K^n$ in basis $\mathcal{B}$, *i.e.* if $x = \mathcal{B}X$, computing $\Phi(x) = \Phi(\mathcal{B}X) = \Phi(\mathcal{B})\sigma(X) = \mathcal{B}A^{-1}\sigma(X)$ shows that $\Phi(x) \in E$ has the coordinate column vector $A^{-1}\sigma(X)$; we thus may identify $(E, \Phi) \simeq (K^n, \Phi_A)$, where $\Phi_A(X) := A^{-1}\sigma(X)$. Morphisms from (K^n, Φ_A) to (K^p, Φ_B) are matrices $F \in M_{p,n}(K)$ such that $(\sigma F)A = BF$ and their composition boils down to matrix product. In particular, isomorphism of modules is described by gauge transformations:

$$(K^n, \Phi_A) \simeq (K^p, \Phi_B) \Longleftrightarrow n = p \text{ et } \exists F \in GL_n(K) \ : \ B = F[A] := (\sigma F)AF^{-1}.$$

2.3.2.2. Matricial description of filtered difference modules. — Let $P_i = (G_i, \Psi_i)$ $(1 \leq i \leq k)$ be difference modules such that each K-module G_i is free of finite rank r_i . For each G_i, choose a basis $\mathcal{D}_i$, and write $B_i \in GL_{r_i}(K)$ the invertible matrix such that $\Psi_i(\mathcal{D}_i) = \mathcal{D}_i B_i$.

Let $\underline{M}$ be a finitely filtered difference module with associated graded module $P = P_1 \oplus \cdots \oplus P_k$; more precisely, $\underline{M} = (0 = M_0 \subset M_1 \subset \cdots \subset M_k = M)$ is equipped with an isomorphism $\underline{u} = (u_i : M_i/M_{i-1} \simeq P_i)_{1 \leq i \leq k}$ from gr$M$ to P.

Letting $M_i = (E_i, \Phi_i)$, one builds a basis $\mathcal{B}_i$ of E_i by induction on $i = 1, \ldots, k$ in such a way that $\mathcal{B}_{i-1} \subset \mathcal{B}_i$ and that $\mathcal{B}'_i := \mathcal{B}_i \setminus \mathcal{B}_{i-1}$ lifts $\mathcal{D}_i$ via u_i (showing by the way that K-modules E_i are free of finite ranks $r_1 + \cdots + r_i$). Write Φ_i (resp. $\overline{\Phi_i}$) the semi-linear automorphism induced by Φ (resp. by Φ_i) on E_i (resp. on E_i/E_{i-1}), and $\mathcal{C}_i$ the basis of E_i/E_{i-1} induced by $\mathcal{B}'_i$, one draws from equality $u_i \circ \overline{\Phi_i} = \Psi_i \circ u_i$ (due to the fact that u_i is a morphism) the relation: $\overline{\Phi_i}(\mathcal{C}_i) = \mathcal{C}_i B_i$, then, from the latter, the relation:

$$\Phi_i(\mathcal{B}'_i) \equiv \mathcal{B}'_i B_i \pmod{E_{i-1}}.$$

Last, one gets that the matrix of Φ in basis $\mathcal{B}$ is block upper-triangular:

$$\Phi(\mathcal{B}) = \mathcal{B} \begin{pmatrix} B_1 & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & B_k \end{pmatrix}.$$

As in 2.3.2.1, we identify P_i $(1 \leq i \leq k)$ with (K^{r_i}, Φ_{A_i}), where $A_i := B_i^{-1} \in GL_{r_i}(K)$. Likewise, P is identified with (K^n, Φ_{A_0}) and M with (K^n, Φ_A), where $n := r_1 + \cdots + r_k$ and:

$$A_0 = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_k \end{pmatrix} \text{ et } A = \begin{pmatrix} A_1 & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & A_k \end{pmatrix}.$$

Note that these relations implicitly presuppose that a filtration on M is given, as well as an isomorphism from $\mathrm{gr}M$ to P. If moreover $M' = (K^n, \Phi_{A'})$, where A' has the same form as A (*i.e.* M' is filtered and equipped with an isomorphism from $\mathrm{gr}M'$ to P), then, a morphism from M to M' respecting filtrations (*i.e.* sending each M_i into M'_i) is described by a matrix F in the following block upper triangular form; and the induced endomorphism of $P \simeq \mathrm{gr}M \simeq \mathrm{gr}M'$ is described by the corresponding block diagonal matrix F_0

$$F = \begin{pmatrix} F_1 & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & F_k \end{pmatrix} \text{ et } F_0 = \begin{pmatrix} F_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & F_k \end{pmatrix}.$$

In particular, a morphism inducing identity on P (thus ensuring that the filtered modules M, M' belong to the same class in $\mathcal{F}(P_1, \ldots, P_k)$) is represented by a matrix in $\mathfrak{G}(K)$, where we denote $\mathfrak{G}$ the algebraic subgroup of GL_n defined by the following shape:

$$\begin{pmatrix} I_{r_1} & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & I_{r_k} \end{pmatrix}.$$

Now write A_U the block upper triangular matrix with block diagonal component A_0 and with upper triangular blocks the $U_{i,j} \in M_{r_i,r_j}(K)$ $(1 \leq i < j \leq k)$; here, U is an abrevation for the family $(U_{i,j})$. For all $F \in \mathfrak{G}(K)$, the matrix $F[A_U]$ is equal to A_V for some family of $V_{i,j} \in M_{r_i,r_j}(K)$. Thus, the group $\mathfrak{G}(K)$ operates on the set $\prod\limits_{1\leq i<j\leq k} M_{r_i,r_j}(K)$. The above discussion can be summarised as follows:

Proposition 2.3.2. — *The map sending* U *to the class of* (K^n, Φ_{A_U}) *induces a bijection from the quotient of the set* $\prod\limits_{1\leq i<j\leq k} M_{r_i,r_j}(K)$ *under the action of the group* $\mathfrak{G}(K)$ *onto the set* $\mathcal{F}(P_1, \ldots, P_k)$*.*

2.3.3. Extensions of difference modules. — Let $0 \to M' \to M \to M'' \to 0$ an exact sequence in $\mathcal{D}_{K,\sigma} - Mod$. If M', M'' are difference modules, so is M (it is projective of finite rank because we have a split sequence of K-modules). The calculus of extensions is therefore the same in $DiffMod(K, \sigma)$ as in $\mathcal{D}_{K,\sigma} - Mod$ and we will simply write $\mathrm{Ext}(M'', M')$ the group $\mathrm{Ext}_{\mathcal{D}_{K,\sigma}}(M'', M')$ of classes of extensions of M'' by M'. According to [**10**, §7], this group is actually endowed with a structure of C-module, which is well described in *loc. cit.*[(5)]. We shall now make explicit that structure in the case that M' and M'' are difference modules.

So let $M = (E, \Phi)$ and $N = (F, \Psi)$. Any extension $0 \to N \to R \to M \to 0$ of M by N gives rise (by restriction of scalars) to an exact sequence of K-modules $0 \to F \to G \to E \to 0$ such that, if $R = (G, \Gamma)$, the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \xrightarrow{i} & G & \xrightarrow{j} & E & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle\Gamma} & & \downarrow{\scriptstyle\Phi} & & \\ 0 & \longrightarrow & F & \xrightarrow{i} & G & \xrightarrow{j} & E & \longrightarrow & 0 \end{array}$$

(5) And, to the best of our knowledge, nowhere else.

(We wrote again i, j the underlying K-linear maps.) Since E is projective, the sequence is split and one can from start identify G with the K-module $F \times E$, thus writing $i(y) = (y, 0)$ and $j(y, x) = x$. The compatibility conditions $\Gamma \circ i = i \circ \Psi$ and $\Phi \circ j = j \circ \Gamma$ then imply:

$$\Gamma(y, x) = \Gamma_u(y, x) := \big(\Psi(y) + u(x), \Phi(x)\big), \text{ with } u \in \mathcal{L}_\sigma(E, F),$$

where we write $\mathcal{L}_\sigma(E, F)$ the set of σ-linear maps from E to F. (This means that u is a group morphism such that $u(\lambda x) = \sigma(\lambda) u(x)$.) Setting moreover $R_u := (F \times E, \Gamma_u)$, which is a difference module naturally equipped with a structure of extension of M by N, we see that we have defined a surjective map:

$$\begin{aligned} \mathcal{L}_\sigma(E, F) &\to \mathrm{Ext}(M, N), \\ u &\mapsto \theta_u := \text{ class of } R_u. \end{aligned}$$

We can make precise the conditions under which $u, v \in \mathcal{L}_\sigma(E, F)$ have the same image $\theta_u = \theta_v$, *i.e.* under which R_u and R_v are equivalent extensions. This happens if there exists a morphism $\phi : R_u \to R_v$ inducing the identity map on M and N, that is, a linear map $\phi : F \times E \to F \times E$ such that $\Gamma_v \circ \phi = \phi \circ \Gamma_u$ (since it is a morphism of difference modules) and having the form $(x, y) \mapsto \big(y + f(x), x\big)$ (since it induces the identity maps on E and on F). Now, the first condition becomes:

$$\forall (y, x) \in F \times E\ ,\ \big(\Psi(y + f(x)) + v(x), \Phi(x)\big) = \big(\Psi(y) + u(x) + f(\Phi(x)), \Phi(x)\big),$$

that is:

$$u - v = \Psi \circ f - f \circ \Phi.$$

Remark by the way that, for all $f \in \mathcal{L}_K(E, F)$, the map $t_{\Phi,\Psi}(f) := \Psi \circ f - f \circ \Phi$ is σ-linear from E to F.

Theorem 2.3.3. — *The map $u \mapsto \theta_u$ from $\mathcal{L}_\sigma(E, F)$ to Ext(M, N) is functorial in M and in N, C-linear, and its kernel is the image of the C-linear map:*

$$\begin{aligned} t_{\Phi,\Psi} : \mathcal{L}_K(E, F) &\to \mathcal{L}_\sigma(E, F), \\ f &\mapsto \Psi \circ f - f \circ \Phi. \end{aligned}$$

Proof. — Functoriality.
We shall only prove (and use) it on the covariant side, *i.e.* in N. We invoke [**10**, §7.1 p. 114, example 3 and §7.4, p. 119, prop. 4]. Let θ be the class

in $\mathrm{Ext}(M,N)$ of the extension $0 \xrightarrow{i} N \to R \xrightarrow{j} M \to 0$ and $g : N \to N'$ a morphism in $DiffMod(K,\sigma)$. Let

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \xrightarrow{i} & R & \xrightarrow{j} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \mathrm{Id}_M} & & \\
0 & \longrightarrow & N' & \xrightarrow{i'} & R' & \xrightarrow{j'} & M & \longrightarrow & 0
\end{array}$$

be a commutative diagram of exact sequences. If θ' is the class in $\mathrm{Ext}(M,N)$ of the extension $0 \xrightarrow{i'} N' \to R' \xrightarrow{j'} M \to 0$, then:

$$\mathrm{Ext}(\mathrm{Id}_M, g)(\theta) = g \circ \theta = \theta' \circ \mathrm{Id}_M = \theta'.$$

We take:

$$R' := R \oplus_N N' = \frac{R \times N'}{\{(i(n), -g(n)) \mid n \in N\}},$$

with i', j' the obvious arrows, and for R the extension R_u; with the previous notations for N, M, R, and also writing $N' = (F', \Psi')$, with compatibility condition $\Psi' \circ g = g \circ \Psi$, we see that the K-module underlying $R \oplus_N N'$ is:

$$G' := \frac{F \times E \times F'}{\{(y, 0, -g(y)) \mid y \in F\}},$$

endowed with the semi-linear automorphism induced by the map $\Gamma_u \times \Psi'$ from $F \times E \times F'$ to itself (the latter does fix the denominator).
The map $(y, x, y') \mapsto (y' + g(y), x)$ from $F \times E \times F'$ to $F' \times E$ induces an isomorphism from G' to $F' \times E$ and the induced semi-linear automorphism on G' is $(y', x) \mapsto (\Psi'(y') + g(u(x)), \Phi(x))$, that is Γ_{gu}, from which it follows that $R' = R_{gu}$. The arrows i', j' are determined as follows: $i'(y')$ is the class of $(0, y')$ in G', that is, under the previous identification, $i'(y') = (y', 0)$; and $j'(y', x)$ is the image of an arbitrary preimage, for instance the class of $(0, x, y')$: that image is $j(0, x) = x$. We have therefore shown that the class of the extension R_u by $\mathrm{Ext}(\mathrm{Id}_M, g)$ is R_{gu}, which is the wanted functoriality. It is expressed by the commutativity of the following diagram:

$$\begin{array}{ccc}
\mathcal{L}_\sigma(E,F) & \longrightarrow & \mathrm{Ext}(M,N) \\
\downarrow{\scriptstyle \mathcal{L}_\sigma(\mathrm{Id}_M, g)} & & \downarrow{\scriptstyle \mathrm{Ext}(\mathrm{Id}_M, g)} \\
\mathcal{L}_\sigma(E,F') & \longrightarrow & \mathrm{Ext}(M,N')
\end{array}$$

Linearity.
According to the remark just before the theorem, the map $t_{\Phi,\Psi}$ indeed sends $\mathcal{L}_K(E,F)$ to $\mathcal{L}_\sigma(E,F)$.
Addition. The reference here is [**10**, §7.6, rem. 2 p. 124]. From the extensions

$0 \to N \xrightarrow{i} R \xrightarrow{p} M \to 0$ and $0 \to N \xrightarrow{i'} R' \xrightarrow{p'} M \to 0$ having classes $\theta, \theta' \in \mathrm{Ext}^1(M, N)$, one computes $\theta + \theta'$ as the class of the extension $0 \to N \xrightarrow{i''} R'' \xrightarrow{p''} M \to 0$, where:

$$R'' := \frac{\{(z, z') \in R \times R' \mid p(z) = p'(z')\}}{\{(-i(y), i'(y)) \mid y \in N\}},$$

and $i''(y)$ is the class of $(0, i'(y))$, *i.e.* the same as the class of $(i(y), 0)$; and p'' sends the class of (z, z') to $p(z) = p'(z')$. Taking $R = R_u$ and $R' = R_{u'}$, the numerator of R'' is identified with $F \times F \times E$ equipped with the semi-linear automorphism $(y, y', x) \mapsto (\Psi(y)+u(x), \Psi(y')+u'(x), \Phi(x))$. The denominator is identified with the subspace $\{(-y, y, 0) \mid y \in F\}$ equipped with the induced map. The quotient is identified with $0 \times F \times E$, through the map $(y, y', x) \mapsto (0, y'', x)$, where $y'' := y' + y$, equipped with the semi-linear automorphism Φ'' which sends $(0, y'', x)$ to

$$(0, \Psi(y') + u'(x) + \Psi(y) + u(x), \Phi(x)) = (0, \Psi(y'') + (u + u')(x), \Phi(x)).$$

This is indeed $R_{u+u'}$.
External multiplication. The reference here is [**10**, §7.6, prop. 4 p. 119]. Let $\lambda \in C$. We apply the invoked proposition to the following commutative diagram of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "\times\lambda"] & R_u \arrow[r] \arrow[d, "(\times\lambda, \mathrm{Id}_M)"] & M \arrow[r] \arrow[d, "\mathrm{Id}_M"] & 0 \\
0 \arrow[r] & N \arrow[r] & R_{\lambda u} \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

If θ, θ' are the classes in $\mathrm{Ext}(M, N)$ of the two extensions, one infers from *loc. cit.* that:

$$\theta' \circ \mathrm{Id}_M = (\times\lambda) \circ \theta \Longrightarrow \theta' = \lambda\theta.$$

The class of the extension $R_{\lambda u}$ is therefore indeed equal to the product of λ by the class of the extension R_u.

<u>Exactness.</u>
It follows immediately from the computation shown just before the statement of the theorem. □

2.3.3.1. The complex of solutions. — The following is sometimes considered as a difference analog of the de Rham complex in one variable, see for instance [**1, 49**].

Definition 2.3.4. — We call *complex of solutions of* M *in* N the following complex of C-modules:

$$\begin{aligned} t_{\Phi,\Psi} : \mathcal{L}_K(E,F) &\to \mathcal{L}_\sigma(E,F), \\ f &\mapsto \Psi \circ f - f \circ \Phi. \end{aligned}$$

concentrated in degrees 0 and 1.

It is indeed clear that the source and target are C-modules, that the map $t_{\Phi,\Psi}$ does sends the source into the target and that it is C-linear.

Corollary 2.3.5. — *The homology of the complex of solutions is* $H^0 = \mathit{Hom}(M,N)$ *and* $H^1 = \mathit{Ext}(M,N)$*, and these equalities are functorial.*

Proof. — The statement about H^1 is the theorem. As regards H^0, the kernel of $t_{\Phi,\Psi}$ is the C-module $\{f \in \mathcal{L}_K(E,F) \mid \Psi \circ f = f \circ \Phi\}$, that is, $\mathrm{Hom}(M,N)$; and functoriality is obvious in that case. □

Corollary 2.3.6. — *From the exact sequence* $0 \to N' \to N \to N'' \to 0$*, one deduces the "cohomology long exact sequence":*

$$\begin{aligned} 0 \to \mathit{Hom}(M,N') \to & \mathit{Hom}(M,N) \to \mathit{Hom}(M,N'') \\ & \to \mathit{Ext}(M,N') \to \mathit{Ext}(M,N) \to \mathit{Ext}(M,N'') \to 0. \end{aligned}$$

Proof. — We keep the previous notations (and moreover adapt them to N', N''). The exact sequence of projective K-modules $0 \to F' \to F \to F'' \to 0$ being split, both lines of the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{L}_K(E,F') & \longrightarrow & \mathcal{L}_K(E,F) & \longrightarrow & \mathcal{L}_K(E,F'') & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle t_{\Phi,\Psi'}} & & \Big\downarrow{\scriptstyle t_{\Phi,\Psi}} & & \Big\downarrow{\scriptstyle t_{\Phi,\Psi''}} & & \\ 0 & \longrightarrow & \mathcal{L}_\sigma(E,F') & \longrightarrow & \mathcal{L}_\sigma(E,F) & \longrightarrow & \mathcal{L}_\sigma(E,F'') & \longrightarrow & 0 \end{array}$$

are exact, and it is enough to call to the snake lemma. □

2.3.3.2. Matricial description of extensions of difference modules. — We now assume E, F to be free of finite rank over K and accordingly identify them with $M = (K^m, \Phi_A)$, $A \in GL_m(K)$ and $N = (K^n, \Phi_B)$, $B \in GL_n(K)$. An extension of N by M then takes the form $R = (K^{m+n}, \Phi_C)$, where $C = \begin{pmatrix} A & U \\ 0_{n,m} & B \end{pmatrix}$ for some rectangular matrix $U \in M_{m,n}(K)$; we shall write $C = C_U$. The injection $M \to R$ and the projection $R \to N$ have as respective matrices $\begin{pmatrix} I_m \\ 0_{n,m} \end{pmatrix}$ and $\begin{pmatrix} 0_{n,m} & I_n \end{pmatrix}$. The extension thus defined will be denoted

R_U.

A morphism of extensions $R_U \to R_V$ is a matrix of the form $F = \begin{pmatrix} I_m & X \\ 0_{n,m} & I_n \end{pmatrix}$ for some rectangular matrix $X \in M_{m,n}(K)$. The compatibility condition with the semi-linear automorphisms writes:

$$(\sigma F)C_U = C_V F \Longleftrightarrow U + (\sigma X)B = AX + V \Longleftrightarrow V - U = (\sigma X)B - AX.$$

Corollary 2.3.7. — *The C-module $Ext^1(N, M)$ is thereby identified with the cokernel of the endomorphism $X \mapsto (\sigma X)B - AX$ of $M_{m,n}(K)$.*

Proof. — The above construction provides us with a bijection, but it follows from theorem 2.3.3 that it is indeed an isomorphism. □

2.3.4. Extension of scalars. — We want to see $\mathcal{F}(P_1, \ldots, P_k)$ as a scheme over C, that is as a representable functor $C' \rightsquigarrow \mathcal{F}(C' \otimes_C P_1, \ldots, C' \otimes_C P_k)$ from commutative C-algebras to sets. To that end, we shall extend what we did to a "relative" situation.

Let C' be a commutative C-algebra. We set:

$$K' := C' \otimes_C K \text{ and } \sigma' := 1 \otimes_C \sigma.$$

Then K' is a commutative C'-algebra and σ' an automorphism of that C'-algebra. Moreover:

$$\mathcal{D}_{K',\sigma'} := K' < \sigma', \sigma'^{-1} >= K' \otimes_K \mathcal{D}_{K,\sigma} = C' \otimes_C \mathcal{D}_{K,\sigma}.$$

These equalities should be interpreted as natural (functorial) isomorphisms.

From any difference module $M = (E, \Phi)$ over (K, σ), one gets a difference module $M' = (E', \Phi')$ over (K', σ') by putting:

$$E' = K' \otimes_K E = C' \otimes_C E \text{ and } \Phi' = \sigma' \otimes_K \Phi = 1 \otimes_C \Phi.$$

(This is indeed a left $\mathcal{D}_{K',\sigma'}$-module and it is projective of finite rank over K'.) We shall write it $M' = C' \otimes_k M$ to emphasize the dependency on C'. The following proposition is the tool to tackle the case $k = 2$.

Proposition 2.3.8. — *Let M, N be two difference modules over (K, σ). One has a functorial isomorphism of C'-modules:*

$$Ext_{\mathcal{D}_{K',\sigma'}}(C' \otimes_C M, C' \otimes_C N) \simeq C' \otimes_C Ext_{\mathcal{D}_{K,\sigma}}(M, N),$$

and a functorial epimorphism of C'-modules:

$$C' \otimes_C Hom_{\mathcal{D}_{K,\sigma}}(M, N) \to Hom_{\mathcal{D}_{K',\sigma'}}(C' \otimes_C M, C' \otimes_C N).$$

Proof. — We shall write $M' = C' \otimes_C M$, $E' = K' \otimes_K E$ etc. The K-modules E, F beeing projective of finite rank, there are natural isomorphisms:

$$C' \otimes_C \mathcal{L}_K(E, F) = \mathcal{L}_{K'}(E', F') \text{ et } C' \otimes_C \mathcal{L}_\sigma(E, F) = \mathcal{L}_{\sigma'}(E', F').$$

(This is immediate if E and F are free, the general case follows.) By tensoring the (functorial) exact sequence:

$$0 \to \mathrm{Hom}_{\mathcal{D}_{K,\sigma}}(M, N) \to \mathcal{L}_K(E, F) \to \mathcal{L}_\sigma(E, F) \to \mathrm{Ext}_{\mathcal{D}_{K,\sigma}}(M, N) \to 0,$$

we get the exact sequence:

$$C' \otimes_C \mathrm{Hom}_{\mathcal{D}_{K,\sigma}}(M, N) \to C' \otimes_C \mathcal{L}_K(E, F) \to C' \otimes_C \mathcal{L}_\sigma(E, F) \to C' \otimes_C \mathrm{Ext}_{\mathcal{D}_{K,\sigma}}(M, N) \to 0.$$

Both conclusions then come by comparison with the exact sequence:

$$0 \to \mathrm{Hom}_{\mathcal{D}_{K',\sigma'}}(M', N') \to \mathcal{L}_{K'}(E', F') \to \mathcal{L}_{\sigma'}(E', F') \to \mathrm{Ext}_{\mathcal{D}_{K',\sigma'}}(M', N') \to 0.$$

□

Proposition 2.3.9. — *Let $0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$ be a k-filtration with associated graded module $P_1 \oplus \cdots \oplus P_k$. Then, setting $M'_i := C' \otimes_C M_i$ and $P'_i := C' \otimes_C P_i$, we get a k-filtration $0 = M'_0 \subset M'_1 \subset \cdots \subset M'_k = M'$ with associated graded module $P'_1 \oplus \cdots \oplus P'_k$.*

Proof. — The P_i being projective as K-modules, the exact sequences $0 \to M_{i-1} \to M_i \to P_i \to 0$ are K-split, so they give rise by the base change $K' \otimes_K$ to exact sequences $0 \to M'_{i-1} \to M'_i \to P'_i \to 0$. □

If $(\underline{M}, \underline{u})$ denotes the pair made up of the above k-filtered object and of a fixed isomorphism from $\mathrm{gr}M$ to $P_1 \oplus \cdots \oplus P_k$, we shall write $(C' \otimes_C \underline{M}, 1 \otimes_C \underline{u})$ the corresponding pair deduced from the proposition.

Definition 2.3.10. — We define as follows a functor F from the category of commutative C-algebras to the category of sets. For any commutative C-algebra C', we set:

$$F(C') := \mathcal{F}(C' \otimes_C P_1, \ldots, C' \otimes_C P_k).$$

For any morphism $C' \to C''$ of commutative C-algebras, the map $F(C') \to F(C'')$ is given by:

$$\text{class of } (\underline{M'}, \underline{u'}) \mapsto \text{class of } (C'' \otimes_{k'} \underline{M'}, 1 \otimes_C \underline{u'}).$$

The set $F(C')$ is well defined according to the previous constructions. The map $F(C') \to F(C'')$ is well defined on pairs thanks to the proposition, and the reader will check that it goes to the quotient. Last, the functoriality (preservation of the composition of morphisms) comes from the contraction rule of tensor products:

$$C''' \otimes_{C''} (C'' \otimes_{C'} \underline{M'}) = C''' \otimes_{C'} \underline{M'}.$$

2.3.5. Our moduli space. — To simplify, herebelow, instead of saying "the functor F is represented by an affine space over C (with dimension d)", we shall say "the functor F is an affine space over C (with dimension d)". This is just the usual identification of a scheme with the space it represents.

Theorem 2.3.11. — *Assume that, for* $1 \leq i < j \leq k$, *one has* $Hom(P_j, P_i) = 0$ *and that the* C*-module* $Ext(P_j, P_i)$ *is free of finite rank* $\delta_{i,j}$. *Then the functor* $C' \rightsquigarrow F(C') := \mathcal{F}(C' \otimes_C P_1, \ldots, C' \otimes_C P_k)$ *is an affine space over* C *with dimension* $\sum\limits_{1 \leq i < j \leq k} \delta_{i,j}$.

Proof. — When $k = 1$, it is trivial. When $k = 2$, writing V the free C-module of finite rank $\mathrm{Ext}(P_2, P_1)$, and appealing to proposition 2.3.8, we see that this is the functor $C' \rightsquigarrow C' \otimes_C V$, which is represented by the symetric algebra of the dual of V, an algebra of polynomials over C. For $k \geq 3$, we use an induction based on a lemma of Babbitt and Varadarajan [**2**, lemma 2.5.3, p. 139]:

Lemma 2.3.12. — *Let* $u : F \to G$ *be a natural transformation between two functors from commutative* C*-algebras to sets. Assume that* G *is an affine space over* C *and that, for any commutative* C*-algebra* C', *and for any* $b \in G(C')$, *the following functor "fiber above* b*" from commutative* C'*-algebras to sets:*

$$C'' \rightsquigarrow u_{C''}^{-1}\big(G(C' \to C'')(b)\big)$$

is an affine space over C'. *Then* F *is an affine space over* C.

In *loc. cit.*, this theorem is proved for $C = \mathbf{C}$, but the argument is plainly valid for any commutative ring. Here is its skeleton. Choose $B = C[T_1, \ldots, T_d]$ representing G. Take for b the identity of $G(B) = \mathrm{Hom}(B, B)$ ("general point"); the fiber is represented by $B[S_1, \ldots, S_e]$. One then shows that $C[T_1, \ldots, T_d, S_1, \ldots, S_e]$ represents F. This gives by the way a computation of $\dim F$ as $\dim G + \dim$ of the general fiber. In our case, all fibers will have the same dimension.

2.3.5.1. Structure of the fibers. — Before going to the proof of the theorem, we need an auxiliary result.

Proposition 2.3.13. — *Let C' be a commutative C-algebra and let M' be a difference module over $K' := C' \otimes_C K$, equipped with a $(k-1)$-filtration: $0 = M'_0 \subset M'_1 \subset \cdots \subset M'_{k-1} = M'$ such that $grM' \simeq P'_1 \oplus \cdots \oplus P'_{k-1}$ (as usual, $P'_i := C' \otimes_C P_i$). Then the functor in commutative C'-algebras $C'' \rightsquigarrow Ext(C'' \otimes_C P_k, C'' \otimes_{C'} M')$ is an affine space over C' with dimension $\sum\limits_{1 \leq i \leq k} \delta_{i,k}$.*

Proof. — After proposition 2.3.8, this is the functor $C'' \rightsquigarrow C'' \otimes_{C'} \mathrm{Ext}(P'_k, M')$. From each exact sequence $0 \to M'_{i-1} \to M'_i \to P'_i \to 0$ one draws the cohomology long exact sequence of corollary 2.3.6; but, from proposition 2.3.8, one draws that, for any commutative C-algebra C', one has $\mathrm{Hom}(C' \otimes_C P_j, C' \otimes_C P_i) = 0$ and that the C'-module $\mathrm{Ext}(C' \otimes_C P_j, C' \otimes_C P_i)$ is free of finite rank $\delta_{i,j}$. According to the equalities $\mathrm{Hom}(P'_j, P'_i) = 0$, the long exact sequence is here shortened as:

$$0 \to \mathrm{Ext}(P'_k, M'_{i-1}) \to \mathrm{Ext}(P'_k, M'_i) \to \mathrm{Ext}(P'_k, P'_i) \to 0,$$

and, for $i = 1, \ldots, k-1$, these sequences are split, the term at the right being free. So, in the end:

$$\mathrm{Ext}(P'_k, M') \simeq \bigoplus_{1 \leq i \leq k} \mathrm{Ext}(P'_k, P'_i),$$

which is free of rank $\sum\limits_{1 \leq i \leq k} \delta_{i,k}$. As in the case $k = 2$ (which is a particular case of the proposition), the functor mentioned is represented by the symetric algebra of the dual of this module. $\square$

For all ℓ such that $1 \leq \ell \leq k$, let us write:

$$V_\ell := \bigoplus_{1 \leq i \leq \ell} \mathrm{Ext}(P_\ell, P_i),$$
$$W_\ell := \bigoplus_{1 \leq i < j \leq \ell} \mathrm{Ext}(P_j, P_i),$$
$$V'_\ell := \bigoplus_{1 \leq i \leq \ell} \mathrm{Ext}(P'_\ell, P'_i),$$
$$W'_\ell := \bigoplus_{1 \leq i < j \leq \ell} \mathrm{Ext}(P'_j, P'_i).$$

We consider V_ℓ, W_ℓ as affine schemes over C and V'_ℓ, W'_ℓ as affine schemes over C', so that:

$$V'_\ell = C' \otimes_C V_\ell, \quad W'_\ell = C' \otimes_C W_\ell.$$

We improperly write $C' \otimes_C V$ the base change of affine schemes Spec $C' \otimes_{\mathrm{Spec}\ C} V$. Also, we do not distinguish between the direct sums of the free C-modules $\mathrm{Ext}(P_j, P_i)$ and the product of the corresponding affine schemes. Note that, in the proof of the proposition, the isomorphism $\mathrm{Ext}(P'_k, M') \simeq V'_\ell$ is functorial in C'. This entails:

Corollary 2.3.14. — *Each fiber $Ext(P'_k, M')$ is isomorphic to V'_k as a scheme over C'.*

2.3.5.2. End of the proof of theorem 2.3.11. — Now we end the proof of the theorem. Besides functor $F(C')$, we consider the functor $C' \rightsquigarrow G(C') := \mathcal{F}(C' \otimes_C P_1, \dots, C' \otimes_C P_{k-1})$, of which we assume, by induction, that it is an affine space of dimension $\sum\limits_{1 \leq i < j \leq k-1} \delta_{i,j}$. The natural transformation from F to G is the one described in 2.3.1.2. An element $b \in G(C')$ is the class of a pair $(\underline{M'}, \underline{u'})$, a $(k-1)$-filtered object over C', and the corresponding fiber is the one studied in the above auxiliary proposition 2.3.13. The lemma 2.3.12 of Babbitt and Varadarajan then allows us to conclude. □

Looking at the proof of 2.3.12, one moreover sees that, if all fibers of $u : F \to G$ are isomorphic to a same affine space V (up to obvious extension of the base), then there is an isomorphism $F \simeq V \times_C G$ such that u corresponds to the second projection. With the previous notations, we see that, writing $\mathcal{F}_\ell$ the functor $C' \rightsquigarrow \mathcal{F}(C' \otimes_C P_1, \dots, C' \otimes_C P_\ell)$, corollary 2.3.13 gives an isomorphism of schemes:

$$\mathcal{F}_\ell \simeq \mathcal{F}_{\ell-1} \times V_\ell.$$

By induction, we conclude:

Corollary 2.3.15. — *The functor in C-algebras $C' \rightsquigarrow \mathcal{F}(C' \otimes_C P_1, \dots, C' \otimes_C P_k)$ is isomorphic to the functor $C' \rightsquigarrow \bigoplus_{1 \leq i < j \leq \ell} Ext(C' \rightsquigarrow P_k, C' \rightsquigarrow P_i)$. That is, we have an isomorphism of affine schemes over C:*

$$\mathcal{F}(P_1, \dots, P_k) \simeq \prod_{1 \leq i < j \leq \ell} Ext(P_k, P_i).$$

2.3.6. Extension classes of difference modules. — We shall now describe more precisely extension spaces in the case of a difference field (K,σ). We keep all the previous notations. Our main tools are theorem 2.3.3 and its corollaries 2.3.5 and 2.3.7. We shall give more "computational" variants of these results.

We use the cyclic vector lemma (lemma 2.1.1): the proof given in [**14**] is valid for general difference modules, under the assumption that the characteristic of K is 0 and that σ is not of finite order (*e.g.* q is not a root of unity). Although most of what follows remains true for $\mathcal{D}$-modules of arbitrary length (because $\mathcal{D}$ is principal [**10**]), the proofs, inspired by [**42**, II.1.3], are easier for $\mathcal{D}/\mathcal{D}P$. Recall that, the center of $\mathcal{D}$ being C, all functors considered here are C-linear and produce C-vector spaces. For a difference module $M = (E,\Phi)$, we still write E the C-vector space underlying E.

Proposition 2.3.16. — *Let $M := \mathcal{D}/\mathcal{D}P$ and $N := (F,\Psi)$ two difference modules. Then $Ext^i(M,N) = 0$ for $i \geq 2$ and there is an exact sequence of C-vector spaces:*

$$0 \to Hom(M,N) \to F \stackrel{P(\Psi)}{\to} F \to Ext^1(M,N) \to 0.$$

Proof. — From the presentation (in the category of left $\mathcal{D}$-modules):

$$\mathcal{D} \stackrel{\times P}{\to} \mathcal{D} \to M \to 0,$$

one draws the long exact sequence:

$$\begin{aligned} 0 \to \mathrm{Hom}(M,N) \to &\mathrm{Hom}(\mathcal{D},M) \to \mathrm{Hom}(\mathcal{D},M) \\ &\to \mathrm{Ext}^1(M,N) \to \mathrm{Ext}^1(\mathcal{D},M) \to \mathrm{Ext}^1(\mathcal{D},M) \to \cdots \\ &\cdots \to \mathrm{Ext}^i(M,N) \to \mathrm{Ext}^i(\mathcal{D},M) \to \mathrm{Ext}^i(\mathcal{D},M) \to \cdots \end{aligned}$$

Since $\mathcal{D}$ is free, $\mathrm{Ext}^i(\mathcal{D},M) = 0$ for $i \geq 1$ and the portion $\mathrm{Ext}^{i-1}(\mathcal{D},M) \to \mathrm{Ext}^i(M,N) \to \mathrm{Ext}^i(\mathcal{D},M)$ of the long exact sequence gives the first conclusion.

We are left to identify the portion $\mathrm{Hom}(\mathcal{D},M) \to \mathrm{Hom}(\mathcal{D},M)$. Of course,

$$x \mapsto \lambda_x := \big(Q \mapsto Q.x = Q(\Psi)(x)\big) \text{ and } f \mapsto f(1)$$

are isomorphisms from M to $\mathrm{Hom}(\mathcal{D},M)$ and return reciprocal to each other. Then, the map to be identified is $f \mapsto f \circ (\times P)$, where, of course, $\times P$ denotes the map $Q \mapsto QP$. Conjugating it with our isomorphisms yields $x \mapsto \big(\lambda_x \circ (\times P)\big)(1) = (P.1)(\Psi)(x)$, that is, $P(\Psi)$. □

Remark 2.3.17. — The complex $F \overset{P(\Psi)}{\to} F$ (in degrees 0 and 1) thus has cohomology $\mathrm{Hom}(M, N)$, $\mathrm{Ext}^1(M, N)$. This is clearly functorial in N. On the other hand, there is a part of arbitrariness in the choice of P, but, after [**10**] §6.1, the homotopy class of the complex depends on M alone.

Corollary 2.3.18. — *Let* $M = (E, \Phi)$. *The* complex of solutions of M*:*

$$E \overset{\Phi - Id}{\longrightarrow} E$$

has cohomology $\Gamma(M)$*,* $\Gamma^1(M)$*.*

Proof. — In the proposition, take $M := \underline{1}$, $P := \sigma_q - 1$ and $N := (E, \Phi)$. □

Note that this is functorial in M.

Remark 2.3.19. — Applying the corollary to $M^\vee \otimes N$ implies that the map $f \mapsto \Psi \circ f \circ \Phi^{-1} - f$ has kernel $\mathrm{Hom}(M, N)$ (which is obvious) and that its cokernel is in one-to-one correspondance with $\mathrm{Ext}^1(M, N)$, which is similar to the conclusion of theorem 2.3.3. However, in this way, we do not get the identification of the operations on extensions: this would be possible using [**10**, §6.3].

Corollary 2.3.20. — *Let* $M = \mathcal{D}/\mathcal{D}P$. *Then, for any dual* $P^\vee$ *of* P*, the complex:*

$$K \overset{P^\vee(\sigma)}{\longrightarrow} K$$

has cohomology $\Gamma(M)$*,* $\Gamma^1(M)$*.*

Proof. — In the proposition, take $N := \underline{1} = (K, \sigma)$. This gives $\Gamma(M^\vee)$, $\Gamma^1(M^\vee)$ as cohomology of $K \overset{P(\sigma)}{\longrightarrow} K$. Now, replace M by $M^\vee$. (Of course, no functoriality here!) □

Example 2.3.21. — For $u \in K^* = \mathrm{GL}_1(K)$, where $K = \mathbf{C}(\{z\})$ or $\mathbf{C}((z))$, put $M_u := (K, \Phi_u) = \mathcal{D}_q/\mathcal{D}_q(\sigma_q - u^{-1})$. A dual of $\sigma_q - u^{-1}$ is, for instance, $\sigma_q - u$. One has $M_u^\vee = M_{u^{-1}}$ and $M_u \otimes M_v = M_{uv}$.
From the corollaries above, we draw that $K \overset{u^{-1}\sigma_q - 1}{\longrightarrow} K$, resp. $K \overset{\sigma_q - u}{\longrightarrow} K$ has cohomology $\Gamma(M_u)$, $\Gamma^1(M_u)$. From the proposition, we draw that $K \overset{v^{-1}\sigma_q - u^{-1}}{\longrightarrow} K$ has cohomology $\mathrm{Hom}(M_u, M_v)$, $\mathrm{Ext}^1(M_u, M_v)$.

2.4. The cohomological equation

We here prepare the grounds for the examples of chapter 7 by giving some practical recipes.

2.4.1. Some inhomogeneous equations. — Like in the theory of linear differential equations, many interesting examples come in dimension 2. So let $a, b \in K^*$ and $u \in K$. Extensions of $M_b := (K, \Phi_b)$ by $M_a := (K, \Phi_a)$ have form $N_{a,b,u} := (K^2, \Phi_{A_u})$ where $A_u := \begin{pmatrix} a & u \\ 0 & b \end{pmatrix}$. An isomorphism of extensions from $N_{a,b,u}$ to $N_{a,b,v}$ would be a matrix $F := \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}$ such that $b\sigma f - af = v - u$. More generally, the space $\mathrm{Ext}(M_b, M_a)$ is isomorphic to the cokernel of the C-linear map $b\sigma - a$ from K to itself. We shall call *cohomological equation* the following first order inhomogeneous equation:

$$b\sigma f - af = u. \tag{7}$$

This can be seen as the obstruction to finding an isomorphism F from A_0 to A_u.

More generally, let $L := a_0 + \cdots + a_n\sigma^n \in \mathcal{D}$. By vectorializing the corresponding equation we get a system with matrix:

$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0/a_n & -a_1/a_n & -a_2/a_n & -\ldots & -a_{n-1}/a_n \end{pmatrix}.$$

Actually, $M := (K^n, \Phi_A) \simeq (\mathcal{D}/\mathcal{D}L)^\vee$. From corollary 2.3.20, we deduce:

$$\mathrm{Coker}\ L \simeq \Gamma^1(M) \simeq \mathrm{Ext}(\underline{1}, M).$$

This can be seen as an obstruction to finding an isomorphism $\begin{pmatrix} I_n & X \\ 0 & 1 \end{pmatrix}$ from $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ to $\begin{pmatrix} A & U \\ 0 & 1 \end{pmatrix}$. In fact, an isomorphism $\begin{pmatrix} I_n & X \\ 0 & 1 \end{pmatrix}$ from $\begin{pmatrix} A & U \\ 0 & 1 \end{pmatrix}$ to $\begin{pmatrix} A & V \\ 0 & 1 \end{pmatrix}$ would correspond to a solution of $\sigma X - AX = V - U$. Assuming for instance $a_n = 1$ and writing x_i the components of X, this gives the equations $\sigma x_i - x_i = v_i - u_i$ $(i = 1, \ldots, n-1)$ and $\sigma x_n + a_0 x_1 + \cdots + a_{n-1}x_n = v_n - u_n$. From this, one can solve trivially to get an equivalent of U with components $0,\dots,0,u$ (exercice for the reader). And, if U has this form, finding an isomorphism $\begin{pmatrix} I_n & X \\ 0 & 1 \end{pmatrix}$ from $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ to $\begin{pmatrix} A & U \\ 0 & 1 \end{pmatrix}$ is equivalent to solving $Lx_1 = u$. Still more generally, it is easy to see that finding an isomorphism $\begin{pmatrix} I_n & X \\ 0 & I_p \end{pmatrix}$ from $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ to $\begin{pmatrix} A & U \\ 0 & B \end{pmatrix}$ is equivalent to solving $(\sigma X)B - AX = U$, and to obtain anew an identification of the cokernel of $X \mapsto (\sigma X)B - AX$ with $\operatorname{Ext}\big((K^p,\Phi_B),(K^n,\Phi_A)\big)$.

