Appendix 6

Algebraic properties of Hopf \( G \)-coalgebras

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Let \( G \) be a group. The notion of a (ribbon) Hopf \( G \)-coalgebra was first introduced by Turaev [Tu4], as the prototype algebraic structure whose category of representations is a (ribbon) \( G \)-category (see Section VIII.1). Recall from Chapter VII that ribbon \( G \)-categories give rise to invariants of 3-dimensional \( G \)-manifolds and to 3-dimensional HQFTs with target \( K(G,1) \). Moreover, Hopf \( G \)-coalgebras may be used directly (without involving their representations) to construct further topological invariants of 3-dimensional \( G \)-manifolds, see Appendix 7.

Here we review the algebraic properties of Hopf \( G \)-coalgebras and provide examples. Most of the results are given without proof, see [Vir1]–[Vir4] for details.

In Section 1, we study the algebraic properties of Hopf \( G \)-coalgebras, in particular the existence of integrals, the order of the antipode (a generalization of the Radford \( S^4 \)-formula), and the (co)semisimplicity (a generalization of the Maschke theorem).

In Section 2, we focus on quasitriangular and ribbon Hopf \( G \)-coalgebras. In particular, we construct \( G \)-traces for ribbon Hopf \( G \)-coalgebras, which are used to construct invariants of 3-dimensional \( G \)-manifolds in Appendix 7.

In Section 3, we give a method for constructing a quasitriangular Hopf \( G \)-coalgebra starting from a Hopf algebra endowed with an action of \( G \) by Hopf automorphisms. This leads to non-trivial examples of quasitriangular Hopf \( G \)-coalgebras for all finite \( G \) and for some infinite \( G \) such as \( \text{GL}_n(K) \). In particular, we define graded quantum groups.

Throughout this appendix, \( G \) is a group (with neutral element 1) and \( K \) is a field. All algebras are supposed to be over \( K \), associative, and unital. The tensor product \( \otimes = \otimes_K \) of \( K \)-vector spaces is always taken over \( K \). If \( U \) and \( V \) are \( K \)-vector spaces, then \( \sigma_{U,V} : U \otimes V \rightarrow V \otimes U \) denotes the flip defined by \( \sigma_{U,V}(u \otimes v) = v \otimes u \) for all \( u \in U \) and \( v \in V \).

6.1 Hopf \( G \)-coalgebras

1.1 Hopf \( G \)-coalgebras. We recall, for completeness, the definition of a Hopf \( G \)-coalgebra from Section VIII.1, but with a minor change: we do not suppose the antipode to be bijective.

A Hopf \( G \)-coalgebra (over \( K \)) is a family \( H = \{H_a\}_{a \in G} \) of \( K \)-algebras endowed with a family \( \Delta = \{\Delta_{\alpha,\beta} : H_{a\beta} \rightarrow H_a \otimes H_\beta\}_{a,\beta \in G} \) of algebra homomorphisms...
(the \textit{comultiplication}), an algebra homomorphism \( \varepsilon : H_1 \to K \) (the \textit{counit}), and a family \( S = \{ S_{\alpha} : H_\alpha \to H_{\alpha^{-1}} \}_{\alpha \in G} \) of \( K \)-linear maps (the \textit{antipode}) such that, for all \( \alpha, \beta, \gamma \in G \),
\[
\begin{align*}
(\Delta_{\alpha, \beta} \otimes \text{id}_{H_\gamma}) \Delta_{\alpha, \beta, \gamma} &= (\text{id}_{H_\alpha} \otimes \Delta_{\beta, \gamma}) \Delta_{\alpha, \beta, \gamma}, \\
(\text{id}_{H_\alpha} \otimes \varepsilon) \Delta_{\alpha, 1} &= \text{id}_{H_\alpha} = (\varepsilon \otimes \text{id}_{H_\alpha}) \Delta_{1, \alpha}, \\
m_{\alpha}(S_{\alpha^{-1}} \otimes \text{id}_{H_\alpha}) \Delta_{\alpha^{-1}, \alpha} &= \varepsilon 1_\alpha = m_\alpha(\varepsilon \otimes S_{\alpha^{-1}}) \Delta_{\alpha^{-1}, \alpha},
\end{align*}
\]
where \( m_\alpha : H_\alpha \otimes H_\alpha \to H_\alpha \) and \( 1_\alpha \in H_\alpha \) denote multiplication in \( H_\alpha \) and the unit element of \( H_\alpha \).

When \( G = 1 \), one recovers the usual notion of a Hopf algebra. In particular, \( H_1 \) is a Hopf algebra.

Remark that the notion of a Hopf \( G \)-coalgebra is not self-dual (the dual notion obtained by reversing the arrows in the definition may be called a Hopf \( G \)-algebra).

If \( H = \{ H_{\alpha} \}_{\alpha \in G} \) is a Hopf \( G \)-coalgebra, then the set \( \{ \alpha \in G \mid H_\alpha \neq 0 \} \) is a subgroup of \( G \). Also, if \( G' \) is a subgroup of \( G \), then \( H = \{ H_{\alpha} \}_{\alpha \in G'} \) is a Hopf \( G' \)-coalgebra.

The antipode \( S \) of a Hopf \( G \)-coalgebra \( H = \{ H_{\alpha} \}_{\alpha \in G} \) is anti-multiplicative (in the sense that each \( S_{\alpha} : H_\alpha \to H_{\alpha^{-1}} \) is an anti-homomorphism of algebras) and anti-comultiplicative in the sense that \( \Delta_{\beta^{-1}, \alpha^{-1}} S_{\alpha \beta} = \delta_{H_\alpha, 1} H_{\beta^{-1}} (S_\alpha \otimes S_{\beta}) \Delta_{\alpha, \beta} \) for all \( \alpha, \beta \in G \) and \( \varepsilon S_1 = \varepsilon \); see [Vir2], Lemma 1.1.

A Hopf \( G \)-coalgebra \( H = \{ H_{\alpha} \}_{\alpha \in G} \) is said to be of \textit{finite type} if, for all \( \alpha \in G \), \( H_\alpha \) is finite-dimensional (over \( K \)). Note that the direct sum \( \bigoplus_{\alpha \in G} H_\alpha \) is finite-dimensional if and only if \( H \) is of finite type and \( H_\alpha = 0 \) for all but a finite number of \( \alpha \in G \).

The antipode \( S = \{ S_\alpha \}_{\alpha \in G} \) of \( H = \{ H_{\alpha} \}_{\alpha \in G} \) is said to be \textit{bijective} if each \( S_\alpha \) is bijective. Unlike in Section VIII.1, we do not suppose that the antipode of a Hopf \( G \)-coalgebra is bijective. As for Hopf algebras, the antipode of a Hopf \( G \)-coalgebra \( H \) is necessarily bijective if \( H \) is of finite type (see Section 1.5) or \( H \) is quasitriangular (see Section 2.4).

\section*{1.2 The case of finite \( G \).}

Suppose that \( G \) is a finite group. Recall that the Hopf algebra \( K^G \) of functions on \( G \) has a basis \( \{ e_\alpha : G \to K \}_{\alpha \in G} \) defined by \( e_\alpha(\beta) = \delta_{\alpha, \beta} \) where \( \delta_{\alpha, \alpha} = 1 \) and \( \delta_{\alpha, \beta} = 0 \) if \( \alpha \neq \beta \). The structure maps of \( K^G \) are given by
\[
\begin{align*}
e_\alpha e_\beta &= \delta_{\alpha, \beta} e_\alpha, \quad 1_{K^G} = \sum_{\alpha \in G} e_\alpha, \quad \Delta(e_\alpha) = \sum_{\beta \gamma = \alpha} e_\beta \otimes e_\gamma, \quad \varepsilon(e_\alpha) = \delta_{\alpha, 1},
\end{align*}
\]
and \( S(e_\alpha) = e_{\alpha^{-1}} \). A \textit{central prolongation} of \( K^G \) is a Hopf algebra \( A \) endowed with a morphism of Hopf algebras \( K^G \to A \), called the \textit{central map}, which carries \( K^G \) into the center of \( A \).
Since $G$ is finite, any Hopf $G$-coalgebra $H = \{H_\alpha\}_{\alpha \in G}$ gives rise to a Hopf algebra $\tilde{H} = \bigoplus_{\alpha \in G} H_\alpha$ with structure maps given by

$$\tilde{\Delta}|_{H_\alpha} = \sum_{\beta \gamma = \alpha} \Delta_{\beta, \gamma}, \quad \tilde{\varepsilon}|_{H_\alpha} = \delta_{\alpha, 1} \varepsilon, \quad \tilde{m}|_{H_\alpha \otimes H_\beta} = \delta_{\alpha, \beta} m_\alpha, \quad \tilde{1} = \sum_{\alpha \in G} 1_\alpha,$$

and $\tilde{S} = \sum_{\alpha \in G} S_\alpha$. The $K$-linear map $K^G \to \tilde{H}$ defined by $e_\alpha \mapsto 1_\alpha$ gives rise to a morphism of Hopf algebras which carries $K^G$ into the center of $\tilde{H}$. Hence $\tilde{H}$ is a central prolongation of $K^G$.

