On hydrodynamic models for LEO spacecraft charging *

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Abstract

This paper is devoted to hydrodynamic models intended to describe charging phenomena the spacecrafts evolving in Low Earth Orbits (LEO) are subject to. The models we are interested in couple the stationary Euler equations to the Poisson equation which defines the electric potential. Furthermore, the charging dynamics is embodied into the boundary conditions where the time derivative of the potential appears. We point out the main mathematical difficulties by restricting to a 1D caricature model for which we present rigorous existence results and numerical simulations.

1 Introduction

A spacecraft evolves in the space plasma and interacts with it. These complex interactions, due to the different dielectric properties of the materials on the surface of the spacecraft, can induce the apparition of severe potential differences which, in turn, produce electric arcing. These phenomena are sources of in-orbit failures since the arcing can lead to irreversible damages on the in-board devices or on the solar arrays. Therefore, the prevention of the apparition of excessive electric charges has motivated an intense research in space engineering in order to design efficient procedures of numerical simulations see e. g. [21], [6], [22].

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This effort requires an important preliminary step on modeling issues. A basis model is clearly based on the Vlasov-Maxwell-Boltzmann (or Fokker-Planck) equations for describing both the motion of the charged particles and the variations of the electro-magnetic fields. The nonlinear system of PDEs is completed by suitable boundary conditions on the surfaces of the satellite and equilibrium conditions at infinity. The charging phenomenon is precisely driven by the boundary conditions on the spacecraft surface for the electromagnetic field and the densities. We shall see that their expression, which involve the dielectric properties of the different materials on the surface, makes the problem highly non standard. Moreover, taking into account the specific features of the plasma environment can help in reducing the complexity of the model, and we can actually decline a hierarchy of possible models. In the next Section we describe some aspects of the derivation of the models, emphasizing the specificities of GEostationary Orbits (GEO) and Low Earth Orbits (LEO) environments. In Section 3 we derive a simpler one dimensional model which helps in pointing out several interesting features of the problem. This is completed by theoretical results in Section 4 and numerical simulations in Section 5.

2 Kinetic, Hydrodynamics Models and Potential Boundary Conditions

2.1 Generalities

We suppose that the plasma consists in two charged particles species: ions H^+ and electrons. We denote by f_i and f_e respectively, the distribution functions of these species: $f_{i/e}(t,x,v)\,\mathrm{d}v\,\mathrm{d}x$ stands for the number of ions (respectively electrons) in the domain centered at the point (x,v) of the phase space with infinitesimal volume $\mathrm{d}v\,\mathrm{d}x$ at time $t \geq 0$. Let $q_i = -q_e = q > 0$ be the elementary charge, let m_i and m_e be the ion mass and the electron mass, respectively. The evolution of the charged particles obeys the following PDEs

$$\partial_t f_{i/e} + v \cdot \nabla_x f_{i/e} + \frac{q_{i/e}}{m_{i/e}}(E + v \wedge B) \cdot \nabla_v f_{i/e} = C_{i/e}(f_i, f_e), \qquad (2.1)$$

which is coupled to the Maxwell equations for the electro-magnetic field (E,B):

$$\epsilon_0\big(-\partial_t E + c^2 \mathrm{curl}_x B\big) = J_i + J_e, \qquad (2.2)$$

$$\mathrm{div}_x(\epsilon_0 E) = q(n_i - n_e), \qquad (2.3)$$

$$c^2 \partial_t B + \mathrm{curl}_x E = 0, \qquad (2.4)$$

$$\mathrm{div}_x B = 0, \qquad (2.5)$$

where ε_0 and c stand for the vacuum permittivity and the light speed respectively and we denote

$$n_{i/e} = \int_{\mathbb{R}^3} f_{i/e}\,\mathrm{d}v, \qquad J_{i/e} = q_{i/e}\int_{\mathbb{R}^3} v\ f_{i/e}\,\mathrm{d}v.$$