2.4.2. A homotopy. — The following is intended to be an explanation of the equivalence of various computations above. Let (K,σ) be a difference field with constant field C. The σ-difference operator $P := \sigma^n + a_1\sigma^{n-1} + \cdots + a_n$ gives rise to a C-linear complex $K \xrightarrow{P} K$. Writing A_P the companion matrix described in 2.1.2.1, we also have a complex $K^n \xrightarrow{\Delta} K^n$, where we have set $\Delta := \sigma - A_P$. We then have a morphism of complexes:

$$\begin{array}{ccc} K & \xrightarrow{P} & K \\ {\scriptstyle V}\big\downarrow & & {\scriptstyle I}\big\downarrow \\ K^n & \xrightarrow{\Delta} & K^n \end{array}, \text{ where } V(f) := \begin{pmatrix} f \\ \sigma_q f \\ \vdots \\ \sigma_q^{n-1} f \end{pmatrix} \text{ and } I(g) := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ g \end{pmatrix}.$$

(The equality $\Delta \circ V = I \circ P$ is obvious.) We now introduce the operators $P_i := \sigma^{n-i} + a_1\sigma^{n-i-1} + \cdots + a_{n-i}$ (they are related to the Horner scheme for P). We then have a morphism of complexes in the opposite direction:

$$\begin{array}{ccc} K & \xrightarrow{P} & K \\ {\scriptstyle \pi_1}\big\uparrow & & {\scriptstyle \Pi}\big\uparrow \\ K^n & \xrightarrow{\Delta} & K^n \end{array}, \text{ where } \pi_1 \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} := f_1 \text{ and } \Pi \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} := \sum_{i=1}^n P_i g_i.$$

(Of course, one must check that $\Pi \circ \Delta = P \circ \pi_1$.) Clearly $(\pi_1,\Pi) \circ (V,I)$ is the identity of the first complex. We are going to see that $(V,I) \circ (\pi_1,\Pi)$ is homotopic to the identity of the second complex, whence their homological equivalence (see [**10**, §2.4, def. 4,5 and prop. 5]). To that end, we introduce a backward operator $K^n \xleftarrow{\Delta'} K^n$ defined by the following relation:

$$\Delta' \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} := \begin{pmatrix} g'_1 \\ \vdots \\ g'_n \end{pmatrix}, \text{ where } g'_i := \sum_{j+k=i-1} \sigma^j g_k \Longrightarrow \begin{cases} V \circ \pi_1 - \mathrm{Id}_{K^n} = \Delta' \circ \Delta, \\ I \circ \Pi - \mathrm{Id}_{K^n} = \Delta \circ \Delta'. \end{cases}$$

(The computations are mechanical and again left to the reader.) This implies that the two complexes are indeed homotopic.

CHAPTER 3

THE AFFINE SPACE OF ISOFORMAL ANALYTIC CLASSES

3.1. Isoformal analytic classes of analytic q-difference modules

We shall now specialize section 2.3, in particular subsection 2.3.6, to the case of q-difference modules.

3.1.1. Extension classes of analytic q-difference modules. — From here on, we consider only analytic q-difference modules and the base field is $\mathbf{C}(\{z\})$ (except for brief indications about the formal case).

Theorem 3.1.1. — *Let M, N be pure modules of ranks $r, s \in \mathbf{N}^*$ and slopes $\mu < \nu \in \mathbf{Q}$. Then $\dim_{\mathbf{C}} \mathcal{F}(M,N) = rs(\nu - \mu)$.*

Proof. — Since $\mathcal{F}(M,N) \simeq \mathrm{Ext}^1(N,M) \simeq \Gamma^1(N^\vee \otimes M)$, and since $N^\vee \otimes M$ is pure isoclinic of rank rs and slope $\mu - \nu < 0$, the theorem is an immediate consequence of the following lemma. □

Lemma 3.1.2. — *Let M be a pure module of ranks r and slope $\mu < 0$. Then $\dim_{\mathbf{C}} \Gamma^1(M) = -r\mu$.*

Proof. — We give four different proofs, of which two require that $\mu \in \mathbf{Z}$.

1) Write $d := -r\mu \in \mathbf{N}^*$. If $M = \mathcal{D}_q/\mathcal{D}_q P$, the module $M^\vee$ is pure of rank r and slope $-\mu$, and can be written as $\mathcal{D}_q/\mathcal{D}_q P^\vee$ for some dual $P^\vee = a_0 + \cdots + a_r \sigma_q^r$ of P, such that $v_0(a_0) = 0$, $v_0(a_r) = d$ and $v_0(a_i) \geq id/r$ for all i (lemma 2.2.3). We want to apply proposition 2.5 of [**6**], but the latter assumes $|q| < 1$, so we consider $L := P^\vee \sigma_q^{-r} = b_0 + \cdots + b_r \sigma^r$, where $\sigma := \sigma_q^{-1}$ and $b_i := a_{r-i}$. After *loc.cit*, the operator $L : \mathbf{C}\{z\} \to \mathbf{C}\{z\}$ has index d. More generally, for all $m \in \mathbf{N}^*$, the operator $L : z^{-m}\mathbf{C}\{z\} \to z^{-m}\mathbf{C}\{z\}$ is conjugate to $z^m L z^{-m} = \sum q^{im} b_i \sigma^i : \mathbf{C}\{z\} \to \mathbf{C}\{z\}$, which also has index d. Hence $L : \mathbf{C}(\{z\}) \to \mathbf{C}(\{z\})$ has index d, and so has $P^\vee$. After corollary

2.3.20, this index is $\dim_{\mathbf{C}} \Gamma^1(M) - \dim_{\mathbf{C}} \Gamma(M)$. But, M being pure isoclinic of non null slope, $\Gamma(M) = 0$, which ends the first proof.

2) In the following proof, we assume $\mu \in \mathbf{Z}$. From theorem 1.2.3 of [**44**], we know that we may choose the dual operator such that:

$$P^{\vee} = (z^{-\mu}\sigma_q - c_1)u_1 \cdots (z^{-\mu}\sigma_q - c_r)u_r,$$

where $c_1, \ldots, c_r \in \mathbf{C}^*$ and $u_1, \ldots, u_r \in \mathbf{C}\{z\}$, $u_1(0) = \cdots = u_r(0) = 1$. We are thus left to prove that each $z^{-\mu}\sigma_q - c_1$ has index $-\mu$: the indexes add up and the sum will be $-r\mu = d$. Since the kernels are trivial, we must compute the cokernel of an operator $z^m\sigma_q - c$, $m \in \mathbf{N}^*$. But it follows from lemma 3.1.3 that the image of $z^m\sigma_q - c : \mathbf{C}(\{z\}) \to \mathbf{C}(\{z\})$ admits the supplementary space $\mathbf{C} \oplus \cdots \oplus \mathbf{C}z^{m-1}$.

3) In the following proof, we assume again $\mu \in \mathbf{Z}$. After lemma 2.2.3, we can write $M = (\mathbf{C}(\{z\})^r, \Phi_{z^\mu A})$, $A \in \mathrm{GL}_r(\mathbf{C})$. After corollary 2.3.18, we must consider the cokernel of $\mathbf{C}(\{z\})^r \overset{\Phi_{z^\mu A} - \mathrm{Id}}{\to} \mathbf{C}(\{z\})^r$. But, after lemma 3.1.3, the image of $\Phi_{z^\mu A} - \mathrm{Id}$ admits the supplementary space $(\mathbf{C} \oplus \cdots \oplus \mathbf{C}z^{-\mu-1})^r$.

4) A similar proof, but for arbitrary μ, can be deduced from [**51**]. It follows indeed from this paper that each isoclinic module of slope μ can be obtained by successive extensions of modules admitting a dual of the form $\mathcal{D}_q/\mathcal{D}_q(z^a\sigma_q^b - c)$, where $b/a = -\mu$ and $c \in \mathbf{C}^*$. □

Lemma 3.1.3. — *Let $d, r \in \mathbf{N}^*$ and $A \in GL_r(\mathbf{C})$. Let $D \subset \mathbf{Z}$ be any set of representatives modulo d, for instance $\{a, a+1, \ldots, a+d-1\}$ for some $a \in \mathbf{Z}$. Then, the image of the $\mathbf{C}$-linear map $F : X \mapsto z^d A\sigma_q X - X$ from $\mathbf{C}(\{z\})^r$ to itself admits as a supplementary $\left(\sum\limits_{i \in D} \mathbf{C}z^i\right)^r$.*

Proof. — For all $i \in \mathbf{Z}$, write $K_i := z^i\mathbf{C}(\{z^d\})$, so that $\mathbf{C}(\{z\}) = \bigoplus_{i \in D} K_i$. Each of the K_i^r is stable under F. We write $w := z^d$, $L := \mathbf{C}(\{w\})$, $\rho := q^d$ and define σ on L by $\sigma f(w) = f(\rho w)$. Multiplication by z^i sends L^r to K_i^r and conjugates the restriction of F to K_i^r to the mapping $G_i : Y \mapsto wq^i A\sigma Y - Y$ from L^r to itself. We are left to check that the image of G_i admits $\mathbf{C}^r$ as a supplementary. But this is just the case $d = 1$, $D = \{0\}$ of the lemma. So we tackle this case under these assumptions with the notations of the lemma.

So write $F_A : X \mapsto zA\sigma_q X - X$ from $\mathbf{C}(\{z\})^r$ to itself. Also write $X = \sum X_n z^n$ and $Y = F_A(X) = \sum Y_n z^n$ (all sums here have at most a finite number of negative indices), so that $Y_n = q^{n-1}AX_{n-1} - X_n$. Putting $\tilde{X} := \sum A^{-n}X_n z^n$ and $\tilde{Y} := \sum A^{-n}Y_n z^n$, from the relation $A^{-n}Y_n = q^{n-1}A^{-(n-1)}X_{n-1} - A^{-n}X_n$, we draw $\tilde{Y} = z\sigma_q\tilde{X} - X$. Of course, $X \mapsto \tilde{X}$ is an automorphism of $\mathbf{C}(\{z\})^r$ and we just saw that it conjugates F_A to the map $F : X \mapsto z\sigma_q X - X$ from $\mathbf{C}(\{z\})^r$ to itself. Moreover, the same automorphism leaves $\mathbf{C}^r \subset \mathbf{C}(\{z\})^r$, so the question boils down to prove that the image of F has supplementary $\mathbf{C}^r$. And this, in turn, splits into r times the same problem for $r = 1$.

So we are left with the case of the map $f \mapsto z\sigma_q f - f$ from $\mathbf{C}(\{z\})$ to itself, which is the very heart of the theory! To study the equation $z\sigma_q f - f = g$, one introduces the *q-Borel transform of level* 1, defined by the formula:

$$\mathcal{B}_{q,1}\left(\sum f_n z^n\right) = \sum \frac{f_n}{q^{n(n-1)/2}} z^n.$$

For properties of $\mathcal{B}_{q,1}$ see [**33**]. It sends $\mathbf{C}(\{z\})$ isomorphically to a subspace of the space $\mathcal{O}(\mathbf{C})[z^{-1}]$, namely, the $\mathbf{C}$-vector space:

$$\mathbf{C}(\{z\})_{q,1} := \left\{\sum f_n z^n \in \mathbf{C}(\{z\}) \mid f_n \prec q^{-n(n-1)/2}\right\}.$$

Here, $u_n \prec v_n$ means: $u_n = O(A^n v_n)$ for some $A > 0$. Introduction of the q-Borel transform is motivated by the following easily checked property:

$$\begin{aligned} z\sigma_q f - f = g &\iff \forall n\ ,\ q^{n-1} f_{n-1} - f_n = g_n \\ &\iff \forall n\ ,\ \frac{f_{n-1}}{q^{(n-1)(n-2)/2}} - \frac{f_n}{q^{n(n-1)/2}} = \frac{g_n}{q^{n(n-1)/2}} \\ &\iff (z-1)\mathcal{B}_{q,1} f = \mathcal{B}_{q,1} g. \end{aligned}$$

We therefore get a commutative diagram:

$$\begin{array}{ccc} \mathbf{C}(\{z\}) & \xrightarrow{1 - z\sigma_q} & \mathbf{C}(\{z\}) \\ \downarrow{\scriptstyle \mathcal{B}_{q,1}} & & \downarrow{\scriptstyle \mathcal{B}_{q,1}} \\ \mathbf{C}(\{z\})_{q,1} & \xrightarrow{\times(1-z)} & \mathbf{C}(\{z\})_{q,1} \end{array}$$

The vertical arrows are isomorphisms and the right one sends $\mathbf{C} \subset \mathbf{C}(\{z\})$ onto $\mathbf{C} \subset \mathbf{C}(\{z\})_{q,1}$. We are going show that the image of the lower horizontal arrow is $\{\phi \in \mathbf{C}(\{z\})_{q,1} \mid \phi(1) = 0\}$. (The fact that $\times(1-z)$ does send $\mathbf{C}(\{z\})_{q,1}$ into itself is easily checked.) It will then follow that the image of the upper horizontal arrow is $\{g \in \mathbf{C}(\{z\}) \mid \mathcal{B}_{q,1} g(1) = 0\}$, that it admits $\mathbf{C}$ as a supplementary and that the corresponding projection operators are $g \mapsto \mathcal{B}_{q,1} g(1)$ and $g \mapsto g - \mathcal{B}_{q,1} g(1)$; thus, the only *obstruction* in solving $z\sigma_q f - f = g$ analytically is $\mathcal{B}_{q,1} g(1)$ (solving formally is always possible).

It is obvious that $\phi(1)$ makes sense for any $\phi \in \mathbf{C}(\{z\})_{q,1}$ and so that any ϕ in the image satisfies $\phi(1) = 0$. If $\phi(1) = 0$, it is also clear that

$\psi := \dfrac{\phi}{1-z} \in \mathcal{O}(\mathbf{C})[z^{-1}]$, and we are left to prove that $\psi \in \mathbf{C}(\{z\})_{q,1}$. So we write $\phi = \sum a_n z^n$, $\psi = \sum b_n z^n$, so that we have $\sum a_n = 0$ and:

$$b_n = \sum_{p \leq n} a_p = -\sum_{p>n} a_p.$$

Suppose that $|a_n| \leq CA^n \, |q|^{-n(n-1)/2}$ for all n, where $C, A > 0$. Then:

$$|b_n| \leq C \sum_{p>n} A^p \, |q|^{-p(p-1)/2} \leq CA^n \, |q|^{-n(n-1)/2} \sum_{\ell>0} A^\ell \, |q|^{-\ell(2n+\ell-1)/2} \leq C' A^n \, |q|^{-n(n-1)/2},$$

where $C' := \sum\limits_{\ell>0} A^\ell \, |q|^{-\ell(\ell-1)/2} < \infty$. This achieves the proof. □

Note that from the proof, one draws explicit projection maps from $\mathbf{C}(\{z\})^r$ to the image of F_A and its supplementary $\mathbf{C}^r$: these are the maps $Y \mapsto Y - \mathcal{B}_{q,1}Y(A^{-1})$ and $Y \mapsto \mathcal{B}_{q,1}Y(A^{-1})$, where:

$$\mathcal{B}_{q,1}Y(A^{-1}) := \sum q^{-n(n-1)/2} A^{-n} Y_n.$$

3.1.2. The affine scheme $\mathcal{F}(P_1, \ldots, P_k)$. — Now let $P_1, \ldots, P_k$ be pure isoclinic q-difference modules over $\mathbf{C}(\{z\})$, of ranks $r_1, \ldots, r_k \in \mathbf{N}^*$ and slopes $\mu_1 < \cdots < \mu_k$. From section 2.3, we have a functor from commutative $\mathbf{C}$-algebras to sets:

$$C \rightsquigarrow F(C) := \mathcal{F}(C \otimes_{\mathbf{C}} P_1, \ldots, C \otimes_{\mathbf{C}} P_k).$$

Here, the base change $C \otimes_{\mathbf{C}} -$ means that we extend the scalars of q-difference modules from $\mathbf{C}(\{z\})$ to $C \otimes_{\mathbf{C}} \mathbf{C}(\{z\})$.

Theorem 3.1.4. — *The functor F is representable and the corresponding affine scheme is an affine space over $\mathbf{C}$ with dimension:*

$$\dim \mathcal{F}(P_1, \ldots, P_k) = \sum_{1 \leq i < j \leq k} r_i r_j (\mu_j - \mu_i).$$

Proof. — We apply theorem 2.3.11. We need two check the assumptions:

$$\forall i, j \text{s.t.} 1 \leq i < j \leq k \text{ , } \mathrm{Hom}(P_j, P_i) = 0 \text{ and } \dim_{\mathbf{C}} \mathrm{Ext}(P_j, P_i) = r_i r_j (\mu_j - \mu_i).$$

The first fact comes from theorem 2.2.4 and the second fact from theorem 3.1.1; the fact that the extension modules are free is here obvious since $\mathbf{C}$ is a field. □

Remark 3.1.5. — The use of base changes $C \otimes_{\mathbf{C}} \mathbf{C}(\{z\})$ is too restrictive to consider this as a true space of moduli: we should like to consider families of modules the coefficients of which are arbitrary analytic functions of some parameter. This will be done in a further work.

Now, to get coordinates on our affine scheme, we just have to find bases of the extension modules $\mathrm{Ext}(P_j, P_i)$ for $1 \leq i < j \leq k$: this follows indeed from corollary 2.3.15. An example will be given by corollary 3.3.7.

3.2. Index theorems and irregularity

We shall now give some complements to the results in 3.1.1, in the form of index theorems. They mostly originate in the works [**6**] of Bézivin and also in [**33**]. Although not strictly necessary for what follows, they allow for an interpretation of $\dim \mathcal{F}(P_1, \ldots, P_k)$ (theorem 3.1.4) in terms of "irregularity", as in the paper [**26**] of Malgrange.

In most of this section, we write K for (indifferently) $\mathbf{C}((z))$ or $\mathbf{C}(\{z\})$ and respectively speak of the *formal* or *convergent* case. We intend to compute the index of an analytic q-difference operator $P \in \mathcal{D}_q$ acting upon $\mathbf{C}((z))$ and $\mathbf{C}(\{z\})$ (and, in the end, upon $\mathbf{C}((z))/\mathbf{C}(\{z\})$). To begin with, we do not assume P to have analytic coefficients.

3.2.1. Kernel and cokernel of $\sigma_q - u$.

— We start with the case $\deg P = 1$. Up to an invertible factor in $\mathcal{D}_q$, we may assume that $P = \sigma_q - u$, where $u = dq^k z^\nu v$, $v \in K$, $v(0) = 1$, with $k \in \mathbf{Z}$ and $d \in \mathbf{C}^*$ such that $1 \leq |d| < |q|$. We consider P as a $\mathbf{C}$-linear operator on K.

Fact. — *The dimensions over* $\mathbf{C}$ *of the kernel and cokernel of* P *depend only of the class of* $u \in K^*$ *modulo the subgroup* $\{\frac{\sigma_q(w)}{w} \mid w \in K^*\}$ *of* K^*. *This class is equal to that of à* dz^ν.

Proof. — Conjugating the $\mathbf{C}$-linear endomorphism $\sigma_q - u$ of K by $w \in K^*$ (undestood as the automorphism $\times w$), one finds:

$$w \circ (\sigma_q - u) \circ w^{-1} = \frac{w}{\sigma_q(w)} \circ (\sigma_q - u'), \text{ with } u' = u\frac{\sigma_q(w)}{w},$$

whence the first statement. Then, $q^k v = \frac{\sigma_q(w)}{w}$, where $w = z^k \prod_{i \geq 1} \sigma_q^{-i}(v) \in K$ (also well defined in the convergent case), whence the second statement. □

So we now set $u := dz^\nu$.

Fact. — *The kernel of $\sigma_q - u : K \to K$ has dimension* 1 *if* $(\nu, d) = (0, 1)$, *and* 0 *otherwise.*

Proof. — The series $f = \sum f_n z^n$ belongs to the kernel if, and only if $\forall n\ ,\ q^n f_n = d f_{n-\nu}$. Since we deal with Laurent series having poles, this implies $f = 0$ except maybe if $\nu = 0$. In the latter case, it implies $f = 0$, except maybe if $d \in q^{\mathbf{Z}}$. From the condition on $|d|$, this is only possible if $d = 1$, thus $u = 1$. In that case, the kernel is plainly $\mathbf{C}$. □

Fact. — *The map $\sigma_q - u : K \to K$ is onto if $\nu > 0$ and also if $\nu = 0,\ d \neq 1$.*

Proof. — Let $g \in K$. We look for f solving the equation $\sigma_q f - d z^\nu f = g$ in K. This is equivalent to: $\forall n\ ,\ q^n f_n - d f_{n-\nu} = g_n$, that is, $\forall n\ ,\ f_n = q^{-n}(d f_{n-\nu} + g_n)$. If $\nu \geq 1$, one computes the coefficients by induction from the ν first among them. Moreover, in the convergent case, inspection of the denominators show that f converges if g does; and then, the functional equation ensures meromorphy. If $\nu = 0$ and $d \neq 1$ (so that $d \notin q^{\mathbf{Z}}$), one gets rightaway $f_n = \frac{g_n}{q^n - d}$. Convergence (resp. meromorphy) is then immediate in the convergent case. (When $\nu \geq 1$, one can also consider the fixpoint equation $F(f) = f$, where $F(f) := \sigma_q^{-1}(g + d z^\nu f)$: it is easy to see that this is a contracting operator as well for the formal as for the transcendant topology.) □

Fact. — *If $(\nu, d) = (0, 1)$, the cokernel of $\sigma_q - u : K \to K$ has dimension* 1.

Proof. — One checks that $\sigma_q - u$ vanishes on $\mathbf{C}$ and induces an automorphism of the subspace $K^\bullet$ of K made up of series without constant term. □

Fact. — *Assume $\nu < 0$. Then $\sigma_q - u : K \to K$ is onto in the formal case.*

Proof. — Put $F'(f) = d^{-1} z^{-\nu}(\sigma_q(f) - g)$. Since $\nu > 0$, this is a z-adically contracting operator, whence the existence (and unicity) of a fixed point. □

Remark 3.2.1. — This is the first place where the formal and convergent cases differ. The operator is (rather strongly) *expanding* for the transcendant topology, it produces Stokes phenomena ! So this is where we need an argument from analysis in the convergent case.

Fact. — *Assume $\nu = -r, r \in \mathbf{N}^*$. Then, in the convergent case, the cokernel of $\sigma_q - u : K \to K$ has dimension r.*

Proof. — This is a consequence of lemma 3.1.2 (and therefore relies on the use of the q-Borel transformation). □

We now summarize our results:

Proposition 3.2.2. — *Let $u = dz^\nu v$, $v \in K$, $v(0) = 1$, with $d \in \mathbf{C}^*$. Write $\overline{d}$ the class of d modulo $q^{\mathbf{Z}}$. The following table shows the ranks of the kernel and cokernel, as well as the index $\chi(P) := \dim Ker\ P - \dim Coker\ P$ of the $\mathbf{C}$-linear operator $P := \sigma_q - u : K \to K$:*

$(\nu, \overline{d})$	*Kernel*	*Cokernel*	*Index*
$(0, 1)$	1	1	0
$(0, \neq 1)$	0	0	0
$(> 0, -)$	0	0	0
$(< 0, -)$	0	$\begin{cases} 0 \text{ (formal case)} \\ -\nu \text{ (convergent case)} \end{cases}$	$\begin{cases} 0 \text{ (formal case)} \\ \nu \text{ (convergent case)} \end{cases}$

3.2.2. The index in the general case. — Generally speaking, there is no simple formula for the dimensions of the kernel and cokernel of a q-difference operator P. However, if P has integral slopes, we can factor it into operators of degree 1, and then use proposition 3.2.2 and linear algebra to deduce:

1. The $\mathbf{C}$-linear map $P : f \mapsto P.f$ from K to itself has an index, that is finite dimensional kernel and cokernel.
2. The index of P, that is the integer $\chi(P) := \dim \mathrm{Ker}\ P - \dim \mathrm{Coker}\ P$ is the sum of indices of factors of P.

Note that $\chi(P)$ is the Euler-Poincaré characteristic of the complex of solutions of P.

Corollary 3.2.3. — *Let P be an operator of order n and pure of slope $\mu \neq 0$.*
(i) In the formal case, $\dim Ker\ P = \dim Coker\ P = 0$.
(ii) In the convergent case $\dim Ker\ P = 0$ *and* $\dim Coker\ P = n \max(0, \mu)$.

Proof. — If $\mu \in \mathbf{Z}$, all the factors of P have the form $(z^\mu \sigma_q - c).u$. If $\mu \neq 0$, they are all bijective in the formal case, whence the first statement. They are also all injective in the convergent case, and onto if $\mu < 0$, whence the second statement in this case. If $\mu > 0$, they all have index μ (in the convergent case), whence the second statement in this case by additivity of the index.
The case of an arbitrary slope is easily reduced to this case by ramification. □

By similar arguments:

Corollary 3.2.4. — *Let P be an operator pure of slope* 0. *Then, the index of P (in the formal or convergent case) is* 0.

Now, using the additivity of the index:

Corollary 3.2.5. — *The index of an arbitrary operator P is 0 in the formal case and $-\sum\limits_{\mu>0} r_P(\mu)\mu$ in the convergent case.*

Corollary 3.2.6. — *Let $P \in \mathcal{D}_q$ be an analytic q-difference operator. The index of P acting as a $\mathbf{C}$-linear endomorphism of $\mathbf{C}((z))/\mathbf{C}(\{z\})$ is equal to $\sum\limits_{\mu>0} r_P(\mu)\mu$.*

Of course, r_P denotes here the Newton function of P introduced in subsection 2.2.1.

3.2.3. Irregularity and the dimension of $\mathcal{F}(P_1,\dots,P_k)$. — The following definition is inspired by that of [**26**].

Definition 3.2.7. — The *irregularity* of a q-difference operator P, resp. of q-difference module M, is defined by the formulas:

$$\mathrm{Irr}\ (P) := \sum_{\mu>0} r_P(\mu)\mu, \quad \mathrm{Irr}\ (M) := \sum_{\mu>0} r_M(\mu)\mu.$$

Graphically, this is the height of the right part of the Newton polygon, from the bottom to the upper right end. It is clearly a formal invariant. From its interpretation as an index in $\mathbf{C}((z))/\mathbf{C}(\{z\})$, the irregularity of operators is additive with respect to the product. From theorem 2.2.1, the irregularity of modules is additive with respect to exact sequences. Moreover, with the notations of 2.2.4, writing

$$M^{>0} := M/M_{\leq 0}$$

the "positive part" of M, one has:

$$\mathrm{Irr}\ (M) = \mathrm{Irr}\ (M^{>0}).$$

Write $M_0 := P_1 \oplus \cdots \oplus P_k$. Then the "internal End" $M := \underline{\mathrm{End}}(M_0) = M_0^\vee \otimes M_0$ of this module (subsection 2.1.1) has as Newton function:

$$r_M(\mu) = \sum_{\mu_j-\mu_i=\mu} r_i r_j.$$

(This follows from theorem 2.2.1.) From theorem 3.1.4 and definition 3.2.7, we therefore get the equality:

$$\dim \mathcal{F}(P_1,\dots,P_k) = \mathrm{Irr}\ \left(\underline{\mathrm{End}}(M_0)\right).$$

In subsection 6.3.2, we shall give a sheaf theoretical interpretation of this formula.

3.3. Explicit description of $\mathcal{F}(P_1,\dots,P_k)$ in the case of integral slopes

FROM NOW ON, WE ASSUME THAT THE SLOPES ARE INTEGRAL: $\mu_1,\dots,\mu_k \in \mathbf{Z}$.

It is then possible to make this result more algorithmic by using the matricial description of paragraph 2.2.4.2. Indeed, as explained in paragraph 1.1.3.1, our construction of normal forms and of explicit Stokes operators depends on the normal form (8) given herebelow for pure modules.

For $i = 1,\dots,k$, we thus write $P_i = (\mathbf{C}(\{z\})^{r_i}, \Phi_{z^{\mu_i}A_i})$, with $A_i \in \mathrm{GL}_{r_i}(\mathbf{C})$. Then, putting $n := r_1 + \cdots + r_k$, we have $M_0 := P_1 \oplus \cdots \oplus P_k = (K^n, \Phi_{A_0})$, with A_0 as in equation (5):

$$A_0 := \begin{pmatrix} z^{\mu_1}A_1 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 0 & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & z^{\mu_k}A_k \end{pmatrix}. \tag{8}$$

Any class in $\mathcal{F}(P_1,\dots,P_k)$ can be represented by a module $M_U := (K^n, \Phi_{A_U})$ for some $U := (U_{i,j})_{1\le i<j\le k} \in \prod\limits_{1\le i<j\le k} \mathrm{Mat}_{r_i,r_j}(\mathbf{C}(\{z\}))$, with:

$$A_U := \begin{pmatrix} z^{\mu_1}A_1 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & U_{i,j} & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & z^{\mu_k}A_k \end{pmatrix}, \tag{9}$$

and M_U, M_V represent the same class if, and only if, the equation $(\sigma_q F)A_V = A_U F$ can be solved with $F \in \mathfrak{G}(\mathbf{C}(\{z\}))$. Such an F is then unique.

3.3.1. Analytic classification with 2 integral slopes. — From the results of 3.1.1, we get the following result for the case of two slopes. Here, and after, we write, for $\mu < \nu \in \mathbf{Z}$:

$$K_{\mu,\nu} := \sum_{i=0}^{\nu-\mu-1} \mathbf{C}z^i \text{ or, at will: } K_{\mu,\nu} := \sum_{i=\mu}^{\nu-1} \mathbf{C}z^i.$$

(The second choice is motivated by good tensor properties, see for instance [**32**].) Then, for $r, s \in \mathbf{N}^*$, we denote $\mathrm{Mat}_{r,s}(K_{\mu,\nu})$ the space of $r \times s$ matrices with coefficients in $K_{\mu,\nu}$.

Proposition 3.3.1. — *Let $U \in Mat_{r_1,r_2}(\mathbf{C}(\{z\}))$. Then, there exists a unique pair:*

$$Red(\mu_1, A_1, \mu_2, A_2, U) := (F_{1,2}, V) \in Mat_{r_1,r_2}(\mathbf{C}(\{z\})) \times Mat_{r_1,r_2}(K_{\mu_1,\mu_2})$$

such that:

$$(\sigma_q F_{1,2}) z^{\mu_2} A_2 - z^{\mu_1} A_1 F_{1,2} = U - V,$$

that is:

$$\begin{pmatrix} I_{r_1} & F_{1,2} \\ 0 & I_{r_2} \end{pmatrix} : \begin{pmatrix} z^{\mu_1} A_1 & V \\ 0 & z^{\mu_2} A_2 \end{pmatrix} \simeq \begin{pmatrix} z^{\mu_1} A_1 & U \\ 0 & z^{\mu_2} A_2 \end{pmatrix}.$$

Proof. — Using the reductions of lemma 3.1.2, this is just a rephrasing of lemma 3.1.3. □

We consider V as a *polynomial normal form* for the class of M_U and $F_{1,2}$ as the corresponding reduction datum. Of course, they depend on the choice of the space of coefficients K_{μ_1,μ_2}.

Example 3.3.2. — This is the archetypal example, from which much of the theory is built. Fix $c \in \mathbf{C}^*$. For any $f, u, v \in \mathbf{C}(\{z\})$, the isomorphism: $\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} : \begin{pmatrix} 1 & v \\ 0 & cz \end{pmatrix} \simeq \begin{pmatrix} 1 & u \\ 0 & cz \end{pmatrix}$ is equivalent to the inhomogeneous q-difference equation: $cz\sigma_q f - f = u - v$. Writing $f = \sum f_n z^n$, etc, we find the conditions:

$$cq^{n-1} f_{n-1} - f_n = u_n - v_n \Longleftrightarrow \frac{f_{n-1}}{c^{n-1} q^{(n-1)(n-2)/2}} - \frac{f_n}{c^n q^{n(n-1)/2}} = \frac{u_n - v_n}{c^n q^{n(n-1)/2}},$$

which admit an analytic solution f if, and only if, $\mathcal{B}_{q,1} u(c^{-1}) = \mathcal{B}_{q,1} v(c^{-1})$. The analytic class of the module with matrix $\begin{pmatrix} 1 & u \\ 0 & cz \end{pmatrix}$ within the formal class $\begin{pmatrix} 1 & 0 \\ 0 & cz \end{pmatrix}$ is therefore $\mathcal{B}_{q,1} u(c^{-1}) \in \mathbf{C}$. Writing f_u the unique solution in $\mathbf{C}(\{z\})$ of the equation $cz\sigma_q f - f = u - \mathcal{B}_{q,1} u(c^{-1})$, we have $Red(0, 1, 1, c, u) = \big(f_u, \mathcal{B}_{q,1} u(c^{-1})\big)$.
Note that equation $cz\sigma_q f - f = u$ always admits a unique formal solution $\hat{f}_u$. For instance, for $u \in \mathbf{C}$, we find that $\hat{f}_u = -u\mathrm{Ch}(cz)$, where Ch is the Tshakaloff series defined by (4) page 16). We infer that, for a general $u \in \mathbf{C}(\{z\})$, one has $\hat{f}_u = f_u - \mathcal{B}_{q,1} u(c^{-1}) \mathrm{Ch}(cz)$.

***Example 3.3.3* (A direct computation)**. — In proposition 3.3.1, we meet the equation $(\sigma_q F_{1,2}) z^{\mu_2} A_2 - z^{\mu_1} A_1 F_{1,2} = U - V$. Here is an explicit resolution that does not go through all the reductions of 3.1.1. We write $X = \sum X_n z^n$ for $F_{1,2}$ and $Y = \sum Y_n z^n$ for $U - V$. We have:

$$\forall n \ , \ q^{n-\mu_2} X_{n-\mu_2} A_2 - A_1 X_{n-\mu_1} = Y_n \Longleftrightarrow \forall n \ , \ q^{n-d} A_1^{-1} X_{n-d} A_2 - X_n = Z_n := A_1^{-1} Y_{n+\mu_1},$$

where we denote $d := \mu_2 - \mu_1 \in \mathbf{N}^*$ the *level* of the above inhomogeneous equation. It follows from [**6**], [**33**] that, for any analytic Y, this has a formal solution of q-Gevrey level d (this was defined in paragraph 1.3.2 of the general notations, in the introduction); it also follows from *loc.cit* that, if there is a solution of q-Gevrey level $d' > d$ (which is a stronger condition), then, there is an analytic solution. To solve our equation, we introduce a q-Borel transform of level d. This depends on an arbitrary family $(t_n)_{n \in \mathbf{Z}}$ in $\mathbf{C}^*$, subject to the condition:

$$\forall n \ , \ t_n = q^{n-d} t_{n-d}.$$

For $d = 1$, the natural choice is $t_n := q^{n(n-1)/2}$. In general, such a family can be built from Jacobi theta functions as in [**32**]. At any rate, t_n has the order of growth of $q^{n^2/2d}$. The q-Borel transform is then defined by:

$$\mathcal{B}_{q,d} \left(\sum f_n z^n \right) := \sum \frac{f_n}{t_n} z^n.$$

We must also choose d^{th} roots B_1, B_2 of A_1, A_2. Then, writing $\mathcal{B}_{q,d} X = \sum \tilde{X}_n z^n$, etc, we find the relations:

$$\forall n \ , \ B_1^{-d} \tilde{X}_{n-d} B_2^d - \tilde{X}_n = \tilde{Z}_n \Longleftrightarrow \forall n \ , \ B_1^{n-d} \tilde{X}_{n-d} B_2^{-(n-d)} - B_1^n \tilde{X}_n B_2^{-n} = B_1^n \tilde{Z}_n B_2^{-n}.$$

Thus, a necessary condition for the existence of an analytic solution X is that, for all i in a set of representatives modulo d, one has:

$$\sum_{n \equiv i \pmod d} B_1^n \tilde{Z}_n B_2^{-n} = 0.$$

This provides us with d obstructions in $\mathrm{Mat}_{r_1,r_2}(\mathbf{C})$ and it is not hard to prove, along the same lines as what has been done, that these form a complete set of invariants. More precisely, the map which sends U to the d-uple of matrices $\displaystyle\sum_{n \equiv i \pmod d} \frac{1}{t_n} B_1^{n-d} U_{n+\mu_1} B_2^{-n}$ yields an isomorphism of $\mathcal{F}(P_1, P_2)$ with $\mathrm{Mat}_{r_1,r_2}(\mathbf{C})^d$. Different choices of the family (t_n) and the matrices B_1, B_2 induce different isomorphisms.

3.3.2. The Birkhoff-Guenther normal form. — Going from two slopes to general case rests on the following remark. The functor $M \rightsquigarrow M' := M_{\leq \mu_{k-1}}$ induces an onto mapping $\mathcal{F}(P_1, \ldots, P_k) \to \mathcal{F}(P_1, \ldots, P_{k-1})$. The inverse image in $\mathcal{F}(P_1, \ldots, P_k)$ of the class of M' in $\mathcal{F}(P_1, \ldots, P_{k-1})$ is in natural one-to-one correspondance with the space $\mathrm{Ext}^1(P_k, M')$. The latter is a $\mathbf{C}$-vector space of dimension $\sum\limits_{1 \leq j < k} r_j r_k(\mu_k - \mu_j)$. This follows from the slope filtration $0 = M_0 \subset \cdots \subset M_{k-1} = M'$ and the resulting exact sequences, for $1 \leq j < k$:

$$0 \to M_{j-1} \to M_j \to P_j \to 0 \Longrightarrow 0 \to \mathrm{Ext}^1(P_k, M_{j-1}) \to \mathrm{Ext}^1(P_k, M_j) \to \mathrm{Ext}^1(P_k, P_j) \to 0.$$

(Recall that $\mathrm{Hom}(P_k, P_j) = 0$.) Actually, there is a non canonical $\mathbf{C}$-linear isomorphism $\mathrm{Ext}^1(P_k, M') \simeq \bigoplus_{1 \leq j < k} \mathrm{Ext}^1(P_k, P_j)$. One can go further using the matricial description of paragraph 2.2.4.2. We keep the notations recalled at the beginning of subsection 3.3.1 and moreover denote $\overline{M_U}$ the class in $\mathcal{F}(P_1, \ldots, P_k)$ of the module $M_U := (K^n, \Phi_{A_U})$ for $U := (U_{i,j})_{1 \leq i < j \leq k} \in \prod\limits_{1 \leq i < j \leq k} \mathrm{Mat}_{r_i, r_j}(\mathbf{C}(\{z\}))$.

Proposition 3.3.4. — *The map $U \mapsto \overline{M_U}$ from $\prod\limits_{1 \leq i < j \leq k} Mat_{r_i, r_j}(K_{\mu_i, \mu_j})$ to $\mathcal{F}(P_1, \ldots, P_k)$ is one-to-one.*

Proof. — We already now that the map is onto and the conclusion will follow from the following fact: for all $U \in \prod\limits_{1 \leq i < j \leq k} \mathrm{Mat}_{r_i, r_j}(\mathbf{C}(\{z\}))$, there exists a unique pair $(F, V) \in \mathfrak{G}(\mathbf{C}(\{z\})) \times \prod\limits_{1 \leq i < j \leq k} \mathrm{Mat}_{r_i, r_j}(K_{\mu_i, \mu_j})$ such that $F[A_V] = A_U$. Writing F as in equation (6) and $V = (V_{i,j})$, this is equivalent to the following system:

$$\forall (i,j), 1 \leq i < j \leq k \;,\; V_{i,j} + \sum_{\ell = i+1}^{j-1} (\sigma_q F_{i,\ell}) V_{\ell,j} + (\sigma_q F_{i,j})(z^{\mu_j} A_j) = (z^{\mu_i} A_i) F_{i,j} + \sum_{\ell = i+1}^{j-1} U_{i,\ell} F_{\ell,j} + U_{i,j}.$$

It is understood that $\sum\limits_{\ell = i+1}^{j-1} = 0$ when $j = i+1$. Then, U being given, the system is (uniquely) solved by induction on $j - i$ by the following formulas:

$$(F_{i,j}, V_{i,j}) := Red\left(\mu_i, A_i, \mu_j, A_j, U_{i,j} + \sum_{\ell = i+1}^{j-1} U_{i,\ell} F_{\ell,j} - \sum_{\ell = i+1}^{j-1} (\sigma_q F_{i,\ell}) V_{\ell,j}\right).$$

□

Theorem 3.3.5. — *The set $\mathcal{F}(P_1, \ldots, P_k)$ is an affine space of dimension $\sum\limits_{1 \leq i < j \leq k} r_i r_j (\mu_j - \mu_i)$.*

Proof. — This is an immediate consequence of the proposition. □

Note that this is the area contained in the lower finite part of the Newton polygon. This result is the natural continuation of a normalisation process found by Birkhoff and Guenther in [**9**].

Definition 3.3.6. — A matrix A_U such that $U := (U_{i,j})_{1\leq i<j\leq k} \in \prod\limits_{1\leq i<j\leq k} \mathrm{Mat}_{r_i,r_j}(\mathbf{C}(\{z\}))$ will be said to be in *Birkhoff-Guenther normal form*.

Now, using the argument that follows theorem 3.1.4, we draw from corollary 2.3.15:

Corollary 3.3.7. — *The components of the matrices* $(U_{i,j})_{1\leq i<j\leq k}$ *make up a complete system of coordinates on the space* $\mathcal{F}(P_1,\dots,P_{k-1})$.

3.3.3. Computational consequences of the Birkhoff-Guenther normal form. — Let A_0 be as in (8) and $A_U = A$ as in (9). Looking for $\hat{F} \in \mathfrak{G}(\mathbf{C}((z)))$ such that $\hat{F}[A_0] = A$ amounts, with notation (6), to solving the system:

$$\forall (i,j), 1 \leq i < j \leq k \ , \ (\sigma_q F_{i,j})(z^{\mu_j} A_j) = (z^{\mu_i} A_i) F_{i,j} + \sum_{\ell=i+1}^{j-1} U_{i,\ell} F_{\ell,j} + U_{i,j}.$$

This is triangular in the sense that the equation in $F_{i,j}$ depends on previously found $F_{\ell,j}$ with $\ell > i$.