The correspondence assigning to every Hopf $G$-coalgebra $H = \{H_\alpha\}_{\alpha \in G}$ the central prolongation $K^G \to \tilde{H}$ is bijective. Given a Hopf algebra $(A, m, 1, \Delta, \varepsilon, S)$ which is a central prolongation of $K^G$, set $H_\alpha = A\{1\}_\alpha$, where $1_\alpha \in A$ is the image of $e_\alpha \in K^G$ under the central map $K^G \to A$. Then the family $\{H_\alpha\}_{\alpha \in G}$ is a Hopf $G$-coalgebra with structure maps given by

$$m_\alpha = 1_\alpha \cdot m|_{H_\alpha \otimes H_\alpha}, \quad \Delta_{\alpha, \beta} = (1_\alpha \otimes 1_\beta) \cdot \Delta|_{H_\alpha \otimes H_\beta}, \quad \varepsilon = \varepsilon|_{H_1}, \quad S_\alpha = 1_{\alpha^{-1}} \cdot S|_{H_\alpha}.$$

### 1.3 Integrals

Recall that a left (resp. right) integral for a Hopf algebra $(A, \Delta, \varepsilon, S)$ is an element $\Lambda \in A$ such that $x\Lambda = \varepsilon(x)\Lambda$ (resp. $\Lambda x = \varepsilon(x)\Lambda$) for all $x \in A$. A left (resp. right) integral for the dual Hopf algebra $A^*$ is a $K$-linear form $\lambda \in A^*$ such that $(\text{id}_A \otimes \lambda)\Delta(x) = \lambda(x)1_A$ (resp. $(\lambda \otimes \text{id}_A)\Delta(x) = \lambda(x)1_A$) for all $x \in A$.

A left (resp. right) $G$-integral for a Hopf $G$-coalgebra $H = \{H_\alpha\}_{\alpha \in G}$ is a family of $K$-linear forms $\lambda = (\lambda_\alpha)_{\alpha \in G} \in \Pi_{\alpha \in G} H_\alpha^*$ such that

$$(\text{id}_{H_\alpha} \otimes \lambda_\beta)\Delta_{\alpha, \beta}(x) = \lambda_{\alpha \beta}(x)1_\alpha \quad \text{(resp.} \quad (\lambda_\alpha \otimes \text{id}_{H_\beta})\Delta_{\alpha, \beta}(x) = \lambda_{\alpha \beta}(x)1_\beta)$$

for all $\alpha, \beta \in G$ and $x \in H_{\alpha \beta}$. Note that $\lambda_1$ is a usual left (resp. right) integral for the Hopf algebra $H_1^*$.

A $G$-integral $\lambda = (\lambda_\alpha)_{\alpha \in G}$ is said to be non-zero if $\lambda_\beta \neq 0$ for some $\beta \in G$. Given a non-zero $G$-integral $\lambda = (\lambda_\alpha)_{\alpha \in G}$, we have $\lambda_\alpha \neq 0$ for all $\alpha \in G$ such that $H_\alpha \neq 0$. In particular $\lambda_1 \neq 0$.

It is known that the $K$-vector space of left (resp. right) integrals for a finite-dimensional Hopf algebra is one-dimensional. This extends to Hopf $G$-coalgebras as follows.

**Theorem A** ([Vir2], Theorem 3.6). *Let $H$ be a Hopf $G$-coalgebra of finite type. Then the vector space of left (resp. right) $G$-integrals for $H$ is one-dimensional.*

The proof of this theorem is based on the fact that a Hopf $G$-comodule has a canonical decomposition generalizing the fundamental decomposition theorem in the theory of Hopf modules.
1.4 Grouplike elements. A family \( g = (g_\alpha)_{\alpha \in G} \in \prod_{\alpha \in G} H_\alpha \) such that \( \Delta_{\alpha,\beta}(g_\alpha g_\beta) = g_\alpha \otimes g_\beta \) for all \( \alpha, \beta \in G \) and \( \varepsilon(g_1) = 1_k \) is called a G-grouplike element of a Hopf \( G \)-coalgebra \( H = \{H_\alpha\}_{\alpha \in G} \). Note that \( g_1 \) is then a grouplike element of the Hopf algebra \( H_1 \) in the usual sense of the word.

One easily checks that the set \( \text{Gr}(H) \) of G-grouplike elements of \( H \) is a group with respect to coordinate-wise multiplication in the product monoid \( \prod_{\alpha \in G} H_\alpha \). If \( g = (g_\alpha)_{\alpha \in G} \in \text{Gr}(H) \), then \( g^{-1} = (S_{\alpha^{-1}}(g_{\alpha^{-1}}))_{\alpha \in G} \). The group \( \text{Hom}(G, K^*) \) of homomorphisms \( G \to K^* \) acts on \( \text{Gr}(H) \) by \( \phi g = (\phi(\alpha)g_\alpha)_{\alpha \in G} \) for arbitrary \( \phi \in \text{Hom}(G, K^*) \) and \( g = (g_\alpha)_{\alpha \in G} \in \text{Gr}(H) \).

1.5 The distinguished G-grouplike element. Throughout this subsection, \( H = \{H_\alpha\}_{\alpha \in G} \) is a Hopf \( G \)-coalgebra of finite type with antipode \( S = \{S_\alpha\}_{\alpha \in G} \). Using Theorem A, one verifies that there is a unique G-grouplike element \( g = (g_\alpha)_{\alpha \in G} \) of \( H \), called the distinguished group-like element of \( H \), such that \( (\text{id}_{H_\alpha} \otimes \lambda_\beta)\Delta_{\alpha,\beta} = \lambda_{\alpha\beta} g_\alpha \) for any right \( G \)-integral \( \lambda = (\lambda_\alpha)_{\alpha \in G} \) and all \( \alpha, \beta \in G \). Note that \( g_1 \) is the distinguished grouplike element of \( H_1 \).

Since \( H_1 \) is a finite-dimensional Hopf algebra, there exists a unique algebra morphism \( \nu: H_1 \to K \) such that if \( \Lambda \) is a left integral for \( H_1 \), then \( \Lambda x = \nu(x)\Lambda \) for all \( x \in H_1 \). This morphism is a group-like element of the Hopf algebra \( H_1^* \), called the distinguished group-like element of \( H_1^* \). It is invertible in \( H_1^* \) and its inverse \( \nu^{-1} \) is also an algebra morphism. Moreover, if \( \Lambda \) is a right integral for \( H_1 \), then \( x\Lambda = \nu^{-1}(x)\Lambda \) for all \( x \in H_1 \).

For all \( \alpha \in G \), we define a left and a right \( H_1^* \)-action on \( H_\alpha \) by setting, for all \( f \in H_1^* \) and \( a \in H_\alpha \),

\[
    f \rightarrow a = (\text{id}_{H_\alpha} \otimes f)\Delta_{\alpha,1}(a) \quad \text{and} \quad a \leftarrow f = (f \otimes \text{id}_{H_\alpha})\Delta_{1,\alpha}(a).
\]

The next assertion generalizes Theorem 3 of [Rad4]. This is a key result in the theory of Hopf \( G \)-coalgebras. It is used in particular to prove the existence of traces (see Section 2.8).

**Theorem B** ([Vir2], Theorem 4.2). Let \( \lambda = (\lambda_\alpha)_{\alpha \in G} \) be a right \( G \)-integral for \( H \). Then, for all \( \alpha \in G \) and \( x, y \in H_\alpha \),

(a) \( \lambda_\alpha(xy) = \lambda_\alpha(S_{\alpha^{-1}}S_\alpha(y \leftarrow \nu)x) \);

(b) \( \lambda_\alpha(xy) = \lambda_\alpha(y S_{\alpha^{-1}}S_\alpha(v^{-1} \leftrightarrow g_\alpha^{-1}xg_\alpha)) \);

(c) \( \lambda_{\alpha^{-1}}(S_\alpha(x)) = \lambda_\alpha(g_\alpha x) \).

As a corollary we obtain a generalization of the celebrated Radford \( S^4 \)-formula to Hopf \( G \)-coalgebras:

**Corollary C** ([Vir2], Lemma 4.6). Let \( H = \{H_\alpha\}_{\alpha \in G} \) be a Hopf \( G \)-coalgebra of finite type. Then for all \( \alpha \in G \) and \( x \in H_\alpha \),

\[
    (S_{\alpha^{-1}}S_\alpha)^2(x) = g_\alpha (\nu \leftrightarrow x \leftarrow \nu^{-1})g_\alpha^{-1}.
\]
This formula implies in particular that the antipode $S$ of $H$ is bijective (i.e., each $S_a$ is bijective).