In (2.1) the right hand side contains the collision dynamics between the particles (electron/electron, ion/ion and electron/ion), the operator $C_{i/e}$ being of Boltzmann or Fokker-Planck type, see [10, 13]. However, except for very specific flights (e. g. in Polar Earth Orbits), the magnetic effects can be neglected so that the Maxwell equations (2.2)–(2.5) can be replaced by a mere Poisson equation for the electric potential. Indeed, let us introduce the electric potential $\Phi(t,x)$: the electric field is defined by $E = -\nabla_x\Phi$. Then, (2.1) reduces to

$$\partial_t f_{i/e} + v\cdot\nabla_x f_{i/e} - \frac{q_{i/e}}{m_{i/e}}\nabla_x\Phi\cdot\nabla_v f_{i/e} = C_{i/e}(f_i, f_e), \tag{2.6}$$

where (2.3) leads to the Poisson equation

$$-\mathrm{div}_x(\epsilon_0\nabla_x\Phi) = q(n_i - n_e). \tag{2.7}$$

Eq. (2.1)–(2.5) hold for $t \geq 0$, $x \in \Omega$, $v \in \mathbb{R}^3$, where $\Omega \subset \mathbb{R}^3$ represents the exterior of the satellite. Therefore the problem should be completed with boundary conditions for the potential and the distribution functions. First of all, far from the spacecraft the plasma is supposed to be in an equilibrium state thus, at infinity, we assume that

$$\begin{aligned} &\Phi(t,x) \xrightarrow[|x|\to\infty]{} 0,\\ &f_{i/e}(t,x,v) \xrightarrow[|x|\to\infty]{} \frac{n^\infty_{i/e}}{(2\pi\Theta^\infty_{i/e})^{3/2}}\ \exp\Big(-\frac{v^2}{2\Theta^\infty_{i/e}}\Big) \end{aligned} \tag{2.8}$$

holds with $n^\infty_{i/e} > 0$ and $\Theta^\infty_{i/e} > 0$ given densities and temperatures for the ions and the electrons.

Second of all, on the spacecraft the particles distributions obeys

$$\gamma_{inc} f_{i/e} = \mathcal{R}\big(\gamma_{out} f_{i/e}\big) + S \quad \text{for } v\cdot\nu(x) < 0 \tag{2.9}$$

where $\nu(x)$ stands for the outward unit vector at point $x \in \partial\Omega$, γ_{inc} denotes the trace operator on the incoming set $\big\{(x,v) \in \partial\Omega\times\mathbb{R}^3 \text{ s. t. } v\cdot\nu(x) < 0\big\}$, and γ_{out} denotes the trace operator on the outgoing set $\big\{(x,v) \in \partial\Omega\times\mathbb{R}^3 \text{ s. t. } v\cdot\nu(x) > 0\big\}$. The linear operator $\mathcal{R}$ describes how impinging particles are reflected by the walls; for instance we can use the simple specular reflection law

$$\mathcal{R}f(x,v) = \alpha\ f\big(x, v - 2(v\cdot\nu(x))\nu(x)\big)$$

with $\alpha \in (0,1)$ an accommodation coefficient. Varying the value of α can be seen as a model of the photo-emission. When flying in darkness the spacecraft surfaces are absorbing ($\alpha = 0$) whereas exposition to light causes emission of particles ($\alpha > 0$). Finally, S is a source term accounting for possible emission of charged particles by the surface. Let us now describe, according to [5], the boundary condition for the potential which is the most original part of the model.

The spacecraft can be seen as a perfect conductor, partially covered by an assembly of dielectric materials. We denote by $\mathcal{O}_0$ the conductor, and $\mathcal{O}_k$, $k \in \{1, ..., N_d\}$ the dielectrics which are characterized by their permittivity $\varepsilon_k > 0$ and conductivity $\sigma_k > 0$. The height of the kth dielectric layer is denoted by d_k. The plasma fills the domain $\Omega = \mathbb{R}^3 \setminus \overline{\bigcup_{k=0}^{N_d} \mathcal{O}_k}$. We set $\Gamma = \bigcup_{k=0}^{N_d} \partial\mathcal{O}_k$ and for a given point $x \in \Gamma$, $\nu(x)$ stands the normal vector at the surface Γ (pointing outward the considered domain). We consider the following interfaces (see figure 2.1):

- $\Gamma_{c/v} = \Gamma \setminus \bigcap_{k=0}^{N_d} \partial\mathcal{O}_k$ the interface between the conductor and the vacuum,
- $\Gamma_{c/d} = \partial\mathcal{O}_0 \setminus \Gamma_{c/v}$ the interface between the conductor and the dielectrics,
- $\Gamma_{d/v} = \partial\Omega \setminus \Gamma_{c/v}$ the interface between the dielectrics and the vacuum,
- $\Gamma_{d/d} = \Gamma \setminus (\partial\mathcal{O}_0 \cup \partial\Omega)$ the interface between neighbors dielectrics.