Assume that A is in Birkhoff-Guenther normal form (definition 3.3.6). Then we can write $U_{i,j} = z^{\mu_i} U'_{i,j}$, where $U'_{i,j}$ has coefficients in $\mathbf{C}[z]$. The previous equation becomes:

$$(\sigma_q F_{i,j})(z^{\mu_j-\mu_i} A_j) = A_i F_{i,j} + \sum_{\ell=i+1}^{j-1} U'_{i,\ell} F_{\ell,j} + U'_{i,j}.$$

Assume by induction that all $F_{\ell,j}$ have coefficients in $\mathbf{C}[[z]]$. Then one may write the above equation as:

$$F_{i,j} = U'' + z^{\delta} A_i^{-1} (\sigma_q F_{i,j}) A_j,$$

where U'' has coefficients in $\mathbf{C}[[z]]$ and $\delta \in \mathbf{N}^*$. This is a fixpoint equation for an operator that is contracting in the z-adic topology of $\mathbf{C}[[z]]$ and so it admits a unique formal solution. Therefore we find a unique $\hat{F} \in \mathfrak{G}(\mathbf{C}[[z]])$ such that $\hat{F}[A_0] = A$.

A noteworthy consequence is that the inclusions $\mathbf{C}\{z\} \subset \mathbf{C}(\{z\})$ and $\mathbf{C}[[z]] \subset \mathbf{C}((z))$ induce a natural identification:

$$\mathfrak{G}^{A_0}(\mathbf{C}[[z]])/\mathfrak{G}(\mathbf{C}\{z\}) \simeq \mathfrak{G}^{A_0}(\mathbf{C}((z)))/\mathfrak{G}(\mathbf{C}(\{z\})),$$

and, as a corollary, a bijection:

$$\mathfrak{G}^{A_0}(\mathbf{C}[[z]])/\mathfrak{G}(\mathbf{C}\{z\}) \to \mathcal{F}(P_1, \ldots, P_k).$$

3.4. Interpolation by q-Gevrey classes

We shall now extend the previous results to q-Gevrey classification. We use here notations from paragraph 1.3.2 of the general notations in the introduction.

3.4.1. q-Gevrey extension spaces. —

3.4.1.1. q-Gevrey extensions for arbitrary slopes. — If one repaces the field $\mathbf{C}(\{z\})$ by the field $\mathbf{C}((z))_{q;s}$ of q-Gevrey series of level $s > 0$ and $\mathcal{D}_q$ by $\mathcal{D}_{q,s} := \mathbf{C}((z))_{q;s}\langle T, T^{-1}\rangle$, one gets the abelian category $DiffMod(\mathbf{C}((z))_{q;s}, \sigma_q)$ and the following extension of the results of 3.1.1.

Proposition 3.4.1. — *Let M, N be pure modules of ranks $r, s \in \mathbf{N}^*$ and slopes $\mu < \nu \in \mathbf{Q}$ in $DiffMod(\mathbf{C}((z))_{q;s}, \sigma_q)$. Then one has:*

$$\dim_{\mathbf{C}} Ext^1(M, N) = \begin{cases} 0 \text{ if } \nu - \mu \geq 1/s, \\ rs(\nu - \mu) \text{ if } \nu - \mu < 1/s. \end{cases}$$

Proof. — This follows indeed, by the same arguments as before, from propositions 3.2 and 3.3 of [**6**]. □

This remains true in the extreme case that $s = \infty$, since, over the field $\mathbf{C}((z))$, the slope filtration splits and $\mathrm{Ext}^1(M, N) = 0$; and also in the extreme case that $s = 0$, by theorem 3.1.1. Therefore, the above proposition is an interpolation between the analytic and formal settings.

3.4.1.2. q-Gevrey extensions for integral slopes. — Here, we extend the results of 3.3.1. If one replaces $\mathbf{C}(\{z\})$ by $\mathbf{C}((z))$, one gets $V = 0$ in all relations $(F, V) = Red(\mu_1, A_1, \mu_2, A_2, U)$ and the corresponding space of classes has dimension 0. The previous algorithm then produces $F = \hat{F}_U$ and the "formal normal form" A_0 of A. If one replaces $\mathbf{C}(\{z\})$ by $\mathbf{C}((z))_{q;s}$, one gets $V = 0$ and a matrix F of g-Gevrey level $\mu_2 - \mu_1$ if $\mu_2 - \mu_1 \geq 1/s$, and the same $Red(\mu_1, A_1, \mu_2, A_2, U)$ as before otherwise.

3.4.2. Isoformal q-Gevrey classification. —

3.4.2.1. q-Gevrey classification for arbitrary slopes. — One can state the moduli problem for q-Gevrey classification at order s, that is, over the field $\mathbf{C}((z))_{q;s}$ of q-Gevrey series of order $s > 0$ (see subsection 3.4.1). From proposition 3.4.1, one gets, by the same argument as before:

Proposition 3.4.2. — *Over $\mathbf{C}((z))_{q;s}$, the functor F is representable and the corresponding affine scheme is an affine space over $\mathbf{C}$ with dimension* $\sum\limits_{\substack{1\leq i<j\leq k \\ \mu_j-\mu_i<1/s}} r_i r_j (\mu_j - \mu_i)$.

This remains true for $s = \infty$ (formal setting, the space of moduli is a point) and for $s = 0$ (analytic setting, this is theorem 3.1.4).

3.4.2.2. q-Gevrey classification for integral slopes. — We fix $s > 0$ and write for short q-Gevrey for q-Gevrey of order s. Every matrix is q-Gevrey equivalent to a matrix A_U such that $U_{i,j} = 0$ for $\mu_j - \mu_i \geq 1/s$. The slopes being assumed to be integral, there is moreover a unique normal form with $U_{i,j} = 0$ for $\mu_j - \mu_i \geq 1/s$ and $U_{i,j} \in \mathrm{Mat}_{r_i,r_j}(K_{\mu_i,\mu_j})$ for $\mu_j - \mu_i < 1/s$.

3.4.3. Another kind of q-Gevrey interpolation. — A somewhat symmetric problem is to describe the space of analytic classes within a fixed q-Gevrey class. This can be done in a similar way. We fix a matrix A_{U_0} where $U_0 \in \prod\limits_{1\leq i<j\leq k} \mathrm{Mat}_{r_i,r_j}(K_{\mu_i,\mu_j})$ is such that $U_{i,j} = 0$ for $\mu_j - \mu_i \geq 1/s$ (any q-Gevrey class contains such a matrix). This characterizes a well defined q-Gevrey class and the space of analytic classes within this q-Gevrey class is an affine space of dimension $\sum\limits_{\substack{1\leq i<j\leq k \\ \mu_j-\mu_i\geq 1/s}} r_i r_j (\mu_j - \mu_i)$. If the slopes are integral, one can assume that U_0 is in normal form, *i.e.* each component $U_{i,j}$ such that $\mu_j - \mu_i < 1/s$ belongs to $\mathrm{Mat}_{r_i,r_j}(K_{\mu_i,\mu_j})$. Then each analytic class admits a unique normal form A_U where U is in normal form and its components such that $\mu_j - \mu_i < 1/s$ are the same as those of U_0.

CHAPTER 4

THE q-ANALOGS OF BIRKHOFF-MALGRANGE-SIBUYA THEOREMS

4.1. Asymptotics

We shall need a q-analogue of asymptotics *in the sense of Poincaré.* In chapter 5 we shall develop a more restrictive notion of asymptotics.

The underlying idea, coming from the thesis [**29**] of Jose-Luis Martins, is that an asymptotic theory is related to a dynamical system. In the "classical" case of Poincaré asymptotics for ordinary differential equations, say locally at 0 in $\mathbf{C}^*$, the dynamical system is given by the action of the semi-group $\Sigma := e^{]-\infty,0[}$; in our case, the semi-group is plainly $\Sigma := q^{-\mathbf{N}}$ (in the case of difference equations there are two semi-groups: $\Sigma := \mathbf{N}$ and $\Sigma := -\mathbf{N}$; this was used by J. Roques in [**41**]). Likewise, the sheaves of functions admitting an asymptotic expansion will be defined on the *horizon*, that is on the quotient space $\mathbf{C}^*/\Sigma$. In the classical case, this is the circle S^1 of directions (rays from 0) in $\mathbf{C}^*$; in our case, this is the elliptic curve $\mathbf{E}_q = \mathbf{C}^*/q^{\mathbf{Z}} = \mathbf{C}^*/q^{-\mathbf{N}}$ (in the difference case this is a pair of cylinders).

4.1.1. q-Asymptotics. — We set $\Sigma := q^{-\mathbf{N}}$ and for $A \subset \mathbf{C}$, $\Sigma(A) := \bigcup_{a \in A} q^{-\mathbf{N}}\{a\}$.

Let U be an open set of $\mathbf{C}$ invariant by the semi-group $\Sigma := q^{-\mathbf{N}}$. We shall say that a Σ-invariant subset K of U is a *strict subset* if there exists a compact subset K' of U such that $K = \Sigma(K')$; then $K \cup \{0\}$ is compact in $\mathbf{C}$.

Let f be a function holomorphic on U and $\hat{f} = \sum_{n \geq 0} a_n z^n \in \mathbf{C}[[z]]$. We shall say that f is *q-asymptotic* to $\hat{f}$ on U if, for all $n \in \mathbf{N}$ and every Σ-invariant strict subset $K \subset U$:

$$z^{-n}\big(f(x) - S_{n-1}\hat{f}\big)$$

is bounded on K. Here, $S_{n-1}\hat{f} := \sum_{p=0}^{n-1} a_p z^p$ stands for "truncation to n terms" of $\hat{f}$. If f admits a q-asymptotic expansion it is plainly unique. In the following we will say for simplicity "asymptotic expansion" for "q-asymptotic expansion". For $n \in \mathbf{N}$ we shall denote $f^{(n)}(0) := n!a_n$.

We shall denote $\mathscr{A}(U_\infty)$ the space of holomorphic functions admitting a q-asymptotic expansion on U, $\mathscr{A}_0(U_\infty)$ the subspace of $\mathscr{A}(U_\infty)$ made up of (infinitely) flat functions, *i.e.* those such that $\hat{f} = 0$, $\mathscr{B}(U_\infty)$ the space of holomorphic functions bounded on every stable strict subset of U. If f is holomorphic on U, then $f \in \mathscr{A}(U_\infty)$ if and only if $z^{-n}\big(f(x) - S_{n-1}\hat{f}\big) \in \mathscr{B}(U_\infty)$ for all $n \in \mathbf{N}$. We check easily that $\mathscr{A}(U_\infty)$ is a $\mathbf{C}$-algebra and that $\mathscr{A}_0(U_\infty)$ is an ideal of $\mathscr{A}(U_\infty)$.

Lemma 4.1.1. — *Let $U \subset \mathbf{C}$ be a Σ-invariant open set and $K \subset V$ be a Σ-invariant strict subset of U. There exists a real number $\rho > 0$ such that the closed disc of radius $\rho\,|z|$ and center z lies in U for all $z \in K$. Moreover it is possible to choose $\rho > 0$ such that $\bigcup_{z \in K} \bar{D}(z, \rho\,|z|)$ be a Σ-invariant strict subset of U.*

Proof. — There exists by definition a subset K' of K, compact in U, such that $K = \Sigma(K')$. By compacity the result of the lemma is true for every $z \in K'$; using the action of Σ we get that it is also true for all $z \in K$. □

Let $f \in \mathscr{A}(U_\infty)$ and $\hat{f} := \sum_{n \geq 0} a_n z^n$ its asymptotic expansion. We set $R_n := z^{-n}\big(f - S_{n-1}\hat{f}\big)$. We have $f = \sum_{p=0}^{n} a_p z^p + z^{n+1} R_{n+1}$ therefore:

$$f' = \sum_{p=1}^{n} p a_p z^{p-1} + (n+1) z^n R_{n+1} + z^{n+1} R'_{n+1}.$$

If $K \subset U$ is a Σ-invariant strict subset, then R_{n+1} is bounded by $M_{K,n+1}$ on K. Therefore, using Cauchy formula for the derivative and lemma 4.1.1, R'_n is bounded by some positive constant $\frac{M_{K,n+1}}{\rho|z|}$ on K.

We have $f' = \sum_{n\geq 1} na_n z^{n-1}$ and:

$$z^{-n}\big(f'(x) - S_{n-1}\hat{f}'\big) = z^{-n}\left(f'(x) - \sum_{p=1}^{n} na_p z^{p-1}\right) = (n+1)R_{n+1} + zR'_{n+1}.$$

Since $\left|(n+1)R_{n+1} + zR'_{n+1}\right| \leq \big(n+1+\frac{1}{\rho}\big)M_{K,n+1}$, then $z^{-n}\big(f'(x) - S_{n-1}\hat{f}'\big)$ is bounded on K, so that f' admits $\hat{f}'$ as a q-asymptotic expansion on U. Hence $\mathscr{A}(V_\infty)$ is a differential algebra and $\mathscr{A}_0(V_\infty)$ is a differential ideal of $\mathscr{A}(V_\infty)$.

4.1.2. Asymptotic expansions and $\mathcal{C}^\infty$ functions in Whitney sense. —

Lemma 4.1.2. — *Let U be a Σ-invariant open set of $\mathbf{C}$ and $f \in \mathscr{A}(U_\infty)$. Let $K \subset U$ be an invariant strict subset.*

(i) *There exists a positive constant C_K (depending on f) such that, for all $z_1, z_2 \in K \cup \{0\}$:*

$$|f(z_1) - f(z_2)| \leq C_K\,|z_1 - z_2|\,.$$

(ii) *For all $n \in \mathbf{N}$ there exists a positive constant $C_{K,n}$ (depending on f) such that, for all $z_1, z_2 \in K \cup \{0\}$:*

$$\left|f(z_1) - \sum_{p=0}^{n} \frac{f^{(p)}(z_2)}{p!}(z_1 - z_2)^p\right| \leq C_{K,n}\,|z_1 - z_2|^{n+1}\,.$$

(iii)

$$\lim_{\substack{z_1\to 0, z_2\to 0\\ z_1,z_2\in K\cup\{0\}, z_1\neq z_2}} \frac{1}{|z_1 - z_2|^n}\left(f(z_1) - \sum_{p=0}^{n} \frac{f^{(p)}(z_2)}{p!}(z_1 - z_2)^p\right) = 0.$$

Proof. — (i) We choose $\rho > 0$ (depending on K) as in lemma 4.1.1; then $K_1 := \bigcup_{z\in K} \bar{D}(z, \rho\,|z|)$ is an invariant strict subset of U. We set:

$$M_K := \sup_{z\in K} \frac{|f(z) - f(0)|}{|z|}\,, \quad M'_{K_1} := \sup_{z\in K_1} \left|f'(z)\right|\,.$$

Let $z_1, z_2 \in K \cup \{0\}$. We shall consider separately three cases: (a) $z_2 = 0$; (b) $z_2 \neq 0$ and $|z_1 - z_2| \leq \rho\,|z_2|$; (c) $z_2 \neq 0$ and $|z_1 - z_2| > \rho\,|z_2|$.

The cases (a) and (b) are easy. In fact, if $z_2 = 0$, then:

$$|f(z_1) - f(z_2)| = |f(z_1) - f(0)| \leq M_K\,|z_1| = M_K\,|z_1 - z_2|\,.$$

If $z_2 \neq 0$ and $|z_1 - z_2| \leq \rho\,|z_2|$, then $z_2 \in K$ and the closed interval $[z_1, z_2]$ is contained in K_1; therefore:

$$|f(z_1) - f(z_2)| \leq M'_{K_1}\,|z_1 - z_2|\,.$$

Now consider case (c). If $z_2 \in K$ and $|z_1 - z_2| > \rho\,|z_2|$, then we write $f(z_1) - f(z_2) = f(z_1) - f(0) - \big(f(z_2) - f(0)\big)$ and we have:

$$|f(z_1) - f(0)| \leq M_K\,|z_1| \leq M_K\big(|z_2| + |z_1 - z_2|\big) \leq \left(\frac{1}{\rho} + 1\right) M_K\,|z_1 - z_2|$$

and

$$|f(z_2) - f(0)| \leq M_K\,|z_2| \leq \frac{1}{\rho} M_K\,|z_1 - z_2|\,,$$

whence:

$$|f(z_1) - f(z_2)| \leq \frac{2+\rho}{\rho} M_K\,|z_1 - z_2|\,.$$

Last, let $C_K := \max\left(M'_{K_1}, \frac{2+\rho}{\rho} M_K\right)$; then, for all $z_1, z_2 \in K \cup \{0\}$, we have:

$$|f(z_1) - f(z_2)| \leq C_K\,|z_1 - z_2|\,.$$

(ii) We define ρ, K_1, C_K as in (i). We set:

$$C_{K,n} := \sup_{z \in K} \left| f(z) - \sum_{p=0}^{n} \frac{f^{(p)}(0)}{p!} z^p \right| \left| z^{-n-1} \right| \quad , \quad M'_{K_1,n} := \sup_{z \in K_1} \left| f^{(n+1)}(z) \right| \ .$$

The result is obviously true if f is a polynomial. Therefore, fixing $n \in \mathbf{N}$ it suffices to prove the result when $f = z^{n+1} g$, $g \in \mathscr{A}(U_\infty)$.

As in (i) we shall consider separately the three cases (a), (b), (c). If $z_2 = 0$, then the result merely comes from the following relation:

$$\left| f(z_1) - \sum_{p=0}^{n} \frac{f^{(p)}(z_2)}{p!} (z_1 - z_2)^p \right| = \left| f(z_1) - \sum_{p=0}^{n} \frac{f^{(p)}(0)}{p!} z_1^p \right| \leq C_{K,n}\,|z_1|^{n+1}\,.$$

Next, if $z_2 \neq 0$ and $|z_1 - z_2| \leq \rho\,|z_2|$, then the closed interval $[z_1, z_2]$ is contained in K_1, so that:

$$\left| f(z_1) - \sum_{p=0}^{n} \frac{f^{(p)}(z_2)}{p!} (z_1 - z_2)^p \right| \leq M'_{K_1,n}\,|z_1 - z_2|^{n+1}\,.$$

It remains to deal with the case (c), that is, when $z_2 \neq 0$ and $|z_1 - z_2| > \rho\,|z_2|$. If we set:

$$\lambda_{K,n} := \sup_{\substack{z\in K \\ p=0,\dots,n}} \left|\frac{g^{(p)}(z)}{p!}\right| \;, \quad \mu_K := \sup_{z\in K} |z|\,,$$

then, for any integer $p \in [0, n]$, we find:

$$\left|\frac{f^{(p)}(z_2)}{p!}\right| = \left|\frac{(z^{n+1}g)^{(p)}(z_2)}{p!}\right| \leq \lambda_{K,n} \sum_{k=0}^{p} \binom{n+1}{k} |z_2|^{n+1-k} \leq \tilde{\lambda}_{K,n}\; |z_2 - z_1|^{n+1-p}\,,$$

where:

$$\tilde{\lambda}_{K,n} := \lambda_{K,n}\; \rho^{p-n-1} \sum_{k=0}^{p} \binom{n+1}{k} \mu_K^{p-k}\;.$$

From this, it follows:

$$\begin{aligned}\left|f(z_1) - \sum_{p=0}^{n} \frac{f^{(p)}(z_2)}{p!}(z_1 - z_2)^p\right| &\leq \left|z_1^{n+1} g(z_1)\right| + (n+1)\tilde{\lambda}_{K,n}\; |z_2 - z_1|^{n+1} \\ &\leq \left(\left(1 + \frac{1}{\rho}\right)^{n+1} \lambda_{K,n} + (n+1)\tilde{\lambda}_{K,n}\right)\; |z_2 - z_1|^{n+1}\;.\end{aligned}$$

To summarize, in the three cases cases we have:

$$\left|f(z_1) - \sum_{p=0}^{n} \frac{f^{(p)}(z_2)}{p!}(z_1 - z_2)^p\right| \leq\; C_{K,n}\,|z_1 - z_2|^{n+1}\,,$$

where

$$C_{K,n} := \max\left(M'_{K_1,n}, \left(1 + \frac{1}{\rho}\right)^{n+1} \lambda_{K,n} + (n+1)\tilde{\lambda}_{K,n}\right)\;.$$

(iii) The result follows immediately from (ii). □

Proposition 4.1.3. — *Let U be a Σ-invariant open set of $\mathbf{C}$. Then, $f \in \mathscr{A}(U_\infty)$ if, and only if, for every invariant strict subset $K \subset U$, one has:*

$$f_{|K} \in Ker\;\; \left(\frac{\partial}{\partial \bar{z}} : \mathcal{C}^\infty_{Whitney}\big(K \cup \{0\}, \mathbf{C}\big) \to \mathcal{C}^\infty_{Whitney}\big(K \cup \{0\}, \mathbf{C}\big)\right).$$

Proof. — We first recall Whitney conditions and Whitney theorem [**25**], [**24**].

We set $z := x + \mathrm{i}y$. Let f be a function of (x, y) with complex values, $\mathcal{C}^\infty$ in Whitney sense on $K \cup \{0\}$. For $k := (k_1, k_2) \in \mathbf{N}^2$, we set $f^{(k)} := \frac{\partial^{k_1}}{\partial x^{k_1}} \frac{\partial^{k_2}}{\partial y^{k_2}} f$. We associate to f the set of its "Taylor fields" on K: $(f^{(k)})_{k\in\mathbf{N}^2}$; the functions $f^{(k)}$ are continuous on $K \cup \{0\}$ and for all $n \in \mathbf{N}$, the functions $\big(f^{(k)}\big)_{|k|\leq n}$

satisfy the following conditions, that are called *Whitney conditions of order* $n \in \mathbf{N}$ and that we shall denote $W_n(K)$:
(10)

$$\lim \frac{1}{|\,|(x_1-x_2, y_1-y_2)|\,|^{n-|k|}} \left(f^{(k)}(x_1,y_1) - \sum_{|h|\le n} \frac{f^{(k+h)}(x_2,y_2)}{h!}(x_1-x_2, y_1-y_2)^h \right) = 0,$$

where the limit is taken for $(x_1,y_1) \to 0$, $(x_2,y_2) \to (0,0)$, with $(x_1,y_1),(x_2,y_2) \in K \cup \{0\}$ and $(x_1,y_1) \neq (x_2,y_2)$.

Conversely a family $\big(f^{(k)}\big)_{k\in\mathbf{N}^2}$ of continuous functions on $K \cup \{0\}$ is the family of Taylor jets of an element of $\mathcal{C}^\infty_{Whitney}\big(K \cup \{0\}\big)$ if and only if it satisfies Whitney conditions $W_n(K)$ for all $n \in \mathbf{N}$ (this is Whitney theorem).

Using $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial \bar z}$ instead of $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, for all $n \in \mathbf{N}$ we can replace Whitney conditions $W_n(K)$ by equivalent conditions $\tilde W_n(K)$.

We now return to the proof of the proposition.

Let $f \in \mathcal{C}^\infty_{Whitney}\big(K \cup \{0\}, \mathbf{C}\big)$. If for all invariant strict subset $K \subset U$, one has:

$$f_{|K} \in \mathrm{Ker} \quad \left(\frac{\partial}{\partial \bar z} : \mathcal{C}^\infty_{Whitney}\big(K \cup \{0\}, \mathbf{C}\big) \to \mathcal{C}^\infty_{Whitney}\big(K \cup \{0\}, \mathbf{C}\big) \right),$$

then, the function f is holomorphic on V, its Taylor jet $\hat f$ at 0 is formally holomorphic, $\hat f = \sum\limits_{n\ge 0} a_n z^n$ and f is asymptotic to $\hat f$ on V, $f \in \mathscr{A}(U_\infty)$.

Conversely let $f \in \mathscr{A}(U_\infty)$; then, for all $p \in \mathbf{N}$, $f^{(p)} \in \mathscr{A}(U_\infty)$ and for all $r \in \mathbf{N}^*$, we have $\frac{\partial^r}{\partial \bar z^r} f^{(p)} = 0$.

We can apply lemma 4.1.2 to each function $f^{(p)}$, $p \in \mathbf{N}$, to the effect that f satisfies conditions $\tilde W_n(K)$ for all $n \in \mathbf{N}$ (this is assertion (iii) of the lemma); therefore it satisfies also Whitney conditions $W_n(K)$ for all $n \in \mathbf{N}$ and, using Whitney theorem, we get $f \in \mathcal{C}^\infty_{Whitney}\big(K \cup \{0\}, \mathbf{C}\big)$. □

4.1.3. Asymptotics and sheaves on $\mathbf{E}_q$. — Recall from the introduction that we denote $p : \mathbf{C}^* \to \mathbf{E}_q$ the quotient map.

A stable open subset U of $\mathbf{C}^*$ has a "horizon" U_∞ and *two* compactifications $\underline{U}$, $\tilde{U}$ defined as follows:

$$\begin{aligned} U_\infty &:= p(U) \subset \mathbf{E}_q, \\ \underline{U} &:= U \cup \{0\} \subset \mathbf{C}, \\ \tilde{U} &:= U \cup U_\infty, \end{aligned}$$

An invariant subset $K \subset U$ is strict if and only if $p(K)$ is a compact subset of $\mathbf{E}_q$.

We then define sheaves on $\mathbf{E}_q$ by putting:

- $\mathscr{B}(U_\infty) =$ the space holomorphic functions bounded on every stable strict subset of U.
- $\mathscr{A}(U_\infty) =$ the space of holomorphic functions admitting an asymptotic expansion $\hat{f} = \sum a_p z^p$, *i.e.*:

$$\forall n \ , \ z^{-n}(f - S_{n-1}\hat{f}) \in \mathscr{B}(U_\infty).$$

- $\mathscr{A}_0(U_\infty) =$ the subspace of $\mathscr{A}(U_\infty)$ made up of flat functions, *i.e.* those such that $\hat{f} = 0$.

Although much weaker than the q-Gevrey theorem of Borel-Ritt 5.3.3, the following result does not directly flow from it, and we give here a direct proof.

Theorem 4.1.4 **(Weak q-analogue of Borel-Ritt)**
The natural map from $\mathscr{A}$ to the constant sheaf $\mathbf{C}[[z]]$ is onto.

Proof. — Starting from an open set $V \subset \mathbf{E}_q$ small enough that $U := p^{-1}(V) \subset \mathbf{C}^*$ is a disjoint union of open sets $U_n := q^n U_0$, where U_0 is mapped homeomorphically to V by p, we divide $\mathbf{C}^*$ in a finite number of sectors S_i such that each U_n is contained in one of the S_i. Now, let $\hat{f} \in \mathbf{C}[[z]]$. Apply the classical Borel-Ritt theorem within each sector S_i, yielding a map f_i holomorphic and admitting the asymptotic expansion $\hat{f}$. Glueing the f_i provides a section of $\mathscr{A}$ over V having image $\hat{f}$. □

4.2. Existence of asymptotic solutions

We call *adequate* an open subset $\mathbf{C}^*$ of the form $q^{-\mathbf{N}}U_0$, where the $q^{-k}U_0, k \geq 0$ are pairwise disjoint. Clearly, $\mathscr{A}_0(U_\infty)$ is a $\mathbf{C}(\{z\})$-vector space stable under σ_q. In particular, any q-difference operator $P := \sigma_q^n + a_1\sigma_q^{n-1} + \cdots + a_n \in \mathcal{D}_q$ defines a $\mathbf{C}$-linear endomorphism $P : f \mapsto Pf = \sigma_q^n(f) + a_1\sigma_q^{n-1} + \cdots + a_n f$ of $\mathscr{A}_0(U_\infty)$. Likewise, any matrix

$A \in \mathrm{GL}_n(\mathbf{C}(\{z\}))$, defines a $\mathbf{C}$-linear endomorphism $\sigma_q - A : X \mapsto \sigma_q X - AX$ of $\mathscr{A}_0(U_\infty)^n$.

Theorem 4.2.1 **(Existence of asymptotic solutions)**
Let $A \in GL_n(\mathbf{C}(\{z\}))$ and let $\hat{X} \in \mathbf{C}[[z]]^n$ a formal solution of the system $\sigma_q X = AX$. For any adequate open subset U, there exists a solution $X \in \mathscr{A}(U_\infty)^n$ of that system which is asymptotic to $\hat{X}$.

Proof. — From theorem 4.1.4, there exists $Y \in\in \mathscr{A}(U_\infty)^n$ which is asymptotic to $\hat{X}$ (without however being a solution of the system). Set:

$$W := \sigma_q Y - AY = \sigma_q(Y - \hat{X}) - A(Y - \hat{X}) \in \mathscr{A}_0(U_\infty)^n.$$

From proposition 4.2.2 below, there exists $Z \in \mathscr{A}_0(U_\infty)^n$ such that $\sigma_q Z - AZ = W$. Then $X := Y - Z \in \mathscr{A}(U_\infty)^n$ is asymptotic to $\hat{X}$ and solution of the system. □

Proposition 4.2.2. — *Let $A \in GL_n(\mathbf{C}(\{z\}))$ and let U be an adequate open subset. Then, the endomorphism $\sigma_q - A$ of $\mathscr{A}_0(U_\infty)^n$ is onto.*

Proof. — Let $F \in \mathrm{GL}_n(\mathbf{C}(\{z\}))$ and $B := F[A]$. From the commutative diagram:

$$\begin{array}{ccc} \mathscr{A}_0(U_\infty)^n & \xrightarrow{\sigma_q - A} & \mathscr{A}_0(U_\infty)^n \\ \Big\downarrow{\scriptstyle F} & & \Big\downarrow{\scriptstyle \sigma_q F} \\ \mathscr{A}_0(U_\infty)^n & \xrightarrow{\sigma_q - B} & \mathscr{A}_0(U_\infty)^n \end{array}$$

in which the vertical arrows are isomorphisms, we just have to prove that $\sigma_q - B$ is onto. After the cyclic vector lemma (lemma 2.1.1), we can take $B := A_P$, the companion matrix of $P := \sigma_q^n + a_1 \sigma_q^{n-1} + \cdots + a_n$ described in 2.1.2.1. We then apply lemma 4.2.3 (herebelow) and theorem 4.2.5 (further below). □

Lemma 4.2.3. — *Let R a $\mathbf{C}(\{z\})$ vector space on which σ_q operates. With the notations of the previous proposition, $P : R \to R$ is onto if, and only if, $\sigma_q - A_P : R^n \to R^n$ is onto.*

Proof. — (i) Assume P is onto. To solve $(\sigma_q - A_P) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$, put $x_{i+1} := \sigma_q x_i - y_i$; one sees that x_1 just has to be solution of an equation $Px_1 = z$, where z is some explicit expression of the y_i.

(ii) Assume $\sigma_q - A_P$ is onto. To solve $Pf = g$, it is enough to solve $\sigma_q X - A_P X = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ g \end{pmatrix}$ and to set f to the first component of X. □

Remark 4.2.4. — This computation is directly related to the homotopy of complexes discussed in 2.4.2.

Theorem 4.2.5. — *The endomorphism P of $\mathscr{A}_0(U_\infty)$ is onto.*

Proof. — We give the proof under the assumption that the slopes are integral, which is the only case we need in this paper; the reader will easily deduce the general case by ramification. The counterpart of the slope filtration on the side of q-difference operators is the existence of a factorisation [**27, 47, 53**] of P as a product of non commuting factors: elements of $\mathbf{C}\{z\}^*$ (which induce automorphisms of $\mathscr{A}_0(U_\infty)$) and first order operators $z^\mu \sigma_q - a$, $\mu \in \mathbf{Z}$, $a \in \mathbf{C}^*$. We are therefore reduced to the following lemma. (This is where some analysis comes in !) □

Lemma 4.2.6. — *The endomorphism $f \mapsto z^\mu \sigma_q f - af$ of $\mathscr{A}_0(U_\infty)$ is onto.*

Proof. — Write L this endomorphism. Let $h \in \mathscr{A}_0(U_\infty)$; for all $n \in \mathbf{N}$ and each strict stable subset V of U, there exists $C_{n,V} > 0$ such that $|h(z)| < C_{n,V}\,|z|^n$ over V. We distinguish two cases: (i) $\mu \leq 0$; (ii) $\mu > 0$.

Case (i): $\mu \leq 0$ (easy case). Let Φ the automorphism of $\mathscr{A}_0(U_\infty)$ defined by $\Phi(f) := aq^\mu z^{-\mu} \sigma_{q^{-1}} f$; one has $\Phi^m f = a^m q^{\mu m(m+1)/2} z^{-m\mu} \sigma_{q^{-m}} f$ for all $m \geq 0$. Since $|h(z)| \leq C_{n,V}\,|z|^n$ on each stable strict subset V of U and all integers $n \geq 0$, one finds:

$$\forall z \in V\ ,\ \sum_{m \geq 0} |\Phi^m h(z)| \leq C_{n,V} \sum_{m \geq 0} |a|^m\,|q|^{\mu m(m+1)/2}\ |q|^{-nm}\ |z|^n\,.$$

One deduces that the series $\sum_{m \geq 0} \Phi^m h$ defines a flat function over U, write it $H \in \mathscr{A}_0(U_\infty)$. Observing that $LH = z^\mu \sigma_q (H - \Phi H) = z^\mu \sigma_q h$, we see that L is onto over $\mathscr{A}_0(U_\infty)$.

Case (ii): $\mu > 0$ (hard case). For all $j \in \mathbf{N}$, set $U_j := q^{-j} U_0 = \{z \in \mathbf{C}^* \mid q^j z \in U_0\}$; these are pairwise disjoint since U was assumed to be adequate. We are going to build successively $f_0, f_1, f_2, \ldots$ over $U_0, U_1, U_2, \ldots$, which will give rise to a function f defined over $U = q^{-\mathbf{N}} U_0$, with $f_j = f_{|U_j}$ and $Lf = h$.

For all $j \in \mathbf{N}$, let $h_j := h_{|U_j} : U_j \to \mathbf{C}$. Since $(\sigma_q f)_{|U_j} = \sigma_q f_{j-1}$, equation $Lf = h$ is equivalent to system:

$$\forall j \in \mathbf{N}^* \ , \ z^\mu \sigma_q f_{j-1} - a f_j = h_j,$$

also written:

$$\forall j \in \mathbf{N}^* \ , \ f_j = (z^\mu \sigma_q f_{j-1} - h_j)/a. \tag{11}$$

We agree that $f_{-1} \equiv 0$ and $f_0 = h_0/a$. Iterating the last relation (11) above, we find:

$$\forall j \in \mathbf{N}^* \ , \ f_j = -\sum_{\ell=0}^{j} q^{\mu\ell(\ell-1)/2} z^{\ell\mu} \sigma_q^\ell h_{j-\ell}\, a^{-\ell-1}. \tag{12}$$

Write f the function defined on U by the relations $f_{|U_j} = f_j$, $j \in \mathbf{N}$; Clearly, $Lf = h$ on U. We are left to show that $f \in \mathscr{A}_0(U_\infty)$.

Let $n \in \mathbf{N}$ and V a strict stable subset of U; let $V_j := U_j \cap V$ and choose a $K > 0$ such that $|z| \leq K\,|q|^{-j}$ for all $z \in U_j$; one has $|z| \leq K|q|^{-j}$ on each V_j. Bounding by above the right hand side of (12), one gets:

$$\forall z \in V_j \ , \ |f_j(z)| \leq \frac{C_{n,V}}{a}\, P_j(|q|^n\, \alpha)\ |z|^n\,, \tag{13}$$

where $\alpha := K^\mu/a$ and where P_j denotes the polynomial with positive coefficients:

$$P_j(X) := \sum_{\ell=0}^{j} |q|^{\mu\ell(\ell-1)/2 - j\ell\mu}\ X^\ell.$$

Taking in account the relation

$$P_{j+1}(X) = P_j(|q|^{-\mu}\, X) + |q|^{-\mu(j+1)(j+2)/2}\ X^{j+1}\,,$$

one finds that $P_j(X) \leq P(X; \mu, q)$ for all $X > 0$, where we write $P(z; \mu, q)$ the entire function defined by:

$$P(z; \mu, q) := \sum_{j=0}^{\infty} |q|^{-\mu j(j+1)/2}\ z^j.$$

Said otherwise, (13) can be written in the form:

$$\forall z \in V_j \ , \ |f_j(z)| \leq \frac{C_{n,V}}{a}\, P(|q|^n\, \alpha; \mu, q)\ |z|^n\,,$$

which allows us to conclude that f is flat. □

We now apply theorem 4.2.1 to a situation that we shall meet when we prove theorem 4.4.1.

Let $A_0 \in \mathrm{GL}_n(\mathbf{C}(\{z\}))$ the matrix of a pure module and let $A \in \mathrm{GL}_n(\mathbf{C}(\{z\}))$ a matrix formally equivalent to A_0. Let $\hat{F} \in \mathrm{GL}_n(\mathbf{C}((z)))$ a gauge transform proving this equivalence, *i.e.* $\hat{F}[A_0] = A$. (Note that we do not assume $\hat{F} \in \mathfrak{G}(\mathbf{C}((z)))$.) The conjugation relation is equivalent to $\sigma_q \hat{F} = A\hat{F}A_0^{-1}$, which we regard as a q-difference system of rank n^2 in the space $\mathrm{Mat}_n(\mathbf{C}(\{z\})) \simeq K^{n^2}$. After theorem 4.2.1, there exists over each adequate open set U a solution $F_U \in \mathrm{Mat}_n(\mathbf{C}(\{z\}))$ of the system $\sigma_q F = AFA_0^{-1}$ that is asymptotic to $\hat{F}$. Since the latter is invertible, and since $\det F_U$ is obviously asymptotic to $\det \hat{F}$, whence non zero, F_U is invertible.

Now let U' another adequate open set which meets U. Then $U'' := U \cap U'$ is adequate and the matrix $G := F_U^{-1} F_{U'}$ satisfies the two following properties:

1. It is an automorphism of A_0, *i.e.* $G[A_0] = A_0$.
2. It is asymptotic to $\hat{F}^{-1}\hat{F} = I_n$, *i.e.* the coefficients of $G - I_n$ belong to $\mathscr{A}_0(U''_\infty)$.

Lemma 4.2.7. — *The matrix G belongs to $\mathfrak{G}(\mathscr{A}_0(U''_\infty))$.*

Proof. — With the usual notations for A_0, each block of G satisfies:

$$\sigma_q G_{i,j} = (z^{\mu_j} A_j)^{-1}\, G_{i,j}\, (z^{\mu_i} A_i) = z^{\mu_i - \mu_j} A_j^{-1} G_{i,j} A_i.$$

The order of growth or decay of $G_{i,j}$ near 0 is therefore the same as ${\theta_q}^{\mu_i - \mu_j}$. Since $G_{i,j}$ is flat for $i \neq j$, this implies $G_{i,j} = 0$ for $i > j$. For $i = j$, we apply the same argument to $G_{i,i} - I_{r_i}$, which therefore vanishes. □

4.3. The fundamental isomorphism

We write $\Lambda_I := I_n + \mathrm{Mat}_n(\mathscr{A}_0)$ for the subsheaf of groups of $\mathrm{GL}_n(\mathscr{A})$ made up of matrices infinitely tangent to the identity and, if G is an algebraic subgroup of $\mathrm{GL}_n(\mathbf{C})$, we put $\Lambda_I^G := \Lambda_I \cap G(\mathscr{A})$. In particular $\Lambda_I^{\mathfrak{G}}$ is a sheaf of triangular matrices of the form (6) with all the $F_{i,j}$ infinitely flat.

For a q-difference module $M = (\mathbf{C}(\{z\})^n, \Phi_A)$ (section 2.1.1), we consider the set of *automorphisms of M infinitely tangent to the identity*: this is the subsheaf $\Lambda_I(M)$ of Λ_I whose sections satisfy the equality: $F[A] = A$ (recall that by definition $F[A] = (\sigma_q F)AF^{-1}$); this is also called the *Stokes sheaf* of the module M.

If the matrix A has the form (5), then $\Lambda_I(M)$ is a subsheaf of $\Lambda_I^{\mathfrak{G}}$. Up to an analytic gauge transformation, we can assume that we are in this case. As a consequence, $\Lambda_I(M)$ is a sheaf of *unipotent* (and even triangular) groups.

4.3.1. Study of $H^1(\mathbf{E}_q;\Lambda_I)$. — We set $q = e^{-2\mathrm{i}\pi\tau}$, $\Im(\tau) > 0$ and for $v \in \mathbf{R}$, we denote $q^v = e^{-2\mathrm{i}\pi\tau v}$. We get a splitting:

$$\mathbf{C}^* = \mathbf{U} \times q^{\mathbf{R}}$$

and the map $p_0 : (u, v) \mapsto e^{2\mathrm{i}\pi(u+v\tau)}$ induces an isomorphism between $\mathbf{C}/(\mathbf{Z}+\mathbf{Z}\tau)$ and $\mathbf{E}_q = \mathbf{C}^*/q^{\mathbf{Z}}$.

We will call *open parallelogram* of $\mathbf{E}_q$ the image by the map p_0 of an open parallelogram:

$$\{u + v\tau \in \mathbf{C} \mid u_1 < u < u_2, v_1 < v < v_2\},$$

where $u_2 - u_1 \leq 1, v_2 - v_1 \leq 1$ and *closed parallelogram* the closure of an open parallelogram such that $u_2 - u_1 < 1, v_2 - v_1 < 1$. We will say that a parallelogram is *small* if $u_2 - u_1 < 1/4$ and $v_2 - v_1 < 1/4$.

For every open covering $\mathfrak{V}$ of $\mathbf{E}_q$, there exists a finite open covering $\mathfrak{U} = (U_i)_{i\in I}$, finer than $\mathfrak{V}$, such that:

(i) the U_i are open parallelograms,

(ii) the open sets U_i are four by four disjoint.

We will call *good* such a covering. There is a similar definition for closed coverings.

- Every element of $H^j(\mathbf{E}_q;\mathrm{GL}_n(\mathscr{A})), j = 0, 1$ can be represented by an element of $H^j(\mathfrak{U};\mathrm{GL}_n(\mathscr{A}))$, where $\mathfrak{U}$ is a good covering.
- The natural map $H^j(\mathfrak{U};\mathrm{GL}_n(\mathscr{A})) \to H^j(\mathbf{E}_q;\mathrm{GL}_n(\mathscr{A}))$ is always injective.