1.6 The order of the antipode. It is known that the order of the antipode of a finite-dimensional Hopf algebra is finite ([Rad1], Theorem 1) and divides four times the dimension of the algebra ([NZ], Proposition 3.1). We apply this result to study a Hopf $G$-coalgebra of finite type $H = \{H_a\}_{a \in G}$ with antipode $S = \{S_a\}_{a \in G}$. Let $a$ be an element of $G$ of finite order $d$. Denote by $\langle a \rangle$ the subgroup of $G$ generated by $a$. By considering the finite-dimensional Hopf algebra $\bigoplus_{\beta \in \langle a \rangle} H_\beta$ (determined by the Hopf $\langle a \rangle$-coalgebra $\{H_\beta\}_{\beta \in \langle a \rangle}$, see Section 1.2), we obtain that the order of $S_a^{-1}S_a$ divides $2d \dim H_1$; see [Vir2], Corollary 4.5.

1.7 Semisimplicity. A Hopf $G$-coalgebra $H = \{H_a\}_{a \in G}$ is said to be semisimple if each algebra $H_a$ is semisimple. For $H$ to be semisimple it is necessary that $H_1$ be finite-dimensional (since an infinite-dimensional Hopf algebra over a field is not semisimple, see [Sw], Corollary 2.7). When $H$ is of finite type, $H$ is semisimple if and only if $H_1$ is semisimple, see [Vir2], Lemma 5.1.

1.8 Cosemisimplicity. The notion of a comodule over a coalgebra may be extended to the setting of Hopf $G$-coalgebras. A right $G$-comodule over a Hopf $G$-coalgebra $H = \{H_a\}_{a \in G}$ is a family $M = \{M_a\}_{a \in G}$ of $K$-vector spaces endowed with a family of $K$-linear maps $\rho = \{\rho_{a, \beta}: M_a \to M_\beta \otimes H_\alpha\}_{a, \beta \in G}$ such that

$$(\rho_{a, \beta} \otimes \text{id}_{H_\gamma})\rho_{a, \gamma, \beta} = (\text{id}_{M_a} \otimes \Delta_{\beta, \gamma})\rho_{a, \beta, \gamma}$$

and

$$(\text{id}_{M_a} \otimes \varepsilon)\rho_{a, 1} = \text{id}_{M_a}$$

for all $a, \beta, \gamma \in G$. A $G$-subcomodule of $M$ is a family $N = \{N_a\}_{a \in G}$, where $N_a$ is a $K$-subspace of $M_a$ such that $\rho_{a, \beta}(N_\beta) \subseteq N_a \otimes H_\beta$ for all $a, \beta \in G$. The sums and direct sums for families of $G$-subcomodules of a right $G$-comodule are defined in the obvious way.

A right $G$-comodule $M = \{M_a\}_{a \in G}$ is said to be simple if it is non-zero (i.e., $M_a \neq 0$ for some $a \in G$) and if it has no $G$-subcomodules other than itself and the trivial one $0 = \{0\}_{a \in G}$. A right $G$-comodule which is a direct sum of a family of simple $G$-subcomodules is said to be cosemisimple. Note that all $G$-subcomodules and all quotients of a cosemisimple right $G$-comodule are cosemisimple.

A Hopf $G$-coalgebra is cosemisimple if it is cosemisimple as a right $G$-comodule over itself (with comultiplication as comodule map). By [Vir2], a Hopf $G$-coalgebra
H = \{H_\alpha\}_{\alpha \in G} is cosemisimple if and only if every reduced right G-comodule over H is cosemisimple.

We state a Hopf G-coalgebra version of the dual Maschke theorem.

**Theorem D** ([Vir2], Theorem 5.4). A Hopf G-coalgebra H = \{H_\alpha\}_{\alpha \in G} is cosemisimple if and only if there exists a right G-integral \( \lambda = (\lambda_\alpha)_{\alpha \in G} \) for H such that \( \lambda_\alpha(1_\alpha) = 1_K \) for some \( \alpha \in G \) (and then \( \lambda_\alpha(1_\alpha) = 1_K \) for all \( \alpha \in G \) with \( H_\alpha \neq 0 \)).

As corollaries, we obtain that a Hopf G-coalgebra H = \{H_\alpha\}_{\alpha \in G} of finite type is cosemisimple if and only if the Hopf algebra \( H_1 \) is cosemisimple, and that the distinguished G-grouplike element of a cosemisimple Hopf G-coalgebra of finite type is trivial.

**1.9 Involutory Hopf G-coalgebras.** A Hopf G-coalgebra \( H = \{H_\alpha\}_{\alpha \in \pi} \) is involutory if its antipode \( S = \{S_\alpha\}_{\alpha \in \pi} \) satisfies the identity \( S_\alpha^{-1} S_\alpha = \text{id}_{H_\alpha} \) for all \( \alpha \in \pi \).

Involutory Hopf G-coalgebras of finite type have special properties. For example, each of their G-integrals \( \lambda = (\lambda_\alpha)_{\alpha \in G} \) is two sided, S-invariant (\( \lambda_\alpha^{-1} S_\alpha = \lambda_\alpha \) for all \( \alpha \in G \)), and symmetric (\( \lambda_\alpha(xy) = \lambda_\alpha(yx) \) for all \( \alpha \in G \) and \( x, y \in H_\alpha \)). Also if the ground field K of H is of characteristic 0, then \( \dim H_\alpha = \dim H_1 \) whenever \( H_\alpha \neq 0 \).

Finally, if \( H = \{H_\alpha\}_{\alpha \in G} \) is an involutory Hopf G-coalgebra of finite type over a field whose characteristic does not divide \( \dim H_1 \), then \( H \) is semisimple and cosemisimple; see [Vir4], Lemma 3.

### 6.2 Quasitriangular Hopf G-coalgebras

**2.1 Crossed Hopf G-coalgebras.** A Hopf G-coalgebra \( H = \{H_\alpha\}_{\alpha \in G} \) is crossed if it is endowed with a crossing, that is, a family of algebra isomorphisms \( \varphi = \{\varphi_\beta : H_\alpha \to H_{\beta \alpha^{-1}}\}_{\alpha, \beta \in G} \) such that

\[
(\varphi_\beta \otimes \varphi_\delta) \Delta_{\alpha, \gamma} = \Delta_{\beta \alpha \gamma^{-1}, \beta \gamma^{-1}} \varphi_\gamma. \quad \varepsilon \varphi_\beta = \varepsilon, \quad \text{and} \quad \varphi_{\alpha \beta} = \varphi_\alpha \varphi_\beta
\]

for all \( \alpha, \beta, \gamma \in G \). One easily verifies that a crossing preserves the antipode, that is, \( \varphi_\beta S_\alpha = S_{\beta \alpha^{-1}} \varphi_\beta \) for all \( \alpha, \beta \in G \). Therefore this definition of a crossed Hopf G-coalgebra is equivalent to the one in Chapter VIII.

A crossing \( \varphi \) in \( H \) yields a group homomorphism \( \varphi : G \to \text{Aut}_{\text{hopf}}(H_1) \) and determines thus an action of G on \( H_1 \) by Hopf algebra automorphisms. Here for a Hopf algebra A, we denote \( \text{Aut}_{\text{hopf}}(A) \) the group of Hopf automorphisms of A.

If G is an abelian group, then any Hopf G-coalgebra admits a trivial crossing \( \varphi_\beta = \text{id} \) for all \( \beta \in G \).

When G is a finite group, the notion of a crossing can be described in terms of central prolongations of \( K^G \) (see Section 1.2): a crossing of a central prolongation \( A \)

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1A right G-comodule \( M = \{M_\alpha\}_{\alpha \in G} \) over H is reduced if \( M_\alpha = 0 \) whenever \( H_\alpha = 0 \).
of $K^G$ is a group homomorphism $\varphi : G \to \text{Aut}_\text{Hopf}(A)$ such that $\varphi_\beta(1_a) = 1_{\beta a \beta^{-1}}$ for all $\alpha, \beta \in G$, where $1_\alpha$ is the image of $e_\alpha \in K^G$ under the central map $K^G \to A$.

2.2 The distinguished character. Let $H = \{H_\alpha\}_{\alpha \in G}$ be a crossed Hopf $G$-coalgebra of finite type with crossing $\varphi$. Using the uniqueness of $G$-integrals (see Theorem A), one can show the existence of a unique group homomorphism $\hat{\varphi} : G \to K^*$, called the distinguished character of $H$, such that $\lambda_{\beta \varphi^{-1}} \varphi_{\beta} = \hat{\varphi}(\beta) \lambda_\alpha$ for any left or right $G$-integral $\lambda = (\lambda_\alpha)_{\alpha \in G}$ for $H$ and all $\alpha, \beta \in G$.

Lemma E ([Vir2], Lemma 6.3). For any $\beta \in G$,

(a) If $\Lambda$ is a left or right integral for $H_1$, then $\varphi_\beta(\Lambda) = \hat{\varphi}(\beta) \Lambda$.
(b) If $v$ is the distinguished grouplike element of $H_1^*$, then $v \varphi_\beta = v$.
(c) If $g = (g_\alpha)_{\alpha \in G}$ is the distinguished $G$-grouplike element of $H$, then $\varphi_\beta(g_\alpha) = g_{\beta \varphi^{-1}} \beta^{-1}$ for all $\alpha \in G$.