The boundary conditions for Φ can be deduced from the Maxwell equations considered in the whole space $\mathbb{R}^3$ and bearing in mind that the different parts of the spacecraft have different electric behavior. At any place of the conductor, the electric potential remains at a constant value: we denote by $\phi_{abs}(t)$, the so-called "absolute potential", this value of the potential at time t. In particular, we have

$$\Phi(t,x) = \phi_{abs}(t) \text{ on } \Gamma_{c/v}. \tag{2.10}$$

In the dielectrics, there exists a runaway current, proportional to the electric field $J_k = -\sigma_k \nabla_x \Phi_{diel}$. Then, we consider the jump relations associated to the Ampère law (2.2) recasts as $\partial_t \text{div}_x(\varepsilon \nabla_x \Phi) = \text{div}_x(J)$. Denoting $J_{ext} = J_i + J_e$, we get

$$\partial_t(\epsilon_k \partial_\nu \Phi_{diel} - \epsilon_0 \partial_\nu \Phi) + J_{ext} \cdot \nu + \sigma_k \partial_\nu \Phi_{diel} = 0 \quad \text{on } \Gamma_{d/v} \tag{2.11}$$

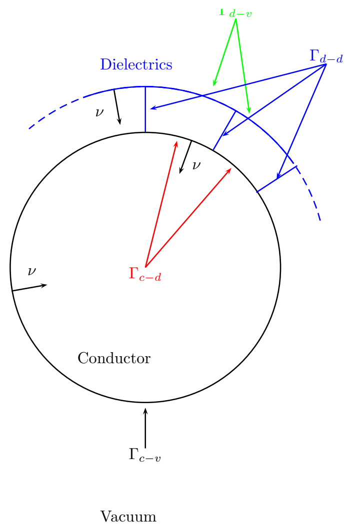

Figure 2.1: Domain and notations of interfaces

together with the relation

$$\int_{\Gamma_{c/v}} \left[\partial_t(-\epsilon_0 \partial_\nu \Phi) + J_{ext} \cdot \nu\right] \,\mathrm{d}\gamma + \int_{\Gamma_{c/d}} \left[\partial_t(-\epsilon_k \partial_\nu \Phi_{diel}) - \sigma_k \partial_\nu \Phi_{diel}\right] \,\mathrm{d}\gamma = 0. \tag{2.12}$$

Since the dielectric layer is very thin, which means that the d_k's are small compared to the characteristic lengths of the spacecraft, the normal derivative of the dielectric potential on $\Gamma_{c/d}$ and $\Gamma_{d/v}$ is approached by

$$\partial_\nu \Phi_{diel}(t, x) \simeq \frac{\phi_{abs}(t) - \Phi(t, x)}{d_k}. \tag{2.13}$$

Finally (2.10), (2.11), (2.12) and (2.13) defines the boundary conditions for the potential.

2.2 From GEO to LEO

The most studied environment relies on the geostationary orbits (GEO) which yields further simplifications, based on asymptotic considerations. There, the plasma can be considered as collisionless, that is $C_{i/e} = 0$ in (2.6). Furthermore, the Debye length is large and the evolution of the charged particles holds on a larger time scale than the time scale of evolution of the electric potential on the boundary. Eventually, the

GEO charging of a spacecraft is thus described by the stationary Vlasov-Poisson equations

$$\begin{cases} v \cdot \nabla_x f_{i/e} - \dfrac{q_{i/e}}{m_{i/e}} \nabla_x \Phi \cdot \nabla_v f_{i/e} = 0, \\ \Delta_x \Phi = 0, \end{cases}$$

with the boundary conditions (2.8), (2.9) and (2.10)–(2.12). Note that the problem remains time-dependent due to the time derivative in (2.11) which governs the evolution of the charging phenomena. We refer to [5] for an introduction to this model, in particular for the discussion of the potential boundary conditions. The model is currently used in GEO codes, see [3, 6, 4, 1]. In this paper we are rather interested in Low Earth Orbits (it means orbits with an altitude between 100 and 2000 km whereas GEO is around 36.000 km). Since the plasma is more dense with a smaller mean free path, the use of hydrodynamic models becomes reasonable, at least in a first approximation. This is interesting for numerical purposes since by getting rid of the velocity variable, it allows to reduce the size of the unknowns. The model can be derived as follows. Bearing in mind the standard collision operators in plasma physics, electron/electron and ion/ion collisions preserve mass, impulsion and energy and relax towards equilibrium states which are the Maxwellian functions. After integration of (2.6) we obtain