Let $\mathfrak{U}$ be a good open covering of $\mathbf{E}_q$; we denote by $\mathscr{C}^0(\mathfrak{U};\mathrm{GL}_n(\mathscr{A}))$ the subspace of the space of 0-cochains $(g_i) \in C^0(\mathfrak{U};\mathrm{GL}_n(\mathscr{A}))$ whose coboundary $(g_{ij} = g_i g_j^{-1})$ belongs to $C^1(\mathfrak{U};\Lambda_I)$. It is equivalent to say that the natural image of the cochain (g_i) in $C^0(\mathbf{E}_q;\mathrm{GL}_n(\mathbf{C}[[z]]))$ is a cocycle, or that the g_i $(i \in I)$ admit the same asymptotic expansion $\hat{g} \in \mathrm{GL}_n(\mathbf{C}[[z]])$. We have two natural maps:

$$\mathscr{C}^0(\mathfrak{U};\mathrm{GL}_n(\mathscr{A})) \to Z^1(\mathfrak{U};\Lambda_I)$$

$$\mathscr{C}^0(\mathfrak{U};\mathrm{GL}_n(\mathscr{A})) \to \mathrm{GL}_n(\mathbf{C}[[z]]).$$

Two cochains $(g_i), (g'_i) \in \mathscr{C}^0(\mathfrak{U}; \mathrm{GL}_n(\mathscr{A}))$ define the same element of $H^1(\mathfrak{U}; \Lambda_I)$ if and only if:

$$g'_i {g'}_j^{-1} = h_i g_i g_j^{-1} h_j^{-1}, \text{ with } (h_i) \in C^0(\mathfrak{U}; \Lambda_I).$$

This is equivalent to ${g'}_j^{-1} h_j g_j = {g'}_i^{-1} h_i g_i$, which thus defines an element $f \in H^0\big(\mathfrak{U}; \mathrm{GL}_n(\mathscr{A})\big)$ and we have $g' = hgf^{-1}$.

Lemma 4.3.1. — (i) *Let $\mathfrak{U}$ be a good open covering of $\mathbf{E}_q$. The coboundary map ∂ induces an injection on the double quotient:*

$$C^0(\mathfrak{U}; \Lambda_I)\backslash \mathscr{C}^0(\mathfrak{U}; GL_n(\mathscr{A}))/H^0\big(\mathfrak{U}; GL_n(\mathscr{A})\big) \to H^1(\mathfrak{U}; \Lambda_I).$$

(ii) *We have a canonical injection:*

$$\varinjlim_{\mathfrak{U}} C^0(\mathfrak{U}; \Lambda_I)\backslash \mathscr{C}^0(\mathfrak{U}; GL_n(\mathscr{A}))/GL_n\big(\mathbf{C}\{z\}\big) \to H^1(\mathbf{E}_q; \Lambda_I),$$

where the direct limit is indexed by the good coverings.

Proposition 4.3.2. — *For every good open covering $\mathfrak{U}$ of $\mathbf{E}_q$ we have a natural isomorphism:*

$$C^0((\mathfrak{U} : \Lambda_I)\backslash \mathscr{C}^0(\mathfrak{U}; GL_n(\mathscr{A}))/H^0\big(\mathfrak{U}; GL_n(\mathscr{A})\big) \to GL_n\big(\mathbf{C}[[z]]\big)/GL_n\big(\mathbf{C}\{z\}\big).$$

Let $\mathfrak{U}$ be a good open covering of $\mathbf{E}_q$. Let $\hat{g} \in \mathrm{GL}_n\big(\mathbf{C}[[z]]\big)$. Using the q-analog of Borel-Ritt (theorem 4.1.4), we can represent it by an element g_i on each open set U_i such that g_i admits $\hat{g}$ as asymptotic expansion, therefore the natural map:

$$\mathscr{C}^0(\mathfrak{U}; \mathrm{GL}_n(\mathscr{A})) \to \mathrm{GL}_n\big(\mathbf{C}[[z]]\big)$$

is onto. The proposition follows.

Corollary 4.3.3. — *For every good open covering $\mathfrak{U}$ of $\mathbf{E}_q$, we have natural injections:*

$$GL_n\big(\mathbf{C}[[z]]\big)/GL_n\big(\mathbf{C}\{z\}\big) \to H^1(\mathfrak{U}; \Lambda_I) \to H^1(\mathbf{E}_q; \Lambda_I).$$

We will see later that these maps are bijections.

4.3.2. Geometric interpretation of the elements of $H^1(\mathbf{E}_q; \Lambda_I)$. — We need some results on regularly separated subsets of the euclidean space $\mathbf{R}^2$. We first recall some definitions.

If $X \subset \mathbf{R}^n$ is a closed subset of the euclidean space $\mathbf{R}^n$, we denote by $\mathscr{E}(X)$ the algebra of $\mathcal{C}^\infty$ functions in Whitney sense on X (*cf.* [**25**]).

We recall the *Whitney extension theorem*: if $Y \subset X$ are closed subset of $\mathbf{R}^n$, the restriction map $\mathscr{E}(X) \to \mathscr{E}(Y)$ is surjective.

Let A, B two closed sets of $\mathbf{R}^n$; we define the following maps:

$$\alpha : f \in \mathscr{E}(A \cup B) \mapsto (f_{|A}, f_{|B}) \in \mathscr{E}(A) \oplus \mathscr{E}(B),$$

$$\beta : (f, g) \in \mathscr{E}(A) \oplus \mathscr{E}(B) \mapsto f_{|A\cap B} - g_{|A\cap B} \in \mathscr{E}(A \cap B),$$

and we consider the sequence:

$$0 \to \mathscr{E}(A \cup B) \xrightarrow{\alpha} \mathscr{E}(A) \oplus \mathscr{E}(B) \xrightarrow{\beta} \mathscr{E}(A \cap B) \to 0. \tag{14}$$

Definition 4.3.4. — Two subsets A and B of the euclidean space $\mathbf{R}^n$ are regularly separated if the above sequence is exact.

The following result is due to Łojasiewicz (*cf.* [**25**]).

Proposition 4.3.5. — *Let A,B two closed subsets A and B of the euclidean space $\mathbf{R}^n$; the following conditions are equivalent:*

(i) *A and B are regularly separated;*

(ii) *$A \cap B = \emptyset$ or for all $x_0 \in A \cap B$, there exists a neighborhood U of x_0 and constants $C > 0$, $\lambda > 0$ such that for all $x \in U$:*

$$d(x, A) + d(x, B) \geq C d(x, A \cap B)^{\lambda};$$

(iii) *$A \cap B = \emptyset$ or for all $x_0 \in A \cap B$, there exists a neighborhood U of x_0 and a constant $C' > 0$ such that for all $x \in A \cap U$:*

$$d(x, B) \geq C' d(x, A \cap B)^{\lambda}.$$

If we can choose $\lambda = 1$ in the above condition(s), we will say that the closed subsets A and B are *transverse.* Transversality is in fact a *local* condition.

Lemma 4.3.6. — *Let K_1, K_2 two closed small parallelograms of $\mathbf{E}_q$. The sets $p^{-1}(K_1)$ and $p^{-1}(K_2)$ are transverse.*

Proof. — We set:

$$A := p^{-1}(K_1), A_0 := p_0^{-1}(A), B := p^{-1}(K_2) \text{ and } B_0 := p_0^{-1}(B)$$

Since K_1 and K_2 are small parallelograms, both A_0 and B_0 are formed of a family of disjoint rectangles, that is to say:

$$A_0 = \bigcup_{m,n \in \mathbf{Z}} A_{m,n} \text{ and } B_0 = \bigcup_{m,n \in \mathbf{Z}} B_{m,n},$$

where the families $(A_{m,n})_{m,n\in\mathbf{Z}}$, $(B_{m,n})_{m,n\in\mathbf{Z}}$ are such that $d(A_{m,n}, A_{m',n'}) \geq 3/4$ and $d(B_{m,n}, B_{m',n'}) \geq 3/4$ for any $(m', n') \neq (m, n)$. We only need to deal with the case when $A \cap B \neq \emptyset$, that is, when there exists (m, n), (m', n') such that $A_{m,n} \cap B_{m',n'} \neq \emptyset$.

Consider the function f defined over $\mathbf{C}^* \setminus A \cap B$ by

$$f(x) = \frac{d(x, A) + d(x, B)}{d(x, A \cap B)}$$

and observe that f is q-invariant; therefore, it is enough to prove the following statement: for any $x_0 \in A \cap B$ such that $1 < |x_0| \leq |q|$, there exists an open disk $D(x_0, r)$, $r > 0$, and a constant $C > 0$ such that $f(x) \geq C$ for all $x \in D(x_0, r) \setminus (A \cap B)$.

On the other hand, the mapping p_0 (introduced at the beginning of 4.3.1) is a local diffeomorphism from $\mathbf{R}^2$ onto $\mathbf{C}^*$ inducing a local equivalence between the natural metrics. Therefore the proof of the lemma boils down to the transversality between A_0 and B_0, which is merely deduced from the transversality of each pair of rectangles $(A_{m,n}, B_{m',n'})$ of the euclidean plane. □

Let $\gamma \in H^1(\mathbf{E}_q; \Lambda_I)$, we can represent it by an element $g \in Z^1(\mathfrak{U}; \Lambda_I)$ where $\mathfrak{U}$ is a good open covering of $\mathbf{E}_q$. We can suppose that all the parallelograms of the covering are small. We will associate to g a germ of fibered space $\big(M_g, \pi, (\mathbf{C}, 0)\big)$. We will see that if we change the choice of g, then these germs of linear fibered space correspond by a canonical isomorphism. This construction mimicks a construction of [**28**] inspired by an idea of Malgrange (*cf.* [**24**]), it is a central point for the proof of one of our main results.

Let $g \in Z^1(\mathfrak{U}; \Lambda_I)$. We interpret the g_i as germs of holomorphic functions on the germ of $q^{-\mathbf{N}}$-invariant open set defined by U_i, that is the germ of $p^{-1}(U_i) \cap D$ (D being an open disk centered at the origin). For sake of simplicity we will denote also by U_i this germ. We consider the disjoint sum $\coprod_{i \in I} U_i \times \mathbf{C}^n$ and (using representatives of germs) we identify the points $(x, Y) \in U_i \times \mathbf{C}^n$ and $(x, Z) \in U_j \times \mathbf{C}^n$ if $x \in U_{ij}$ and $Z = g_{ij}(x)(Y)$ (we verify that we have an equivalence relation using the cocycle condition), we denote the quotient by $\mathbf{M}_g$. If $x = 0$, $g_i(0)$ is the identity of $\mathbf{C}^n$ therefore the quotient $\mathbf{M}_g$ is a germ of topological space along $\{0\} \times \mathbf{C}^n$.

Lemma 4.3.7. — (i) *The projections $U_i \times \mathbf{C}^n \to U_i$ induce a germ of continuous fibration $\pi : \mathbf{M}_g \to (\mathbf{C}, 0)$, the fiber $\pi^{-1}(0)$ is identified with $(\mathbf{C}^n, 0)$.*

(ii) *The germ $\dot{\mathbf{M}}_g := \mathbf{M}_g \setminus \pi^{-1}(0)$ admits a natural structure of germ of complex manifold of dimension $n + 1$, it is the unique structure such that the natural injections $(U_i \setminus \{0\}) \times \mathbf{C}^n \to \dot{\mathbf{M}}_g$ are holomorphic. The restriction of π to $\dot{\mathbf{M}}_g$ is holomorphic.*

The proof of the lemma is easy.

Lemma 4.3.8. — *There exists on $\mathbf{M}_g$ a unique structure of germ of differentiable manifold ($\mathcal{C}^\infty$) such that the natural injections $(U_i \setminus \{0\}) \times \mathbf{C}^n \to \dot{\mathbf{M}}_g$ are $\mathcal{C}^\infty$ (in Whitney sense). The induced structure on $\dot{\mathbf{M}}_g$ is the underlying structure of the holomorphic structure on $\dot{\mathbf{M}}_g$. The map π is $\mathcal{C}^\infty$ of rank one, $\big(\mathbf{M}_g, \pi, (\mathbf{C}, 0)\big)$ is a germ of vector bundle.*

Proof. — This lemma is the central point. The proof uses the Whitney extension theorem and the notion of regularly separated closed sets (*cf.* above).

We replace the good open covering $(U_i)_{i\in I}$ by a good closed covering $(V_i)_{i\in I}$ with $V_i \subset U_i$. We interpret the V_i as $q^{-\mathbf{N}}$-invariant germs of compact sets $p^{-1}(V_i) \cap \bar{D} \subset \mathbf{C}^*$ (we take $\bar{D} := \bar{D}(r)$, $r > 0$). The differentiable functions on $V_i \times \mathbf{C}^n$ are by definition the $\mathcal{C}^\infty$ functions in Whitney sense. We can built a germ of set as a quotient of $\coprod_{i\in I} V_i \times \mathbf{C}^n$ using g as above, the natural map from this quotient to $\mathbf{M}_g$ is an homeorphism and we can identify these two sets. If we consider a representative of the germ $\mathbf{M}_g$ and $x \in \mathbf{C}^*$ sufficiently small, there exists a structure of differentiable manifold along $\pi^{-1}(x)$ and it satisfies all the conditions. Let $Y_0 \in \mathbf{C}^n$, we set $m_0 = (0, Y_0)$ and we denote by $\mathscr{E}_{m_0}$ the ring of germs of real functions on $\mathbf{M}_g$ represented by germs at $m_0 \in V_i \times \mathbf{C}^n$ of functions f_i compatible with glueing applications, that is $f_j \circ g_{ji} = f_i$.

We will prove that each cordinate y_h ($h = 1, \dots, n$) on $\mathbf{C}^n$ extends in an element η_h of $\mathscr{E}_{m_0}$. It suffices to consider y_1.

We start with $i \in I$ We extend y_1 in a $\mathcal{C}^\infty$ function f_i on $V_i \times \mathbf{C}^n$ (we can choose $f_i = y_1$).

Let $j \in I$, $j \neq i$. The glueing map g_{ij} gives a $\mathcal{C}^\infty$ function h_j on $V_{ij} \times \mathbf{C}^n \subset V_j \times \mathbf{C}^n$, using Whitney extension theorem, we can extend it in a $\mathcal{C}^\infty$ function f_j on $V_j \times \mathbf{C}^n$.

Let $k \in I$, $k \neq i, j$. The glueing maps g_{ik} and g_{jk} give functions h_{ik} and h_{jk} respectively on $V_{ik} \times \mathbf{C}^n \subset V_k \times \mathbf{C}^n$ and $V_{jk} \times \mathbf{C}^n \subset V_k \times \mathbf{C}^n$; these functions coincide on $V_{ijk} \times \mathbf{C}^n \subset V_k \times \mathbf{C}^n$. The closed sets $V_{ik} \times \mathbf{C}^n$ and $V_{jk} \times \mathbf{C}^n$ are regularly separated (V_{ik} and V_{jk} are, as subsets of $\mathbf{E}_q$, closed parallelograms and we can apply lemma 4.3.6), therefore the $\mathcal{C}^\infty$ functions h_{ik} and h_{jk} define a $\mathcal{C}^\infty$ function h_k on $(V_{ik} \cup V_{jk}) \times \mathbf{C}^n \subset V_k \times \mathbf{C}^n$, using Whitney extension theorem, we can extend it in a $\mathcal{C}^\infty$ function f_k on $V_k \times \mathbf{C}^n$.

Let $h \in I$, $h \neq i, j, k$, then $V_{ih} \cap V_{jh} \cap V_{kh} = V_{ijkh} = \emptyset$ and we can do as in the preceding step, using moreover the fact that to be $\mathcal{C}^\infty$ is a local property.

We can end the proof along the same lines. $\square$

The n functions $\eta_1, \dots, \eta_m \in \mathscr{E}_{m_0}$ and π, interpreted as an element of $\mathscr{E}_{m_0}$, define a system of local coordinates on $\mathbf{M}_g$ in a neighborhood of m_0; an element of $\mathscr{E}_{m_0}$ is a $\mathcal{C}^\infty$ function of these coordinates. Therefore $\mathbf{M}_g$ admits a structure of differentiable manifold, this structure is fixed on $\dot{M}_g$, therefore unique.

Proposition 4.3.9. — (i) *The germ of differentiable manifold $\mathbf{M}_g$ admits a unique structure of germ of complex analytic manifold extending the complex analytic structure of $\dot{\mathbf{M}}_g$. For this structure the map π is holomorphic and its rank is one, $\big(\mathbf{M}_g, \pi, (\mathbf{C}, 0)\big)$ is a germ of holomorphic vector bundle.*

(ii) *We consider the formal completion $\hat{\mathbf{M}}_g$ of $\mathbf{M}_g$ along $\pi^{-1}(0)$ (for the holomorphic structure defined by (i)) and the formal completion $\hat{\mathbf{F}}$ of $(\mathbf{C}, 0) \times \mathbf{C}^n$ along $\{0\} \times \mathbf{C}^n$. We denote by $\hat{\pi} : \hat{M}_g \to \big((\mathbf{C}, 0)\hat{|}\{0\}\big)$ the completion of π and by $\hat{\pi}_0 : \hat{\mathbf{F}} \to \big((\mathbf{C}, 0)\hat{|}\{0\}\big)$ the natural projection. The formal vector bundle $\Big(\hat{\mathbf{M}}_g, \hat{\pi}, \big((\mathbf{C}, 0)\hat{|}\{0\}\big)\Big)$ is naturally isomorphic to the formal vector bundle $\big(\hat{\mathbf{F}}, \hat{\pi}_0, ((\mathbf{C}, 0)\hat{|}\{0\})\big)$.*

Proof. — The proof of (i) is based on the Newlander-Niremberg integrability theorem, it is similar, mutatis mutandis to a proof in [**28**], page 77. The proof of (ii) is easy (the $\hat{g}_{ij}$ are equal to identity). □

If g, g' are two representatives of γ, the complex analytic manifolds $\mathbf{M}_g$ and $\mathbf{M}_{g'}$ are isomorphic (*cf.* [**28**], page 77)

4.3.3. The formal trivialisation of the elements of $H^1(\mathbf{E}_q; \Lambda_I)$ and the fundamental isomorphism. — Using the geometric interpretation of the cocycles of $Z^1(\mathfrak{U}; \Lambda_I)$ we will build a map:

$$Z^1(\mathfrak{U}; \Lambda_I) \to \mathrm{GL}_n\big(\mathbf{C}[[z]]\big)$$

defined up to composition on the right by an element of $\mathrm{GL}_n\big(\mathbf{C}\{z\}\big)$. Hence we will get a map:

$$H^1(\mathbf{E}_q; \Lambda_I) \to \mathrm{GL}_n\big(\mathbf{C}[[z]]\big)/\mathrm{GL}_n\big(\mathbf{C}\{z\}\big)$$

and we will verify that it inverses the natural map:

$$\mathrm{GL}_n\big(\mathbf{C}[[z]]\big)/\mathrm{GL}_n\big(\mathbf{C}\{z\}\big) \to H^1(\mathbf{E}_q; \Lambda_I).$$

We consider the germ of trivial bundle $(\mathbf{C}, 0) \times \mathbf{C}^n, \pi_0, (\mathbf{C}, 0)$. We choose a holomorphic trivialisation H of the germ of fiber bundle $\big(\mathbf{M}_g, \pi, (\mathbf{C}, 0)\big)$: $\pi_0 \circ H = \pi$ (H is defined up to composition on the right by an element of $\mathrm{GL}_n\big(\mathbf{C}\{z\}\big)$). From the proposition 4.3.9 (ii) and H we get an *automorphism*

$\hat{\varphi}$ of the formal vector bundle $\big(\hat{\mathbf{F}}, \hat{\pi}_0, (\widehat{\mathbf{C}|}\{0\}\big)$. We can interpet $\hat{\varphi}$ as an element of $\mathrm{GL}_n\big(\mathbf{C}[[z]]\big)$, this element is well defined up to the choice of H, that is up to composition on the left by an element of $\mathrm{GL}_n\big(\mathbf{C}\{z\}\big)$. Hence $\hat{\varphi}^{-1}$ is defined up to composition on the right by an element of $\mathrm{GL}_n\big(\mathbf{C}\{z\}\big)$.

The map:

$$Z^1(\mathfrak{U}; \Lambda_I) \to \mathrm{GL}_n\big(\mathbf{C}[[z]]\big)/\mathrm{GL}_n\big(\mathbf{C}\{z\}\big)$$

induced by $g \mapsto \hat{\varphi}^{-1}$ induces a map:

$$H^1(\mathbf{E}_q; \Lambda_I) \to \mathrm{GL}_n\big(\mathbf{C}[[z]]\big)/\mathrm{GL}_n\big(\mathbf{C}\{z\}\big).$$

We will verify that this map is the inverse of the natural map:

$$\mathrm{GL}_n\big(\mathbf{C}[[z]]\big)/\mathrm{GL}_n\big(\mathbf{C}\{z\}\big) \to H^1(\mathbf{E}_q; \Lambda_I).$$

From the natural injections $(U_i, 0) \times \mathbf{C}^n \to M_g$ $(i \in I)$ and the trivialisation H, we get automorphisms of the vector bundles $\big((U_i, 0) \times \mathbf{C}^n \to (U_i, 0)\big)$, that we can interpret as elements $\varphi_i \in \Gamma\big(U_i; \mathrm{GL}_n(\mathscr{A})\big)$ compatible with the glueing maps g_{ij}, *i.e.* $\varphi_i = \varphi_j \circ g_{ji}$. We have:

$$g_{ij} = \varphi_i^{-1} \circ \varphi_j = \varphi_i^{-1} \circ (\varphi_j^{-1})^{-1} \text{ and } \partial(\varphi_i^{-1}) = (g_{ij}).$$

The element $\hat{\varphi}_i \in \mathrm{GL}_n\big(\mathbf{C}[[z]]\big)$ is independant of $i \in I$; we will denote it by $\hat{\varphi}$.

We have $(\varphi_i^{-1}) \in \mathscr{C}^0\big(\mathfrak{U}; \mathrm{GL}_n(\mathscr{A})\big)$. The coboundary of (φ_i^{-1}) is g and its natural image in $\mathrm{GL}_n\big(\mathbf{C}[[z]]\big)$ is $\hat{\varphi}^{-1}$. This ends the proof of our contention.

It is possible to do the same constructions replacing $\mathrm{GL}_n(\mathbf{C})$ by an algebraic subgroup (vector bundles are replaced by vector bundles admitting G as structure group). We leave the details to the reader.

Theorem 4.3.10 **(First q-Birkhoff-Malgrange-Sibuya theorem)**
Let G be an algebraic subgroup of $GL_n(\mathbf{C})$. The natural maps:

$$GL_n\big(\mathbf{C}[[z]]\big)/GL_n\big(\mathbf{C}\{z\}\big) \to H^1(\mathbf{E}_q; \Lambda_I),$$

$$G\big(\mathbf{C}[[z]]\big)/G\big(\mathbf{C}\{z\}\big) \to H^1(\mathbf{E}_q; \Lambda_I^G)$$

are bijective. □

4.4. The isoformal classification by the cohomology of the Stokes sheaf

We will see that the constructions of the preceding paragraph are compatible with discrete dynamical systems data and deduce a new version of the analytic isoformal classification. We use notations from 2.2.4.

4.4.1. The second main theorem. — Before going on, let us make some preliminary remarks. Let $M_0 := P_1 \oplus \cdots \oplus P_k$ be a pure module. Represent it by a matrix A_0 with the same form as in (8). Assume moreover that A_0 is in Birkhoff-Guenther normal form as found in 3.3.2.

We saw in proposition 2.2.9 that there is a natural bijective mapping between $\mathcal{F}(M_0) = \mathcal{F}(P_1, \ldots, P_k)$ and the quotient $\mathfrak{G}^{A_0}(\mathbf{C}((z)))/\mathfrak{G}(\mathbf{C}(\{z\}))$. It also follows from the first part of 3.3.3 that this, in turn, is equal to $\mathfrak{G}^{A_0}(\mathbf{C}[[z]])/\mathfrak{G}(\mathbf{C}\{z\})$. Last, the inclusion $\mathfrak{G} \subset \mathrm{GL}_n$ induces a bijection from $\mathfrak{G}^{A_0}(\mathbf{C}[[z]])/\mathfrak{G}(\mathbf{C}\{z\})$ to the quotient $\mathrm{GL}_n^{M_0}\big(\mathbf{C}[[z]]\big)/\mathrm{GL}_n\big(\mathbf{C}\{z\}\big)$, where we take $\mathrm{GL}_n^{M_0}\big(\mathbf{C}[[z]]\big)$ to denote the set of those formal gauge transformation matrices that send A_0 into $\mathrm{GL}_n\big(\mathbf{C}\{z\}\big)$.

***Theorem 4.4.1* (Second q-Birkhoff-Malgrange-Sibuya theorem)**
Let M_0 be a pure module represented by a matrix A_0 in Birkhoff-Guenther normal form. There is a natural bijective mapping:

$$GL_n^{M_0}\big(\mathbf{C}[[z]]\big)/GL_n\big(\mathbf{C}\{z\}\big) \to H^1\big(\mathbf{E}_q; \Lambda_I(M_0)\big),$$

whence a natural bijective mapping:

$$\mathcal{F}(M_0) \to H^1\big(\mathbf{E}_q; \Lambda_I(M_0)\big).$$

Proof. — *The map λ*
Let $\mathfrak{U} := (U_i)_{i \in I}$ be a good open covering of $\mathbf{E}_q$.
Let $\hat{F} \in \mathrm{GL}_n^{M_0} \subset \mathrm{GL}_n\big(\mathbf{C}[[z]]\big)$; by definition there exists an unique $A \in \mathrm{GL}_n\big(\mathbf{C}\{z\}\big)$ such that $\hat{F}[A_0] = A$.
For all $i \in I$, there exists $g_i \in \mathrm{GL}_n\big(\mathcal{A}(U_i)\big)$ asymptotic to $\hat{F}$ on U_i (*cf.* theorem 4.2.1). The coboundary (g_{ij}) of the cochain (g_i) belongs to $Z^1(\mathfrak{U}, \Lambda_I(M_0))$ (the reason for that is lemma 4.2.7 which can be applied because the condition of good covering implies that the open sets are adequate); it does not change if we multiply $\hat{F}$ on the right by an element of $\mathrm{GL}_n\big(\mathbf{C}\{z\}\big)$. If we change the representatives g_i, we get another cochain in $Z^1(\mathfrak{U}, \Lambda_I(M_0))$ inducing the same

element of $H^1(\mathfrak{U}, \Lambda_I(M_0))$. We thus get a natural map:

$$\lambda : \mathrm{GL}_n^{M_0}\big(\mathbf{C}[[z]]\big)/\mathrm{GL}_n\big(\mathbf{C}\{z\}\big) \to H^1\big(\mathbf{E}_q; \Lambda_I(M_0)\big).$$

This map is injective.

Surjectivity of λ Let $\mathfrak{U} := (U_i)_{i\in I}$ be a good open covering of $\mathbf{E}_q$; assume moreover that the compact covering $(\overline{U}_i)_{i\in I}$ is good and that the U_i are small parallelograms.

Let $\gamma \in H^1(\mathbf{E}_q; \Lambda_I(M_0))$; we can represent it by an element $g \in Z^1(\mathfrak{U}; \Lambda_I(M_0)$, $\mathfrak{U} = (U_i)_{i\in I}$ being a good open covering of $\mathbf{E}_q$. Moreover we can choose $\mathfrak{U}$ such that the compact covering $(\overline{U}_i)_{i\in I}$ is good and such that the U_i are small parallelograms.

As in section 4.3.1, using g, we can built a germ of analytic manifold $\mathbf{M}_g$ and a holomorphic fibration $\pi : \mathbf{M}_g \to (\mathbf{C}, 0)$. But now there is a new ingredient: we have a holomorphic automorphism $\Theta_0 : (\mathbf{C}, 0) \times \mathbf{C}^n \to (\mathbf{C}, 0) \times \mathbf{C}^n$ of the trivial bundle $\big((\mathbf{C}, 0) \times \mathbf{C}^n, \pi_0, (\mathbf{C}, 0)\big)$ defined by:

$$(z, X) \mapsto (qz, A_0 X).$$

It induces, for all $i \in I$ a holomorphic automorphism of fibre bundles $\Theta_{0,i} : U_i \times \mathbf{C}^n \to U_i$.

By definition the $g_{ij} \in \Lambda_I(M_0)(U_{ij})$ commute with Θ_0, therefore the automorphisms $\Theta_{0,i}$ glue together into a holomorphic automorphism Ψ of the germ of fibered manifold $\big(\mathbf{M}_g, \pi, (\mathbf{C}, \{0\})\big)$. The map Ψ is linear on the fibers and the map on the basis is the germ of $z \mapsto qz$.

We choose a holomorphic trivialisation H of the germ of fiber bundle $\big(\mathbf{M}_g, \pi, (\mathbf{C}, 0)\big)$: $\pi_0 \circ H = \pi$. The map $\Phi := H \circ \Psi \circ H^{-1}$ is a holomorphic automorphism of the germ of trivial bundle $\big((\mathbf{C}, 0) \times \mathbf{C}^n, \pi_0, (\mathbf{C}, 0)\big)$, it corresponds to an element $A \in \mathrm{GL}_n\big(\mathbf{C}\{z\}\big)$. Using the results and notations of section 4.3.1, we see that there exists an element $\hat{\varphi} \in \mathrm{GL}_n\big(\mathbf{C}\{z\}\big)$ such that $g \mapsto \hat{\varphi}^{-1}$ induces the inverse of the natural map $\mathrm{GL}_n\big(\mathbf{C}[[z]]\big)/\mathrm{GL}_n\big(\mathbf{C}\{z\}\big) \to H^1\big(\mathbf{E}_q; \Lambda_I\big)$. If we set $\hat{F} := \hat{\varphi}^{-1}$, it is easy to check that $\hat{F}[A_0] = A$. We have $\hat{F} \in \mathrm{GL}_n^{M_0}$ and $\lambda(\hat{F}) = g$, therefore λ is surjective. $\square$

CHAPTER 5

SUMMATION AND ASYMPTOTIC THEORY

To find analytic solutions having a given formal solution as asymptotic expansion is a rather old subject in the theory of q-difference equations: see for instance the paper [**50**] by Trjitzinsky in 1933. The first author had already suggested in [**34**] the use of the gaussian function to formulate a q-analogue for Laplace transform; this allowed to find in [**53**] the Gq-summation, a a q-analog of the exponential summation method of Borel-Laplace. In [**27**], the Gq-summation was extended to the case of multiple levels, and it was shown that any formal power series solution of a linear equation with analytic coefficients is Gq-multisummable, and thus can be seen as the asymptotic expansion of an analytical solution in a sector with infinite opening in the Riemann surface of the logarithm. The work in [**39**] [**38**], [**56**], [**54**] was undertaken with the goal of obtaining a summation over the elliptic curve, or a finite covering of it. We used these results in the work presented in [**38**].

5.1. Some preparatory notations and results

In this section, we will introduce some notations related elliptic curves $\mathbf{E}_q$ and to Jacobi theta function.

5.1.1. Divisors and sectors on the elliptic curve $\mathbf{E}_q = \mathbf{C}^*/q^{\mathbf{Z}}$. — The projection $p : \mathbf{C}^* \to \mathbf{E}_q$ and the discrete logarithmic q-spiral $[\lambda; q]$ were defined in the general notations, section 1.3. In this chapter, we shall usually shorten the latter notation into $[\lambda] := [\lambda; q]$. We call *divisor* a finite formal sum of weighted such q-spirals:

$$\Lambda = \nu_1[\lambda_1] + \cdots + \nu_m[\lambda_m], \text{ where } \nu_i \in \mathbf{N}^* \text{ and } [\lambda_i] \neq [\lambda_j] \text{ if } i \neq j.$$

This can be identified with the *effective* divisor $\sum \nu_i \left[\overline{\lambda_i}\right]$ on $\mathbf{E}_q$ and, to simplify notations, we shall also write $\sum \nu_i[\lambda_i]$ that divisor on $\mathbf{E}_q$. The *support* of Λ is the union of the q-spirals $[\lambda_1], \ldots, [\lambda_m]$. We write:

$$|\Lambda| := \nu_1 + \cdots + \nu_m$$

the *degree* of Λ (an integer) and:

$$\|\Lambda\| := (-1)^{|\Lambda|} \lambda_1^{\nu_1} \cdots \lambda_m^{\nu_m} \pmod{q^{\mathbf{Z}}}$$

the *weight* of Λ, an element of $\mathbf{E}_q$. It is equal, in additive notation, to $\sum \nu_i p(-\lambda_i)$ evaluated in $\mathbf{E}_q$.

For any two non zero complex numbers z and λ, we put:

$$d_q(z, [\lambda]) := \inf_{\xi \in [\lambda]} \left|1 - \frac{z}{\xi}\right|.$$

This is a kind of "distance" from z to the q-spiral $[\lambda]$: one has $d_q(z, [\lambda]) = 0$ if, and only if $z \in [\lambda]$.

Lemma 5.1.1. — *Let* $\|q\|_1 := \inf\limits_{n \in \mathbf{Z}^*} |1 - q^n|$ *and* $M_q := \frac{\|q\|_1}{2 + \|q\|_1}$. *Let* $a \in \mathbf{C}$ *be such that* $|1 - a| \leq M_q$. *Then:*

$$d_q(a; [1]) = |1 - a|.$$

Proof. — This follows at once from the inequalities:

$$|1 - aq^n| \geq |a| \; |1 - q^n| - |1 - a| \geq (1 - M_q)\|q\|_1 - |1 - a| \; \geq |1 - a|.$$

□

For $\rho > 0$, we put:

$$D([\lambda]; \rho) := \{z \in \mathbf{C}^* \mid d_q(z, [\lambda]) \leq \rho\}, \quad D^c([\lambda]; \rho) := \mathbf{C}^* \setminus D([\lambda]; \rho),$$

so that $D([\lambda]; 1) = \mathbf{C}^*$ and $D^c([\lambda]; 1) = \emptyset$. From lemma 5.1.1, it follows that for any $\rho < M_q$:

$$D([\lambda]; \rho) = \bigcup_{n \in \mathbf{Z}} q^n D(\lambda; |\lambda|\rho),$$

where $D(\lambda; |\lambda|\rho)$ denotes the closed disk with center λ and radius $|\lambda|\rho$.

Let $\lambda, \mu \in \mathbf{C}^*$; since:

$$\left|1 - \frac{z}{\mu} q^n\right| \geq \left|\frac{z}{\lambda}\right| \; \left|1 - \frac{\lambda}{\mu} q^n\right| - \left|1 - \frac{z}{\lambda}\right|,$$

one gets:

$$d_q(z,[\mu]) \geq d_q(\lambda,[\mu]) - \rho\big(1 + d(\lambda,[\mu])\big) \tag{15}$$

when $z \in D([\lambda];\rho)$ and $\rho < M_q$.

Proposition 5.1.2. — *Given two distinct q-spirals $[\lambda]$ and $[\mu]$, there exists a constant $N > 1$ such that, for all $\rho > 0$ near enough from 0, if $d_q(z,[\lambda]) > \rho$ and $d_q(z,[\mu]) > \rho$, then:*

$$d_q(z,[\lambda])\, d_q(z,[\mu]) > \frac{\rho}{N}\cdot$$

Proof. — By contradiction, assume that, for all $N > 1$, there exists $\rho_N > 0$ and $z \in \mathbf{C}^*$ such that $d_q(z,[\lambda]) > \rho_N$ and $d_q(z,[\mu]) > \rho_N$ but $d_q(z,[\lambda])\, d_q(z,[\mu]) \leq \dfrac{\rho_N}{N}$; this would entail:

$$\rho_N < d_q(z,[\lambda]) < \frac{1}{N},$$

whence, after (15):

$$d_q(z,[\mu]) \geq d_q(\lambda,[\mu]) - \big(1 + d_q(\lambda,[\mu])\big)/N.$$

In other words, when $N \to \infty$, one would get:

$$d_q(z,[\lambda])\, d_q(z,[\mu]) \gtrsim d_q(\lambda,[\mu])\ \rho_N$$

(*i.e.* there is an inequality up to an $o(1)$ term), contradicting the assumption $d_q(z,[\lambda])\ d_q(z,[\mu]) < \frac{\rho_N}{N}$. □

Let $\Lambda := \nu_1[\lambda_1] + \cdots + \nu_m[\lambda_m]$ be a divisor and let $\rho > 0$. We put:

$$d_q(z,\Lambda) := \prod_{1\leq j\leq m} \left(d_q(z,[\lambda_j])\right)^{\nu_j} \tag{16}$$

and

$$D(\Lambda;\rho) := \{z \in \mathbf{C}^* \mid d_q(z,\Lambda) \leq \rho\}\ ,\quad D^c(\Lambda;\rho) := \mathbf{C}^* \setminus D(\Lambda;\rho). \tag{17}$$

Proposition 5.1.2 implies that, if $\Lambda = [\lambda] + [\mu]$ with $[\lambda] \neq [\mu]$, one has:

$$D^c([\lambda];\rho) \cap D^c([\mu];\rho) \subset D^c\left(\Lambda; \frac{\rho}{N}\right)$$

as long as ρ is near enough from 0.

Proposition 5.1.3. — *Let $\Lambda := \nu_1[\lambda_1] + \cdots + \nu_m[\lambda_m]$ be a divisor such that $[\lambda_i] \neq [\lambda_j]$ for $i \neq j$. There exists a constant $N > 1$ such that, for all $\rho > 0$ near enough from 0, one has:*

$$D^c(\Lambda;\rho) \subset \bigcap_{1\leq j\leq m} D^c([\lambda_j];\rho^{1/\nu_j}) \subset D^c\left(\Lambda; \frac{\rho}{N}\right). \tag{18}$$

Proof. — The first inclusion comes from the fact that, if $d_q(z,[\lambda_j]) \leq \epsilon^{1/\nu_j}$ for some index j, then $d_q(z,\Lambda) \leq \epsilon$, because $d_q(z,[\lambda]) \leq 1$ for all $\lambda \in \mathbf{C}^*$.

To prove the second inclusion, we use (15) in the same way as in the proof of proposition 5.1.2; details are left to the reader. □

Taking the complementaries in (18), one gets:

$$D\left(\Lambda;\frac{\rho}{N}\right) \subset \bigcup_{1\leq j\leq m} D\left([\lambda_j];\rho^{1/\nu_j}\right) \subset D(\Lambda;\rho). \tag{19}$$

Modifying the definition (16) of $d_q(z,\Lambda)$ as follows:

$$\delta(z,\Lambda) := \min_{1\leq j\leq m} \{d_q(z,[\lambda_j])\}^{\nu_j}, \tag{20}$$

one gets, for all small enough ρ:

$$D_\delta(\Lambda;\rho) := \{z \in \mathbf{C}^* : \delta(z,\Lambda) \leq \rho\} = \bigcup_{1\leq j\leq m} D([\lambda_j];\rho^{1/\nu_j})$$

Relation (19) shows that the maps $z \mapsto d_q(z,\Lambda)$ and $z \mapsto \delta(z,\Lambda)$ define equivalent systems of small neighborhoods relative to the divisor Λ; precisely, for $\rho \to 0$:

$$D\left(\Lambda;\frac{\rho}{N}\right) \subset D_\delta(\Lambda;\rho) \subset D(\Lambda;\rho).$$

Definition 5.1.4. — Let $\epsilon \geq 0$. We call *germ of sector within distance* ϵ *from divisor* Λ any set:

$$S(\Lambda,\epsilon;R) = \{z \in \mathbf{C}^* : d_q(z,\Lambda) > \epsilon, |z| < R\}, \text{ where } R > 0.$$

When $R = +\infty$, we write $S(\Lambda,\epsilon)$ instead of $S(\Lambda,\epsilon;+\infty)$.

Note that $S(\Lambda,0;R)$ is the punctured open disk $\{0 < |z| < R\}$ deprived of the q-spirals pertaining to divisor Λ. Since $d_q(z,\Lambda) \leq 1$ for all $z \in \mathbf{C}^*$, one has $S(\Lambda,1;R) = \emptyset$; that is why we shall assume that $\epsilon \in]0,1[$. Last, each sector within a short enough distance to Λ represents, on the elliptic curve $\mathbf{E}_q = \mathbf{C}^*/q^{\mathbf{Z}}$, the curve $\mathbf{E}_q$ deprived of a family of ovals centered on the λ_j with sizes varying like ϵ^{1/ν_j}.

By a *(germ of) analytic function in* 0 *out of* Λ, we shall mean any function defined and analytic in some germ of sector $S(\Lambda,0;R)$ with $R > 0$. Such a function f is said to be *bounded* (at 0) if, for all $\epsilon > 0$ and all $r \in]0,R[$, one has:

$$\sup_{z\in S(\Lambda,\epsilon;r)} |f(z)| < \infty.$$

We write $\mathbb{B}^\Lambda$ the set of such functions.

Proposition 5.1.5. — *If $f \in \mathbb{B}^{\Lambda}$, then for any $R > 0$ small enough and for all $a \in S(\Lambda, 0; R)$ such that $|a| \notin \bigcup_{n\in\mathbf{N}} |\lambda_i|\,|q^{-n}|$, there exists $K > 0$ such that*

$$\sup_{n\in\mathbf{N}} \max_{|z|=|aq^{-n}|} |f(z)| < K.$$

Proof. — From the assumption on a, it follows that:

$$\epsilon := \frac{\max_{\tau\in[0,2\pi]} d_q(ae^{i\tau}, \Lambda)}{2} > 0,$$

whence all circles centered at 0 and with respective radii $|aq^{-n}|$, $n \in \mathbf{N}$ are contained in the sector $S(\Lambda, \epsilon; R)$, which concludes the proof. □

5.1.2. Jacobi Theta function, q-exponential growth and q-Gevrey series.

— Jacobi's theta function $\theta(z; q) = \theta(z)$ was defined by equation (2) page 8. From its functional equation, one draws:

$$\forall (z, n) \in \mathbf{C}^* \times \mathbf{Z}\ ,\ \theta(q^n z) = q^{n(n+1)/2}\ z^n\ \theta(z). \tag{21}$$

Moreover, for all $z \in \mathbf{C}^*$, one has:

$$|\theta(z)| \leq e(z) := e(z; q) := \theta\left(|z|\,; |q|\right).$$

Lemma 5.1.6. — *There exists a constant $C > 0$, depending on q only, such that, for all $\epsilon > 0$:*

$$|\theta(z)| \geq C\ \epsilon\ e(z),$$

as long as $d_q(z, [-1]) > \epsilon$.