2.3 Quasitriangular Hopf $G$-coalgebras. Following Chapter VIII, we call a crossed Hopf $G$-coalgebra $(H = \{H_\alpha\}_{\alpha \in G}, \varphi)$ quasitriangular if it is endowed with an $R$-matrix, that is, a family $R = \{R_{\alpha, \beta}\} \in H_\alpha \otimes H_\beta$ of invertible elements such that, for all $\alpha, \beta, \gamma \in G$ and $x \in H_{\alpha \beta}$,

$$R_{\alpha, \beta} \cdot \Delta_{\alpha, \beta}(x) = \sigma_{\beta, \alpha}(\varphi_\gamma^{-1} \otimes \text{id}_{H_\gamma}) \Delta_{\alpha \varphi^{-1}, \alpha}(x) \cdot R_{\alpha, \beta},$$

$$(\text{id}_{H_\alpha} \otimes \Delta_{\beta, \gamma})(R_{\alpha, \beta}) = (R_{\alpha, \gamma})_{\beta \varphi} \cdot (R_{\beta, \gamma})_{\alpha \varphi^{-1}},$$

$$(\Delta_{\alpha, \beta} \otimes \text{id}_{H_\gamma})(R_{\alpha, \beta}) = [(\text{id}_{H_\alpha} \otimes \varphi^{-1})(R_{\alpha, \beta \varphi^{-1}})]_{\beta \varphi} \cdot (R_{\beta, \gamma})_{\alpha \varphi^{-1}},$$

$$(\varphi_\gamma \otimes \varphi_\beta)(R_{\alpha, \gamma}) = R_{\beta \varphi^{-1}, \gamma \varphi^{-1}} \cdot \Delta_{\alpha, \beta}.\gamma.$$ Here $\sigma_{\beta, \alpha}$ denotes the flip $H_\beta \otimes H_\alpha \to H_\alpha \otimes H_\beta$ and, for $K$-vector spaces $P, Q$ and $r = \sum_j p_j \otimes q_j \in P \otimes Q$, we set

$$r_{\alpha \varphi^{-1}} = r \otimes 1_{\gamma} \in P \otimes Q \otimes H_\gamma, \quad r_{\alpha \varphi^{-1}} = 1_{\alpha \varphi^{-1}} \otimes r \in H_\alpha \otimes P \otimes Q,$$

and $r_{\alpha \varphi^{-1}} = \sum_j p_j \otimes 1_{\gamma} \otimes q_j \in P \otimes H_\gamma \otimes Q$. Note that $R_{1, 1}$ is an $R$-matrix for the Hopf algebra $H_1$ is the usual sense of the word.

When $G$ is abelian and $\varphi$ is the trivial crossing, we recover the definition of a quasitriangular $G$-colored Hopf algebra due to Ohtsuki [Oh1].

An $R$-matrix for a crossed Hopf $G$-coalgebra provides a solution of the $G$-colored Yang–Baxter equation

$$(R_{\beta, \gamma})_{\alpha \varphi^{-1}} \cdot (R_{\alpha, \beta})_{\varphi} \cdot (R_{\beta, \gamma})_{\alpha \varphi^{-1}} = (R_{\alpha, \beta})_{\varphi} \cdot [(\text{id}_{H_\alpha} \otimes \varphi^{-1})(R_{\alpha, \beta \varphi^{-1}})]_{\beta \varphi} \cdot (R_{\beta, \gamma})_{\alpha \varphi^{-1}},$$

where $\varphi_\alpha$ denotes the distinguished character of $H$.
and satisfies the following identities (see [Vir2], Lemma 6.4): for all $\alpha, \beta, \gamma \in G$,
\[
(e \otimes \text{id}_{H_\alpha})(R_{1,\alpha}) = 1_\alpha = (\text{id}_{H_\alpha} \otimes e)(R_{\alpha,1}),
\]
\[
(S_{\alpha^{-1}} \varphi_\alpha \otimes \text{id}_{H_\alpha})(R_{\alpha^{-1},\beta}) = R_{\alpha,\beta}^{-1} \quad \text{and} \quad (\text{id}_{H_\alpha} \otimes S_{\beta})(R_{\alpha,\beta}^{-1}) = R_{\alpha,\beta}^{-1},
\]
\[
(S_{\alpha} \otimes S_{\beta})(R_{\alpha,\beta}) = (\varphi_\alpha \otimes \text{id}_{H_{\beta^{-1}}})(R_{\alpha,\beta^{-1}}^{-1}).
\]

2.4 The Drinfeld element. The Drinfeld element of a quasitriangular Hopf $G$-coalgebra $H = \{H_\alpha\}_{\alpha \in G}$ is the family $u = (u_\alpha)_{\alpha \in G} \in \Pi_{\alpha \in G} H_\alpha$, where

\[
u_\alpha = m_\alpha(S_{\alpha^{-1}} \varphi_\alpha \otimes \text{id}_{H_\alpha}) \sigma_{\alpha,\alpha^{-1}}(R_{\alpha,\alpha^{-1}}).
\]

Observe that $u_1$ is the Drinfeld element of the quasitriangular Hopf algebra $H_1$ (see [Drin2]). By [Vir2], Lemma 6.5, each $u_\alpha$ is invertible in $H_\alpha$ and

\[
u_\alpha^{-1} = m_\alpha(\text{id}_{H_\alpha} \otimes S_{\alpha^{-1}} \varphi_\alpha) \sigma_{\alpha,\alpha}(R_{\alpha,\alpha}).
\]

Moreover, for any $\alpha \in G$ and $x \in H$,
\[
S_{\alpha^{-1}} \varphi_\alpha(x) = u_\alpha \varphi_\alpha(x) u_\alpha^{-1},
\]
where $\varphi$ is the crossing in $H$. This implies that the antipode of $H$ is bijective.

Note also the identities $e(u_1) = 1$, $\psi_\beta(u_\alpha) = u_\beta \varphi_{\alpha \beta^{-1}}$, and
\[
\Delta_{\alpha,\beta}(u_{\alpha \beta}) = [\sigma_{\beta,\alpha}(\text{id}_{H_\beta} \otimes \varphi_\alpha)(R_{\beta,\alpha}) \cdot R_{\alpha,\beta}]^{-1} \cdot (u_\alpha \otimes u_\beta).
\]

2.5 Ribbon Hopf $G$-coalgebras. Following Chapter VIII, we call a quasitriangular Hopf $G$-coalgebra $H = \{H_\alpha\}_{\alpha \in G}$ ribbon if it is endowed with a twist, that is, a family of invertible elements $\theta = \{\theta_\alpha \in H_\alpha\}_{\alpha \in G}$ such that for all $\alpha, \beta \in G$ and $x \in H_\alpha$,

\[
\varphi_\alpha(x) = \theta_{\alpha^{-1}} x \theta_\alpha, \quad S_\alpha(\theta_\alpha) = \theta_\alpha^{-1}, \quad \psi_\beta(\theta_\alpha) = \theta_{\beta \alpha \beta^{-1}},
\]
\[
\Delta_{\alpha,\beta}(\theta_{\alpha \beta}) = (\theta_\alpha \otimes \theta_\beta) \cdot \sigma_{\beta,\alpha}(\text{id}_{H_\beta} \otimes \varphi_\alpha)(R_{\beta,\alpha}) \cdot R_{\alpha,\beta}.
\]

Note that $\theta_1$ is a twist of the quasitriangular Hopf algebra $H_1$, and so $e(\theta_1) = 1$. If $\alpha \in G$ has a finite order $d$, then $\delta_{\alpha}^d$ is a central element of $H_\alpha$. In particular, $\delta_1$ is central in $H_1$.