$$\begin{aligned} \partial_t \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ v \\ v^2 \end{pmatrix} f_{i/e} \, \mathrm{d}v + \nabla_x \int_{\mathbb{R}^3} v \begin{pmatrix} 1 \\ v \\ v^2 \end{pmatrix} f_{i/e} \, \mathrm{d}v \\ + \frac{q_{i/e}}{m_{i/e}} \nabla_x \Phi \cdot \int_{\mathbb{R}^3} \begin{pmatrix} 0 \\ 1 \\ 2v \end{pmatrix} f_{i/e} \, \mathrm{d}v = \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ v \\ v^2 \end{pmatrix} C_{i/e}(f_i, f_e) \, \mathrm{d}v. \end{aligned} \tag{2.14}$$

Actually, the right hand side only retains the momentum and energy exchanges between the two species due to the electron/ion collisions. Of course, this set of moment equations is not closed since higher moments appear in the convection terms. However, dealing with collision-dominated flows, the distribution functions relax to Maxwellians and replacing $f_{i/e}$ by the corresponding $\frac{n_{i/e}}{(2\pi\Theta_{i/e})^{3/2}} \exp\left(-\frac{|v-u_{i/e}|^2}{2\Theta_{i/e}}\right)$ we are led to the Euler equations satisfied by the density $n_{i/e}$, velocity $u_{i/e}$ and temperature $\Theta_{i/e}$

$$\begin{cases} \partial_t n_{i/e} + \mathrm{div}_x(n_{i/e}u_{i/e}) = 0, \\ m_{i/e}\big(\partial_t(n_{i/e}u_{i/e}) + \mathrm{Div}_x((n_{i/e}u_{i/e}) \otimes u_{i/e})\big) + \nabla_x p_{i/e} \\ \qquad = -q_{i/e}n_{i/e}\nabla\Phi - kq_{i/e}n_e n_i(u_i - u_e), \\ \partial_t w_{i/e} + \mathrm{div}_x(w_{i/e}u_{i/e} + p_{i/e}u_{i/e}) \\ \qquad = -q_{i/e}n_{i/e}\nabla_x\Phi \cdot u_{i/e} - kq_{i/e}n_e n_i(u_i - u_e)\cdot u_{i/e} \\ \qquad\qquad -\kappa q_{i/e}n_e n_i(\Theta_i - \Theta_e) \end{cases} \tag{2.15}$$

with $w_{i/e} = \frac{m_{i/e}}{2}n_{i/e}|u_{i/e}|^2 + \frac{3}{2}n_{i/e}\Theta_{i/e}$. Here we denote by div the standard divergence of a vector and by Div the divergence of a matrix.

In the right hand side, the term $kq_{i/e}n_e n_i(u_i - u_e)$ is a drag force associated to the momentum exchanges between the two species, due to the ion-electron collisions. Similarly $\kappa q_{i/e}n_e n_i(\Theta_i - \Theta_e)$ represents the energy exchanges due to the ion-electron collisions. We refer on this derivation to classical textbooks in plasmas physics [2, 10, 13]. The equation is completed by the perfect gas law

$$p_{i/e} = n_{i/e}\Theta_{i/e}.$$

The force field is still given by the Poisson equation

$$-\varepsilon_0\Delta_x\Phi = q(n_i - n_e) \tag{2.16}$$

endowed with the boundary conditions

$$\begin{aligned} &\int_{\Gamma_{c/v}} \big[\partial_t(-\epsilon_0\partial_\nu\Phi) + J_{ext}\cdot\nu\big]\,\mathrm{d}\gamma \\ &\qquad + \int_{\Gamma_{c/d}} \big[\partial_t(-\epsilon_k\frac{\phi_{abs}-\Phi}{d_k}) - \sigma_k\frac{\phi_{abs}-\Phi}{d_k}\big]\,\mathrm{d}\gamma = 0, \\ &\partial_t\big(\epsilon_k\frac{\phi_{abs}-\Phi}{d_k} - \epsilon_0\partial_\nu\Phi\big) + \sigma_k\frac{\phi_{abs}-\Phi}{d_k} + J_{ext}\cdot\nu = 0 \quad \text{on } \Gamma_{d/v}, \\ &\Phi(t,x) = \phi_{abs}(t) \text{ on } \Gamma_{c/v}, \\ &\lim_{\|x\|\to+\infty} \Phi(t,x) = 0, \end{aligned} \tag{2.17}$$

with

$$J_{ext} = q(n_i u_i - n_e u_e) + J_S,$$

where J_S describes the possible emission current of particles from the boundary.