Proof. — It is enough to see that the function $z \mapsto |\theta(z)|\,/e(z)$ is q-invariant, which allows one to restrict the problem to the (compact) closure of a fundamental annulus, for instance $\{z \in \mathbf{C} \mid 1 \leq |z| \leq |q|\}$. See [**54**, Lemma 1.3.1] for more details. □

Recall that $\theta(z) = \theta\left(\dfrac{1}{qz}\right)$; one draws from this that $e(z) = e\left(\dfrac{1}{qz}\right)$. To each subset W of $\mathbf{C}$, one associates two sets $W_{(\infty)}$, $W_{(0)}$ as follows:

$$W_{(\infty)} := \bigcup_{n\geq 0} q^n\ W\ ,\quad W_{(0)} := \bigcup_{n\leq 0} q^n\ W\ .$$

Definition 5.1.7. — Let f be a function analytic over an open subset Ω of $\mathbf{C}^*$ and let $k \geq 0$.

1. We say that f has q-exponential growth of order (at most) k at infinity (resp. at 0) over Ω if, for any compact $W \subset \mathbf{C}$ such that $W_{(\infty)} \subset \Omega$ (resp. such that $W_{(0)} \subset \Omega$), the exists $C > 0$ and $\mu > 0$ such that $|f(z)| \leq C\big(e(\mu z)\big)^k$ for all $z \in W_{(\infty)}$ (resp. $z \in W_{(0)}$).
2. We say that f has q-exponential decay of order (at least) k at infinity (resp. at 0) over Ω if the condition $|f(z)| \leq C\big(e(\mu z)\big)^k$ can be replaced by $|f(z)| \leq C\big(e(\mu z)\big)^{-k}$.

Lemma 5.1.8. — *Let $k > 0$ and f an entire function with Taylor expansion $f(z) = \sum\limits_{n \geq 0} a_n z^n$ at 0. Then f has q-exponential growth of order (at most) k at infinity over $\mathbf{C}$, if, and only if, the sequence (a_n) is q-Gevrey of order $(-1/k)$.*

Proof. — The q-Gevrey order of a sequence was defined in paragraph 1.3.2 of the general notations, in the introduction. A variant of this result was given in [**33**, Proposition 2.1]. The proof shown below rests on the functional equation $\theta(qz) = q\ z\ \theta(z)$.

If (a_n) is q-Gevrey of order $(-k)$, one has $|f(z)| \leq C\ \theta(\mu\,|z|\,; |q|^{1/k})$, which shows that f has q-exponential growth of order (at most) k at infinity over $\mathbf{C}$.

On the other hand, by Cauchy formula, one has:

$$|a_n| \leq \min_{m \in \mathbf{Z}} \left(\max_{|z| = \mu |q|^m} |f(z)|\,(\mu\ \ |q|^m)^{-n} \right),$$

where $\mu > 0$ is an arbitrary parameter. If f is an entire function with q-exponential growth of order (at most) k at infinity, one has $|f(z)| \leq C\ \big(e(z)\big)^k$ for all $z \in \mathbf{C}^*$. From relation $e(q^m z) = |q|^{m(m+1)/2}\ \ |z|^m\ \ e(z)$, one draws, for all integers $n \geq 0$:

$$|a_n| \leq C\ e^k(\mu)\ \mu^{-n}\ \big(\mu^k\ \ |q|^{-n+k/2}\big)^m\ \ |q|^{km^2/2}\,.$$

Last, we note that the function $t \mapsto A^t\,|q|^{kt^2/2}$ reaches its minimum at $t = -\dfrac{\ln A}{k \ln |q|}$, with:

$$\min_{t \in \mathbf{R}} A^t\,|q|^{kt^2/2} = e^{-\frac{\ln^2 A}{2k \ln |q|}}.$$

From this, one deduces:

$$\min_{m \in \mathbf{Z}} A^m\,|q|^{km^2/2} \leq |q|^{\frac{k}{8}}\ \ e^{-\frac{\ln^2 A}{2k \ln |q|}} \tag{22}$$

for all $A > 0$. Therefore, if we set $A = \mu^k |q|^{-n+k/2}$, we get the wanted q-Gevrey estimates for $|a_n|$, which concludes the proof. □

5.2. Asymptotics relative to a divisor

In this section, we fix a divisor $\Lambda := \nu_1[\lambda_1] + \cdots + \nu_m[\lambda_m]$ and we write $\nu := |\Lambda| = \nu_1 + \cdot + \nu_m$. Without loss of generality, we make the following

ASSUMPTION:
The complex numbers λ_i are pairwise distinct and such that:

$$1 \leq |\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_m| < |q\lambda_1| \,. \tag{23}$$

Definition 5.2.1. — Let $f \in \mathbb{B}^\Lambda$. We write[(1)] $f \in \mathbb{A}_q^\Lambda$ and we say that f admits a *q-Gevrey asymptotic expansion of level* ν *(or order* $s := 1/\nu$*) along divisor* Λ *at* 0, if there exists a power series $\sum\limits_{n\geq 0} a_n z^n$ and constants C, $A > 0$ such that, for any $\epsilon > 0$ and any integer $N \geq 0$, one has, for some small enough $R > 0$ and for all $z \in S(\Lambda, \epsilon; R)$:

$$\left| f(z) - \sum_{0\leq n<N} a_n z^n \right| < \frac{C}{\epsilon} A^N \left|q\right|^{N^2/(2\nu)} \left|z\right|^N \quad . \tag{24}$$

Recall that condition $z \in S(\Lambda, \epsilon; R)$ means that $0 < |z| < R$ and $d_q(z, \Lambda) > \epsilon$; see definition 5.1.4. One sees at once that, if $f \in \mathbb{A}_q^\Lambda$, its asymptotic expansion, written $J(f)$, belongs to the space of q-Gevrey series of level ν, whence the following linear map:

$$J : \quad \mathbb{A}_q^\Lambda \to \mathbf{C}[[z]]_{q;1/\nu}, \quad f \mapsto J(f).$$

(For the notation $\mathbf{C}[[z]]_{q;s}$, see paragraph 1.3.2 of the general notations, in the introduction.) In section 5.3, we shall see that this map is onto, providing a "meromorphic q-Gevrey version" of classical Borel-Ritt theorem. In order to get the surjectivity of J, we shall obtain a characterization of $\mathbb{A}_q^\Lambda$ in terms of the residues along poles belonging to the divisor Λ. This description will be shared among the following two paragraphs.

5.2.1. Asymptotics and residues (I). — When $\nu = 1$, divisor Λ reduces to $[\lambda]$ and the above definition coincides with the definition of $\mathbb{A}_{q;1}^{[\lambda;q]}$ in [**54**], [**39**]. After [**39**], one has $f \in \mathbb{A}_q^{[\lambda]}$ if, and only if, there exists an integer $N \in \mathbf{N}$, a q-Gevrey sequence $(c_n)_{n\geq N}$ of order (-1) and a function h holomorphic near

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(1) The set $\mathbb{A}_q^\Lambda$ was previously written $\mathbb{A}_{q;|\Lambda|}^\Lambda$ in our Note [**38**].

$z = 0$ such that, for any $z \in S([\lambda], 0; |\lambda q^{-N}|)$:

$$f(z) = \sum_{n \geq N} \frac{c_n}{z - q^{-n}\lambda} + h(z). \tag{25}$$

In the sequel, we shall give a caracterisation of all elements of $\mathbb{A}_q^\Lambda$ with help of its partial fraction decomposition along divisor Λ.

Theorem 5.2.2. — *Let $f \in \mathbb{B}^\Lambda$. The following assertions are equivalent:*

(i) : *One has $f \in \mathbb{A}_q^\Lambda$.*

(ii) : *There exists an integer N_0 and ν sequences $(\alpha_{i,j,n})_{n \geq N_0}$ ($1 \leq i \leq m$, $0 \leq j < \nu_i$) of q-Gevrey level $(-\nu)$ and a function h holomorphic at $0 \in \mathbf{C}$ such that the following equality holds over the sector $S(\Lambda, 0; |q^{-N_0}\lambda_m|)$:*

$$f(z) = \sum_{n \geq N_0} \sum_{1 \leq i \leq m} \sum_{0 \leq j < \nu_i} \frac{\alpha_{i,j,n}}{(z - \lambda_i q^{-n})^{j+1}} + h(z). \tag{26}$$

Proof. — Recall that $\nu := |\Lambda| \in \mathbf{N}^*$. Let $f \in \mathbb{B}^\Lambda$. Under assumption (23), page 85 about the λ_i, we can choose a real $R > 0$ and an integer N_0 such that, on the one hand, one has $\left|q^{-N_0}\lambda_m\right| < R < \left|q^{-N_0+1}\lambda_1\right|$ and, on the other hand, f is defined and analytic in the open sector $S(\Lambda, 0; R)$. Therefore, the only possible singularities of f in the punctured disc $0 < |z| < R$ belong to the half q-spirals $\lambda_i q^{-N_0 - \mathbf{N}}$, $1 \leq i \leq m$.

Proof of (i) $\Rightarrow$ (ii): It will rest on lemmas 5.2.3, 5.2.4 and 5.2.5.

Lemma 5.2.3. — *We keep the above notations R, N_0, and moreover assume that $f \in \mathbb{A}_q^\Lambda$. Then f has a pole with multiplicity at most ν_i at each point of the half q-spiral $\lambda_i q^{-N_0 - \mathbf{N}}$.*

Proof. — Let $n \geq N_0$ be an integer. When z tends to $z_{i,n} := \lambda_i q^{-n}$, one has $d_q(z, \Lambda) = \left|1 - \dfrac{z}{z_{i,n}}\right|^{\nu_i} \to 0$. Taking $\epsilon := \dfrac{d_q(z, \Lambda)}{2}$ and $N := 0$ in relation (24), definition 5.2.1, one finds:

$$|f(z)| < \frac{2C}{\left|1 - \frac{z}{z_{i,n}}\right|^{\nu_i}},$$

which allows one to conclude. □

Write $P_{i,n}$ the polar part of f at point $\lambda_i q^{-n}$:

$$P_{i,n}(z) = \sum_{0 \leq j < \nu_i} \frac{\alpha_{i,j,n}}{(z - \lambda_i q^{-n})^{j+1}}. \tag{27}$$

We are going to study the growth of coefficients with respect to index n when n goes to infinity. For that, we choose some small $\eta \in \left(0, \frac{|q|}{1+|q|}\right)$ so that, putting $r_{i,n} := |\lambda_i q^{-n}|\,\eta$, all the open disks $D^{\eta}_{i,n} := \{z \in \mathbf{C} \mid |z - \lambda_i q^{-n}| < r_{i,n}\}$ $(1 \le i \le m,\ n \ge N_0)$ are pairwise disjoint. Also write $\mathcal{C}^{\eta}_{i,n} = \partial^+ D^{\eta}_{i,n}$ the boundary of disk $D_{i,n}$, with positive orientation.

Lemma 5.2.4. — *Let* $\mathcal{R}_N := f(z) - \sum_{k=0}^{N-1} a_n z^n$*, the* N*-th remainder of* f*. For all* $z \in \mathcal{C}^{\eta}_{i,n}$*, one has:*

$$|\mathcal{R}_N(z)| < \frac{2C}{\eta^{\nu}}\ \left(2A\left|\lambda_i q^{-n}\right|\right)^N |q|^{N^2/(2\nu)}\ , \tag{28}$$

where C *and* A *are the constants in relation* (24) *of definition 5.2.1.*

Proof. — Since $\eta < \frac{|q|}{1+|q|}$, one has $d_q(z, [\lambda_i]) = \eta$ and $d_q(z, [\lambda_j]) \ge \eta$ for $j \ne i$. One deduces that $d_q(z, \Lambda) \ge \eta^{\max(\nu_1,\dots,\nu_m)} \ge \eta^{\nu}$, which allows one to write, taking $\epsilon := \dfrac{\eta^{\nu}}{2}$ in (24):

$$|\mathcal{R}_N(z)| < \frac{2C}{\eta^{\nu}}\ A^N\ |q|^{N^2/(2\nu)}\ |z|^N\,.$$

Relation (28) follows, using the fact that $|z| \le (1+\eta)\,|\lambda_i q^{-n}| < 2\,|\lambda_i q^{-n}|$. □

We now express the coefficients $\alpha_{i,j,n}$ in (27) with the help of Cauchy formula in the following way:

$$\alpha_{i,j,n} = \frac{1}{2\mathrm{i}\pi} \oint_{\mathcal{C}^{\eta}_{i,n}} f(z)\ (z - \lambda_i q^{-n})^j\ dz,$$

that is, for any integer $N \ge 0$:

$$\alpha_{i,j,n} = \frac{1}{2\mathrm{i}\pi} \oint_{\mathcal{C}^{\eta}_{i,n}} \mathcal{R}_N(z)\ (z - \lambda_i q^{-n})^j\ dz.$$

Taking in account estimation (28) above entails:

$$|\alpha_{i,j,n}| \le 2C\ \eta^{j+1-\nu}\ \left|\lambda_i q^{-n}\right|^{j+1}\ X^N\ Q^{N^2/2},$$

where $X := 2A\,|\lambda_i q^{-n}|$ and $Q := |q|^{1/\nu}$. Relation (22) implies that, if $X < 1$, then:

$$\min_{N \in \mathbf{N}} \left(X^N\ Q^{N^2/2}\right) \le\ Q^{1/8}\ e^{-\frac{\ln^2 X}{2\ln Q}},$$

which implies that:

$$|\alpha_{i,j,n}| \le 2C\eta^{j+1-\nu}\ |\lambda_i|^{j+1}\ |q|^{1/(8\nu)-(j+1)n}\ e^{-\frac{\nu \ln^2(2A|\lambda_i q^{-n}|)}{2\ln|q|}},$$

so that all the sequences $(\alpha_{i,j,n})_{n\geq N_0}$ are q-Gevrey of level $(-\nu)$. Thus, for each pair (i,j) of indexes, the series:

$$S_{i,j}(z) := \sum_{n\geq N_0} \frac{\alpha_{i,j,n}}{(z-\lambda_i q^{-n})^{j+1}}$$

converges normally on each compact subset of the sector $S(\Lambda, 0)$.

Last, if $0 < |z| < R$ and if $z \notin \bigcup_{1\leq i\leq m} \lambda_i q^{-N_0+\mathbf{N}}$, let us put:

$$h(z) := f(z) - \sum_{1\leq i\leq m} \sum_{0\leq j<\nu_i} S_{i,j}(z).$$

From normal convergence of the series $S_{i,j}(z)$ on every compact subset of $S(\Lambda, 0)$, we draw:

$$h(z) = f(z) - \sum_{n\geq N_0} \sum_{1\leq i\leq m} P_{i,n}(z). \tag{29}$$

Lemma 5.2.5. — *The function h represents the germ of an analytic function at $z = 0$.*

Proof. — Since all non zero singularities of f in the disk $|z| < R$ are necessarily situated on the union of half q-spirals $\bigcup_{1\leq i\leq m} \lambda_i q^{-N_0+\mathbf{N}}$ and each $P_{i,n}(z)$ represents its polar part at each point $\lambda_i q^{-n}$ of the latter, the function h has an analytic continuation to the punctured disk $0 < |z| < R$. Moreover, if $\Omega := \bigcup_{n>N_0} \mathcal{C}(0; R\,|q|^{-n})$ denotes the union of the circles centered at $z = 0$ and with respective radii $R\,|q|^{-n}$ with $n > N_0$, all the series $S_{i,j}(z)$ converge normally over Ω, and therefore remain bounded on that family of circles. Besides, from proposition 5.1.5, f is also bounded on Ω because $f \in \mathbb{B}^{\Lambda}$. It follows that h is bounded on Ω, which entails the analyticity of h at $z = 0$. $\square$

Combining relations (27) and (29), we get decomposition (26) of f, thus achieving the proof of (i) $\Rightarrow$ (ii).

Proof of (ii) $\Rightarrow$ (i): It will rest on lemma 5.2.6.

Lemma 5.2.6. — *Let $j \in \mathbf{N}$, $N \in \mathbf{N}$ and write $\mathcal{R}_{j,N}$ the rational fraction defined by relation:*

$$\frac{1}{(1-t)^{j+1}} = \sum_{n=0}^{N-1} \binom{n+j}{n} t^n + \mathcal{R}_{j,N}(t). \tag{30}$$

Let $\delta > 0$. If $t \in \mathbf{C}$ satisfies $|1-t| \geq \delta$, then:

$$|\mathcal{R}_{j,N}(t)| < C^{\delta,r}_{j,N}\ \frac{|t|^N}{\delta^{j+1}}, \quad C^{\delta,r}_{j,N} = \frac{1}{2-r}\ \big(\frac{1+r\delta}{r-1}\big)^j\ \big(\frac{1+r\delta}{1+\delta}\big)^N \tag{31}$$

for all $r \in (1,2)$.

Proof. — The relation (31) being trivially satisfied at point $t=0$, we assume that $t \neq 0$. Taking the j-th derivative for each side of relation $\frac{1}{1-t} = \sum_{n=0}^{N+j-1} t^n + \frac{t^{N+j}}{1-t}$, one finds:

$$\mathcal{R}_{j,N}(t) = \frac{1}{j!}\ \big(\frac{t^{N+j}}{1-t}\big)^{(j)} = \frac{1}{2i\pi}\ \oint_{\mathcal{C}(t,\rho)} \frac{s^{N+j}}{1-s}\ \frac{ds}{(s-t)^{j+1}}\ ,$$

where $\mathcal{C}(t,\rho)$ denotes the positively oriented boundary of the circle centered at $s=t$ and with radius $\rho\,|t|$, with $\rho < \left|1-\frac{1}{t}\right|$. One deduces that:

$$|\mathcal{R}_{j,N}(t)| \leq \frac{(1+\rho)^{N+j}\ \ |t|^N}{\big(|1-t| - \rho\,|t|\big)\ \rho^j}, \tag{32}$$

where ρ denotes a real number located between 0 and $\left|1-\frac{1}{t}\right|$.

If $|1-t| > \delta$, with $t = 1+Re^{i\alpha}$ ($\alpha \in \mathbf{R}$), one gets: $\left|1-\frac{1}{t}\right| \geq \frac{R}{1+R} > \frac{\delta}{1+\delta}$. Choose $\rho := \frac{r'\delta}{1+\delta}$ with $r' \in (0,1)$, so that $|1-t| - \rho\,|t| \geq |1-t|\,(1-r') > (1-r')\ \delta$; relation (31) follows at once (along with lemma 5.2.6) with the help of (32), with $\rho := \frac{(r-1)\ \delta}{1+\delta}$ and $r := r'+1 \in (1,2)$. □

We shall now use lemma 5.2.6 to prove that (ii) ⇒ (i); by linearity, it suffices to check that a series of the form:

$$S(z) := \sum_{n\geq 0} \frac{\alpha_n}{(z-\lambda q^{-n})^{j+1}} \tag{33}$$

defines a function in space A_q^Λ, where $\lambda = \lambda_i$ for some $i \in \{1,2,\ldots,m\}$, $0 \leq j < \nu_i$ and where (α_n) denotes a q-Gevrey sequence of order $(-\nu)$: there are C, $A > 0$ such that:

$$\forall n \in \mathbf{N}\ ,\ |\alpha_n| \leq CA^n\,|q|^{-\nu n(n-1)/2}\,. \tag{34}$$

Let $z \in \mathbf{C}^*$ such that $d_q(z,\Lambda) > \epsilon$; we have $d_q(z,[\lambda])^{\nu_i} > \epsilon$, hence $\left|1-\frac{z}{\lambda q^{-n}}\right| > \sqrt[\nu_i]{\epsilon}$. Let $N \in \mathbf{N}$ and apply to each term of the series $S(z)$

formula (30) with $t = \dfrac{z}{\lambda q^{-n}}$ to obtain formally $S(z) = S_N(z) + R_N(z)$, where one puts:

$$S_N(z) := \sum_{n \geq 0} \sum_{k=0}^{N-1} \alpha_n \binom{k+j}{k} (-\lambda q^{-n})^{-j-1} \left(\frac{z}{\lambda q^{-n}}\right)^k$$

and:

$$R_N(z) := \sum_{n \geq 0} \alpha_n \ (-\lambda q^{-n})^{-j-1} \ \mathcal{R}_{j,N}(\frac{z}{\lambda q^{-n}}).$$

Since (α_n) has q-Gevrey decay of order ν (see (34)), we see that $S_N(z)$ and $R_N(z)$ are normally convergent series on any domain Ω such that $\inf\limits_{z \in \Omega} d_q(z, [\lambda]) > 0$.

Define:

$$\mathcal{A}(z) := \sum_{n \geq 0} \alpha_n \ z^n, \quad \tilde{\mathcal{A}}(z) := \sum_{n \geq 0} |\alpha_n| \ z^n. \tag{35}$$

From relation (34), one gets:

$$|\mathcal{A}(z)| \leq \tilde{\mathcal{A}}(|z|) \leq C \sum_{n \geq 0} |q|^{-\nu n(n-1)/2} \ |Az|^n < C \ e(Az; |q|^\nu).$$

On the other hand, one can express the series $S_N(z)$ in the following way:

$$S_N(z) = \sum_{n=0}^{N-1} a_n z^n, \quad a_n := (-\lambda)^{-j-1} \binom{n+j}{n} \mathcal{A}(q^{j+1+n}) \ \lambda^{-n} \ ; \tag{36}$$

besides, the relation (31) implies that, putting $\delta := \sqrt[\nu]{\epsilon}$:

$$|R_N(z)| \leq K_{j,N}^{\delta,r} \ |z|^N \ , \quad K_{j,N}^{\delta,r} := \frac{C_{j,N}^{\delta,r}}{|\lambda|^{j+1+N} \ \delta^{j+1}} \ \tilde{\mathcal{A}}(|q|^{j+1+N}) \ , \tag{37}$$

with $\tilde{\mathcal{A}}(z)$ as in (35) and $r \in (1, 2)$.

To sum up, we have seen that, if $d_q(z, \Lambda) > \epsilon > 0$ and $\delta = \sqrt[\nu]{\epsilon}$, then:

$$\left| S(z) - \sum_{n=0}^{N-1} a_n \ z^n \right| \leq K_{j,N}^{\delta,r} \ |z|^N \ ,$$

the coefficients a_n and $K_{j,N}^{\delta,r}$ being defined as in (36) and (37) respectively. To conclude that $S(z)$ admits $\sum\limits_{n \geq 0} a_n z^n$ as a q-Gevrey asymptotic expansion of level ν in Λ at 0, it only remains to be checked that one can find $C_0 > 0$, $A_0 > 0$ with:

$$\min_{r \in (1,2)} K_{j,N}^{\delta,r} \leq \frac{C_0}{\epsilon} \ A_0^N \ |q|^{N^2/(2\nu)} \tag{38}$$

for all $N \in \mathbf{N}$ and $j = 0, \ldots, \nu_i - 1$. To do that, note that, after lemma 5.1.8 and relation (22), one finds:

$$\tilde{\mathcal{A}}\left(|q|^{j+1+N}\right) \leq C' \ \left(|q|^{\frac{j+1}{\nu}-\frac{1}{2}} \ A^{\frac{1}{\nu}}\right)^N \ q^{N^2/(2\nu)} \ ,$$

where C' denotes a constant independant of N; this, along with definition (37) of $K^{\delta,r}_{j,N}$, entails (38) and we leave the details to the reader. The proof of theorem 5.2.2 is now complete. □

5.2.2. Asymptotics and residues (II). — After [**39**], for any $f \in \mathbb{A}_q^{[\lambda]}$, the function $z \mapsto F(z) := \theta(-\frac{z}{\lambda}) \ f(z)$ is analytic in some punctured disk $0 < |z| < \left|\lambda q^{-N}\right|$, with (at most) a first order q-exponential growth when z goes to 0, and such that its restriction to the q-spiral $[\lambda]$ has at most a growth of q-Gevrey order $0^{(2)}$. In order to extend this result to a general divisor Λ (with degree $\nu > 1$), we introduce the following definition.

Definition 5.2.7. — Let F be a function defined and analytic in a neighborhood of each of the points of the support of the divisor Λ near 0.

1. We call *values of* F *on* Λ *at* 0, and write $\Lambda F(0)$, the ν (germs at infinity of) sequences:
$$\Lambda F(0) := \left\{\left(F^{(j)}(\lambda_i q^{-n})\right)_{n\gg 0} \ : \ 1 \leq i \leq m, 0 \leq j < \nu_i\right\},$$
where $F^{(j)}$ is the j-th derivative of F (with $F^{(0)} = F$) and where "$n \gg 0$" means "n great enough".
2. We say that F has *q-Gevrey order k along divisor* Λ *at* 0 if all its values are q-Gevrey sequences of order k.
3. We write $F \in \mathbb{E}_0^{\Lambda}$ if F is analytic in a neighborhood of 0 punctured at 0, has q-Gevrey order ν at 0 and q-Gevrey order 0 along Λ at 0.

To simplify, we shall write:

$$\theta_\Lambda(z) := \prod_{j=1}^{m} \left(\theta\left(-\frac{z}{\lambda_j}\right)\right)^{\nu_j} \ . \tag{39}$$

Thus, if $f \in \mathbb{B}^{\Lambda}$, $\theta_\Lambda f$ represents an analytic function in a punctured disk $0 < |z| < R$.

Theorem 5.2.8. — *Let $f \in \mathbb{B}^{\Lambda}$ and $F := \theta_\Lambda f$. Then, $f \in \mathbb{A}_q^{\Lambda}$ if, and only if, $F \in \mathbb{E}_0^{\Lambda}$.*

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(2) That is, there exists constants $C, A > 0$ such that, for any integer $n \gg 0$, one has $\left|F(q^{-n}\lambda)\right| \leq CA^n$.

Proof. — We prove separately the two implications.

Proof of $f \in \mathbb{A}_q^\Lambda \Rightarrow F \in \mathbb{E}_0^\Lambda$: It will rest on lemma 5.2.9.

Lemma 5.2.9. — *Let $\lambda \in \mathbf{C}^*$, $k \in \mathbf{N}^*$ and let $g(z) := (\theta_\lambda(z))^k$ for $z \in \mathbf{C}^*$. For $\ell \geq 0$, consider the ℓ-th derivative $g^{(\ell)}$ of g.*

1. *The function $g^{(\ell)}$ has q-exponential growth of order k both at 0 and infinity.*
2. *If $\ell < k$, then $g^{(\ell)}(\lambda q^{-n}) = 0$ for all $n \in \mathbf{N}$; else, $\left(g^{(\ell)}(\lambda q^{-n})\right)_{n \geq 0}$ is a q-Gevrey sequence of order k.*
3. *Let $n \in \mathbf{Z}$ and put $u(z) := \dfrac{g(z)}{(z - \lambda q^{-n})^k}$ for all $z \in \mathbf{C}^* \setminus \{\lambda q^{-n}\}$. Then u has an analytic continuation at λq^{-n}, with:*

$$u^{(\ell)}(\lambda q^{-n}) = \frac{(\ell + k)!}{k!}\; g^{(\ell+k)}(\lambda q^{-n}). \tag{40}$$

In particular:

$$u(\lambda q^{-n}) = (-1)^{(n-1)k}\; (q^{-1}; q^{-1})_\infty^{3k}\; \Big(\frac{1}{\lambda}\Big)^k\; q^{kn(n+1)/2}\;, \tag{41}$$

where $(q^{-1}; q^{-1})_\infty = \prod\limits_{r \geq 0} (1 - q^{-r-1})$ (Pochhammer symbol).

Proof. — Writing $g^{(\ell)}$ as the sum of a power series in z and one in $\frac{1}{z}$, one draws from lemma 5.1.8 assertion (1), because derivation does not affect the q-Gevrey character of a series. Assertion (2) is obvious.

The function u can be analytically continuated at λq^{-n}, because θ_λ has a simple zero at each point of the q-spiral $[\lambda]$; formula (40) comes by differentiating $(\ell + k)$ times the equality $g(z) = u(z)(z - \lambda q^{-n})^k$, while (41) follows from a direct evaluation using Jacobi triple product formula (2). $\square$

We now come to the proof of the direct implication; that is, we assume condition (ii) of theorem 5.2.2. Since $\theta_\Lambda h \in \mathbb{E}_0^\Lambda$, by linearity, we just have to check that the series $S(z)$ defined by relation (33) satisfies $\theta_\Lambda S \in \mathbb{E}_0^\Lambda$. To that, put $T(z) := \theta_\Lambda(z)\, S(z)$ for all $t \in \mathbf{C}^* \setminus \lambda q^{-\mathbf{N}}$; this clearly has an analytic continuation to $\mathbf{C}^*$. On the other hand, $T^{(\ell)}(\lambda_{i'} q^{-n}) = 0$ for all $i' \neq i$, $\ell < \nu_{i'}$ and $n \in \mathbf{Z}$.

In order to evaluate $T^{(\ell)}(\lambda q^{-n})$, put $\Lambda' := \Lambda - (j+1)[\lambda]$ and:

$$T_0(z) := \sum_{n \geq 0} \alpha_n\; \frac{(\theta_\lambda(z))^{j+1}}{(z - \lambda q^{-n})^{j+1}};$$

one has $T(z) = \theta_{\Lambda'}(z)\ T_0(z)$, where T_0 is analytic on $\mathbf{C}^*$. First note that, if $\ell < \nu_i - j - 1$, one has $T^{(\ell)}(\lambda q^{-n}) = 0$ for all $n \in \mathbf{N}$; when $\nu_i - j - 1 \leq \ell < \nu_i$, using relation (40), we get:

$$T^{(\ell)}(\lambda q^{-n}) = \alpha_n \sum_{k=0}^{\ell-\nu_i+j+1} \binom{\ell}{k}\ \theta_{\Lambda'}^{(\ell-k)}(\lambda q^{-n})\ \frac{(k+j+1)!}{(j+1)!}\ g^{(k+j+1)}(\lambda q^{-n}),$$

where $n \in \mathbf{N}$, $g = (\theta_\lambda)^{j+1}$. Taking in account assertions (1) and (2) of lemma 5.2.9, the sequences $(\theta_{\Lambda'}^{(\ell-k)}(\lambda q^{-n}))_{n\geq 0}$ and $(g^{(k+j+1)}(\lambda q^{-n}))_{n\geq 0}$ respectively have q-Gevrey order $(\nu - j - 1)$ and $(j+1)$, which implies that $(T^{(\ell)}(\lambda q^{-n}))_{n\geq 0}$ is bounded above by a geometric sequence, for $(\alpha_n)_{n\geq 0}$ has q-Gevrey order $(-\nu)$.

Moreover, like θ_Λ, the function T has q-exponential growth of order ν at 0. Indeed, let $a \in (1, |q|)$ and put $\delta := \min\limits_{|z|=a} d_q(z, [1])$; one has $\delta > 0$, so that:

$$\max_{|z|=a|\lambda q^{-k}|} |S(z)| \leq \frac{1}{\delta\ |\lambda|} \sum_{n\geq 0} |\alpha_n q^n| < \infty. \tag{42}$$

This shows that $S(z)$ stays uniformly bounded over the family of circles centered at 0 and such that each one goes through a point of the q-spiral with basis $a\lambda$. It follows first that the restriction of function T to the circles has q-exponential growth of order ν at 0; then that function T itself has such growth. To summarize, we see that $T \in \mathbb{E}_0^\Lambda$, whence $F \in \mathbb{E}_0^\Lambda$.

Proof of $F \in \mathbb{E}_0^\Lambda \Rightarrow f \in \mathbb{A}_q^\Lambda$: It will rest on lemma 5.2.10, which gives a minoration on θ as in lemma 5.1.6.

Lemma 5.2.10. — *Let $a > 0$ and $k \in \mathbf{Z}$. Then:*

$$\min_{|z|=a|q|^{-k}} |\theta(z)| \geq \frac{\left|(q^{-1};q^{-1})_\infty\right|}{(|q|^{-1};|q|^{-1})_\infty}\ \ |\theta(-a;|q|)|\ \ a^{-k}\ \ |q|^{k(k-1)/2}. \tag{43}$$

Proof. — Let $z \in \mathbf{C}$ be such that $|z| = a\,|q|^{-k}$. From a minoration of each factor of the triple product formula (2), we get:

$$\begin{aligned}\min_{|z|=a|q|^{-k}} |\theta(z)| \quad &\geq \quad \left|(q^{-1};q^{-1})_\infty\right| \prod_{n\geq 0} \left|1 - a\,|q|^{-n-k}\right| \left|1 - \frac{|q|^{-n-1}}{a\,|q|^{-k}}\right| \\ &= \quad \frac{\left|(q^{-1};q^{-1})_\infty\right|}{(|q|^{-1};|q|^{-1})_\infty}\ \left|\theta(-a\,|q|^{-k};|q|)\right|,\end{aligned}$$

yielding relation (43). □

Now we assume $F \in \mathbb{E}_0^\Lambda$. We put $f := \dfrac{F}{\theta_\Lambda}$ and keep the notations R, N_0 introduced at the beginning of the proof of the theorem; also, we choose $a \in (|\lambda_m|, R)$. According to relation (43), there is $C > 0$ and $\mu > 0$ such that, if $|z| = aq^{-k}$ and $k \geq N_0$, then $|f(z)| \leq C\,|z|^\mu$; that is, f has moderate growth at 0 on the circles $|z| = a\,|q|^{-k}$, $k \geq N_0$.

Partial fraction decomposition of f at each point of Λ contained in the punctured disk $0 < |z| < R$, will produce an expression of the form (26), in which the coefficients $\alpha_{i,j,n}$ have q-Gevrey order $(-\nu)$. Indeed, let:

$$F_{i,k,n} := \frac{F^{(k)}(\lambda_i q^{-n})}{k!}, \quad \Theta_{i,k,n} := \frac{\theta_\Lambda^{k+\nu_i}(\lambda_i q^{-n})}{(k+\nu_i)!};$$

then:

$$\alpha_{i,\nu_i-1,n} = \frac{F_{i,0,n}}{\Theta_{i,0,n}}\,, \quad \alpha_{i,\nu_i-2,n} = \frac{F_{i,1,n} - \alpha_{i,\nu_i-1,n}\,\Theta_{i,1,n}}{\Theta_{i,0,n}}\,, \quad \ldots,$$

$$\alpha_{i,0,n} = \frac{F_{i,\nu_i-1,n} - \sum_{k=1}^{\nu_i-1} \alpha_{i,\nu_i-k,n}\,\Theta_{i,\nu_i-k,n}}{\Theta_{i,0,n}}.$$

According to relation (41), one finds:

$$\Theta_{i,0,n} = (q^{-1};q^{-1})_\infty^{3\nu_i}\,(-\lambda_i q^{-n})^{-\nu_i}\,\theta_{\Lambda_i'}(\lambda_i) \prod_{j=1}^m \left(-\frac{\lambda_j}{\lambda_i}\right)^{n\nu_j} q^{\nu n(n-1)/2}.$$

Since $(F_{i,0,n})_{n\geq N_0}$ has q-Gevrey order 0, it follows that the sequence $(\alpha_{i,\nu_i-1,n})_{n\geq N_0}$ has q-Gevrey order $(-\nu)$. As for the other coefficients $\alpha_{i,j,n}$, $0 \leq j < \nu_i - 1$, since the sequences $(\Theta_{i,j,n})_{n\geq N_0}$ all have q-Gevrey order ν (see lemma 5.2.9), one successively shows that the sequences $(\alpha_{i,\nu_i-2,n})_{n\geq N_0}$, $\ldots$, $(\alpha_{i,0,n})_{n\geq N_0}$ all have q-Gevrey order $(-\nu)$.

Last, note that after (42), the function P defined by:

$$P(z) := \sum_{n\geq N_0} \sum_{1\leq i\leq m} \sum_{0\leq j<\nu_i} \frac{\alpha_{i,j,n}}{(z-\lambda_i q^{-n})^{j+1}}$$

is uniformly bounded over the family of circles $|z| = a\,|q|^{-k}$, $k \in \mathbf{Z}$ if $a \notin \bigcup_{1\leq i\leq m} [\lambda_i]$. Since $f \in \mathbb{B}^\Lambda$, from proposition 5.1.5 we deduce that $h(z) := f(z) - P(z)$ is analytic in some open disk $0 < |z| < R$ and remains bounded over these circles, which implies that h is holomorphic in a neighborhood of 0, achieving the proof of the converse implication and of the theorem. $\square$

5.3. q-Gevrey theorem of Borel-Ritt and flat functions

The classical theorem of Borel-Ritt says that any formal power series is the asymptotic expansion, in Poincaré's sense, of some germ of analytical function in a sector of the complex plane with vertex 0. We are going to prove a q-Gevrey version of that theorem involving the set of functions $\mathbb{A}_q^\Lambda$ and then give a characterization of the corresponding flat functions.

5.3.1. From a q-Gevrey power series to an entire function. — As before, let $\Lambda = \sum\limits_{1\leq i\leq m} \nu_i[\lambda_i]$ be a divisor and let θ_Λ be the associated theta function given by the relation (39).

Lemma 5.3.1. — *Consider the Laurent series expansion $\sum\limits_{n\in\mathbf{Z}} \beta_n\, z^n$ of θ_Λ.*

1. *For all $(k,r)\in\mathbf{Z}^2$, one has:*

$$\beta_{k\nu+r} = \left(\frac{L}{q^r}\right)^k q^{-k(k+1)\nu/2}\, \beta_r\ . \tag{44}$$

2. *For all $n\in\mathbf{Z}$, write T_n the function defined by:*

$$T_n := \theta_{\Lambda,n}(z) = \sum_{\ell\leq n} \beta_\ell\, z^\ell\ .$$

If $j\in\{1,\dots,m\}$, one has:

$$K_{n,j} := \sup_{z\in[\lambda_j]} \left|z^{-n}\, T_n(z)\right| < +\infty\ . \tag{45}$$

Proof. — From the fundamental relation $\theta(qz) = qz\theta(z)$, one draws:

$$\theta_\Lambda(qz) = L\, z^\nu\, \theta_\Lambda(z), \quad L := \prod_{j=1}^m \left(-\frac{q}{\lambda_j}\right)^{\nu_j}\ ,$$

whence, for all $k\in\mathbf{Z}$:

$$\theta_\Lambda(q^k z) = L^k\, z^{k\nu}\, q^{k(k-1)\nu/2}\, \theta_\Lambda(z)\ .$$

The relation (44) follows immediately.

Since $z^{-n}\, T_n(z)$ represents an analytic function over $\mathbf{C}^*\cup\{\infty\}$, one obtains (45) by noticing that, if $z\in[\lambda_j]$, then:

$$T_n(z) = -\sum_{\ell>n} \beta_\ell\, z^\ell\ ,$$

which implies that $z^{-n}\, T_n(z)$ tends to zero as z tends 0 along the half-spiral $\lambda_i q^{-\mathbf{N}}$. □

Consider a q-Gevrey series $\hat{f} = \sum_{n\geq 0} a_n z^n \in \mathbf{C}[[z]]_{q;1/\nu}$ with order $1/\nu$ and, for all $\ell \in \mathbf{Z}$, put $c_\ell := \sum_{n\geq 0} a_n \beta_{\ell-n}$. Set $\ell := k\nu + r$ with $k \in \mathbf{Z}$ and $r \in \mathbf{Z}$, and write:

$$c_\ell = \sum_{j=0}^{\nu-1} \sum_{m\geq 0} a_{m\nu+j}\, \beta_{(k-m)\nu+(r-j)}.$$

As we shall see now, the above infinie series converges when the integer $-\ell$ is large enough. Indeed, using relation (44), one gets:

$$\beta_{(k-m)\nu+(r-j)} = (Lq^{-r})^k\, q^{-k(k+1)\nu/2}\, q^{kj} \left(\frac{q^{r-j+k\nu}}{L}\right)^m q^{-m(m-1)\nu/2}.$$

Since $\hat{f}$ has q-Gevrey order $1/\nu$, there exists $C_1 > 0$, $A_1 > 0$ such that $|a_n|\ leC_1\, A_1^n\, |q|^{n(n-1)/(2\nu)}$ for all $n \in \mathbf{N}$. One has:

$$\left|a_{m\nu+j}\, \beta_{(k-m)\nu+(r-j)}\right| \leq$$

$$C_1(|L||q|^{-r})^k\, |q|^{-k(k+1)\nu/2}\, (|q|^k A_1)^j\, |q|^{j(j-1)/(2\nu)} \left(\frac{|q|^{r+k\nu+(\nu-1)/2}}{|L|A_1^{-\nu}}\right)^m.$$

Therefore, if we set:

$$C_2 := \sum_{j=0}^{\nu-1} A_1^j |q|^{j(j-1)/(2\nu)},$$

then we find ($\ell = k\nu + r$):

$$|c_\ell| \leq C_1\, C_2\, (|Lq^{-r}|)^k\, |q|^{-k(k-1)\nu/2} \sum_{m\geq 0} \left(\frac{|q|^{\ell+(\nu-1)/2}}{|L|A_1^{-\nu}}\right)^m.$$

It is easy to see that the last series converges if $|q|^{\ell+(\nu-1)/2} < |L|A_1^{-\nu}$; moreover, when $\ell \to -\infty$, the sequence (c_ℓ) is q-Gevrey of level $(-\nu)$.

Proposition 5.3.2. — *Let* $\hat{f} := \sum_{n\geq 0} a_n z^n \in \mathbf{C}[[z]]_{q;1/\nu}$ *and let* N_0 *be a negative integer such that any series defining* c_ℓ *converges for* $\ell \leq N_0$. *Let* $F(z) := \sum_{\ell\leq N_0} c_\ell z^\ell$. *Then,* $F \in E_0^\Lambda$.

Proof. — Since (c_ℓ) is a q-Gevrey sequence of order $(-1/\nu)$ for $\ell \to -\infty$, the sum F represents an analytic function in $\mathbf{C}^* \cup \{\infty\}$, with q-exponential growth of order ν at zero.

We are left to estimate $F(z)$ and its derivatives along the spirals $[\lambda_j]$ belonging to the divisor Λ. Taking in account relation (44), one has:

$$T_{-n-k\nu}(z) = (Lz^\nu)^{-k}\, q^{-k(k-1)\nu/2}\, T_{-n}(q^k z).$$

Therefore, when $z \in [\lambda_j]$, relation (45) allows writing $F(z)$ in the following form:

$$F(z) = \sum_{k \geq 0} \sum_{n=0}^{\nu-1} a_{k\nu+n}\ z^{k\nu+n}\ T_{N_0-k\nu-n}(z),$$

and also:

$$F(z) = \sum_{n=0}^{\nu-1} z^n \sum_{k \geq 0} L^{-k} a_{k\nu+n}\ q^{-k(k-1)\nu/2}\ T_{N_0-n}(q^k z). \tag{46}$$

One deduces that F has moderate growth when $z \to 0$ along spirals $[\lambda_j]$ belonging to the divisor Λ.