Example. Let $G$ be a group and $c : G \times G \to K^*$ be a bicharacter of $G$, that is, $c(\alpha, \beta \gamma) = c(\alpha, \beta)c(\alpha, \gamma)$ and $c(\alpha \beta, \gamma) = c(\alpha, \gamma)c(\beta, \gamma)$ for all $\alpha, \beta, \gamma \in G$. Consider the following crossed Hopf algebra $K^c$: for all $\alpha, \beta \in G$, we have $K^c_\alpha = K$ as an algebra and
\[
\Delta_{\alpha,\beta}(1_K) = 1_K \otimes 1_K, \quad e(1_K) = 1_K, \quad S_\alpha(1_K) = 1_K, \quad \psi_\beta(1_K) = 1_K.
\]
Then $K^c$ is a ribbon Hopf $G$-coalgebra of finite type with $R$-matrix and twist given by $R_{\alpha,\beta} = c(\alpha, \beta) 1_K \otimes 1_K$ and $\theta_\alpha = c(\alpha, \alpha)$. The Drinfeld elements of $K^c$ are computed by $u_\alpha = c(\alpha, \alpha)^{-1}$. 
2.6 The spherical $G$-grouplike element. Let $H = \{H_\alpha\}_{\alpha \in G}$ be a ribbon Hopf $G$-coalgebra with Drinfeld element $u = (u_\alpha)_{\alpha \in G}$. For any $\alpha \in G$, set

$$w_\alpha = \theta_\alpha u_\alpha = u_\alpha \theta_\alpha \in H_\alpha.$$ 

Then $w = (w_\alpha)_{\alpha \in G}$ is a $G$-grouplike element, called the spherical $G$-grouplike element of $H$. It satisfies the identities

$$\varphi_\beta(w_\alpha) = w_{\beta \alpha^{-1}}, \quad S_\alpha(u_\alpha) = w_{\alpha^{-1}} u_{\alpha^{-1}} w_{\alpha^{-1}}, \quad \text{and} \quad S_{\alpha^{-1}} S_\alpha(x) = w_{\alpha} x w_{\alpha}^{-1}$$

for all $\alpha, \beta \in G$ and $x \in H_\alpha$. Conversely, any $G$-grouplike element $w = (w_\alpha)_{\alpha \in G}$ of a quasitriangular Hopf $G$-coalgebra $H = \{H_\alpha\}_{\alpha \in G}$ which satisfies these identities gives rise to a twist $\theta = (\theta_\alpha)_{\alpha \in G}$ in $H$ by $\theta_\alpha = w_{\alpha} u_{\alpha^{-1}} = u_{\alpha^{-1}} w_{\alpha}$.

2.7 Further $G$-grouplike elements. Let $H = \{H_\alpha\}_{\alpha \in G}$ be a quasitriangular Hopf $G$-coalgebra of finite type. Set

$$\ell_\alpha = S_{\alpha^{-1}}(u_{\alpha^{-1}})^{-1} u_{\alpha} S_{\alpha^{-1}}(u_{\alpha^{-1}})^{-1} \in H_\alpha,$$

where $u = (u_\alpha)_{\alpha \in G}$ is the Drinfeld element of $H$. The properties of $u$ ensure that $\ell = (\ell_\alpha)_{\alpha \in G}$ is a $G$-grouplike element of $H$. Let $\nu$ be the distinguished grouplike element of $H^*_1$ and $\hat{\varphi}$ be the distinguished character of $H$ (see Sections 1.5 and 2.2). Denoting $R = \{R_{\alpha, \beta} \in H_\alpha \otimes H_{\beta^{-1}}\}_{\alpha, \beta \in G}$ the $R$-matrix of $H$, set

$$h_\alpha = (\text{id}_{H_\alpha} \otimes \nu)(R_{\alpha, 1}) \in H_\alpha.$$

Theorem F ([Vir2], Theorem 6.9). The family $h = (h_\alpha)_{\alpha \in G}$ is a $G$-grouplike element of $H$. The distinguished $G$-grouplike element $(g_\alpha)_{\alpha \in G}$ of $H$ is computed by $g_\alpha = \hat{\varphi}(\alpha)^{-1} \ell_\alpha h_\alpha$ for all $\alpha \in G$.

For ribbon $H$, we obtain as a corollary that $g_\alpha = \hat{\varphi}(\alpha)^{-1} w_\alpha^2 h_\alpha$ for all $\alpha \in G$, where $w = (w_\alpha)_{\alpha \in G}$ is the spherical $G$-grouplike element of $H$.

2.8 Traces. Let $H = \{H_\alpha\}_{\alpha \in G}$ be a crossed Hopf $G$-coalgebra. A $G$-trace for $H$ is a family of $K$-linear forms $\text{tr} = (\text{tr}_\alpha)_{\alpha \in G} \in \Pi_{\alpha \in G} H_{\alpha}^*$ such that

$$\text{tr}_\alpha(x y) = \text{tr}_\alpha(y x), \quad \text{tr}_{\alpha^{-1}}(S_\alpha(x)) = \text{tr}_\alpha(x), \quad \text{and} \quad \text{tr}_{\beta \alpha^{-1}}(\varphi_\beta(x)) = \text{tr}_\alpha(x)$$

for all $\alpha, \beta \in G$ and $x, y \in H_\alpha$. Note that $\text{tr}_1$ is a usual trace for the Hopf algebra $H_1$, which is invariant under the action $\varphi$ of $G$.

A Hopf $G$-coalgebra $H = \{H_\alpha\}_{\alpha \in G}$ is unimodular if the Hopf algebra $H_1$ is unimodular (that is the spaces of left and right integrals for $H_1$ coincide). If $H_1$ is finite-dimensional, then $H$ is unimodular if and only if $\nu = \epsilon$, where $\nu$ is the distinguished grouplike element of $H^*_1$. For example, any finite type semisimple Hopf $G$-coalgebra is unimodular.
6.3 The twisted double construction

Consider in more detail a unimodular ribbon Hopf $G$-coalgebra $H = \{H_\alpha\}_{\alpha \in G}$ of finite type. Let $\lambda = (\lambda_\alpha)_{\alpha \in G}$ be a non-zero right $G$-integral for $H$, $w = (w_\alpha)_{\alpha \in G}$ be the spherical $G$-grouplike element of $H$, and $\varphi$ be the distinguished character of $H$.

Using Theorems B and F, we obtain that the $G$-traces for $H$ are parameterized by families $z = (z_\alpha)_{\alpha \in G}$ such that $z_\alpha \in H_\alpha$ is central, $S_\alpha(z_\alpha) = \varphi(\alpha)^{-1} z_{\alpha^{-1}}$, and $\varphi_\beta(z_\alpha) = \varphi(\beta) z_{\alpha \beta^{-1}}$ for all $\alpha, \beta \in G$. The $G$-trace corresponding to such a family $z$ is given by $\text{tr}_z(x) = \lambda_\alpha(w_\alpha z_\alpha x)$. We point out two such families.

Let $\Lambda$ be a left integral for $H_1$ such that $\lambda_1(\Lambda) = 1$. Set $z_1 = \Lambda$ and $z_\alpha = 0$ if $\alpha \neq 1$. The resulting family $(z_\alpha)_{\alpha \in G} satisfies all the conditions above since $H$ is unimodular (and so $\Lambda$ is central and $S_1(\Lambda) = \Lambda$) and by Lemma E (a). The corresponding $G$-trace is given by $\text{tr}_1 = \varepsilon$ and $\text{tr}_\alpha = 0$ for all $\alpha \neq 1$.

If $\varphi(\alpha) = 1$ for all $\alpha \in G$, then another possible choice of a family $z$ is $z_\alpha = 1_\alpha$ for all $\alpha$. Note that $\varphi = 1$ if $H$ is semisimple or cosemisimple or if $\lambda_1(\theta_1) \neq 0$, where $\theta = \{\theta_\alpha\}_{\alpha \in G}$ is the twist of $H$. We obtain the following assertion.

Theorem G ([Vir2], Theorem 7.4). Suppose under the assumptions above that at least one of the following four conditions is satisfied: $H$ is semisimple, or $H$ is cosemisimple, or $\lambda_1(\theta_1) \neq 0$, or $\varphi_\beta|_{H_1} = \text{id}_{H_1}$ for all $\beta \in G$. Then the family of $K$-linear maps $\text{tr} = (\text{tr}_\alpha)_{\alpha \in G}$, defined by $\text{tr}_z(x) = \lambda_\alpha(w_\alpha x)$ for all $x \in H_\alpha$, is a $G$-trace for $H$.

6.3 The twisted double construction

Starting from a crossed Hopf $G$-coalgebra $H = \{H_\alpha\}_{\alpha \in G}$, Zunino [Zu1] constructed a double $Z(H) = \{Z(H)_\alpha\}_{\alpha \in G}$ of $H$ which is a quasitriangular Hopf $G$-coalgebra containing $H$ as a Hopf $G$-subcoalgebra. As a vector space, $Z(H)_\alpha = H_\alpha \otimes (\bigoplus_{\beta \in G} H_\beta^*)$. Generally speaking, $Z(H)$ is not of finite type: the components $Z(H)_\alpha$ may be infinite-dimensional.

In this section we provide a method, called the twisted double construction, for deriving quasitriangular Hopf $G$-coalgebras of finite type from finite-dimensional Hopf algebras endowed with action of $G$ by Hopf automorphisms (cf. Section 2.1). We also give an $h$-adic version of this construction. This will lead us to non-trivial examples of quasitriangular Hopf $G$-coalgebras for any finite group $G$ and for infinite groups $G$ such as $GL_n(K)$. In particular, we define the ($h$-adic) graded quantum groups.