As for the derivatives $F^{(\ell)}$, along each spiral $[\lambda_j]$, with $\ell < \nu_j$, one may proceed all the same: the function $T_n^{(\ell)}$ is indeed bounded over $[\lambda_j]$ for all $\ell < \nu_j$ and one checks that relation (46) remains true up to order ℓ, that is, if $z \in [\lambda_j]$,:

$$F^{(\ell)}(z) = \sum_{n=0}^{\nu-1} \sum_{k \geq 0} L^{-k} a_{k\nu+n}\ q^{-k(k-1)\nu/2}\ \left[z^n\ T_{N_0-n}(q^k z)\right]^{(\ell)}.$$

□

5.3.2. The q-Gevrey theorem of Borel-Ritt. — The main result of this paragraph is the following.

Theorem 5.3.3 **(q-Gevrey theorem of Borel-Ritt)**

The mapping J sending an element f to its asymptotic expansion is a surjective linear map from the $\mathbf{C}$-vector space $\mathbb{A}_q^\Lambda$ to the $\mathbf{C}$-vector space $\mathbf{C}[[z]]_{q;1/\nu}$.

Proof. — Let $\hat{f} = \sum_{n \geq 0} a_n z^n$ be a q-Gevrey series of order $1/\nu$ and let F be the function defined in proposition 5.3.2. Maybe at the cost of taking a smaller N_0, we may assume that formula (46) remains true for all $z \in \mathbf{C}^*$; this being granted, one has:

$$F(z) = \sum_{\ell \geq 0} a_\ell\ z^\ell\ T_{N_0-\ell}(z)\ . \tag{47}$$

Put $f := \dfrac{F}{\theta_\Lambda}$, so that $f \in \mathbb{A}_q^\Lambda$, after theorem 5.2.8. We are going to prove that f admits $\hat{f}$ as an asymptotic expansion. To that aim, it is enough to show that, if $\lambda \notin \Lambda$, one has:

$$\lim_{n \to +\infty} f^{(\ell)}(\lambda q^{-n}) = \ell!\ a_\ell \tag{48}$$

for any integer $\ell \geq 0$.

Since $|T_{N_0-n}(z)| \leq e_\Lambda(z)$ and $|\theta_\Lambda(z)| \sim e_\Lambda(z)$ (see lemma 5.2.10), relation (47) immediately leads us to the limit (48) for $\ell = 0$. If $\ell \geq 1$, one has:

$$f^{(\ell)}(z) = \sum_{n\geq 0}\sum_{j=0}^{\ell} a_n \binom{n}{j} \frac{j!}{(\ell-j)!} z^{n-j} \left[\frac{T_{N_0-n}(z)}{\theta_\Lambda(z)}\right]^{(\ell-j)} .$$

Through direct estimates, one gets the limit (48) (omitted details are left to the reader). □

The theorem 5.3.3 can also be interpreted with help of interpolation of a q-Gevrey sequence by entire functions at points in a geometric progression. Indeed, if $f \in \mathbb{A}_q^\Lambda$ is asymptotic to the q-Gevrey series $\sum_{k\geq 0} a_k z^k$ of level ν, then, from the decomposition (26), one draws:

$$a_k = \sum_{n\geq N_0}\sum_{1\leq i\leq m}\sum_{0\leq j<\nu_i} (-1)^{j+1}\binom{k+j}{j}\frac{\alpha_{i,j,n}}{(\lambda_i q^{-n})^{k+j+1}} + h_k ,$$

where h_k denotes the coefficient of z^k in the Taylor series of h. Put:

$$f_{i,j}(z) := \sum_{n\geq N_0} \alpha_{i,j,n}\, z^n ;$$

then:

$$a_k = \sum_{1\leq i\leq m}\sum_{0\leq j<\nu_i} \binom{k+j}{j}\frac{(-1)^{j+1}}{\lambda_i^{k+j+1}} f_{i,j}(q^{k+j+1}) + h_k . \tag{49}$$

Corollary 5.3.4. — *Let (a_n) be a q-Gevrey sequence of order ν and let $\Lambda = \sum_{1\leq i\leq m} \nu_i[\lambda_i]$ be a divisor of degree $\nu := \nu_1+\cdots+\nu_m$. Then, there exists ν entire functions $A_{i,j}$, $1\leq i\leq m$, $0\leq j<\nu_i$ satisfying the following properties:*

1. *Each function $A_{i,j}$ has q-exponential growth of order (at most) ν at infinity.*
2. *There exists $C>0$, $A>0$ such that, for all $n\in\mathbf{N}$:*

$$\left| a_n - \sum_{1\leq i\leq m}\sum_{0\leq j<\nu_i} \binom{n+j}{j}\frac{(-1)^{j+1}}{\lambda_i^{n+j+1}} A_{i,j}(q^{n+j+1})\right| < CA^n . \tag{50}$$

Proof. — The functions $A_{i,j}$ are defined by replacing N_0 by $\max(0,N_0)$ in definition (49); one then uses relation (49) to get the bounds (50). □

Note that if Λ is solely made of simple spirals ($\nu_i = 1$), then corollary 5.3.4 spells as follows:

$$\left| a_n + \sum_{1\leq i\leq \nu} \lambda_i^{-n-1}\ A_i(q^{n+1}) \right| < CA^n \ ,$$

where $C > 0$, $A > 0$ and where the A_i are ν entire functions with q-exponential growth of order at most ν at infinity. In other words, one can interpolate a q-Gevrey sequence of order ν, up to a sequence with at most geometric growth, by ν entire functions on a half-spiral $\lambda q^{\mathbf{N}}$.

5.3.3. Flat functions with respect to a divisor. — In the proof of theorem 5.3.3, function F was defined up to a function of the form $\dfrac{P}{\theta_\Lambda}$, where P is a polynomial in z and z^{-1}. We shall see that, if h denotes a meromorphic function at $z = 0$, then $\dfrac{h}{\theta_\Lambda}$ represents a function with trivial asymptotic expansion, that is, $\dfrac{h}{\theta_\Lambda} \in \operatorname{Ker} J$ with notations from theorem 5.3.3.

Theorem 5.3.5. — *A function $f \in \mathbb{A}_q^\Lambda$ is* flat, i.e. *has trivial expansion, if, and only if, $\theta_\Lambda f$ is meromorphic in a neighborhood of* $0 \in \mathbf{C}$.

Proof. — If $f \in \mathbb{A}_q^\Lambda$ has trivial asymptotic expansion, then, for all $\epsilon > 0$ and $z \in \mathbf{C}^*$ with small enough modulus:

$$d_q(z,\Lambda) > \epsilon \Longrightarrow |f(z)| < \frac{C}{\epsilon}\ A^N\ |q|^{N^2/(2\nu)}\ |z|^N \tag{51}$$

for any integer $N \geq 0$. Using relation (22), one deduces that $\theta_\Lambda f$ has moderate growth at zero, thus is at worst meromorphic at $z = 0$.

Conversely, note that if $f(z) = \dfrac{h(z)}{\theta_\Lambda(z)}$ with $h(z) = O(z^\mu)$, $\mu \in \mathbf{Z}$, then lemma 5.1.6 entails the following relation for all z with small enough modulus:

$$d_q(z,\Lambda) > \epsilon > 0 \Longrightarrow |f(z)| < \frac{M}{\epsilon}\ |z|^\mu\ \prod_{j=1}^m \left(e\left(\frac{z}{\lambda_j}\right)\right)^{-\nu_j} \ ,$$

where M denotes a constant > 0. But for any integer $n \in \mathbf{Z}$:

$$e(z) = \sum_{k\in\mathbf{Z}} |q|^{-k(k-1)/2}\ |z|^k > |q|^{-n(n-1)/2}\ |z|^n \ ,$$

so that:

$$\prod_{j=1}^m \left(e\left(\frac{z}{\lambda_j}\right)\right)^{-\nu_j} < \|\Lambda\|^{-n}\ |q|^{\nu n(n-1)/2}\ |z|^{-\nu n} \ . \tag{52}$$

One thereby gets an estimate of type (51) for f, for a sequence of integral values of N, with $N = \mu + \nu n$, $n \to +\infty$ and, as a consequence, for all integers $N \geq 0$: this achieves the proof of the theorem. □

Corollary 5.3.6. — *Let $f_1, f_2 \in \mathbb{A}_q^\Lambda$. Assume that f_1 et f_2 have the same asymptotic expansion. Then $f_1 = f_2$ if, and only if, one of the following conditions is satisfied:*

1. *One has $f_1(z_n) = f_2(z_n)$ for some sequence of points (z_n) tending to zero outside of divisor Λ.*
2. *On some spiral $[\lambda_i]$ belonging to divisor Λ:*

$$\lim_{z \to \lambda_i q^n} (z - \lambda_i q^n)^{\nu_i}\ f_1(z) = \lim_{z \to \lambda_i q^n} (z - \lambda_i q^n)^{\nu_i}\ f_2(z)$$

as $n \to -\infty$.

Proof. — Immediate. □

5.4. Relative divisors and multisummable functions

Theorems 5.3.3 and 5.3.5 show that, for each divisor Λ with degree $\nu \in \mathbf{N}^*$, the datum of a q-Gevrey series of order $1/\nu$ is equivalent to the datum of an element in space $\mathbb{A}_q^\Lambda$ together with a function of the form $\dfrac{h}{\theta_\Lambda}$ for some $h \in \mathbf{C}(\{z\})$. When studying q-difference equations, we shall observe that the function h in correction term $\dfrac{h}{\theta_\Lambda}$ will have to be determined in some space of asymptotic functions similar to $\mathbb{A}_q^\Lambda$, and that is why we are now going to define a notion of asymptoticity involving more than one divisor level.

5.4.1. Relative divisors and two levels asymptotics. — Let Λ', Λ be two divisors. Assume that $\Lambda' < \Lambda$, meaning that $\Lambda' = \sum\limits_{1 \leq i \leq m} \nu_i'[\lambda_i]$, that $\Lambda = \sum\limits_{1 \leq i \leq m} \nu_i[\lambda_i]$, $0 \leq \nu_i' \leq \nu_i$, $\nu_i > 0$ and that $|\Lambda| > |\Lambda'|$.

Definition 5.4.1. — Let $\Lambda' < \Lambda$ and let F be a function defined and analytic in a neighborhood of each point of the spirals in the support of Λ near 0.

1. We call *values of F over the relative divisor* Λ/Λ' *at* 0, and we write $(\Lambda/\Lambda')F(0)$, the $|\Lambda| - |\Lambda'|$ (germs of) sequences:

$$(\Lambda/\Lambda')F(0) := \left\{ \left(F^{(j)}(\lambda_i q^{-n}) \right)_{n \gg 0}\ :\ 1 \leq i \leq m, \nu_i' \leq j < \nu_i \right\}.$$

2. We say that F *has q-Gevrey order k over Λ/Λ' at* 0 if all its values there constitute q-Gevrey sequences of order k.
3. Let $\Lambda = \Lambda_1 + \Lambda_2$ and $\Lambda' = \Lambda_1$. We write $F \in \mathbb{E}^{\Lambda}_{(\Lambda_1,\Lambda_2)}$ if F is analytic in a neighborhood of 0 punctured at 0, has q-Gevrey order $|\Lambda|$ at 0, has q-Gevrey order $|\Lambda_2|$ over Λ/Λ_2 at 0 and has q-Gevrey order 0 over Λ_2 at 0.

We get the following generalization of theorem 5.2.8.

Proposition 5.4.2. — *Let* $\Lambda_1 < \Lambda = \Lambda_1 + \Lambda_2$. *One has the following decompositions:*

$$\begin{aligned} \mathbb{E}^{\Lambda}_{(\Lambda_1,\Lambda_2)} &= \mathbb{E}^{\Lambda_2}_0 + \theta_{\Lambda_2}\mathbb{E}^{\Lambda_1}_0, \\ \left(\frac{1}{\theta_\Lambda}\,\mathbb{E}^{\Lambda}_{(\Lambda_1,\Lambda_2)}\right) \cap \mathbb{B}^{\Lambda} &= \mathbb{A}^{\Lambda_1}_q + \frac{1}{\theta_{\Lambda_1}}\,\mathbb{A}^{\Lambda_2}_q. \end{aligned}$$

Proof. — We shall only check the first decomposition, the second one following immediately with help of theorem 5.2.8.

Let $\Lambda_2 = \sum\limits_{i=1}^{m} \nu_i[\lambda_i]$, with $[\lambda_i] \neq [\lambda_{i'}]$ for $i \neq i'$. Let $F \in \mathbb{E}^{\Lambda}_{(\Lambda_1,\Lambda_2)}$. Since F has q-Gevrey order 0 over Λ_2 at zero, one gets the following Taylor expansion:

$$F(z) = \sum_{n\gg 0}\sum_{i=0}^{m}\sum_{j=0}^{\nu_i-1} \frac{F^{(j)}(\lambda_i q^{-n})}{j!}\,(z-\lambda_i q^{-n})^j + R_{\Lambda_2}(z)\ , \tag{53}$$

where the convergence of these series for $n \gg 0$ comes from the fact that each sequence $\left(F^{(j)}(\lambda_i q^{-n})\right)_{n\gg 0}$ is bounded. Let $f_2 := \dfrac{F}{\theta_{\Lambda_2}}$ and $F_1 := \dfrac{R_{\Lambda_2}}{\theta_{\Lambda_2}}$ and put $F_2 := \theta_{\Lambda_2}(f_2 - F_1)$; one has $F = F_2 + \theta_{\Lambda_2}\,F_1$ and F_2 is the function represented by the triple summation in the expansion (53). After theorem 5.2.8, one has $(f_2 - F_1) \in \mathbb{A}^{\Lambda_2}_q$ and $F_2 \in \mathbb{E}^{\Lambda_2}_0$.

There remains to check that $F_1 \in \mathbb{E}^{\Lambda_1}_0$. First note that R_{Λ_2} has order at least ν_i at each $z = \lambda_i q^{-n}$ for $n \gg 0$, which implies that F_1 represents a germ of analytic function in a punctured neighborhood of $z = 0$. Moreover, since the values of F over the relative divisor Λ/Λ_2 at 0 have q-Gevrey order $|\Lambda_2|$, the same is true for the function R_{Λ_2} because F_2 has q-Gevrey order $|\Lambda_2|$ at zero. Using lemma 5.2.10, one finds at last that $F_1 \in \mathbb{E}^{\Lambda_1}_0$, achieving the proof of the theorem. □

Note that:

$$\mathbb{A}^{\Lambda_1}_q \subset \mathbb{A}^{\Lambda_1}_q + \frac{1}{\theta_{\Lambda_1}}\mathbb{A}^{\Lambda_2}_q\ ,$$

the inclusion being strict for a non trivial divisor Λ_2.

5.4.2. Multisummable functions. — More generally, let $\Lambda_1 < \Lambda_1 + \Lambda_2 < \cdots < \Lambda_1 + \Lambda_2 + \cdots + \Lambda_m = \Lambda$; for i from 1 to $m+1$, put $\Lambda_{\geq i} := \sum\limits_{j \geq i} \Lambda_j$ and $\Lambda_{\leq i} := \sum\limits_{j \leq i} \Lambda_j$. Also, we will write $\Lambda_{\geq m+1} = \Lambda_{\leq 0} = \mathbf{O}$, where $\mathbf{O}$ stands for the null divisor, and $\Lambda_{\geq 0} = \Lambda_{\leq m} = \Lambda$. For short, we'll call $(\Lambda_1, \Lambda_2, \ldots, \Lambda_m)$ an *ordered partition of* Λ.

Theorem 5.4.3. — *Let $(\Lambda_1, \Lambda_2, \ldots, \Lambda_m)$ be an ordered partition of Λ and define $\mathbb{E}^{\Lambda}_{(\Lambda_1, \Lambda_2, \ldots, \Lambda_m)}$ as the set of functions that are analytic in a punctured neighborhood of* 0*, have q-Gevrey order $|\Lambda|$ at* 0 *and, for i from* 1 *to m, having q-Gevrey order $|\Lambda_{\geq i+1}|$ over $\Lambda_{\geq i}/\Lambda_{\geq i+1}$ at* 0*. Then, one has:*

$$\mathbb{E}^{\Lambda}_{(\Lambda_1, \Lambda_2, \ldots, \Lambda_m)} = \mathbb{E}_0^{\Lambda_m} + \theta_{\Lambda_{\geq m}} \mathbb{E}_0^{\Lambda_{m-1}} + \cdots + \theta_{\Lambda_{\geq 2}} \mathbb{E}_0^{\Lambda_1}.$$

In other words, writing $\mathbb{O}^{\Lambda}_{(\Lambda_1, \Lambda_2, \ldots, \Lambda_m)} := \left(\dfrac{1}{\theta_\Lambda} \mathbb{E}^{\Lambda}_{(\Lambda_1, \Lambda_2, \ldots, \Lambda_m)} \right) \cap \mathbb{B}^{\Lambda}$, *one has:*

$$\mathbb{O}^{\Lambda}_{(\Lambda_1, \Lambda_2, \ldots, \Lambda_m)} = \mathbb{A}_q^{\Lambda_1} + \frac{1}{\theta_{\Lambda_{\leq 1}}} \mathbb{A}_q^{\Lambda_2} + \cdots + \frac{1}{\theta_{\Lambda_{\leq m-1}}} \mathbb{A}_q^{\Lambda_m}.$$

Proof. — One proceeds by induction on the length m of the partition of divisor Λ: the cass $m = 1$ is theorem 5.2.8 while the case $m = 2$ is dealt with in proposition 5.4.2. The other cases follow directly from the next lemma. □

Lemma 5.4.4. — *Let $m \geq 2$. Then:*

$$\mathbb{E}^{\Lambda}_{(\Lambda_1, \Lambda_2, \ldots, \Lambda_m)} = \mathbb{E}_0^{\Lambda_m} + \theta_{\Lambda_m} \mathbb{E}^{\Lambda_{\leq m-1}}_{(\Lambda_1, \Lambda_2, \ldots, \Lambda_{m-1})}.$$

Proof. — Let $F \in \mathbb{E}^{\Lambda}_{(\Lambda_1, \Lambda_2, \ldots, \Lambda_m)}$. As in equation (53) in the course of the proof of proposition 5.4.2, one considers the Taylor expansion of F along the divisor Λ_m, thereby finding that $F \in \mathbb{E}_0^{\Lambda_m} + \theta_{\Lambda_m} \mathbb{E}^{\Lambda_{\leq m-1}}_{(\Lambda_1, \Lambda_2, \ldots, \Lambda_{m-1})}$. □

Definition 5.4.5. — We call $\mathbb{O}^{\Lambda}_{(\Lambda_1, \Lambda_2, \ldots, \Lambda_m)}$ the set of *multisummable functions at* 0 *with respect to partition $(\Lambda_1, \Lambda_2, \ldots, \Lambda_m)$ of divisor* Λ.

By convention, we will set $\mathbb{O}^{\mathbf{O}}_{(\mathbf{O})} = \mathbf{C}\{z\}$, the ring of all germs of analytic functions at $z = 0$.

Proposition 5.4.6. — *The sets $\mathbb{E}^{\Lambda}_{(\Lambda_1, \Lambda_2, \ldots, \Lambda_m)}$ and $\mathbb{O}^{\Lambda}_{(\Lambda_1, \Lambda_2, \ldots, \Lambda_m)}$ have a structure of module over $\mathbf{C}\{z\}$ and are stable under the q-difference operator σ_q.*

Proof. — Immediate. □

If $m \geq 2$, an element $f \in \mathbb{O}^{\Lambda}_{(\Lambda_1,\Lambda_2,\ldots,\Lambda_m)}$ generally has no asymptotic expansion in the sense of definition 5.2.1. Nonetheless, according to theorem 5.4.3, there exists a $f_1 \in \mathbb{A}_q^{\Lambda_1}$ such that $f(z) - f_1(z) = O\left(\dfrac{1}{\theta_{\Lambda_1}(z)}\right)$ as $z \to 0$ outside of Λ, which allows one to prove the following.

Theorem 5.4.7. — *To each $f \in \mathbb{O}^{\Lambda}_{(\Lambda_1,\Lambda_2,\ldots,\Lambda_m)}$ there corresponds a unique power series $\hat{f} = \sum\limits_{n\geq 0} a_n\ z^n \in \mathbf{C}[[z]]_{q;1/|\Lambda_1|}$ satisfying the following property: for all $R > 0$ near enough 0, there are constants $C,\ A > 0$ such that, for all $\epsilon > 0$ and all $z \in S(\Lambda, \epsilon; R)$:*

$$\left| f(z) - \sum_{\ell=0}^{n-1} a_\ell\ z^\ell \right| < \frac{C}{\epsilon}\ A^n\ \ |q|^{n^2/(2|\Lambda_1|)}\ \ |z|^n\ . \tag{54}$$

Proof. — Applying theorem 5.4.3 to the function f, one gets:

$$f = f_1 + \frac{f_2}{\theta_{\Lambda_1}} + \cdots + \frac{f_m}{\theta_{\Lambda_{\leq m-1}}},$$

where $f_i \in \mathbb{A}_q^{\Lambda_i}$ for all indexes i from 1 to m. Choose $R > 0$ in such a way that the $f_i\ \theta_{\Lambda_i}$ have an analytic continuation to the punctured disk $0 < |z| < 2R$; from the definition of each space $\mathbb{A}_q^{\Lambda_i}$, one has:

$$d_q(z, \Lambda_i) > \epsilon \Longrightarrow |f_i(z)| < \frac{C_i}{\epsilon}$$

for all z such that $0 < |z| < R$, where C_i is some positive constant. On the other hand, from lemma 5.1.6 and relation (52), one has, for any divisor $\Lambda' = \Lambda_1, \ldots, \Lambda_{\leq m-1}$:

$$d_q(z, \Lambda') > \epsilon \Longrightarrow \left| \frac{1}{\theta_{\Lambda'}(z)} \right| < \frac{C'}{\epsilon}\ A'^n\ \ |q|^{n^2/(2|\Lambda'|)}\ \ |z|^n$$

for all integers $n \geq 0$, where C' and A' are positive constants depending only on the divisor Λ'. Noting that $d_q(z, \Lambda) \leq d_q(z, \Lambda_i)\ d_q(z, \Lambda_{\leq i-1})$ for $i = 2, \ldots, m$, one gets that, for all integers $n \geq 0$:

$$d_q(z, \Lambda > \epsilon \Longrightarrow |f(z) - f_1(z)| < \frac{C_1}{\epsilon}\ {A_1}^n\ \ |q|^{n^2/(2|\Lambda_1|)}\ \ |z|^n \tag{55}$$

as long as $0 < |z| < R$.

Relation (55) implies that both f and f_1 have $\hat{f}$ as an asymptotic expansion in the classical sense (Poincaré) at 0, outside of divisor Λ. Since f_1 is asymptotic to $\hat{f}$ in the frame of space $\mathbb{A}_q^{\Lambda_1}$, one has $\hat{f} \in \mathbf{C}[[z]]_{q;1/|\Lambda_1|}$. The bounds (54) are directly obtained from (24) and (55), using the obvious relation $d_q(z, \Lambda) \leq d_q(z, \Lambda_1)$. □

It is clear that a function f satisfying (54) does not necessarily belong to the space $\mathbb{O}^{\Lambda}_{(\Lambda_1,\Lambda_2,\dots,\Lambda_m)}$. To simplify, we shall adopt the following definition.

Definition 5.4.8. — When f and $\hat{f}$ satisfy condition (54), we shall say that *f admits $\hat{f}$ as an asymptotic expansion at* 0 *along the divisors* (Λ_1, Λ). If moreover $\hat{f}$ is the null series, we shall say that f *is flat at* 0 *along divisors* (Λ_1, Λ).

The following result is straightforward from definitions 5.2.1 and 5.4.8.

Proposition 5.4.9. — *Let Λ_1, Λ' and Λ be divisors such that $\Lambda_1 < \Lambda' < \Lambda$. If $f \in \mathbb{A}_q^{\Lambda'}$, then its expansion along Λ' is also the expansion of f along (Λ_1, Λ).*

Proof. — Immediate. □

5.5. Analytic classification of linear q-difference equations

We now return to the classification problem, with the notations of chapter 3. In particular, we consider a block diagonal matrice A_0 as in (8) page 51, and the corresponding space $\mathcal{F}(P_1, \dots, P_k)$ of isoformal analytic classes within the formal class defined by A_0.

The space of block upper triangular matrices A_U as in (9) page 51 such that moreover each rectangular block $U_{i,j}$ belongs to $\mathrm{Mat}_{r_i,r_j}(K_{\mu_i,\mu_j})$ (according to the notations of the beginning of subsection 3.3.1) will be written for short $\mathcal{C}_{A_0}$. It follows from proposition 3.3.4 that sending A_U to its class induces an isomorphism of $\mathcal{C}_{A_0}$ with $\mathcal{F}(P_1, \dots, P_k)$.

Let $A \in \mathcal{C}_{A_0}$ et consider the conjugation equation:

$$(\sigma_q F) A_0 = AF; \tag{56}$$

We saw in the first part of 3.3.3 that this admits a unique solution $\hat{F} = (\hat{F}_{ij})$ in $\mathfrak{G}(\mathbf{C}[[z]])$. After J.P. Bézivin [**6**], this solution satisfies the following q-Gevrey condition:

$$\forall i < j \ , \ \hat{F}_{ij} \in \mathrm{Mat}_{r_i,r_j}\big(\mathbf{C}[[z]]_{q;1/(\mu_j-\mu_{j-1})}\big) \ . \tag{57}$$

In this section, we shall first prove that the formal power series $\hat{F}$ is multisummable according to definition 5.4.5. We shall deduce the existence of solutions that are asymptotic to the formal solution $\hat{F}$ at zero along divisors (Λ_1, Λ) (see definition 5.4.8). Last, in 5.5.2, we shall prove that the Stokes multipliers make up a complete set of analytic invariants.

In the following, a matrix-valued function will be said to be summable or multisummable or to have an asymptotic expansion if each coefficient of the matrix is such function. In this way, we will be led to consider spaces $\mathfrak{G}(\mathbb{A}_q^\Lambda)$, $\mathrm{Mat}_{n_1,n_2}(\mathbb{O}^\Lambda_{(\Lambda_1,\ldots,\Lambda_m)})$, etc.

5.5.1. Summability of formal solutions. — Consider the equation (56) and keep the notation (9). To each pair (i,j) such that $i<j$, we associate a divisor $\Lambda_{i,j}$ with degree $\mu_j-\mu_i$; we assume that $\Lambda_{i+1,j}<\Lambda_{i,j}<\Lambda_{i,j+1}$, meaning that $\Lambda_{i,j}=\sum\limits_{\ell=i}^{j-1}\Lambda_{\ell,\ell+1}$. Put $\Lambda_j:=\Lambda_{j-1,j}$ and $\Lambda:=\sum\limits_{j=2}^{k}\Lambda_j$; then $\Lambda_{i,j}=\sum\limits_{\ell=i+1}^{j}\Lambda_j$ and $|\Lambda|=\mu_k-\mu_1$. By convention, we write $\Lambda_{i,i}=\mathbf{O}$, the null divisor.

In order to establish the summability of the formal solution $\hat{F}$, we need a genericity condition on divisors.

Definition 5.5.1. — Let A_0 be as in (8) and let Λ be a divisor of degree $|\Lambda|=\mu_k-\mu_1$. A partition $(\Lambda_1,\ldots,\Lambda_m)$ is called *compatible with* A_0 if $m=k-1$ and if, for i from 1 to $(k-1)$, one has $|\Lambda_i|=\mu_{i+1}-\mu_i$; when this is the case, the partition is called *generic for* A_0 if it moreover satisfies the following non-resonancy condition: $\|\Lambda_i\|\not\equiv\dfrac{\alpha_i}{\alpha_j}\pmod{q^{\mathbf{Z}}}$ for any eigen values α_i of A_i and α_j of A_j, $j>i$.

The following theorem makes more precise the result [**46**, Theorem 3.7]. Here, we prove that the *unique* meromorphic solution $F=F_{(\Lambda_1,\ldots,\Lambda_{k-1})}$, found by both methods, is *asymptotic to the unique formal solution* $\hat{F}$, in the sense of definition 5.4.8). Moreover, its construction is different [**56**] and uses the theorem of Borel-Ritt 5.3.3.

Remark 5.5.2. — The two approaches can be compared with help of the following dictionary (see also subsection 6.1.1):

Here	[**46**]
partition	summation divisor
compatible	adapted
generic	allowed

Theorem 5.5.3 **(Summability)**. — *Let $A\in\mathcal{C}_{A_0}$, let Λ be a divisor and $(\Lambda_1,\ldots,\Lambda_{k-1})$ a generic partition of Λ for A_0. For $i<j$, put $\Lambda_{i,j}:=\sum\limits_{\ell=i}^{j-1}\Lambda_\ell$. Then the conjugacy equation $(\sigma_q F)A_0=AF$ admits a unique solution $F=(F_{ij})$ in $\mathfrak{G}$ such that $F_{ij}\in Mat_{r_i,r_j}(\mathbb{O}^{\Lambda_{i,j}}_{\Lambda_i,\ldots,\Lambda_{j-1}})$ for $i<j$*

Proof. — Take A in the form (9). The conjugacy equation (56) is equivalent to the following system; for$1 \leq i < j \leq k$:

$$z^{\mu_j-\mu_i}(\sigma_q F_{ij})A_j = A_i F_{ij} + z^{-\mu_i} \sum_{\ell=i+1}^{j} U_{i\ell} F_{\ell j}.$$

After J.P. Bézivin [**6**], we know that the formal solution $\hat{F}_{ij}$ has q-Gevrey order $1/(\mu_j - \mu_{j-1})$ for $i < j$.

Let $1 < j \leq k$ and consider the formal solution $\hat{F}_{j-1,j}$ of equation:

$$z^{\mu_j-\mu_{j-1}}(\sigma_q Y)\ A_j = A_{j-1}\ Y + z^{-\mu_{j-1}} U_{j-1,j}. \tag{58}$$

After the theorem of Borel-Ritt 5.3.3, there exists a function $\Phi_{j-1,j} \in \mathbb{A}_q^{\Lambda_{j-1}}$ with $\hat{F}_{j-1,j}$ as an asymptotic expansion at zero. Putting $Y = Z + \Phi_{j-1,j}$ in equation (58), one gets, from theorem 5.3.5:

$$z^{\mu_j-\mu_{j-1}}(\sigma_q Y)\ A_j = A_{j-1}\ Y + \frac{H_{j-1,j}}{\theta_{\Lambda_{j-1}}}\ ,$$

where $H_{j-1,j}$ denotes a meromorphic function in a neighborhood of $z = 0$. Putting $Z := \dfrac{X}{\theta_{\Lambda_{j-1}}}$ one finds:

$$\|\Lambda_{j-1}\|(\sigma_q X)\ A_j = A_{j-1}\ X + H_{j-1,j}\ .$$

By the non-resonancy assumption, this yields a unique solution that is meromorphic at $z = 0$. It follows that the equation admits a unique solution in class $\mathbb{A}_q^{\Lambda_{j-1}}$ which is asymptotic to the series $\hat{F}_{j-1,j}$; the uniqueness is obvious. This solution will be written $F_{j-1,j}$.

Now consider the equation satisfied by $\hat{F}_{j-2,j}$:

$$z^{\mu_j-\mu_{j-2}}(\sigma_q Y)\ A_j = A_{j-2}\ Y + z^{-\mu_{j-2}}\ \big(U_{j-2,j-1} F_{j-1,j} + U_{j-2,j}\big). \tag{59}$$

Choosing a function $\Phi_{j-2,j}$ in $\mathbb{A}_q^{\Lambda_{j-1}}$ with $\hat{F}_{j-2,j}$ as an asymptotic expansion, the change of unknown function $Y = \dfrac{Z + \Phi_{j-2,j}}{\theta_{\Lambda_{j-1}}}$ transforms (59) into:

$$\|\Lambda_{j-1}\|\ z^{\mu_{j-1}-\mu_{j-2}}\ (\sigma_q X)\ A_j = A_{j-2}\ X + H_{j-2,j}\ ,$$

where $H_{j-2,j}$ is a function meromorphic at $z = 0$. We are thus led back to a similar situation as in equation (58), with $\mu_{j-1} - \mu_{j-2}$ instead of $\mu_j - \mu_{j-1}$. One thus gets a solution of (59) in $\mathbb{O}_{\Lambda_{j-2},\Lambda_{j-1}}$ which is asymptotic to $\hat{F}_{j-2,j}$ along the divisors $(\Lambda_{j-2}, \Lambda_{j-1} + \Lambda_{j-2})$. By considering the associated homogeneous equation, one checks that this solution is unique.

Iterating the process, one shows that there is a unique solution $F_{i,j}$ in $\mathbb{O}^{\Lambda_{i,j}}_{\Lambda_i,\ldots,\Lambda_{j-1}}$ which is asymptotic to the formal solution $\hat{F}_{ij}$ for all $i < j$. $\square$

Theorem 5.5.3 can be extended in the following way.

Corollary 5.5.4. — *Let $B_1, B_2 \in \mathfrak{G}(\mathbf{C}\{z\})$ be such that $(A_0)^{-1}\ B_i \in \mathfrak{g}(\mathbf{C}\{z\})$, $i = 1, 2$, and consider the associated q-difference equation:*

$$(\sigma_q Y)\ B_1 = B_2 Y\ . \tag{60}$$

Then, for any given generic partition of divisors $(\Lambda_1, \ldots, \Lambda_{k-1})$ for A_0, there exists, in $\mathfrak{g}$, a unique matrix solution $Y := (Y_{i,j})$ of (60) *and such that, for each pair $i < j$ of indices, $Y_{i,j} \in Mat_{r_i,r_j}(\mathbb{O}^{\Lambda_{i,j}}_{\Lambda_i,\ldots,\Lambda_{j-1}})$.*

Proof. — Recall that (60) is analytically conjugated to an equation of form (56); see section 2.2.4. The result follows from proposition 5.4.6 and the fact that $\mathbb{O}^{\Lambda_{i,j}}_{\Lambda_i,\ldots,\Lambda_{j-1}} \subset \mathbb{O}^{\Lambda_{i,j+1}}_{\Lambda_i,\ldots,\Lambda_j}$) for all $i < j$. $\square$

An obvious consequence of theorem 5.5.3 and corollary 5.5.4 is the following.

Theorem 5.5.5 **(Existence of asymptotic solutions)**
The functions $F = (F_{i,j})$ and $Y = (Y_{i,j})$ respectively considered in theorem 5.5.3 and corollary 5.5.4 both admit an asymptotic expansion in the following sense: for all pairs (i, j) with $i < j$, the blocks $F_{i,j}$ and $Y_{i,j}$ have an asymptotic expansion at 0 *along divisors $(\Lambda_j, \Lambda_{i,j})$, according to definition 5.4.8.*

Proof. — Immediate, with help of theorem 5.4.7. $\square$

5.5.2. Stokes phenomenon and analytic classification. — Write $F^A_{(\Lambda_1,\ldots,\Lambda_{k-1})}$ the solution of (56) obtained in theorem 5.5.3, which can be seen as a sum of the formal solution $\hat{F}$ with respect to generic partition $(\Lambda_1, \ldots, \Lambda_{k-1})$. One thereby deduces a summation process of $\hat{F}$ with respect to each generic partition of divisors for A_0; we shall denote:

$$\mathcal{F}^A : (\Lambda_1, \ldots, \Lambda_{k-1}) \mapsto F^A_{(\Lambda_1,\ldots,\Lambda_{k-1})}.$$

The q-analogue of Stokes phenomenon is displayed by the existence of various "sums" of $\hat{F}$.

Proposition 5.5.6. — *In theorem 5.5.3, one has $A = A_0$ if, and only if, the map $\mathcal{F}^A$ is not constant on the set of generic partitions of divisors for A_0.*

Proof. — If $A = A_0$, the formal solution boils down to the identity matrix, which plainly coincides with $F^A_{(\Lambda_1,\ldots,\Lambda_{k-1})}$ for any compatible partition $(\Lambda_1,\ldots,\Lambda_{k-1})$.

On the other hand, if $A \neq A_0$, the polynomial $U_{i,j}$ in (9) are not all 0. To begin with, assume that $U_{j-1,j} \neq 0$ for some index j. Since $\hat{F}_{j-1,j}$ diverges, its sum along divisor Λ_j will be distinct from its sum along any different divisor. More generally, when $U_{i,j} \neq 0$, the solution $F^A_{(\Lambda_1,\ldots,\Lambda_{k-1})}$ will depend in a one-to-one way on divisor Λ_j; details are left to the reader. □

For any $\lambda \in \mathbf{C}^*$, let $[\lambda; q : A_0]$ be the ordered partition of divisors generated by the spiral $[\lambda; q]$ in the following way:

$$[\lambda; q : A_0] := \big((\mu_2 - \mu_1)[\lambda; q], (\mu_3 - \mu_2)[\lambda; q], \ldots, (\mu_k - \mu_{k-1})[\lambda; q]\big) ,$$

where μ_j are as in (8), *i.e.* $A_0 = \mathrm{diag}\,(z^{\mu_1} A_1, \ldots, z^{\mu_k} A_k)$.

If $[\lambda; q : A_0]$ is generic with respect to A_0, we will write $\lambda \in \{A_0\}$ and say that λ *is generic for* A_0.

***Theorem 5.5.7* (Stokes phenomenon).** — *Fix some $\lambda_0 \in \{A_0\}$ and for any $\lambda \in \{A_0\}$ and $A \in \mathcal{C}_{A_0}$, set:*

$$St_{\lambda_0}(\lambda; A) := \left(F^A_{[\lambda_0;q:A_0]}\right)^{-1} F^A_{[\lambda;q:A_0]} .$$

The following conditions are equivalent.

1. *There exists $\lambda \in \{A_0\}$ such that $[\lambda; q] \neq [\lambda_0; q]$ but $St_{\lambda_0}(\lambda; A) = Id$.*
2. *For all $\lambda \in \{A_0\}$, the equality $St_{\lambda_0}(\lambda; A) = Id$ holds.*
3. *One has $A = A_0$.*

Proof. — It follows immediately from proposition 5.5.6 □

In the previous theorem, each $\mathrm{St}_{\lambda_0}(\lambda; A)$ represents an upper-triangular unipotent matrix-valued function that is analytic in some disk $0 < |z| < R$ except at spirals $[\lambda; q]$ and $[\lambda_0; q]$; moreover, it is infinitely close to the identity matrix as $z \to 0$. If $Y := \mathrm{St}_{\lambda_0}(\lambda; A) - I_n$, then Y is flat at zero and satifies the relation:

$$(\sigma_q Y)\ A_0 = A_0\ Y.$$

Theorem 5.5.7 implies that each matrix A chosen within the isoformal class $\mathcal{C}_{A_0}$ reduces to the normal form A_0 whenever the Stokes matrix $\mathrm{St}_{\lambda_0}(\lambda; A)$ becomes trivial for some couple of generic parameters λ_0, λ such that $\lambda \neq \lambda_0 \bmod q^{\mathbf{Z}}$.

We shall now show that for any given pair of generic parameters λ_0, λ that are not q-congruent, the data $\{\mathrm{St}_{\lambda_0}(\lambda; A), A_0\}$ constitutes a complete set of analytical invariants associated to the q-difference module determined by A.

Indeed, the notation $\mathrm{St}_{\lambda_0}(\lambda; A)$ introduced for any $A \in \mathcal{C}_{A_0}$ can be naturally generalized for any matrix $B \in \mathfrak{G}(\mathbf{C}\{z\})$ that belongs to the same formal class as A_0. To simplify, assume that $(A_0)^{-1}B \in \mathfrak{G}(\mathbf{C}\{z\})$; this is for instance the case if B is taken in Birkhoff-Guenther normal form (definition 3.3.6). From corollary 5.5.4 it follows that the corresponding conjugacy equation:

$$(\sigma_q Y)\ B = A_0\ Y$$

admits, in $\mathfrak{G}$, a unique solution in the space of multi-summable functions associated to any given generic ordered partition of divisors $[\lambda; q : A_0]$. Thus, we are led to the notations $\mathcal{F}^{B}_{[\lambda;q:A_0]}$ and $\mathrm{St}_{\lambda_0}(\lambda; B)$ as in the case of $A \in \mathcal{C}_{A_0}$.

Corollary 5.5.8. — *Suppose B_1 and B_2 be two matrices such that $(A_0)^{-1}B_i \in \mathfrak{G}(\mathbf{C}\{z\})$. Then the following assertions are equivalent.*

1. *There exists $(\lambda_0, \lambda) \in \{A_0\} \times \{A_0\}$ such that $[\lambda; q] \neq [\lambda_0; q]$ but $St_{\lambda_0}(\lambda; B_1) = St_{\lambda_0}(\lambda; B_2)$.*
2. *The equality $St_{\lambda_0}(\lambda; B_1) = St_{\lambda_0}(\lambda; B_2)$ holds for all λ_0, $\lambda \in \{A_0\}$.*
3. *The matrices B_1 and B_2 give rise to analytically equivalent q-difference modules.*

Proof. — The only point to prove is that (1) implies (3). To do that, notice that from (1) we obtain the equality

$$\mathcal{F}^{B_2}_{[\lambda_0;q:A_0]} \left(\mathcal{F}^{B_1}_{[\lambda_0;q:A_0]}\right)^{-1} = \mathcal{F}^{B_2}_{[\lambda;q:A_0]} \left(\mathcal{F}^{B_1}_{[\lambda;q:A_0]}\right)^{-1},$$

in which the left hand side ant the right hand side are both solution to the same equation (60). Since these solutions are analytical in some disk $0 < |z| < R$ except maybe respectively on the q-spiral $[\lambda_0; q]$ or $[\lambda; q]$ and that they have a same asymptotic expansion at zero, they must be analytic at $z = 0$ and we arrive at assertion (3). □

CHAPTER 6

GEOMETRY OF THE SPACE OF CLASSES

In this chapter we describe to some extent the geometry of the space $\mathcal{F}(M_0)$ of analytic isoformal classes in the formal class M_0. In subsection 3.3.2, we already used the Birkhoff-Guenther normal form to find coordinates on $\mathcal{F}(M_0)$. Here, we rather use the identification of $\mathcal{F}(M_0)$ with $H^1(\mathbf{E}_q, \Lambda_I(M_0))$ proved in theorem 4.4.1 and completed in section 5.5. The description given here will be further pursued in a separate work [**48**].