3.1 Hopf pairings. Recall that a Hopf pairing between two Hopf algebras $A$ and $B$ (over $K$) is a bilinear pairing $\sigma: A \times B \to K$ such that, for all $a, a' \in A$ and $b, b' \in B$,

\[ \sigma(a, b b') = \sigma(a a', b) \sigma(a, b'), \quad \sigma(a, 1) = \varepsilon(a), \]

\[ \sigma(a a', b) = \sigma(a, b_2) \sigma(a', b_1), \quad \sigma(1, b) = \varepsilon(b). \]

Such a pairing always preserves the antipode: $\sigma(S(a), S(b)) = \sigma(a, b)$ for all $a \in A$ and $b \in B$. 
A Hopf pairing $\sigma : A \times B \to K$ determines two annihilator ideals $I_A = \{a \in A \mid \sigma(a, b) = 0 \text{ for all } b \in B\}$ and $I_B = \{b \in B \mid \sigma(a, b) = 0 \text{ for all } a \in A\}$. It is easy to check that $I_A$ and $I_B$ are Hopf ideals of $A$ and $B$, respectively. The pairing $\sigma$ is non-degenerate iff $I_A = I_B = 0$. Any Hopf pairing $\sigma : A \times B \to K$ induces a non-degenerate Hopf pairing $\tilde{\sigma} : A/I_A \times B/I_B \to K$.

### 3.2 The twisted double.

Let $\sigma : A \times B \to K$ be a Hopf pairing between two Hopf algebras $A$ and $B$, and let $\phi : A \to A$ be a Hopf algebra endomorphism of $A$. Set

$$D(A, B; \sigma, \phi) = A \otimes B$$

as a $K$-vector space. We provide $D(A, B; \sigma, \phi)$ with a structure of an algebra with unit $1_A \otimes 1_B$ and multiplication

$$(a \otimes b) \cdot (a' \otimes b') = \sigma(\phi(a''(1)), S(b''(1))) \sigma(a''(3), b''(3)) a''(2) \otimes b''(2) b'$$

for any $a, a' \in A$ and $b, b' \in B$.

Note that if $\phi$ and $\phi'$ are different Hopf algebra endomorphisms of $A$, then the algebras $D(A, B; \sigma, \phi)$ and $D(A, B; \sigma, \phi')$ are in general not isomorphic (see Remark in Section 3.4 below).

**Theorem H** ([Vir3], Theorem 2.6). Let $\sigma : A \times B \to K$ be a Hopf pairing between Hopf algebras $A$ and $B$, and let $\phi$ be an action of $G$ on $A$ by Hopf algebra automorphisms, that is, $\phi$ is a group homomorphism $G \to \text{Aut}_{\text{Hopf}}(A)$. Then the family of algebras

$$D(A, B; \sigma, \phi) = \{D(A, B; \sigma, \phi_a)\}_{a \in G}$$

has a structure of a Hopf $G$-coalgebra given by

$$\Delta_{a, \beta}(a \otimes b) = (\phi_\beta(a''(1)) \otimes b''(1)) \otimes (a''(2) \otimes b''(2)),$$

$$\epsilon(a \otimes b) = \epsilon_A(a) \epsilon_B(b),$$

$$S_a(a \otimes b) = \sigma(\phi_\alpha(a''(1)), b''(1)) \sigma(a''(3), S(b''(3))) \phi_\alpha S(a''(2)) \otimes S(b''(2))$$

for all $a \in A, b \in B$ and $\alpha, \beta \in G$. Furthermore, if $\sigma$ is non-degenerate and $A, B$ are finite dimensional, then the Hopf $G$-coalgebra $D(A, B; \sigma, \phi)$ is quasitriangular with crossing $\psi$ and $R$-matrix $R = \{R_{a, \beta}\}_{a, \beta \in G}$ given by

$$\psi_\beta(a \otimes b) = \phi_\beta(a) \otimes \phi_\beta^*(b) \quad \text{and} \quad R_{a, \beta} = \sum_i (e_i \otimes 1_B) \otimes (1_A \otimes f_i).$$

where $\phi^* : G \to \text{Aut}_{\text{Hopf}}(B)$ is the unique action such that $\sigma(\phi_\beta, \phi^*_\beta) = \sigma$ for all $\beta \in G$, and $(e_i)_i$ and $(f_i)_i$ are dual bases of $A$ and $B$ with respect to $\sigma$.

**Corollary I.** Let $A$ be a finite-dimensional Hopf algebra and $\phi$ be an action of $G$ on $A$ by Hopf algebra automorphisms. Then the duality bracket $(\cdot, \cdot)_{A \otimes A^*}$ is a non-degenerate Hopf pairing between $A$ and $A^{\text{cop}}$ and $D(A, A^{\text{cop}}; (\cdot, \cdot)_{A \otimes A^*})$ is a quasitriangular Hopf $G$-coalgebra.
Note that the group of Hopf automorphisms of a finite-dimensional semisimple Hopf algebra $A$ over a field of characteristic $0$ is finite (see [Rad2]). To obtain quasitriangular Hopf $G$-coalgebras with infinite $G$ using the twisted double method, one has to start from non-semisimple Hopf algebras or from Hopf algebras over fields of non-zero characteristic.

In the next three sections, we use Theorem H to give examples of quasitriangular Hopf $G$-coalgebras.

### 3.3 Example: finite $G$

Let $G$ be a finite group. In this section, we describe the ribbon Hopf $G$-coalgebras obtained by the twisted double construction from the group algebra $K[G]$. The standard Hopf algebra structure on $K[G]$ is given by $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$, and $S(g) = g^{-1}$ for all $g \in G$. The dual of $K[G]$ is the Hopf algebra $F(G) = K^G$ of functions $G \to K$ with structure maps and basis $(e_g : G \to K)_{g \in G}$ described in Section 2.1. Let $\phi : G \to \text{Aut}_{\text{Hopf}}(K[G])$ be the homomorphism defined by $\phi_{\alpha}(h) = \alpha h a^{-1}$ for $\alpha \in G, h \in K[G]$. Corollary I yields a quasitriangular Hopf $G$-coalgebra

$$D_G(G) = D(K[G], F(G)^{\text{cop}}, (\cdot)_K[G] \times F(G), \phi).$$

Let us describe $D_G(G) = \{D_\alpha(G)\}_{\alpha \in G}$ more precisely. For $\alpha \in G$, the algebra $D_\alpha(G)$ is equal to $K[G] \otimes F(G)$ as a $K$-vector space, has unit $1_{D_\alpha(G)} = \sum_{g \in G} 1 \otimes e_g$ and multiplication

$$(g \otimes e_h) \cdot (g' \otimes e_{h'}) = \delta_{ag'\alpha^{-1},1g'g} e_{g'g} \otimes e_{h'}$$

for all $g, g', h, h' \in G$. The structure maps of $D_G(G)$ are

$$\Delta_{\alpha,\beta}(g \otimes e_h) = \sum_{x,y = h} \beta g \beta^{-1} \otimes e_y \otimes g \otimes e_x, \quad \varepsilon(g \otimes e_h) = \delta_{h,1},$$

$$S_{\alpha}(g \otimes e_h) = \alpha g^{-1} \otimes e_{ag\alpha^{-1}g^{-1}h^{-1}}, \quad \varphi_{\alpha}(g \otimes e_h) = \alpha g \alpha^{-1} \otimes e_{a\alpha h},$$

for all $\alpha, \beta, g, h \in G$. The crossed Hopf $G$-coalgebra $D_G(G)$ is quasitriangular and furthermore ribbon with $R$-matrix and twist

$$R_{\alpha,\beta} = \sum_{g,h \in G} g \otimes e_h \otimes 1 \otimes e_g$$

and

$$\theta_{\alpha} = \sum_{g \in G} \alpha^{-1} g \alpha \otimes e_g$$

for all $\alpha, \beta \in G$. The spherical $G$-grouplike element of $D_G(G)$ is $w = (1_{D_\alpha(G)})_{\alpha \in G}$. The family $\lambda = (\lambda_{\alpha})_{\alpha \in G}$, defined by $\lambda_{\alpha}(g \otimes e_h) = \delta_{h,1}$, is a two-sided $G$-integral for $D_G(G)$.

### 3.4 An example of a quasitriangular Hopf $GL_n(K)$-coalgebra

In this section, $K$ is a field of characteristic $\neq 2$ and $n$ is a positive integer. Let $A$ be the $K$-algebra with generators $g, x_1, \ldots, x_n$ subject to the relations

$$g^2 = 1, \quad x_i^2 = 0, \quad gx_i = -x_ig, \quad x_i x_j = -x_j x_i.$$
The algebra $A$ is $2^{n+1}$-dimensional and has a Hopf algebra structure given by

$$
\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad \Delta(x_i) = x_i \otimes g + 1 \otimes x_i, \quad \varepsilon(x_i) = 0, \quad S(g) = g,
$$

and $S(x_i) = gx_i$ for all $i$. The group of Hopf automorphisms of $A$ is isomorphic to the group $GL_n(K)$ of invertible $n \times n$-matrices with coefficients in $K$ (see [Rad2]). An explicit isomorphism $\phi: GL_n(K) \rightarrow \text{Aut}_H(A)$ carries any $\alpha = (\alpha_{i,j}) \in GL_n(K)$ to the automorphism $\phi_\alpha$ of $A$ given by

$$
\phi_\alpha(g) = g \quad \text{and} \quad \phi_\alpha(x_i) = \sum_{k=1}^n \alpha_{k,i} x_k.
$$

We apply Corollary I to these $A$ and $\phi$. Observing that $A^* \cong A$ as Hopf algebras, we can quotient the resulting quasitriangular Hopf $GL_n(K)$-coalgebra to eliminate one copy of the generator $g$ (which appears twice), see [Vir3], Proposition 4.1. This gives a quasitriangular Hopf $GL_n(K)$-coalgebra $H = \{H_\alpha\}_{\alpha \in GL_n(K)}$. We give here a direct description of $H$. For $\alpha = (\alpha_{i,j}) \in GL_n(K)$, let $H_\alpha$ be the $K$-algebra generated $g, x_1, \ldots, x_n, y_1, \ldots, y_n$, subject to the relations

$$
\begin{align*}
g^2 &= 1, \quad x_i^2 = \cdots = x_n^2 = 0, \quad gx_i = -x_i g, \quad x_ix_j = -x_j x_i, \\
y_1^2 &= \cdots = y_n^2 = 0, \quad gy_i = -y_i g, \quad y_iy_j = -y_j y_i, \\
x_iy_j - y_j x_i &= (\alpha_{j,i} - \delta_{i,j}) g,
\end{align*}
$$

where $1 \leq i, j \leq n$. The family $\{H_\alpha\}_{\alpha \in GL_n(K)}$ has the following structure of a crossed Hopf $GL_n(K)$-coalgebra:

$$
\begin{align*}
\Delta_{\alpha,\beta}(g) &= g \otimes g, \quad \varepsilon(g) = 1, \quad S_{\alpha}(g) = g, \\
\Delta_{\alpha,\beta}(x_i) &= 1 \otimes x_i + \sum_{k=1}^n \beta_{k,i} x_k \otimes g, \quad \varepsilon(x_i) = 0, \quad S_{\alpha}(x_i) = \sum_{k=1}^n \alpha_{k,i} g x_k, \\
\Delta_{\alpha,\beta}(y_i) &= y_i \otimes 1 + g \otimes y_i, \quad \varepsilon(y_i) = 0, \quad S_{\alpha}(y_i) = -gy_i, \\
\varphi_{\alpha}(g) &= g, \quad \varphi_{\alpha}(x_i) = \sum_{k=1}^n \alpha_{k,i} x_k, \quad \varphi_{\alpha}(y_i) = \sum_{k=1}^n \alpha_{i,k} y_k,
\end{align*}
$$

where $\alpha = (\alpha_{i,j}), \beta = (\beta_{i,j})$ run over $GL_n(K)$, $(\alpha_{i,j}) = \alpha^{-1}$, and $1 \leq i \leq n$. The crossed Hopf $GL_n(K)$-coalgebra $H$ is quasitriangular with $R$-matrix

$$
R_{\alpha,\beta} = \frac{1}{2} \sum_{S \subseteq \{1, \ldots, n\}} x_S \otimes y_S + x_S \otimes gy_S + gx_S \otimes y_S - gx_S \otimes gy_S
$$

for all $\alpha, \beta \in GL_n(K)$. Here $x_0 = 1, y_0 = 1$, and for a nonempty subset $S$ of $\{1, \ldots, n\}$, we set $x_S = x_{i_1} \cdots x_{i_k}$ and $y_S = y_{i_1} \cdots y_{i_k}$, where $i_1 < \cdots < i_k$ are the elements of $S$. 


Remark. Generally speaking, for distinct \( \alpha, \beta \in \text{GL}_n(K) \), the algebras \( \mathcal{H}_\alpha \) and \( \mathcal{H}_\beta \) are not isomorphic. For example, \( \mathcal{H}_\alpha \not\cong \mathcal{H}_1 \) for any \( \alpha \in \text{GL}_n(K) \) \( \setminus \{1\} \). It suffices to prove that

\[
\mathcal{H}_\alpha / [\mathcal{H}_\alpha, \mathcal{H}_\alpha] \not\cong \mathcal{H}_1 / [\mathcal{H}_1, \mathcal{H}_1].
\]

Indeed, \( \mathcal{H}_\alpha / [\mathcal{H}_\alpha, \mathcal{H}_\alpha] = 0 \) since \( g = \frac{1}{a_{i,j}-a_{i,j}}(x_i y_j - y_j x_i) \in [\mathcal{H}_\alpha, \mathcal{H}_\alpha] \) (for some \( 1 \leq i, j \leq n \) such that \( a_{i,j} \neq \delta_{i,j} \)) and so \( 1 = g^2 \in [\mathcal{H}_\alpha, \mathcal{H}_\alpha] \). In \( \mathcal{H}_1 / [\mathcal{H}_1, \mathcal{H}_1] \), we have \( x_k = x_k g^2 = 0 \) (since \( x_k g = g x_k = -x_k g \) and so \( x_k g = 0 \)) and likewise \( y_k = 0 \). Hence \( \mathcal{H}_1 / [\mathcal{H}_1, \mathcal{H}_1] = K \langle g \mid g^2 = 1 \rangle \neq 0 \).

### 3.5 Graded quantum groups.

Let \( \mathfrak{g} \) be a finite-dimensional complex simple Lie algebra of rank \( l \) with Cartan matrix \( (a_{i,j}) \). Let \( \{d_i\}_{i=1}^l \) be coprime integers such that the matrix \( (d_i a_{i,j}) \) is symmetric. Let \( q \) be a fixed non-zero complex number and \( q_i = q^{d_i} \) for \( i = 1, 2, \ldots, l \). We suppose that \( q_i^2 \neq 1 \) for all \( i \).

Recall that the (usual) quantum group \( U_q(\mathfrak{g}) \) can be obtained as a quotient of the quantum double of \( U_q(\mathfrak{b}_+) \), where \( \mathfrak{b}_+ \) is the (positive) Borel subalgebra of \( \mathfrak{g} \) (the quotient is needed to eliminate the second copy of the Cartan subalgebra). Applying Theorem H to the Hopf algebra \( U_q(\mathfrak{b}_+) \) endowed with an action of \((\mathbb{C}^*)^l \) by Hopf automorphisms, we obtain the “graded quantum group” introduced in [Vir3], Proposition 5.1. It can be directly described as follows.

Set \( G = (\mathbb{C}^*)^l \). For \( \alpha = (\alpha_1, \ldots, \alpha_l) \in G \), let \( U^G_q(\mathfrak{g}) \) be the \( \mathbb{C} \)-algebra generated by \( K_i^{\pm 1}, E_i, F_i, 1 \leq i \leq l \), subject to the following defining relations:

\[
K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1, \\
K_i E_j = q_i^{a_{i,j}} E_j K_i, \\
K_i F_j = q_i^{-a_{i,j}} F_j K_i, \\
E_i F_j - F_j E_i = \delta_{i,j} (a_{i,j} K_i - K_i^{-1}), \\
\sum_{r=0}^{1-a_{i,j}} (-1)^r \binom{1-a_{i,j}}{r} q_i^{1-a_{i,j}-r} E_i^{1-a_{i,j}-r} E_i = 0 \quad \text{if } i \neq j, \\
\sum_{r=0}^{1-a_{i,j}} (-1)^r \binom{1-a_{i,j}}{r} q_i^{1-a_{i,j}-r} F_i^{1-a_{i,j}-r} F_i = 0 \quad \text{if } i \neq j.
\]

The family \( U^G_q(\mathfrak{g}) = \{U^\alpha_q(\mathfrak{g})\}_{\alpha \in G} \) has a structure of a crossed Hopf \( G \)-coalgebra given, for \( \alpha = (\alpha_1, \ldots, \alpha_l) \in G, \beta = (\beta_1, \ldots, \beta_l) \in G \) and \( 1 \leq i \leq l \), by:

\[
\Delta_{\alpha, \beta}(K_i) = K_i \otimes K_i, \\
\Delta_{\alpha, \beta}(E_i) = \beta_i E_i \otimes K_i + 1 \otimes E_i, \\
\Delta_{\alpha, \beta}(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i.
\]
\[ \varepsilon(K_1) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0, \]
\[ S_a(K_1) = K_i^{-1}, \quad S_a(E_i) = -\alpha_i E_i K_i^{-1}, \quad S_a(F_i) = -K_i F_i, \]
\[ \varphi_a(K_1) = K_i, \quad \varphi_a(E_i) = \alpha_i E_i, \quad \varphi_a(F_i) = \alpha_i^{-1} F_i. \]

Note that \((U_q^1(\mathfrak{g}), \Delta_{1,1}, \varepsilon, S_1)\) is the usual quantum group \(U_q(\mathfrak{g})\).

To give a rigorous treatment of \(R\)-matrices for the graded quantum groups, we need \(h\)-adic versions of Hopf \(G\)-coalgebras and of graded quantum groups. This is the content of the next two sections.