6.1. Privileged cocycles

In "applications", it is sometimes desirable to have explicit cocycles to work with instead of cohomology classes. We shall now describe cocycles with nice properties, that can be explicitly computed from a matrix in standard form. Most of the proofs of what follows are consequences of theorem 5.5.3. However, they can also be obtained through the more elementary approach of [**46**], which we shall briefly summarize here in subsection 6.1.1.

Fix A_0 as in (8) and let M_0 the corresponding pure module. *All notations that follow are relative to this* A_0 *and* M_0. Recall from section 1.3 at the end of the introduction the notations $\overline{a} \in \mathbf{E}_q$ for the class of $a \in \mathbf{C}^*$ and $[a;q] := aq^{\mathbf{Z}}$ for the corresponding q-spiral. Also recall the function θ_q, holomorphic over $\mathbf{C}^*$ with simple zeroes on $[-1;q]$ and satisfying the functional equation $\sigma_q\theta_q = z\theta_q$.

6.1.1. "Algebraic summation". — To construct cocycles encoding the analytic class of $A \in \mathrm{GL}_n(\mathbf{C}(\{z\}))$, we use "meromorphic sums" of $\hat{F}_A$. These can be obtained either by the summation process of theorem 5.5.3, or by the "algebraic summation process" of [**46**]. We now summarize this process. We restrict to the case of divisors supported by a point, since it is sufficient for classification.

Let A as in (9) (we do not assume from start that it is in Birkhoff-Guenther normal form). We want to solve the equation $\sigma_q F = AFA_0^{-1}$ with $F \in \mathfrak{G}_{A_0}$ meromorphic in some punctured neighborhood of 0 in $\mathbf{C}^*$. Since A, A_0, A^{-1}, A_0^{-1} are all holomorphic in some punctured neighborhood of 0 in $\mathbf{C}^*$, the singular locus[(1)] of F near 0 is invariant under $q^{-\mathbf{N}}$: it is therefore a finite union of germs of half q-spirals $aq^{-\mathbf{N}}$ (the finiteness flows from the meromorphy of F). If we take A in Birkhoff-Guenther normal form (definition 3.3.6), the functional equation $\sigma_q F = AFA_0^{-1}$ actually allows us to extend F to a meromorphic matrix on the whole of $\mathbf{C}^*$ and its singular locus is a finite union of discrete logarithmic q-spirals $[a; q]$.

To illustrate the process, we start with an example.

Example 6.1.1. — Setting $A_u := \begin{pmatrix} z^{-1} & z^{-1}u \\ 0 & 1 \end{pmatrix}$, with $u \in \mathbf{C}(\{z\})$, and $F = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}$, we saw that $A_u = F[A_0] \Longleftrightarrow z\sigma_q f - f = u$. This admits a unique formal solution $\hat{f}$, which can be computed by iterating the z-adically contracting operator $f \mapsto -u + z\sigma_q f$. If A is in Birkhoff-Guenther normal form, $u \in \mathbf{C}$ and $\hat{f} = -u\,\mathrm{Ch}$ (the Tshakaloff series). We want f to be meromorphic on some punctured neighborhood of 0 in $\mathbf{C}^*$, which we write $f \in \mathcal{M}(\mathbf{C}^*, 0)$, and to have (in germ) half q-spirals of poles. For simplicity, we shall assume that u is holomorphic on $\mathbf{C}^*$, so that f will have to be meromorphic on the whole of $\mathbf{C}^*$ with complete q-spirals of poles. We set $\theta_{q,c}(z) := \theta_q(z/c)$, which is holomorphic on $\mathbf{C}^*$, with simple zeroes on the q-spiral $[-c; q]$ and satisfies the functional equation $\sigma_q \theta_{q,c} = (z/c)\theta_{q,c}$. We look for f with simple poles on $[-c; q]$ by writing it $f = g/\theta_{q,c}$ and looking for g holomorphic on $\mathbf{C}^*$ and satisfying the functional equation:

$$z\sigma_q(g/\theta_{q,c}) - (g/\theta_{q,c}) = u \Longleftrightarrow c\sigma_q g - g = u\theta_{q,c} \Longleftrightarrow \forall n\ ,\ (cq^n - 1)g_n = [u\theta_{q,c}]_n,$$

[(1)]In the context of q-difference equations, the singular locus of a matrix of functions P is made of the poles of P as well as those of P^{-1}. For a unipotent matrix, like F, these are all poles of F since F^{-1} is a polynomial in F.

where we have introduced the Laurent series expansions $g = \sum g_n z^n$ and $u\theta_{q,c} = \sum [u\theta_{q,c}]_n z^n$. If $c \notin q^{\mathbf{Z}}$, we get the solution:

$$f_{\overline{c}} := \frac{1}{\theta_{q,c}} \sum \frac{[u\theta_{q,c}]_n}{cq^n - 1} z^n.$$

This is clearly the unique solution of $z\sigma_q f - f = u$ with only simple poles on $[-c; q]$, a condition that depends on $\overline{c}$ only and justifies the notation $f_{\overline{c}}$. We consider it as *the summation of the formal solution $\hat{f}$ in the "authorized direction of summation" $\overline{c} \in \mathbf{E}_q$*. Accordingly, we write it $S_{\overline{c}}\hat{f}$. The sum $S_{\overline{c}}\hat{f}$ is asymptotic to $\hat{f}$ (see further below). Note that the "directions of summation" are elements of $\mathbf{E}_q = \mathbf{C}^*/q^{\mathbf{Z}}$ which plays here the role of the circle of directions $S^1 := \mathbf{C}^*/\mathbf{R}_+^*$ of the classical theory. Also note that all "directions of summations" $\overline{c} \in \mathbf{E}_q$ are authorized, except for one.

Returning to the general equation $F[A_0] = A \Leftrightarrow \sigma_q F = AFA_0^{-1}$, we want to look for solutions F that are meromorphic on some punctured neighborhood of 0 in $\mathbf{C}^*$, and we want to have unicity by imposing conditions on the half q-spirals of poles: position and multiplicity. Thus, for $f \in \mathcal{M}(\mathbf{C}^*, 0)$, and for a finite family (a_i) of points of $\mathbf{C}^*$ and (n_i) of integers, we shall translate the condition: "The germ f at 0 has all its poles on $\bigcup [a_i; q]$, the poles on $[a_i; q]$ having at most multiplicity n_i" by an inequality of divisors on $\mathbf{E}_q$:

$$\mathrm{div}_{\mathbf{E}_q}(f) \geq -\sum n_i [\overline{a_i}].$$

We introduce a finite subset of $\mathbf{E}_q$ defined by resonancy conditions:

$$\Sigma_{A_0} := \bigcup_{1 \leq i < j \leq k} S_{i,j}, \text{ where } S_{i,j} := \{p(-a) \in \mathbf{E}_q \mid q^{\mathbf{Z}} a^{\mu_i} \mathrm{Sp}(A_i) \cap q^{\mathbf{Z}} a^{\mu_j} \mathrm{Sp}(A_j) \neq \emptyset\}.$$

Here, Sp denotes the spectrum of a matrix.

***Theorem 6.1.2* ("Algebraic summation").** — *For any matrix A in form* (9)*, and for any "authorized direction of summation" $\overline{c} \in \mathbf{E}_q \setminus \Sigma_{A_0}$, there exists a unique meromorphic gauge transform $F \in \mathfrak{G}_{A_0}\big(\mathcal{M}(\mathbf{C}^*, 0)\big)$ satisfying:*

$$F[A_0] = A \text{ and } div_{\mathbf{E}_q}(F_{i,j}) \geq -(\mu_j - \mu_i)[\overline{-c}] \text{ for } 1 \leq i < j \leq k.$$

Proof. — See [**46**, theorem 3.7]. □

Of course, $\mathrm{div}_{\mathbf{E}_q}(F_{i,j}) \geq -(\mu_j - \mu_i)[\overline{-c}]$ means that all coefficients f of the rectangular block $F_{i,j}$ are such that $\mathrm{div}_{\mathbf{E}_q}(f) \geq -(\mu_j - \mu_i)[\overline{-c}]$. The matrix F of the theorem is considered as *the summation of $\hat{F}_A$ in the authorized direction of summation* $\overline{c} \in \mathbf{E}_q$ and we write it $S_{\overline{c}}\hat{F}_A$.

It is actually the very same sum as that obtained in theorem 5.5.3. Indeed, using notations from subsection 5.5.1, if one requires the divisors Λ_i to be supported by a single point $\overline{c} \in \mathbf{E}_q$, the genericity condition of definition 5.5.1 translates here to: $\overline{c} \in \mathbf{E}_q \setminus \Sigma_{A_0}$. As a consequence of subsection 5.5.1, the sum $S_{\overline{c}}\hat{F}_A$ is asymptotic to $\hat{F}_A$.

The algorithm to compute $F := S_{\overline{c}}\hat{F}_A$ is the following. We introduce the gauge transform:

$$T_{A_0,c} := \begin{pmatrix} {\theta_{q,c}}^{-\mu_1} I_{r_1} & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 0 & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & {\theta_{q,c}}^{-\mu_k} I_{r_k} \end{pmatrix}. \quad \text{(Recall that } \theta_{q,c}(z) = \theta_q(z/c).\text{)}$$

Then $F[A_0] = A$, is equivalent to $G[B_0] = B$, where $B = T_{A_0,c}[A]$, $B_0 = T_{A_0,c}[A_0]$, and $G = T_{A_0,c} F T_{A_0,c}^{-1}$. Now, B_0 is block-diagonal with blocks $c^{\mu_i} A_i \in \mathrm{GL}_{r_i}(\mathbf{C})$, and we can solve for $G[B_0] = B$ with $G \in \mathfrak{G}_{A_0}\big(\mathcal{O}(\mathbf{C}^*, 0)\big)$, because this boils down to a system of matricial equations:

$$(\sigma_q X) c^{\mu_j} A_j - c^{\mu_i} A_i X = \text{ some r.h.s. } Y.$$

Expanding in Laurent series $X = \sum X_p z^p$, $Y = \sum Y_p z^p$, we are led to:

$$q^p X_p c^{\mu_j} A_j - c^{\mu_i} A_i X_p = Y_p.$$

Since the spectra of $c^{\mu_i} A_i$ and $c^{\mu_i} A_i$ are non resonant modulo $q^{\mathbf{Z}}$, the spectra of $q^p c^{\mu_i} A_i$ and $c^{\mu_i} A_i$ are disjoint and the endomorphism $V \mapsto q^p V c^{\mu_j} A_j - c^{\mu_i} A_i V$ of $\mathrm{Mat}_{r_i,r_j}(\mathbf{C})$ is actually an automorphism. There is therefore a unique formal solution X, and convergence in $(\mathbf{C}^*, 0)$ is not hard to prove, so we indeed get G with the required properties. Last, putting $F := T_{A_0,c}^{-1} G T_{A_0,c}$ gives $S_{\overline{c}}\hat{F}_A$ in the desired form.

6.1.2. Privileged cocycles. — We have three ways of constructing a cocycle encoding the analytic isoformal class of A in the formal class defined by A_0: the map λ defined in the first part of the proof of theorem 4.4.1; the construction of theorem 5.5.7; and the one of [**46**], based on theorem 6.1.2. The three ways give the same cocycle, which we now describe more precisely.

If M_0 is the pure module with matrix A_0, recall from section 4.3 the sheaf $\Lambda_I(M_0)$ of automorphisms of A_0 infinitely tangent to the identity. We shall also denote $\mathfrak{U}$ the covering of $\mathbf{E}_q$ consisting of the Zariski open subsets $U_c := \mathbf{E}_q \setminus \{c\}$

such that $c \notin \Sigma_{A_0}$ (thus, we drop the upper bar denoting classes in $\mathbf{E}_q$). For $c \in \mathbf{E}_q \setminus \Sigma_{A_0}$ an authorized direction of summation write for short $F_c := S_c \hat{F}_A$. Then F_c is holomorphic invertible over the preimage $p^{-1}(U_c) \subset \mathbf{C}^*$. If $c, d \in \mathbf{E}_q \setminus \Sigma_{A_0}$, $F_{c,d} := F_c^{-1} F_d$ is in $\mathfrak{G}\big(\mathcal{O}(p^{-1}(U_c \cap U_d))\big)$ and it is an automorphism of A_0 infinitely tangent to the identity:

$$F_{c,d} \in \Lambda_I(M_0)(U_c \cap U_d).$$

The cocycle $(F_{c,d}) \in Z^1(\mathfrak{U}, \Lambda_I(M_0))$ involves rectangular blocks $(F_{c,d})_{i,j}$ (for $1 \leq i < j \leq k$) belonging to the space $E_{i,j}$ of solutions of the equation $(\sigma_q X) z^{\mu_j} A_j = z^{\mu_i} A_i X$. For $X \in E_{i,j}$, it makes sense to speak of its poles on $\mathbf{E}_q$, and of their multiplicities. For $c, d \in \mathbf{E}_q \setminus \Sigma_{A_0}$ distinct, we write $E_{i,j,c,d}$ the space of those $X \in E_{i,j}$ that have at worst poles in c, d and with multiplicities $\mu_j - \mu_i$.

Lemma 6.1.3. — *The space $E_{i,j,c,d}$ has dimension $r_i r_j (\mu_j - \mu_i)$.*

Proof. — Writing again $\theta_{q,a}(z) := \theta_q(z/a)$, any $X \in E_{i,j,c,d}$ can be written $(\theta_{q,a}\theta_{q,b})^{-(\mu_j - \mu_i)} Y$, where $p(-a) = c$, $p(-b) = d$ and Y is holomorphic on $\mathbf{C}^*$ and satisfies the equation:

$$\sigma_q Y = \left(\frac{z}{ab}\right)^{\mu_j - \mu_i} A_i Y A_j^{-1}.$$

Taking Y as a Laurent series, one sees that $(\mu_j - \mu_i)$ consecutive terms in $\mathrm{Mat}_{r_i, r_j}(\mathbf{C})$ can be chosen at will. □

Definition 6.1.4. — The cocycle $(F_{c,d}) \in Z^1(\mathfrak{U}, \Lambda_I(M_0))$ is said to be *privileged* if $(F_{c,d})_{i,j} \in E_{i,j,c,d}$ for all distinct $c, d \in \mathbf{E}_q \setminus \Sigma_{A_0}$ and all $1 \leq i < j \leq k$. The space of privileged cocycles is denoted $Z^1_{pr}(\mathfrak{U}, \Lambda_I(M_0))$.

It is not hard to see that, i, j being fixed, the corresponding component $(c, d) \mapsto (F_{c,d})_{i,j}$ of a cocycle is totally determined by its value at any particular choice of c, d. That is, fixing distinct $c, d \in \mathbf{E}_q \setminus \Sigma_{A_0}$ gives an isomorphism:

$$Z^1_{pr}(\mathfrak{U}, \Lambda_I(M_0)) \to \bigoplus_{1 \leq i < j \leq k} E_{i,j,c,d}.$$

Thus, the space of privileged cocycles has the same dimension as $\mathcal{F}(M_0)$. Actually, one has two bijections:

Theorem 6.1.5. — *The following natural maps are bijections:*

$$\mathcal{F}(M_0) \to Z^1_{pr}(\mathfrak{U}, \Lambda_I(M_0)) \to H^1(\mathbf{E}_q, \Lambda_I(M_0)).$$

Proof. — See [**46**, prop. 3.17, th. 3.18]. The second map is the natural one. □

6.1.3. q-Gevrey interpolation. — The results found in section 3.4 can be adapted here *mutatis mutandis*: the two classification problems tackled there lead to the following spaces of classes: $\bigoplus_{\mu_j-\mu_i<1/s} E_{i,j,c,d}$ and $\bigoplus_{\mu_j-\mu_i\geq 1/s} E_{i,j,c,d}$, which have exactly the desired dimensions. This can be most easily checked using the devissage arguments from section 6.2 below.

6.2. Dévissage q-Gevrey

The following results come mostly from [**46**].

6.2.1. The abelian case of two slopes. — When the Newton polygon of M_0 has only two slopes $\mu < \nu$ with multiplicities r, s, so that we can write:

$$A_0 = \begin{pmatrix} z^\mu B & 0 \\ 0 & z^\nu C \end{pmatrix}, \quad B \in \mathrm{GL}_r(\mathbf{C}),\ C \in \mathrm{GL}_s(\mathbf{C}),$$

then the unipotent group $\mathfrak{G}_{A_0}$ is isomorphic to the vector space $\mathrm{Mat}_{r,s}$ through the isomorphism:

$$F \mapsto \begin{pmatrix} I_r & F \\ 0 & I_s \end{pmatrix}$$

from the latter to the former. Accordingly, the sheaf of unipotent groups $\Lambda_I(M_0)$ can be identified with the abelian sheaf Λ on $\mathbf{E}_q$ defined by:

$$\Lambda(U) := \{F \in \mathrm{Mat}_{r,s}\big(\mathcal{O}(p^{-1}(U))\big) \mid (\sigma_q F)(z^\nu C) = (z^\mu B)F\}.$$

This is actually a locally free sheaf, hence the sheaf of sections of a holomorphic vector bundle on $\mathbf{E}_q$, which we also write Λ (more on this in section 6.3). This bundle can be described geometrically as the quotient of the trivial bundle $\mathbf{C}^* \times \mathrm{Mat}_{r,s}(\mathbf{C})$ over $\mathbf{C}^*$ by the equivariant action of $q^{\mathbf{Z}}$ determined by the action $(z, F) \mapsto \big(qz, (z^\mu B)F(z^\nu C)^{-1}\big)$ of the generator q:

$$\Lambda = \frac{\mathbf{C}^* \times \mathrm{Mat}_{r,s}(\mathbf{C})}{(z,F) \sim \big(qz, (z^\mu B)F(z^\nu C)^{-1}\big)} \longrightarrow \frac{\mathbf{C}^*}{z \sim qz} = \mathbf{E}_q.$$

For details, see [**46, 43**]. This bundle is the tensor product of a line bundle of degree $\mu - \nu$ (corresponding to the "theta" factor $\sigma_q f = z^{\mu-\nu} f$) and of a flat bundle (corresponding to the "fuchsian" factor $\sigma_q F = BFC^{-1}$); we shall say that it is *pure (isoclinic)*. Now, the first cohomology group of such a vector bundle is a finite dimensional vector space whose dimension can be easily computed from its rank rs and degree $\mu - \nu$: it is $rs(\nu - \mu)$. Actually, using Serre duality and an explicit frame of the dual bundle (made up of theta functions), one can provide an explicit coordinate system on $H^1(\mathbf{E}_q, \Lambda)$: this is done in [**43**], where the relation to the q-Borel transform is shown. In this section, we shall see how, in general, the non-abelian cohomology set $H^1(\mathbf{E}_q, \Lambda)$ can be described from successive extensions from the abelian situation.

6.2.2. A sequence of central extensions. — For simplicity, we write $\mathfrak{G}$ for $\mathfrak{G}_{A_0}$. The Lie algebra $\mathfrak{g}$ of $\mathfrak{G}$ consists in all nilpotent matrices of the form:

$$\begin{pmatrix} 0_{r_1} & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & F_{i,j} & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots \\ \ldots & 0 & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \ldots & 0_{r_k} \end{pmatrix},$$

where 0_r is the square null matrix of size r and where each $F_{i,j}$ is in Mat_{r_i,r_j} for $1 \leq i < j \leq k$. For each integer δ, write $\mathfrak{g}^{\geq \delta}$ the sub-Lie algebra of matrices whose only non null blocks $F_{i,j}$ have level $\mu_j - \mu_i \geq \delta$; it is actually an ideal of $\mathfrak{g}$ and $\mathfrak{G}^{\geq \delta} := I_n + \mathfrak{g}^{\geq \delta}$ is a normal subgroup of $\mathfrak{G}$. Moreover, one has an exact sequence:

$$0 \to \mathfrak{G}^{\geq \delta}/\mathfrak{G}^{\geq \delta+1} \to \mathfrak{G}/\mathfrak{G}^{\geq \delta+1} \to \mathfrak{G}/\mathfrak{G}^{\geq \delta} \to 1,$$

actually, a central extension. The map $g \mapsto I_n + g$ induces an isomorphism from the vector space $\mathfrak{g}^{\geq \delta}/\mathfrak{g}^{\geq \delta+1}$ to the kernel $\mathfrak{G}^{\geq \delta}/\mathfrak{G}^{\geq \delta+1}$ of this exact sequence. We write $g^{(\delta)}$ this group: it consists of matrices in $\mathfrak{g}$ whose only non null blocks $F_{i,j}$ have level $\mu_j - \mu_i = \delta$. (This is the reason why we wrote a 0 instead of a 1 at the left of the exact sequence !)

Restricting the above consideration to $\Lambda := \Lambda_I(M_0)$ considered as a sheaf of subgroups of $\mathfrak{G}$ (with coefficients in function fields), we get a similar central extension:

$$0 \to \lambda^{(\delta)} \to \Lambda/\Lambda^{\geq \delta+1} \to \Lambda/\Lambda^{\geq \delta} \to 1.$$

Now, a fundamental theorem in the non-abelian cohomology of sheaves says that we have an exact sequence of cohomology sets:

Theorem 6.2.1. — *One has an exact sequence of pointed sets:*

$$0 \to H^1(\mathbf{E}_q, \lambda^{(\delta)}) \to H^1(\mathbf{E}_q, \Lambda/\Lambda^{\geq \delta+1}) \to H^1(\mathbf{E}_q, \Lambda/\Lambda^{\geq \delta}) \to 1.$$

Proof. — The exactness should be here understood in a rather strong sense. The sheaf $\lambda^{(\delta)}$ is here a pure isoclinic holomorphic vector bundle of degree $-\delta$ and rank $\sum\limits_{\mu_j - \mu_i = \delta} r_i r_j$. Its first cohomology group $V^{(\delta)}$ is therefore a vector space of dimension $\delta \sum\limits_{\mu_j - \mu_i = \delta} r_i r_j$. The theorem says that $V^{(\delta)}$ operates on the pointed set $H^1(\mathbf{E}_q, \Lambda/\Lambda^{\geq \delta+1})$ with quotient the pointed set $H^1(\mathbf{E}_q, \Lambda/\Lambda^{\geq \delta})$. For a proof, see [**19**, th. 1.4 and prop. 8.1]. □

Corollary 6.2.2. — *There is a natural bijection of $H^1(\mathbf{E}_q, \Lambda_I(M_0))$ with $\bigoplus\limits_{\delta \geq 1} H^1(\mathbf{E}_q, \lambda^{(\delta)})$.*

Proof. — Indeed, the cohomology sets being pointed, the theorem yields a bijection of each $H^1(\mathbf{E}_q, \Lambda/\Lambda^{\geq \delta+1})$ with $H^1(\mathbf{E}_q, \Lambda/\Lambda^{\geq \delta}) \times H^1(\mathbf{E}_q, \lambda^{(\delta)})$. □

We shall write $\mathcal{F}_{\leq \delta}(M_0) := H^1(\mathbf{E}_q, \Lambda/\Lambda^{\geq \delta+1})$. This space is the solution to the following classification problem: two matrices with diagonal part A_0 are declared equivalent if their truncatures corresponding to the levels $\leq \delta$ are analytically equivalent (through a gauge transform in $\mathfrak{G}$); and there is no condition on the components with levels $> \delta$. This is exactly the equivalence under $\mathbf{C}((z))_{q;s}$ for $\delta < s \leq \delta + 1$. The corresponding Birkhoff-Guenther normal forms have been described in section 3.4. The extreme cases are $\delta > \mu_k - \mu_1$, where $\mathcal{F}_{\leq \delta}(M_0)$ is the whole space $\mathcal{F}(M_0)$; and $\delta < 1$, where it is trivial.

6.2.3. Explicit computation. — We want to make more explicit theorem 6.2.1.

So we assume that A, A' represent two classes in $H^1(\mathbf{E}_q, \Lambda/\Lambda^{\geq \delta+1})$ having the same image in $H^1(\mathbf{E}_q, \Lambda/\Lambda^{\geq \delta})$. Up to an analytic gauge transform, the situation just described can be made explicit as follows. Assume that A, A' have graded part A_0 and are in Birkhoff-Guenther normal form. Assume moreover that they coincide up to level $\delta - 1$. Then the same is true for $\hat{F}_A$ and $\hat{F}_{A'}$; therefore, it is also true for $\hat{F}_{A,A'}$ and I_n, that is, $\hat{F}_{A,A'} \in \mathfrak{G}^{\geq \delta}(\mathbf{C}((z)))$. The first non-trivial upper diagonal of $\hat{F}_{A,A'}$ is at level δ and, as a divergent series, generically it actually has q-Gevrey level δ; if it has level $< \delta$, then it is convergent. The classes of A, A' in $\mathcal{F}_{\leq \delta}(M_0) = H^1(\mathbf{E}_q, \Lambda/\Lambda^{\geq \delta+1})$ have the same image in $\mathcal{F}_{\leq \delta - 1}(M_0) = H^1(\mathbf{E}_q, \Lambda/\Lambda^{\geq \delta})$. After theorem 6.2.1, there exists a unique element of $V^{(\delta)}$ which carries the class of A to the class of A'. The Stokes matrices $S_{c,d}\hat{F}_A$ and $S_{c,d}\hat{F}_{A'}$ are congruent modulo $\Lambda^{\geq \delta}$, so that their quotient $S_{c,d}\hat{F}_{A,A'}$ is in $\Lambda^{\geq \delta}$. Its first non trivial upper diagonal is at level δ; call it $f_{c,d}$ and consider it as an element of $\lambda^{(\delta)}$. This defines a cocycle, hence a class in $H^1(\mathbf{E}_q, \lambda^{(\delta)})$. This class is the element of $V^{(\delta)}$ we look for.

Example 6.2.3. — Take $A_0 := \begin{pmatrix} a & 0 & 0 \\ 0 & bz & 0 \\ 0 & 0 & cz^2 \end{pmatrix}$ and $A := \begin{pmatrix} a & u & v_0+v_1z \\ 0 & bz & wz \\ 0 & 0 & cz^2 \end{pmatrix}$ in Birkhoff-Guenther normal form, so that $u, v_0, v_1, w \in \mathbf{C}$. Here, the total space of classes $H^1(\mathbf{E}_q, \Lambda_I(M_0))$ has dimension 4, while each of the components $H^1(\mathbf{E}_q, \lambda^{(1)})$ and $H^1(\mathbf{E}_q, \lambda^{(2)})$ has dimension 2.

To compute the class of A in $H^1(\mathbf{E}_q, \Lambda_I(M_0))$ identified with $H^1(\mathbf{E}_q, \lambda^{(1)}) \oplus H^1(\mathbf{E}_q, \lambda^{(2)})$, we introduce the intermediate point $A' := \begin{pmatrix} a & u & 0 \\ 0 & bz & wz \\ 0 & 0 & cz^2 \end{pmatrix}$ and the intermediate gauge transforms $G := \begin{pmatrix} 1 & f & x \\ 0 & 1 & h \\ 0 & 0 & 1 \end{pmatrix}$ such that $G[A_0] = A'$ and $H := \begin{pmatrix} 1 & 0 & g \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ such that $H[A'] = A$, so that $F := HG = \begin{pmatrix} 1 & f & g+x \\ 0 & 1 & h \\ 0 & 0 & 1 \end{pmatrix}$ is such that $F[A_0] = A$.

The component in $H^1(\mathbf{E}_q, \lambda^{(1)})$ is computed by considering G alone. Its coefficients satisfy the equations:

$$bz\sigma_q f - af = u \text{ and } cz\sigma_q h - bh = w.$$

(The coefficient x is irrelevant here.) We shall see in section 7.1 how to compute the corresponding cocycles $(f_{c,d})$ and $(h_{c,d})$, but what is clear here is that they are respectively linear functions of u and of w. So in the end, the component of the class of A in $H^1(\mathbf{E}_q, \lambda^{(1)})$ is $uL_1(a,b) + wL_1(b,c)$ for some explicit basic classes $L_1(a,b)$, $L_1(b,c)$ (the index 1 is for the level).

Similarly, the component in $H^1(\mathbf{E}_q, \lambda^{(2)})$ is computed by considering H alone. Its coefficient satisfies the equation:

$$cz^2\sigma_q g - ag = v_0 + v_1 z.$$

It is therefore clear that the component we look for is $v_0 L_{2,0}(a,c) + v_1 L_{2,1}(a,c)$ for some explicit base of classes $L_{2,0}(a,c), L_{2,1}(a,c)$.

This example will be pursued in section 7.2.

6.2.4. Various geometries on $\mathcal{F}(P_1, \ldots, P_k)$. — In subsection 3.3.2, we drawed from the Birkhoff-Guenther notrmal form an affine structure on $\mathcal{F}(M_0)$. This structure is made explicit by the coordinates provided by proposition 3.3.4. The dévissage above implies that it is the same as the affine structure on $H^1(\mathbf{E}_q, \Lambda_I(M_0))$ inherited from the vector space structures on the $H^1(\mathbf{E}_q, \lambda^{(\delta)})$ through corollary 6.2.2. More precisely:

Theorem 6.2.4. — *The mapping from* $\prod_{1\leq i<j\leq k} Mat_{r_i,r_j}(K_{\mu_i,\mu_j})$ *to* $\bigoplus_{\delta\geq 1} H^1(\mathbf{E}_q, \lambda^{(\delta)})$ *coming from proposition 3.3.4, corollary 6.2.2 and theorems 4.4.1 and 5.5 is linear.*

Proof. — The computations are in essence the same as in the example. Details will be written in [**48**]. □

There is a third source for the geometry on $\mathcal{F}(M_0)$, namely the identification with the space $\bigoplus E_{i,j,c,d}$ of all privileged cocycles (where c, d are fixed arbitrary) found in subsection 6.1.2. The corresponding geometry is the same, see *loc. cit.*.

6.3. Vector bundles associated to q-difference modules

We briefly recall here the general construction of which the vector bundle in subsection 6.2.1 is an example, and then give an application. This is based on [**43**].

6.3.1. The general construction. — For details on the following, see [**43**]. To any q-difference module M over $\mathbf{C}(\{z\})$, one can associate a holomorphic vector bundle $\mathcal{F}_M$ over $\mathbf{E}_q$ in such a way that the correspondence $M \rightsquigarrow \mathcal{F}_M$ is functorial and that the functor is faithful, exact and compatible with tensor products and duals. If one restricts to pure modules, the functor is moreover *fully* faithful, but this ceases to be true for arbitrary modules, which is one of the reasons why it is not very important in the present work (see however the remark further below).

In order to describe $\mathcal{F}_M$, we shall assume for simplicity that the module M is related to a q-difference system $\sigma_q X = AX$ such that A and A^{-1} are holomorphic all over $\mathbf{C}^*$, for instance, that A is in Birkhoff-Guenther normal form. (In the general case, one just has to speak of germs at 0 everywhere). The sheaf of holomorphic solutions over $\mathbf{E}_q$ is then defined by the relation:

$$\mathcal{F}_M(U) := \{X \in \mathcal{O}\big(p^{-1}(U)\big)^n \mid \sigma_q X = AX\}.$$

This is a locally free sheaf, whence the sheaf of sections of a holomorphic vector bundle over $\mathbf{E}_q$ which we also write $\mathcal{F}_M$. The bundle $\mathcal{F}_M$ can be realized geometrically as the quotient of the trivial bundle $\mathbf{C}^* \times \mathbf{C}^n$ over $\mathbf{C}^*$ by an equivariant action of the subgroup $q^{\mathbf{Z}}$ of $\mathbf{C}^*$:

$$\mathcal{F}_M = \frac{\mathbf{C}^* \times \mathbf{C}^n}{(z,X) \sim \big(qz, AX\big)} \to \frac{\mathbf{C}^*}{z \sim qz} = \mathbf{E}_q.$$

The construction of the bundle Λ in subsection 6.2.1 corresponds to the q-difference system $(\sigma_q F)(z^\nu C) = (z^\mu B)F$, which is the "internal Hom" $\underline{Hom}(N,M)$ of the modules M,N respectively associated to the q-difference systems with matrices $z^\mu B, z^\nu C$. From the compatibilities with tensor products and duals, one therefore draws:

$$\Lambda = \mathcal{F}_N^\vee \otimes \mathcal{F}_M.$$

Remark 6.3.1. — From the exactness and the existence of slope filtrations, one deduces that the vector bundle associated to a q-difference module with integral slopes admits a flag of subbundles such that each quotient is "pure isoclinic", that is, isomorphic to the tensor product of a line bundle with a flat bundle. The functor sending the module M to the vector bundle $\mathcal{F}_M$ *endowed with such a flag* is fully faithful.

6.3.2. Sheaf theoretical interpretation of irregularity. — We interpret here in sheaf theoretical terms the formula relating irregularity and the dimension of $\dim \mathcal{F}(P_1, \ldots, P_k)$ given in subsection 3.2.3.

Actually, the irregularity of $\underline{\mathrm{End}}(M_0)$ comes from its positive part:

$$\underline{\mathrm{End}}^{>0}(M_0) = \bigoplus_{1 \leq i < j \leq k} \underline{Hom}(P_i, P_j).$$

For each pure component $P_{i,j} := \underline{Hom}(P_j, P_i)$, the computations of sections 3.1 and 3.2 give:

$$\dim \Gamma^0(P_{i,j}) - \dim \Gamma^1(P_{i,j}) = r_i r_j (\mu_j - \mu_i).$$

As we know, $\mathcal{F}(P_1, \ldots, P_k) = \mathcal{F}(M_0)$ is isomorphic to the first cohomology set of the sheaf $\Lambda_I(M_0)$. Also, in the previous section, we have seen that this is an affine space with underlying vector space the first cohomology space of the sheaf $\lambda_I(M_0)$ of their Lie algebras. And the latter, from its description as $\bigoplus_{\delta \geq 1} \lambda^{(\delta)}$, is the vector bundle associated to the q-difference module $\underline{\mathrm{End}}^{>0}(M_0)$. So the irregularity is actually an Euler-Poincaré characteristic.

CHAPTER 7

EXAMPLES OF THE STOKES PHENOMENON

In the study of q-special functions, one frequently falls upon series that are *convergent* solutions of *irregular* q-difference equations, the latter thus also admitting *divergent* solutions; and this should be so, since Adams lemma ensures us that irregular equations always have *some* convergent solutions (see 2.2.3). However, in most works, only convergent solutions have been considered, although the other ones are equally interesting[(1)]

The touchstone of any theory of the Stokes phenomenon is Euler series. We shall therefore concentrate on one of its q-analogs, the q-Euler or Tshakaloff series $\mathrm{Ch}(z)$ (see equation (4), page 16). The simplest q-difference equation satisfied by Ch is the q-Euler equation $z\sigma_q f - f = -1$. We shall detail the Stokes phenomenon for a family of similar equations in 7.1 and we shall apply it to some confluent basic hypergeometric series. Then we shall show how such equations naturally appear in some well known historical cases: that of Mock Theta functions in 7.3, that of the enumeration of class numbers of quadratic forms in 7.4; this has been exploited by the third author in [**59, 58**].

7.0.2.1. Notations. — We use some notations from q-calculus for this section only. Let $p \in \mathbf{C}$ be such that $0 < |p| < 1$, *e.g.* $p := q^{-1}$. Let $a, a_1, \ldots, a_k \in \mathbf{C}$

(1) This was of course well known to Stokes himself when he studied the Airy equation, as well as to all those who used divergent series in numerical computations of celestial mechanics, leading to Poincaré work on asymptotics. But, of course, one should above all remember Euler, who used divergent series for numerically computing $\zeta(2) = \pi^2/6$, in flat contradiction to the opinion of Bourbaki in "Topologie Générale", IV, p. 71.

and $n \in \mathbf{N}$. The *q-Pochhammer symbols* are:

$$(a;p)_n := \prod_{0 \le i < n} (1 - ap^i),$$
$$(a;p)_\infty := \prod_{i=0}^{\infty} (1 - ap^i),$$
$$(a_1, \ldots, a_k; p)_n := \prod_{j=1}^{k} (a_j; p)_n,$$
$$(a_1, \ldots, a_k; p)_\infty := \prod_{j=1}^{k} (a_j; p)_\infty.$$

Note that, for all $n \in \mathbf{N}$, we have:

$$(a;p)_n = \frac{(a;p)_\infty}{(a;p^n)_\infty},$$

but, since the right hand side is well defined for all $n \in \mathbf{Z}$, this allows us to extend the definition of q-Pochhammer symbols; and similarly for $(a_1, \ldots, a_k; p)_n$.

With these notations, Jacobi's theta function θ_q defined in equation (3) page 8 admits the factorisation:

$$\theta_q(z) = (q^{-1}, -q^{-1}z, -z^{-1}; q^{-1})_\infty.$$

(This is again Jacobi Triple Product Formula.)

7.1. The q-Euler equation and confluent basic hypergeometric series

Consider a q-difference system of rank 2 and level 1 (that is, its slopes are $\mu, \mu + 1$ for some $\mu \in \mathbf{Z}$).Through some analytic gauge transformation, its matrix can be put in the form $bz^\mu \begin{pmatrix} 1 & u \\ 0 & az \end{pmatrix}$, where $a, b \in \mathbf{C}^*$, $\mu \in \mathbf{Z}$ and $u \in \mathbf{C}(\{z\})$. The bz^μ factor corresponds to a tensor product by a rank one object L, which does not affect the Stokes phenomenon, nor the isoformal classification, *i.e.* the map $M \rightsquigarrow L \otimes M$ induces an isomorphism:

$$\mathcal{F}(P_1, \ldots, P_k) \to \mathcal{F}(L \otimes P_1, \ldots, L \otimes P_k).$$

We therefore assume that $b = 1, \mu = 0$. The associated inhomogeneous equation is the following *q-Euler equation*:

$$\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} : \begin{pmatrix} 1 & 0 \\ 0 & az \end{pmatrix} \simeq \begin{pmatrix} 1 & u \\ 0 & az \end{pmatrix} \Longleftrightarrow az\sigma_q f - f = u. \tag{61}$$

7.1.1. A digest on the q-Euler equation. —

7.1.1.1. Formal solution. — We obtain it, for instance, by iterating the z-adically contracting operator $f \mapsto -u + az\sigma_q f$. One finds the fixed point:

$$\hat{f} = -\sum_{n \geq 0} a^n q^{n(n-1)/2} z^n \sigma_q^n u.$$

If u is a constant, the right hand side is just $-u\mathrm{Ch}(az)$. If $u = \sum\limits_{k>>-\infty} u_k z^k$, then $\hat{f} = -\sum\limits_{k>>-\infty} u_k z^k \mathrm{Ch}(q^k az)$.

7.1.1.2. Birkhoff-Guenther normal form. — There is a unique $\alpha \in \mathbf{C}$ such that, setting $v := u - \alpha$, the unique formal solution of $az\sigma_q f - f = v$ is convergent. Indeed, putting $v = \sum\limits_{k>>-\infty} v_k z^k$, from the relation $aq^{n-1} f_{n-1} - f_n = v_n$, we draw that $\sum\limits_{n \in \mathbf{Z}} a_n q^{-n(n-1)/2} v_n = 0$, which also writes: $\alpha = \mathcal{B}_{q,1} u(a^{-1})$, where, as usual, $\mathcal{B}_{q,1} u(\xi) = \sum\limits_{k>>-\infty} u_k q^{-k(k-1)/2} \xi^k$. (The letter ξ is traditional for the Borel plane.) Note that the value $\mathcal{B}_{q,1} u(a^{-1})$ is well defined, for $\mathcal{B}_{q,1} u$ is an entire function. The Birkhoff-Guenther normal form of $\begin{pmatrix} 1 & u \\ 0 & az \end{pmatrix}$ is therefore $\begin{pmatrix} 1 & \mathcal{B}_{q,1} u(a^{-1}) \\ 0 & az \end{pmatrix}$.