### 3.6 The \(h\)-adic case

In this section, we develop an \(h\)-adic variant of Hopf \(G\)-coalgebras. Roughly speaking, \(h\)-adic Hopf \(G\)-coalgebras are obtained by taking the ring \(\mathbb{C}[h]\) of formal power series as the ground ring and requiring that the algebras (resp. the tensor products) are complete (resp. completed) in the \(h\)-adic topology.

Recall that if \(V\) is a (left) module over \(\mathbb{C}[h]\), then the topology on \(V\) for which the sets \(\{v^n h^n | n \in \mathbb{N}\}\) form a base for neighborhoods of \(v \in V\) is called the \(h\)-adic topology. For \(\mathbb{C}[h]\)-modules \(V\) and \(W\), denote by \(V \hat{\otimes} W\) the completion of \(V \otimes \mathbb{C}[h]\) \(W\) in the \(h\)-adic topology.

If \(V\) is a complex vector space, then the set \(V[h]\) of all formal power series \(\sum_{n=0}^{\infty} v_n h^n\) with coefficients \(v_n \in V\) is a \(\mathbb{C}[h]\)-module called a topologically free module. Topologically free modules are exactly \(\mathbb{C}[h]\)-modules which are complete, separated, and torsion-free. Furthermore, \(V[h] \hat{\otimes} W[h] = (V \otimes W)[h]\) for any complex vector spaces \(V\) and \(W\).

An \(h\)-adic algebra \(A\) is a \(\mathbb{C}[h]\)-module complete in the \(h\)-adic topology and endowed with a \(\mathbb{C}[h]\)-linear map \(m: A \hat{\otimes} A \to A\) and an element \(1 \in A\) such that \(m(id_A \hat{\otimes} m) = m(m \hat{\otimes} id_A)\) and \(m(id_A \hat{\otimes} 1) = id_A = m(1 \hat{\otimes} id_A)\).

By an \(h\)-adic Hopf \(G\)-coalgebra, we mean a family \(H = \{H_{a}\}_{a \in G}\) of \(h\)-adic algebras endowed with \(h\)-adic algebra homomorphisms \(\Delta_{a,\beta}: H_{a} \hat{\otimes} H_{\beta} \to H_{a \beta}\) \((a, \beta \in G)\), \(\varepsilon: A \to \mathbb{C}[h]\), and with \(\mathbb{C}[h]\)-linear maps \(S_{a}: H_{a} \to H_{a^{-1}}\) \((a \in G)\) satisfying formulas of Section 1.1. It is understood that the algebraic tensor product \(\otimes\) is replaced everywhere by its \(h\)-adic completions \(\hat{\otimes}\).

The notions of crossed, quasitriangular, and ribbon \(h\)-adic Hopf \(G\)-coalgebras can be defined similarly following Sections 2.1 and 2.3.

Theorem H carries over to the \(h\)-adic Hopf algebras. The key modifications are that \(\sigma: A \hat{\otimes} B \to \mathbb{C}[h]\) must be \(\mathbb{C}[h]\)-linear and \(D(A, B; \sigma, \phi) = A \hat{\otimes} B\).

**Theorem J.** Let \(\sigma: A \hat{\otimes} B \to \mathbb{C}[h]\) be an \(h\)-adic Hopf pairing between two \(h\)-adic Hopf algebras \(A\) and \(B\). Let \(\phi: G \to Aut_{Hopf}(A)\) be an action of \(G\) on \(A\) by \(h\)-adic Hopf automorphisms. Then the family \(D(A, B; \sigma, \phi) = \{D(A, B; \sigma, \phi_a)\}_{a \in G}\) is an \(h\)-adic Hopf \(G\)-coalgebra. Assume furthermore that \(A\) and \(B\) are topologically free, \(\sigma\) is non-degenerate, and \(\mathbb{R}_{\alpha, \beta} = \sum_i (e_i \otimes 1_B) \otimes (1_A \otimes f_i)\) belongs to the \(h\)-adic completion \(D(A, B; \sigma, \phi_a) \otimes D(A, B; \sigma, \phi_B)\), where \((e_i)\) and \((f_i)\) are bases of \(A\) and \(B\) dual with respect to \(\sigma\). Then \(D(A, B; \sigma, \phi)\) is quasitriangular with \(R\)-matrix \(R = \{\mathbb{R}_{\alpha, \beta}\}_{a, \beta \in G}\).
The condition on $R_{\alpha, \beta}$ in the second part of the theorem means the following. Since $A$ and $B$ are topologically free, $A = V[[h]]$ and $B = W[[h]]$ for some complex vector spaces $V$ and $W$. Then

$$D(A, B; \sigma, \phi_\alpha) \otimes D(A, B; \sigma, \phi_\beta) = (V \otimes W \otimes V \otimes W)[[h]].$$

We require that $R_{\alpha, \beta} = \sum_i (e_i \otimes 1_B) \otimes (1_A \otimes f_i)$ can be expanded as $\sum_{n=0}^{\infty} r_n h^n$ for some $r_n \in V \otimes W \otimes V \otimes W$.

In the next section, we use Theorem J to define $h$-adic graded quantum groups.

### 3.7 $h$-adic graded quantum groups.

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra of rank $l$ with Cartan matrix $(a_{i,j})$. Let $\{d_i\}_{i=1}^l$ be coprime integers such that the matrix $(d_i, a_{i,j})$ is symmetric. Applying Theorem J to the $h$-adic Hopf algebras $U_h(b_+)$ and $\tilde{U}_h(b_-) = \mathbb{C}[[h]]1 + hU_h(b_-)$, we obtain (after appropriate quotienting) quasitriangular "$h$-adic graded quantum groups" (see [Vir3], Proposition 6.1). We give here a direct description of these quantum groups.

Let $G = \mathbb{C}[[h]]^l$ with group operation being addition. For $\alpha = (\alpha_1, \ldots, \alpha_l) \in G$, let $U^G_h(\mathfrak{g})$ be the $h$-adic algebra generated by the elements $H_i, E_i, F_i, 1 \leq i \leq l$, subject to the following defining relations:

$$[H_i, H_j] = 0,$$
$$[H_i, E_j] = a_{ij} E_j,$$
$$[H_i, F_j] = -a_{ij} F_j,$$
$$[E_i, F_j] = \delta_{ij} \frac{e^{d_i h} e^{d_j H_i} - e^{d_j h} e^{d_i h}}{e^{d_i h} - e^{d_j h}},$$
$$\sum_{r=0}^{1-a_{i,j}} (-1)^r \left[ \sum_{r=0}^{1-a_{i,j}} \right] e^{d_i h} E_i^{1-a_{i,j}} F_i^{1-a_{i,j}} E_i^r F_i^r = 0 \quad (i \neq j).$$

The family $U^G_h(\mathfrak{g}) = \{U^G_h(\mathfrak{g})\}_{\alpha \in G}$ has a structure of a crossed $h$-adic Hopf $G$-coalgebra given, for $\alpha = (\alpha_1, \ldots, \alpha_l), \beta = (\beta_1, \ldots, \beta_l) \in G$ and $1 \leq i \leq l$, by

$$\Delta_{\alpha, \beta}(H_i) = H_i \otimes 1 + 1 \otimes H_i, \quad \varepsilon(H_i) = 0,$$
$$\Delta_{\alpha, \beta}(E_i) = e^{d_i h} \beta_i E_i \otimes e^{d_i h} H_i + 1 \otimes E_i, \quad \varepsilon(E_i) = 0,$$
$$\Delta_{\alpha, \beta}(F_i) = F_i \otimes 1 + e^{-d_i h} H_i \otimes F_i, \quad \varepsilon(F_i) = 0,$$
$$S_{\alpha}(H_i) = -H_i, \quad S_{\alpha}(E_i) = -e^{d_i h} \alpha_i E_i e^{-d_i h} H_i, \quad S_{\alpha}(F_i) = -e^{d_i h} H_i F_i,$$
$$\phi_{\alpha}(H_i) = H_i, \quad \phi_{\alpha}(E_i) = e^{d_i h} \alpha_i E_i, \quad \phi_{\alpha}(F_i) = e^{-d_i h} \alpha_i F_i.$$
Furthermore, $U_h^G(g)$ is quasitriangular by Theorem J (the conditions of this theorem are satisfied by $A = U_h(b_+)$ and $B = \tilde{U}_h(b_-)$). For example, for $g = sl_2$ and $G = \mathbb{C}[[h]]$, the $R$-matrix of $U_h^G(sl_2)$ is given by

$$R_{\alpha,\beta} = e^{h (H \otimes H)/2} \sum_{n=0}^{\infty} R_n(h) \ E^n \otimes F^n \in U_h^\alpha(sl_2) \otimes U_h^\beta(sl_2)$$

for all $\alpha, \beta \in \mathbb{C}[[h]]$, where $R_n(h) = q^{n(n+1)/2} \frac{(1-q^{-2})^n}{[n]_q!}$ and $q = e^h$. 