7.1.1.3. "Algebraic" summation. — For any $a \in \mathbf{C}^*$, we shall set:

$$\theta_{q,a}(z) := \theta_q(z/a).$$

(See general notations in section 1.3.) One looks for a solution of (61) in the form $f = g/\theta_{q,\lambda}$, with $g \in \mathcal{O}(\mathbf{C}^*)$ and λ adequately chosen in $\mathbf{C}^*$. We are led to solve the equation: $a\lambda\sigma_q g - g = u\theta_{q,\lambda}$. Identifying the coefficients of the Laurent series, one gets the unique solution:

$$g = \sum_{n \in \mathbf{Z}} \frac{[u\theta_{q,\lambda}]_n}{a\lambda q^n - 1} z^n,$$

which makes sense if, and only if, $\lambda \notin [a^{-1}; q]$. (Note that we write $[\sum a_n z^n]_n := a_n$.) Thus, for all "authorized directions of summation" λ, we get the unique solution for (61) with only simple poles over $[-\lambda; q]$:

$$S_\lambda \hat{f} := \frac{1}{\theta_{q,\lambda}} \sum_{n \in \mathbf{Z}} \frac{[u\theta_{q,\lambda}]_n}{a\lambda q^n - 1} z^n,$$

7.1.1.4. "True" summation. — Since the polar condition that uniquely characterizes the solution $S_\lambda \hat{f}$ only depends on $\overline{\lambda} := \lambda \pmod{q^{\mathbf{Z}}} \in \mathbf{E}_q$, or (equivalently) on $\Lambda := [\lambda; q]$, we shall also write:

$$S_{\overline{\lambda}} \hat{f} := S_\Lambda \hat{f} := S_\lambda \hat{f}.$$

We now make this dependency explicit. From the equality $u\theta_{q,\lambda} = \sum u_k q^{-\ell(\ell+1)/2} z^k (z/\lambda)^\ell$, we draw:

$$[u\theta_{q,\lambda}]_n = \sum_{k+\ell=n} u_k q^{-\ell(\ell+1)/2} \lambda^{-\ell} = \sum_k u_k q^{-(n-k)(n-k+1)/2} \lambda^{k-n},$$

so that:

$$q^{n(n+1)/2} \lambda^n \, [u\theta_{q,\lambda}]_n = \sum_k u_k q^{nk - k(k-1)/2} \lambda^k = \mathcal{B}_{q,1} u(q^n \lambda).$$

On the other hand, iterating the relation $\theta_{q,\lambda}(z) = \frac{z}{q\lambda} \theta_{q,q\lambda}(z)$ yields $\theta_{q,\lambda}(z) = \frac{z^n}{q^{n(n+1)/2}\lambda^n} \theta_{q,q^n\lambda}(z)$, whence:

$$S_{\overline{\lambda}} \hat{f} = S_\Lambda \hat{f} = \sum_{n \in \mathbf{Z}} \frac{q^{-n(n+1)/2} \lambda^{-n} \mathcal{B}_{q,1} u(q^n \lambda)}{z^n q^{-n(n+1)/2} \lambda^{-n} \theta_{q,q^n\lambda}(z)(a\lambda q^n - 1)} \, z^n = \sum_{\mu \in \Lambda} \frac{\mathcal{B}_{q,1} u(\mu)}{(a\mu - 1)\theta_{q,\mu}(z)} \, z^n.$$

Remark 7.1.1. — In [**32, 36, 35**], one computes the residue of the meromorphic function $\overline{\lambda} \mapsto S_{\overline{\lambda}} \hat{f}$ at the pole $\overline{a^{-1}} \in \mathbf{E}_q$. According to the above formula, one finds:

$$\operatorname{Res}_{\overline{\lambda} = \overline{a^{-1}}} S_{\overline{\lambda}} \hat{f} = \frac{1}{2\mathrm{i}\pi} \, \frac{\mathcal{B}_{q,1} u(a^{-1})}{\theta_q(az)}.$$

Indeed, for any $b \in \mathbf{C}^*$ and for any map $f : \mathbf{C}^* \to \mathbf{C}$ analytic in a neighborhood of $[b; q]$, setting:

$$\forall \overline{\lambda} \in \mathbf{E}_q \ , \ F\left(\overline{\lambda}\right) := \sum_{\mu \in [\lambda; q]} \frac{f(\mu)}{\mu - b},$$

defines a meromorphic map F with a simple pole at $\overline{b} \in \mathbf{E}_q$ and the corresponding residue:

$$\operatorname{Res}_{\overline{\lambda} = \overline{b}} S_{\overline{\lambda}} \hat{f} = \frac{1}{2\mathrm{i}\pi} \, \frac{f(b)}{b}.$$

Note that the residue of a function here makes sense, because of the canonical generator $dx = \dfrac{1}{2\mathrm{i}\pi} \, \dfrac{dz}{z}$ of the module of differentials, which allows one to flatly identify maps on $\mathbf{E}_q$ with differentials. (Here, as usual, $z = e^{2\mathrm{i}\pi x}$, where x is the canonical uniformizing parameter of $\mathbf{E}_q = \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau)$.)

7.1.2. Some confluent basic hypergeometric functions. — Usual basic hypergeometrics series have the form:

$$ {}_2\Phi_1(a,b;c;q,z) := \sum_{n \geq 0} \frac{(a,b;q^{-1})_n}{(c,q^{-1};q^{-1})_n} z^n \ , \quad \text{where } a,b,c \in \mathbf{C}^*. $$

(Remember that here, $|q| > 1$.) Writing for short $F(z)$ this series, the rescaling $F(-q^{-1}z/c)$ degenerates, when $c \to \infty$, into a *confluent* basic hypergeometrics series:

$$ \phi(a,b;q,z) := \sum_{n \geq 0} \frac{(a,b;q^{-1})_n}{(q^{-1};q^{-1})_n} q^{n(n+1)/2} z^n \ , \quad \text{where } a,b \in \mathbf{C}^*. $$

Writing $u_n := \dfrac{(a,b;q^{-1})_n}{(q^{-1};q^{-1})_n} q^{n(n+1)/2}$ the general coefficient, we have, for all $n \geq 0$:

$$ (q^{n+1} - 1) u_{n+1} = q^2 (q^n - a)(q^n - b) u_n, $$

whence, multiplying by z^{n+1} and summing:

$$ (\sigma_q - 1)\hat{f} = q^2 z (\sigma_q - a)(\sigma_q - b)\hat{f}, $$

where we write for short $\hat{f}(z)$ the divergent series $\phi(a,b;q,z)$. We shall denote the corresponding q-difference operator as:

$$ L := q^2 z(\sigma_q - a)(\sigma_q - b) - (\sigma_q - 1) = q^2 z \sigma_q^2 - \big(1 + (a+b)q^2 z\big)\sigma_q + (1 + abq^2 z). $$

We are interested in the equation $Lf = 0$. Its Newton polygon (*i.e.* that of L) has slopes 0 and 1. The slope 0 has exponent 1 and gives rise to the divergent solution $\hat{f}$. To tackle the slope 1, we compute:

$$ (z\theta_q) L (z\theta_q)^{-1} = \frac{1}{qz}\Big(\sigma_q^2 - \big(1 + (a+b)q^2 z\big)\sigma_q + qz(1 + abq^2 z)\Big), $$

which has slopes 0 and -1, the latter having exponent 0. According to Adams lemma, we thus get a unique solution $g_0 \in 1 + z\mathbf{C}\{z\}$, whence the "convergent" solution $f_0 := \dfrac{g_0}{z\theta_q}$ of equation $Lf = 0$.

7.1.2.1. Factoring L. — To get $L = q^2 z(\sigma_q - A)(\sigma_q - B)$, we first look for B such that $(\sigma_q - B)f_0 = 0$, that is, $B := \dfrac{\sigma_q f_0}{f_0} = \dfrac{1}{qz}\dfrac{\sigma_q g_0}{g_0}\cdot$ Thus, $L = (\sigma_q - A)(qz\sigma_q - qzB) = (\sigma_q - A)\left(qz\sigma_q - \dfrac{\sigma_q g_0}{g_0}\right)$ and, by identification of the constant terms: $A = \dfrac{(1+abq^2z)g_0}{\sigma_q g_0}$ and in the end:

$$L = \left(\sigma_q - \frac{(1+abq^2z)g_0}{\sigma_q g_0}\right)\left(qz\sigma_q - \frac{\sigma_q g_0}{g_0}\right).$$

The corresponding non homogeneous equation is $qzf\sigma_q - \dfrac{\sigma_q g_0}{g_0} f = v$, where v is a non trivial solution of $\left(\sigma_q - \dfrac{(1+abq^2z)g_0}{\sigma_q g_0}\right) v = 0$. An obvious choice is:

$$v := \frac{1}{g_0}\prod_{n\geq 1}(1+abq^{2-n}z) = \frac{1}{g_0}(-abqz;q^{-1})_\infty.$$

One checks easily that the above non homogeneous equation has indeed a unique solution in $1+z\mathbf{C}[[z]]$, and this has to be $\hat{f}$. This equation is associated with the matrix $\begin{pmatrix} \dfrac{\sigma_q g_0}{g_0} & v \\ 0 & qz \end{pmatrix}$, which can be seen to be equivalent to the matrix $\begin{pmatrix} 1 & u \\ 0 & z \end{pmatrix}$ through the gauge transform $\begin{pmatrix} g_0^{-1} & 0 \\ 0 & z^{-1} \end{pmatrix}$.

7.1.3. Some special cases. — Taking $a := b := q^{-1}$ yields:

$$\hat{f} = \sum_{n\geq 0}(q^{-1};q^{-1})_n q^{n(n+1)/2} z^n.$$

The recurrence relation $\dfrac{u_{n+1}}{u_n} = q^{n+1} - 1$ immediately gives the non homogeneous equation $qz\sigma_q f - (1+z)f = -1$. This, in turn, boils down to the straight q-Euler equation by setting $g := \dfrac{z}{(-q^{-1}z;q^{-1})_\infty} f$, so that $z\sigma_q g - g = \dfrac{-z}{(-z;q^{-1})_\infty}\cdot$

Taking $a := b := 0$ yields:

$$\hat{f} = \sum_{n\geq 0}\frac{1}{(q^{-1};q^{-1})_n} q^{n(n+1)/2} z^n.$$

The recurrence relation $\dfrac{u_{n+1}}{u_n} = \dfrac{q^{2n+2}}{q^{n+1}-1}$ gives the homogeneous equation $q^2 z\sigma_q^2 f - \sigma_q f + f = 0$. The convergent solution is $f_0 := \dfrac{1}{z\theta_q} g_0$, where $g_0 = \sum \gamma_n z^n \in 1 + z\mathbf{C}\{z\}$ is solution of the equation $(\sigma_q^2 - \sigma_q + qz)g = 0$. The corresponding recurrence relation $(q^{2n} - q^n)\gamma_n + q\gamma_{n-1} = 0$ can be solved exactly and entails:

$$g_0 = \sum_{n \geq 0} \frac{(-1)^n q^{-n^2}}{(q^{-1};q^{-1})_n} z^n.$$

The corresponding factorisation is $L = \left(\sigma_q - \dfrac{g_0}{\sigma_q g_0}\right)\left(qz\sigma_q - \dfrac{\sigma_q g_0}{g_0}\right)$. Following [**57**], one can prove the following formula for the Stokes operators:

$$S_\lambda \hat{f} - S_\mu \hat{f} = (q^{-1};q^{-1})_\infty^2 \frac{\theta_q(-\lambda/\mu)\theta_q(z/\lambda\mu)}{\theta_q(-1/\lambda)\theta_q(-1/\mu)\theta_q(\lambda/z)\theta_q(z/\mu)} \sum_{n \in \mathbf{Z}} \frac{(-1)^n q^{-n^2}}{(q^{-1};q^{-1})_n} z^n.$$

We shall not attempt to prove this, but a similar formula is checked in the next case.

Taking $a := 0, b := q^{-1}$ yields:

$$\hat{f} = \sum_{n \geq 0} q^{n(n+1)/2} z^n = \mathrm{Ch}(qz),$$

which is solution of the q-Euler equation $(qz\sigma_q - 1)f = -1$. A solution of the associated homogeneous equation is $\dfrac{1}{\theta_q(qz)} = \dfrac{1}{\theta_q(1/z)}$, so that, for any two $\lambda, \mu \not\in [1;q]$ (authorized directions of summation):

$$S_\lambda \hat{f} - S_\mu \hat{f} = \frac{K(\lambda,\mu,z)}{\theta_q(1/z)},$$

where K is q-invariant in each of the three arguments. We assume $\overline{\lambda} \neq \overline{\mu}$; then, as a function of z, the numerator K is elliptic with simple zeroes over $[-1;q]$ and at most simple poles over $[-\lambda;q]$ and $[-\mu;q]$; thus:

$$K(\lambda,\mu,z) = K'(\lambda,\mu)\, \frac{\theta_q(1/z)\theta_q(z/\lambda\mu)}{\theta_q(\lambda/z)\theta_q(z\mu)},$$

where $K'(\lambda,\mu)$ is independent of z and, as a function of λ, has at most simple poles over $[1;q]$ and $[\mu;q]$; thus:

$$K'(\lambda,\mu)\frac{\theta_q(z/\lambda\mu)}{\theta_q(\lambda/z)} = K'' \frac{\theta_q(z/\lambda\mu)\theta_q(-\lambda/\mu)}{\theta_q(\lambda/z)\theta_q(-1/\lambda)},$$

where K'' is independent of λ, μ, z. Last, we get:

$$S_\lambda \hat{f} - S_\mu \hat{f} = C \frac{\theta_q(-\lambda/\mu)\theta_q(z/\lambda\mu)}{\theta_q(-1/\lambda)\theta_q(-1/\mu)\theta_q(\lambda/z)\theta_q(z/\mu)}\cdot$$

We shall now see that $C = -\theta_q{}'(-1) = (q^{-1};q^{-1})_\infty^3$. The second equality follows immediately from Jacobi Triple Product Formula. Note that, by simple singularity analysis, one may write:

$$S_\lambda \hat{f} = \sum_{n \in \mathbf{Z}} \frac{\alpha_n}{z + \lambda q^n}\cdot$$

Since $S_\lambda \hat{f}(0) = 1$, we have $\sum \alpha_n q^{-n} = \lambda$. On the other hand, from the functional equation, taking residues yields the recurrence relation: $\alpha_{n-1} = -\lambda \alpha_n q^{n-1}$, then $\alpha_n = (-1/\lambda)^n q^{-n(n-1)/2} \alpha_0$ and in the end:

$$\lambda = \alpha_0 \sum_{n \in \mathbf{Z}} (-1/\lambda)^n q^{-n(n-1)/2} = \alpha_0 \theta_q(-1/\lambda) \Longrightarrow \alpha_0 = \frac{\lambda}{\theta_q(-1/\lambda)}\cdot$$

On the other hand:

$$\begin{aligned}
\alpha_0 &= \lim_{z \to -\lambda} (z+\lambda)(S_\lambda \hat{f} - S_\mu \hat{f}) \\
&= \lim_{z \to -\lambda} (z+\lambda) C \frac{\theta_q(-\lambda/\mu)\theta_q(z/\lambda\mu)}{\theta_q(-1/\lambda)\theta_q(-1/\mu)\theta_q(\lambda/z)\theta_q(z/\mu)} \\
&= C \frac{1}{\theta_q(-1/\lambda)} \frac{-\lambda}{\theta_q{}'(-1)},
\end{aligned}$$

whence the desired conclusion.

7.2. The symmetric square of the q-Euler equation

This will be our only example with more than two slopes. Consider the square $\hat{Y} := \mathrm{Ch}^2$ of the Tshakaloff series:

$$\hat{Y}(z) = \left(\sum_{n \geq 0} q^{n(n-1)/2} z^n \right)^2.$$

As we shall see in section 7.2.2, the series $\hat{Y} = \mathrm{Ch}^2$ does not support the same process of analytic summation as Ch itself. This comes from the fact that the Newton polygon of $\hat{Y}$ has three slopes, as shall see, while that of Ch has two slopes. First, however, we want to give some recipes to tackle such examples.

7.2.1. Algebraic aspects. — Remember that the equation $f = 1 + z\sigma_q f$ satisfied by Ch is nothing but the cohomological equation for $\begin{pmatrix} 1 & -1 \\ 0 & z \end{pmatrix}$.

7.2.1.1. Newton polygon of a symmetric square. — To find the equation satisfied by a product of two functions, one uses the tensor product of two systems or modules; for a square, one uses likewise the symmetric square.

Let $M = (V, \Phi)$ be a q-difference module and $T^2M := M \otimes M = (V \otimes V, \Phi \otimes \Phi)$ its tensor square. The linear automorphism $x \otimes y \mapsto y \otimes x$ commutes with $\Phi \otimes \Phi$, so that it actually defines an involutive q-difference automorphism of $M \otimes M$, and a splitting:

$$T^2M = S^2M \oplus \Lambda^2 M.$$

If M has slopes $\mu_1, \dots, \mu_k$ with multiplicities $r_1, \dots, r_k$, then T^2M has slopes the $\mu_i + \mu_j$, $1 \leq i, j \leq k$ with multiplicities the $r_i r_j$. (Of course, if many sums $\mu_i + \mu_j$ are equal, the corresponding multiplicities $r_i r_j$ should be added; the same remark will hold for the following computations.) Said otherwise, the slopes of T^2M are the $2\mu_i$, $1 \leq i \leq k$ with multiplicities the r_i^2; and the $\mu_i + \mu_j$, $1 \leq i < j \leq k$ with multiplicities the $2r_i r_j$.

The repartition of these slopes (breaking of the Newton polygon) among the symmetric and exterior square is as follows:

- S^2M has slopes the $2\mu_i$, $1 \leq i \leq k$ with multiplicities the $\dfrac{r_i^2 + r_i}{2}$; and the $\mu_i + \mu_j$, $1 \leq i < j \leq k$ with multiplicities the $r_i r_j$.
- Λ^2M has slopes the $2\mu_i$, $1 \leq i \leq k$ with multiplicities the $\dfrac{r_i^2 - r_i}{2}$; and the $\mu_i + \mu_j$, $1 \leq i < j \leq k$ with multiplicities the $r_i r_j$.

If there are two slopes $\mu < \nu$, with multiplicities r, s, no confusion of sums $\mu_i + \mu_j$ can arise, and we find:

- T^2M has slopes $2\mu < \mu + \nu < 2\nu$, with multiplicities $r^2, 2rs, s^2$;
- Λ^2M has the same slopes, with multiplicities $\dfrac{r^2 + r}{2}, rs, \dfrac{s^2 + s}{2}$;
- Λ^2M has the same slopes, with multiplicities $\dfrac{r^2 - r}{2}, rs, \dfrac{s^2 - s}{2}$.

We now take $M_0 = \begin{pmatrix} a & 0 \\ 0 & bz \end{pmatrix}$ and $M = \begin{pmatrix} a & u \\ 0 & bz \end{pmatrix}$, where $a, b \in \mathbf{C}^*$ and $u \in \mathbf{C}(\{z\})$. The symmetric squares admit an obvious choice of basis and corresponding matrices:

$$N_0 = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & abz & 0 \\ 0 & 0 & b^2z^2 \end{pmatrix} \text{ and } N = \begin{pmatrix} a^2 & 2au & u^2 \\ 0 & abz & ubz \\ 0 & 0 & b^2z^2 \end{pmatrix}.$$

If $F = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}$ is such that $F[M_0] = M$, then we have $G[N_0] = N$ with G given by:

$$G = S^2F = \begin{pmatrix} 1 & 2f & f^2 \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix}.$$

Actually, if one does not cheat, when looking for $G = \begin{pmatrix} 1 & f_1 & f_2 \\ 0 & 1 & f_3 \\ 0 & 0 & 1 \end{pmatrix}$ such that $G[N_0] = N$, one has to solve the system:

$$\begin{aligned} abz\sigma_q f_1 &= a^2 f_1 + 2au, \\ b^2z^2\sigma_q f_2 &= a^2 f_2 + 2auf_3 + u^2, \\ b^2z^2\sigma_q f_3 &= abzf_3 + ubz. \end{aligned}$$

Since we know from start that $bz\sigma_q f = af + u$, we see that $f_1 := 2f$ and $f_3 := f$ respectively solve the first and third equation; then, we find that $f_2 := f^2$ solve the second equation.

Using the system above, we find a second order inhomogeneous equation for f_2 alone as follows:

$$\begin{aligned} &(bz\sigma_q - a)\,\frac{1}{2au}\,(b^2z^2\sigma_q - a^2)f_2 = (bz\sigma_q - a)\,f_3 + (bz\sigma_q - a)\,\frac{u^2}{2au} \Longrightarrow \\ &\left(\frac{bz}{\sigma_q(u)}\sigma_q - \frac{a}{u}\right)(b^2z^2\sigma_q - a^2)f_2 = bz\sigma_q(u) + au. \end{aligned}$$

We leave to the reader to find a simpler form, as well as the corresponding third order homogeneous equation. At any rate, in the case that $u \in \mathbf{C}$ (Birkhoff-Guenther normal form) we have:

$$(bz\sigma_q - a)\,(b^2z^2\sigma_q - a^2)f_2 = bz + a.$$

In the particular case $a = b = -u = 1$ of the Tshakaloff series, we are led to the following equation:

$$L\hat{Y} = 1 + z, \text{ where } L := q^2z^3\sigma_q^2 - z(1+z)\sigma_q + 1. \tag{62}$$

Remark 7.2.1. — We refer here to the example 6.2.3 and specialize it to the case of the symmetric square N above. We see that, when u varies, the class of N in $H^1(\mathbf{E}_q, \lambda^{(1)}) \oplus H^1(\mathbf{E}_q, \lambda^{(2)})$ has components $2auL_1(a^2, ab) + ubL_1(ab, b^2)$ and $u^2 L_{2,0}(a^2, b^2)$: a nice parabola.

7.2.1.2. Algebraic summation of Ch^2. — The fact that $F[A_0] = A \Rightarrow S^2F[B_0] = B$ is purely algebraic and stays true of the sum in direction $\overline{c}$, so that one gets the following sums:

$$S_{\overline{c}}G = S^2(S_{\overline{c}}F) = \begin{pmatrix} 1 & 2f_{\overline{c}} & f_{\overline{c}}^2 \\ 0 & 1 & f_{\overline{c}} \\ 0 & 0 & 1 \end{pmatrix}.$$

Moreover, observing that $x \mapsto \begin{pmatrix} 1 & 2x & x^2 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}$ is a morphism from $\mathbf{C}$ $\mathbf{C}$ to $\mathrm{GL}_3(\mathbf{C})$, one gets the explicit formula for the cocycle:

$$S_{\overline{c},\overline{d}}G = S^2(S_{\overline{c,\overline{d}}}F) = \begin{pmatrix} 1 & 2f_{\overline{c},\overline{d}} & f_{\overline{c},\overline{d}}^2 \\ 0 & 1 & f_{\overline{c},\overline{d}} \\ 0 & 0 & 1 \end{pmatrix}.$$

The algebraic sums $f_{\overline{c}}$ have been described explicitly un subsection 7.1.1 where their notation was $S_{\overline{\lambda}}\hat{f}$, with $\overline{\lambda} = c$.

7.2.2. Analytic aspects. —

7.2.2.1. The series Ch^2 is not summable with one level. — Consider the square $\hat{Y} := \mathrm{Ch}^2$ of the Tshakaloff series:

$$\hat{Y}(z) = \left(\sum_{n \geq 0} q^{n(n-1)/2} z^n \right)^2.$$

Its q-Borel transform $\mathcal{B}_{q,1}\hat{Y}$ (at level 1) can be computed from the following simple remark:

$$\hat{f}(z) = \sum_{n \geq 0} a_n z^n \text{ and } \hat{g} \in \mathbf{C}[[z]] \Longleftrightarrow \mathcal{B}_{q,1}(\hat{f}\hat{g}) = \sum_{n \geq 0} a_n q^{-n(n-1)/2} \xi^n \mathcal{B}_{q,1}\hat{g}(q^{-n}\xi).$$

It follows that, if $P(\xi) = \mathcal{B}_{q,1}\hat{Y}(\xi) \prod_{n \geq 0}(1 - q^{-n}\xi)$, then P is an entire function such that:

$$P(q^m) = (-1)^m q^{m(3m+1)/2} (q^{-1}; q^{-1})_m (q^{-1}; q^{-1})_\infty. \tag{63}$$

From this, we see that P has at infinity q-exponential growth of order ≥ 3.

The q-Borel transform of $\hat{Y}$ represents a meromorphic function in $\mathbf{C}$ with (simple) poles on $q^{\mathbf{N}}$ and having at infinity in $\mathbf{C} \setminus q^{\mathbf{N}}$ q-exponential growth of order exactly 2.

Thus, $\hat{Y} = \mathrm{Ch}^2$ is not q-Borel-Laplace summable as Ch itself is. This comes from the fact that the Newton polygon of the former (resp. of the latter) has three (resp. two) slopes as we have seen. Moreover, the problem os cousin to a nonlinear problem.

7.2.2.2. Multisummability. —

Proposition 7.2.2. — *Let λ, $\mu \in \mathbf{C}^*$ and let $f \in \mathbb{A}_q^{[\lambda}$ and $g \in_q^{[\mu]}$.*

1. *If $[\lambda] = [\mu]$, then $fg \in \mathbb{O}^{2[\lambda]}_{([\lambda],\lambda])}$.*
2. *If $[\lambda] \neq [\mu]$, then, generically $fg \notin \mathbb{O}^{[\lambda]+[\mu]}_{([\lambda],[\mu])} \cup \mathbb{O}^{[\lambda]+[\mu]}_{([\mu],[\lambda])}$.*

Proof. — Write $f = F/\theta_\lambda$, $g = G/\theta_\mu$; from theorem 5.2.8, one gets $F \in \mathbb{E}_0^{[\lambda]}$ and $G \in \mathbb{E}_0^{[\mu]}$.
If $[\lambda] = [\mu]$, then FG represents an analytic function near 0 in $\mathbf{C}^*$, with q-Gevrey growth of null order on the q-spiral $[\lambda]$ at 0 and q-Gevrey growth of order 2 globally. To obtain the q-Gevrey growth order of its values on $2[\lambda]/[\lambda]$ at 0, write:

$$(FG)'(\lambda q^{-n}) = F'(\lambda q^{-n})G(\lambda q^{-n}) + F(\lambda q^{-n})G'(\lambda q^{-n}),$$

which will provide the q-Gevrey order. Using theorem 5.4.3, we get the first statement.
For the second statement, just note that, generally, FG has null q-Gevrey order at 0 neither on $[\lambda]$, nor on $[\mu]$; indeed, the sequences $(F(\mu q^{-n}))$ and $(G(\lambda q^{-n}))$ have q-Gevrey order one. □

Let λ, $\mu \notin [1] = q^{\mathbf{Z}}$. Write f_λ, resp. f_μ, the solutions of the q-Euler equation satisfied by Ch in $\mathbb{A}_q^{[\lambda]}$, resp. in $\mathbb{A}_q^{[\mu]}$. If $[\lambda] = [\mu]$, from the proposition above, one has:

$$(f_\lambda)^2 \in \mathbb{O}^{2[\lambda]}_{([\lambda],\lambda])},$$

and this is the solution provided by theorem 5.5.3 for equation (62) with $\Lambda = 2[\lambda]$ and $\Lambda_1 = \Lambda_2 = [\lambda]$. However, if $[\lambda] \neq [\mu]$, then $f_\lambda f_\mu$ is *not* the solution provided by that theorem with $\Lambda = [\lambda] + [\mu]$ and $\{\Lambda_1, \Lambda_2\} = \{[\lambda], [\mu]\}$.

7.2.2.3. Possibility of a multisummation process. — No explicit algorithm is presently known to yield the solution of (62) in the spirit of theorem 5.5.3 with $\Lambda = [\lambda] + [\mu]$. A multisummation algorithm does exist in a very different setting, due to the third author. The following result shows its similarity to classical Borel-Laplace summation.

Theorem 7.2.3. — *In* (62), *we have* $L = (z\sigma_q - 1)(z^2\sigma_q - 1)$ *and the series* $\hat{Y}$ *is* $(1, 1/2)$*-summable by the following process:*

$$\hat{f} = \sum_{n \geq 0} a_n z^n \quad \xrightarrow{\mathcal{B}_{q,1}} \quad \varphi := \sum_{n \geq 0} a_n q^{-n(n-1)/2} \xi^n$$
$$\xrightarrow{\mathcal{A}^*_{q;1,1/2}} \quad f_1^* \xrightarrow{\mathcal{L}^*_{q;1/2}} f^*_{(1,1/2)}.$$

For a basic introduction to this method and explanation of the notations above, see [**16**], which contains further references to this work.

7.3. From the Mock Theta functions to the q-Euler equation

Our source here is the famous 1935 paper "The final problem: an account of the Mock Theta Functions" by G. N. Watson, as reproduced, for instance, in [**3**]. On page 330 of *loc. cit.*, seven "mock theta functions of order three" are considered. The first four are called f, ϕ, ψ, χ (after Ramanujan who discovered them); the three last are called ω, ν, ρ (after Watson who added them to the list). In the notation of Ramanujan and Watson, the unique variable of these analytic functions is written q (this tradition goes back to Jacobi) and it is assumed there that $0 < |q| < 1$.

In [**59**], a new variable x is added (this tradition goes back to Euler) and one puts:

$$s(\alpha, \beta; q, x) := \sum_{n \geq 0} a_n x^n,$$

where:

$$\begin{aligned} a_n &:= \frac{q^{n^2}}{\left(\frac{q}{\alpha}, \frac{q}{\beta}; q\right)_n} \left(\frac{1}{\alpha\beta}\right)^n \\ &= \frac{(\alpha, \beta; q)_\infty}{(\alpha q^{-n}, \beta q^{-n}; q)_\infty} q^{-n} \\ &= \frac{q^{n^2}}{(\alpha - q) \cdots (\alpha - q^n)(\beta - q) \cdots (\beta - q^n)}. \end{aligned}$$

Note for further use that the second formula makes sense for all $n \in \mathbf{Z}$ and allows one to define another series:

$$F(\alpha, \beta; q, x) := \sum_{n \in \mathbf{Z}} a_n x^n,$$

so that $F(\alpha, \beta; q, x) = s(\alpha, \beta; q, x) + G(\alpha, \beta; q, x)$, where:

$$G(\alpha, \beta; q, x) = \sum_{n<0} a_n x^n = \sum_{n \geq 0} (\alpha, \beta; q)_n (q/x)^n.$$

The formula giving $s(\alpha, \beta; q, x)$ subsumes all seven mock theta functions, which can be respectively recovered by setting (α, β, x) to be one of the following: $(-1, -1, 1)$, $(\mathrm{i}, -\mathrm{i}, 1)$, $(\sqrt{q}, -\sqrt{q}, 1)$, $(j, j^2, -q)$, $(1/q, 1/q, 1)$, $(\mathrm{i}/\sqrt{q}, -\mathrm{i}/\sqrt{q}, 1)$ and $(-j/q, -j^2/q, 1)$. (Note that in all cases, among other multiplicative relations, α and β map to torsion points of $\mathbf{E}_q$.)

In the following, which is intended to motivate the study of the Stokes phenomenon, we follow recent work by the third author [**59**], skipping most of the proofs. In 7.3.1 and 7.3.2, we use the conventions of *loc. cit.* and use the theta function:

$$\theta(x; q) := \sum_{n \in \mathbf{Z}} q^{n(n-1)/2} x^n = (q, -x, -q/x; q)_\infty = \theta_{q^{-1}}(q^{-1}x).$$

(See the general notations in section 1.3.) In 7.3.3, we shall return to the general conventions of the present paper.

7.3.1. Functional equation for $s(\alpha, \beta; q, z)$. — From the recurrence relation $(\alpha - q^{n+1})(\beta - q^{n+1}) a_{n+1} = q^{2n+1} a_n$, one deduces that s, as a function of x, is solution of the second order non homogeneous q-difference equation:

$$\big((\sigma_q - \alpha)(\sigma_q - \beta) - qx\sigma_q^2\big) s = (1 - \alpha)(1 - \beta). \tag{64}$$

The recurrence relation actually remains valid for all $n \in \mathbf{Z}$, which implies that F is solution of the corresponding homogeneous equation:

$$\big((\sigma_q - \alpha)(\sigma_q - \beta) - qx\sigma_q^2\big) F = 0. \tag{65}$$

This, in turn, entails that $-G = s - F$ is a solution defined at infinity of (64).

We shall assume now that $\alpha, \beta \neq 0$ and that $\beta/\alpha \notin q^{\mathbf{Z}}$. One checks easily that the equation (65) is fuchsian at 0 with exponents α, β (see for instance [**44**]). Likewise, taking in account the general conventions of this paper (*i.e.* using the dilatation factor q^{-1} in order to have a modulus > 1), we see that (65) is pure isoclinic at infinity, with slope $-1/2$; this will explain the appearance of $\theta(-,q^2)$ in the following formulas. We define:

$$M(\alpha,\beta;q,x) := \left(\frac{q}{\alpha},\frac{q}{\beta};q\right)_\infty F(\alpha,\beta;q,x).$$

This is another solution of (65), with more symetries. It admits the following expansions:

$$\begin{aligned} M(\alpha,\beta;q,x) &= \theta(q\alpha\beta x;q^2)U_{\alpha/\beta}(x) - \frac{q}{\alpha}\,\theta(\alpha\beta x;q^2)V_{\alpha/\beta}(x) \\ &= \theta(qx/\alpha\beta;q^2)U_{\alpha/\beta}(1/x) - \frac{q}{\alpha}\,\theta(q^2x/\alpha\beta;q^2)V_{\alpha/\beta}(1/x), \end{aligned}$$

with the following definitions:

$$\begin{aligned} U_\lambda(x) &:= \sum_{m\geq 0} q^{m^2} S_{2m}(-q^{-2m}\lambda;q)(qx/\lambda)^m, \\ V_\lambda(x) &:= \sum_{m\geq 0} q^{m(m+1)} S_{2m+1}(-q^{-2m-1}\lambda;q)(qx/\lambda)^m, \\ S_n(x;q) &:= \sum_{k=0}^{n} \frac{q^{k^2}}{(q;q)_k(q;q)_{n-k}}(-x)^k. \end{aligned}$$

(The S_n are the *Stieltjes-Wiegert polynomials.*)

Remark 7.3.1. — The parameter $\lambda := \alpha/\beta$ is linked to monodromy. Indeed, the local Galois group of (65) at 0 is the set of matrices $\begin{pmatrix}\gamma(\alpha) & 0\\ 0 & \gamma(\beta)\end{pmatrix}$, where γ runs through the group endomorphisms of $\mathbf{C}^*$ that send q to 1; and the local monodromy group is the rank 2 free abelian subgroup with generators corresponding to two particular choices of γ described in [**45**].

7.3.2. Back to the the Mock Theta function. — We can for instance study ϕ,ψ,ν,ρ by setting $x=1$ in $s(\alpha,\beta;q,x)$. (The function χ involves $x=-q$ and f,ω will not comply the condition $\beta/\alpha \notin q^{\mathbf{Z}}$: in [**59**], their study is distinct, though similar.) Up to the knowledge of standard q-functions, one is reduced to the study of:

$$U(\lambda) := U_\lambda(1) \text{ and } V(\lambda) := V_\lambda(1).$$

These are solutions of the q-difference equations:

$$U(q\lambda) - \lambda U(\lambda/q) = (1-\lambda)\frac{\theta(q\lambda;q^2)}{(q,q;q)_\infty},$$
$$V(q\lambda) - q\lambda V(\lambda/q) = (1-\lambda)\frac{\theta(\lambda/q;q^2)}{(q,q;q)_\infty}.$$

Upon setting:

$$U(\lambda) =: \frac{\theta(\lambda;q^2)}{(q,q;q)_\infty} Y(\lambda) \text{ and } V(\lambda) =: \frac{\theta(\lambda/q;q^2)}{(q,q;q)_\infty} Z(\lambda),$$

we find that both Y and Z are solutions of the q-difference equation:

$$X(q\lambda) - \frac{\lambda^2}{q} X(\lambda/q) = 1 - \lambda.$$

7.3.3. Back to the q-Euler equation. — To fit this equation with the convention of this paper, we shall put $z := \lambda$, $f(z) := X(q\lambda)$ and take q^{-2} as the new dilatation coefficient, that we shall denote by q, so that $|q| > 1$ indeed. Our equation becomes:

$$\sqrt{q} z^2 \sigma_q f - f = z - 1.$$

(We have implicitly chosen a square root $\sqrt{q}$.) There are four pathes of attack. The most poweful involves the summation techniques of chapter 5 and it is the one used in [**59**]. We show the other three as an easy application exercise.

7.3.3.1. q-Borel transformation. — Following 3.1.1, we put $Z := z^2$, $Q := q^2$ and $f(z) = g(Z) + zh(Z)$, so that:

$$\sqrt{q} Z \sigma_Q g - g = -1 \text{ and } q\sqrt{q} Z \sigma_Q h - h = 1.$$

The end of the computation, *i.e.* that of the invariants $(\mathcal{B}_{Q,1})(-1)(\sqrt{q})$ and $(\mathcal{B}_{Q,1})(1)(q\sqrt{q})$, is left to the reader.

7.3.3.2. Birkhoff-Guenther normal form. — According to section 3.3, we see that our equation is already in Birkhoff-Guenther normal form. Actually, it is the equation for f such that $\begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}$ is an isomorphism from $\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{q} z^2 \end{pmatrix}$ to $\begin{pmatrix} 1 & z-1 \\ 0 & \sqrt{q} z^2 \end{pmatrix}$.

7.3.3.3. Privileged cocycles. — There is here an obvious isomorphism of $\Lambda_I(M_0)$ with the vector bundle $F_{1/\sqrt{q}z^2}$ (*cf.* 3.1.1). (More generally, if $A_0 = \begin{pmatrix} a & 0 \\ 0 & bz^\delta \end{pmatrix}$, then $\Lambda_I(M_0) \simeq F_{a/bz^\delta}$.) The privileged cocycles of 6.1 are best obtained by the elementary approach of [**46**] as follows. We look for a solution $f_{\overline{c}} = \dfrac{g}{\theta_{q,c}^2}$ with g holomorphic over $\mathbf{C}^*$. The corresponding equation is $\sqrt{q}c^2\sigma_q g - g = (z-1)\theta_{q,c}^2$. Writing ${\theta_q}^2 =: \sum \tau_n z^n$, we see that:

$$f_{\overline{c}} = \frac{1}{\theta_{q,c}^2} \sum_{n \in \mathbf{Z}} \frac{(\tau_{n-1}c - \tau_n)c^{-n}}{\sqrt{q}c^2q^n - 1} z^n.$$

This is the only solution with poles only on $[-c;q]$ and at most double. It makes sense only for $\sqrt{q}c^2 \notin q^{\mathbf{Z}}$, which prohibits four values $\overline{c} \in \mathbf{E}_q$. The components of the Stokes cocycle are the $(f_{\overline{c}} - f_{\overline{d}})$.

7.4. From class numbers of quadratic forms to the q-Euler equation

This topic is related to a paper of Mordell [**30**] and to recent work [**58**] by the third author. We follow their notations, except for the use of the letter q, and also for the dependency on the modular parameter ω, which we do not always make explicit.

We shall have here use for the classical theta functions, defined for $x \in \mathbf{C}$ and $\Im(\omega) > 0$:

$$\begin{aligned} \theta_{0,1}(x) = \theta_{0,1}(x,\omega) &:= \sum_{n \in \mathbf{Z}} (-1)^n e^{\mathrm{i}\pi(n^2\omega + 2nx)}, \\ \theta_{1,1}(x) = \theta_{1,1}(x,\omega) &:= \frac{1}{\mathrm{i}} \sum_{m \ odd} (-1)^{(m-1)/2} e^{\mathrm{i}\pi(m^2\omega/4 + mx)}, \\ &= e^{\mathrm{i}\pi(\omega/4 + x - 1/2)} \sum_{n \in \mathbf{Z}} (-1)^n e^{\mathrm{i}\pi(n(n+1)\omega + 2nx)}. \end{aligned}$$

We shall set $q := e^{-2\mathrm{i}\pi\omega}$ (so that indeed $|q| > 1$) and $z := e^{2\mathrm{i}\pi x}$. The above theta functions are related to θ_q through the formulas:

$$\begin{aligned} \theta_{0,1}(x,\omega) &= \theta_q(-\sqrt{q}z), \\ \theta_{1,1}(x,\omega) &= \frac{\sqrt{z}}{\mathrm{i}q^{1/8}} \theta_q(-z). \end{aligned}$$

(Thus, the latter is multivalued as a function of z.)

7.4.1. The generating series for the class numbers. — Consider the quadratic forms $ax^2+2hxy+by^2$, with $a,b,h \in \mathbf{Z}$, a,b not both even ("uneven forms"), and $D := ab-h^2 > 0$, up to the usual equivalence. For any $D \in \mathbf{N}^*$, write $F(D)$ the (finite) number of classes of such forms.

***Theorem 7.4.1* (Mordell, 1916)).** — *Let $f_{0,1}(x) = f_{0,1}(x,\omega)$ the unique entire function solution of the system:*

$$\begin{cases} f_{0,1}(x+1) = f_{0,1}(x), \\ f_{0,1}(x+\omega) + f_{0,1}(x) = \theta_{0,1}(x). \end{cases}$$

Then, for $\Im(\omega) > 0$:

$$\sum_{n \in \mathbf{N}^*} F(n) e^{i\pi n\omega} = \frac{i}{4\pi} \frac{f'_{0,1}(0)}{\theta_{0,1}(0)}.$$

If one now defines $G_{0,1}(z) := \dfrac{f_{0,1}(x)}{\theta_{0,1}(x)}$ (which does make sense, since the right hand side is 1-periodic), one falls upon the familiar q-difference equation:

$$(\sqrt{q} z \sigma_q - 1) G_{0,1} = \sqrt{q} z.$$

We leave it as an exercise for the reader to characterize $G_{0,1}$ as the unique solution complying some polar conditions.

7.4.2. Modular relations. — In order to obtain modular and asymptotic properties for the generating series of theorem 7.4.1, Mordell generalized his results in 1933. We extract the part illustrating our point. Mordell sets:

$$f(x) = f(x,\omega) := \frac{1}{\mathrm{i}} \sum_{m \ odd} \frac{(-1)^{(m-1)/2} e^{\mathrm{i}\pi(m^2\omega/4 + mx)}}{1 + e^{\mathrm{i}\pi\omega m}}.$$

This the unique entire function solution of the system:

$$\begin{cases} f(x+1) + f(x) = 0, \\ f(x+\omega) + f(x) = \theta_{1,1}(x). \end{cases}$$

The interest for Mordell is the (quasi-)modular relation:

$$f(x,\omega) - \frac{\mathrm{i}}{\omega} f(x/\omega, -1/\omega) = \frac{1}{\mathrm{i}} \theta_{1,1}(x,\omega) \int_{-\infty}^{+\infty} \frac{e^{\mathrm{i}\pi\omega t^2 - 2\pi t x}}{e^{2\pi t} - 1}\, dt.$$

(The path of integration is $\mathbf{R}$ except that one avoids 0 by below.) The interest for us is that one can put $G(z) := \dfrac{f(x)}{\theta_{1,1}(x)}$ (the right hand side is 1-periodic), and get the *same* equation as before:

$$(\sqrt{q}z\sigma_q - 1)G = \sqrt{q}z.$$

This is used in [**58**] to generalize Mordell results: the fundamental idea is to compare two summations of the solutions, one along the lines of the present paper, the other along different lines previously developped by C. Zhang.

7.4.3. Related other examples. — Mordell also mentions that the following formula of Hardy and Ramanujan:

$$\frac{1}{\sqrt{-\mathrm{i}\omega}} \int_{-\infty}^{+\infty} \frac{e^{-\mathrm{i}\pi(t-\mathrm{i}x)^2/\omega}}{\cosh \pi t}\, dt = \int_{-\infty}^{+\infty} \frac{e^{\mathrm{i}\pi\omega t^2 - 2\pi t x}}{\cosh \pi t}\, dt$$

can be proved along similar lines, by noting that both sides are entire solutions of the system:

$$\begin{cases} \Phi(x-1) + \Phi(x) = \dfrac{2e^{\mathrm{i}\pi(x-1/2)^2/\omega}}{\sqrt{-\mathrm{i}\omega}}, \\ \Phi(x+\omega) + e^{\mathrm{i}\pi(2x+\omega)}\Phi(x) = 2e^{\mathrm{i}\pi(3\omega/4+x)} \end{cases}$$

and that the latter admits only one such solution.

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