The derivation of relevant boundary conditions for the macroscopic quantities $(n_{i/e}, u_{i/e}, \Theta_{i/e})$ is an issue. The difficulty is two-fold:

- On the one hand, we deal with an hyperbolic system so that we should prescribe only the incoming fields. We refer to [12] for a deep discussion on this aspect.
- On the other hand, the Maxwellian state is usually not compatible with the kinetic boundary condition (2.9). Hence a kinetic boundary layer, the so-called Knudsen layer, should be taken into account, see [17, 25], or for a more practical viewpoint [11]. Remark that a conservative boundary condition such that $J_{ext} \cdot \nu = 0$, for instance with full reflection $\alpha = 1$ and no source $S = 0$ in (2.9), has no interest for the charging phenomena; we refer to [1] for similar remarks.
This aspect of the problem is particularly relevant, but it belongs beyond the scope of the present paper.

Next, asymptotic considerations allow to derive a hierarchy of possible models. Indeed, for LEO regimes the following reasoning can be applied:

- the charging time can still be considered as small compared to the typical time scale of the fluid evolution. This leads to replace the evolution equation in (2.15) by their stationary version:

$$\begin{cases} \mathrm{div}_x(n_{i/e}u_{i/e}) = 0, \\ m_{i/e}\mathrm{Div}_x((n_{i/e}u_{i/e}) \otimes u_{i/e}) + \nabla p_{i/e} \\ \qquad\qquad = -q_{i/e}n_{i/e}\nabla_x\Phi - kq_{i/e}n_e n_i(u_i - u_e), \\ \mathrm{div}_x(w_{i/e}u_{i/e} + p_{i/e}u_{i/e}) \\ \qquad\qquad = -q_{i/e}n_{i/e}\nabla_x\Phi \cdot u_{i/e} - kq_{i/e}n_e n_i(u_i - u_e) \cdot u_{i/e} \\ \qquad\qquad\qquad -\kappa q_{i/e}n_e n_i(\Theta_i - \Theta_e) \end{cases} \tag{2.18}$$

  coupled to (2.16). Time appears as a parameter in these equations and the problem remains subject to time evolution through the boundary conditions (2.17).

- A further approximation comes by assuming that the ions/electrons temperatures depends only on the densities

$$\Theta_{i/e} = \Theta^0_{i/e}\, n_{i/e}^{\gamma_{i/e}-1}, \qquad \gamma_{i/e} \geq 1, \quad \Theta^0_{i/e} > 0,$$

  which leads to isentropic ($\gamma_{i/e} > 1$) or isothermal ($\gamma_{i/e} = 1$) models.

- Then the classical asymptotics $m_e/m_i \ll 1$ and the quasi-neutral regime where the Debye length is small compared to the characteristic length of the spacecraft make sense for this application. The situation differs completely from the GEO case: in GEO the Debye length is of order 10-100m, but it is of order of a few centimeters in LEO. A rigorous justification of these asymptotics is a very tough piece of analysis; we mention for instance to [18, 24, 27] for the treatment of some specific situations, including a complete description of the boundary layers, and further references on these topics.

3 A Simple 1D Model

In this Section we consider a one-dimensional caricature of the LEO charging problem. Despite its simplicity, this model is interesting since it allows to bring out certain mathematical difficulties and to evaluate easily the efficiency of numerical schemes. In this model the spacecraft is seen as a scatterer occupying the domain $\mathcal{O} = (-h_d, h_c)$ where $\mathcal{O}_1 = (-h_d, 0)$ is occupied by a dielectric material whereas $\mathcal{O}_0 = (0, h_c)$ is the conductor domain. The plasma fills the domain $\Omega = (-L - h_d, -h_d) \cup (h_c, L+h_c)$. Bearing in mind numerical purposes, we consider a bounded domain, characterized by $0 < L < \infty$, but L is thought of as a "large" quantity, far from the scatterer. We consider only the population of positive particles, described by the density $n \geq 0$ and current J. They obey the following stationary Euler equations:

$$\partial_x J = 0, \tag{3.1}$$

$$\partial_x \left(\frac{J^2}{n} + p(n) \right) = -\frac{q}{m_i} n \partial_x \Phi \tag{3.2}$$

for $x \in \Omega$, with the pressure function

$$p(n) = n^\gamma \qquad \gamma > 1.$$

In what follows, we assume the following Dirichlet boundary conditions for the density

$$n(t, -h_d) = n_0^l > 0, \qquad n(t, h_c) = n_0^r > 0, \tag{3.3}$$

$$n(t, L + h_c) = n(t, -L - h_d) = n_\infty > 0. \tag{3.4}$$

The potential Φ is required to satisfy the Poisson equation

$$-\varepsilon_0 \partial_{xx}^2 \Phi = q(n - C), \tag{3.5}$$

for $x \in \Omega$ where $C(x)$ is a given positive function describing the electrons background. The neutrality far from the spacecraft is guaranteed by $C(L + h_c) = C(-L - h_d) = n_\infty$. The potential verifies

$$\Phi(t, -L - h_d) = \Phi(t, L + h_c) = 0. \tag{3.6}$$

This set of equations can be roughly obtained from (2.18) by assuming $k = 0$, $\kappa = 0$ (no impulsion nor energy exchanges), $n_e = C = n_\infty$ is constant, $u_e = 0$ (hence there is no electron current) and $\nabla_x \Theta_e = q \nabla_x \Phi$ with the isentropic approximation for the ions. It remains to write the boundary conditions for the potential on $-h_d$ and h_c. For the sake of completeness, we give the main hints of the derivation, following [1]. The basis of the derivation consists in keeping in mind that the potential is actually defined on the whole domain $(-L - h_d, L + h_c)$ and that electrodynamics relations should be used in the scatterer. We introduce a reference potential Φ_{ref} defined by

$$\begin{cases} -\partial^2_{xx} \Phi_{ref} = 0, \\ \Phi_{ref}(-h_d) = \Phi_{ref}(h_c) = 1, \ \Phi_{ref}(L + h_c) = \Phi_{ref}(-L - h_d) = 0. \end{cases} \tag{3.7}$$

In the conductor domain, the potential is constant: $\Phi(t, x) = \phi_{abs}(t)$ for any $x \in (0, h_c)$ where the absolute potential ϕ_{abs} is a function of time to be determined. We denote J_{cond} the current in the conductor. We split $\Phi(t, x) = \phi_{abs}(t)\Phi_{ref}(x) + \Phi'(t, x)$, so that the differential potential Φ' verifies

$$\begin{cases} -\partial^2_{xx} \Phi'(t, x) = n - C \qquad \text{on } \Omega, \\ \Phi'(t, L + h_c) = \Phi'(t, -L - h_d) = 0, \\ \Phi'(t, h_c) = 0, \ \Phi'(t, -h_d) = \Phi(t, -h_d) - \phi_{abs}(t). \end{cases} \tag{3.8}$$

The boundary condition on the spacecraft will take the form of equations satisfied by $\phi_{abs}(t)$ and $\Phi'(t, -h_d)$. Note that for the spacecraft engineering application, the crucial quantity to be controlled is precisely the differential potential.

Since the dielectric layer is very thin, $h_d \ll h_c \ll L$, there is no volumic charge in the dielectric and the derivative of the potential in the dielectric can be approximated by the finite difference

$$\partial_x \Phi(t, -h_d) \simeq \frac{\phi_{abs}(t) - \Phi(t, -h_d)}{h_d}.$$

The runaway current in the dielectric domain is defined by

$$J_{diel} = -\sigma_d \frac{\phi_{abs}(t) - \Phi(t, -h_d)}{h_d},$$

σ_d being the conductivity of the dielectric. Therefore, the Ampère law yields the following relations

- At $x = -h_d$

$$\partial_t \left(\epsilon_0 \partial_x \Phi(t, -h_d) - \epsilon_d \frac{\phi_{abs}(t) - \Phi(t, -h_d)}{h_d} \right) = J(t, -h_d) + \sigma_d \frac{\phi_{abs}(t) - \Phi(t, -h_d)}{h_d}. \tag{3.9}$$

- At $x = 0$

$$\partial_t \left(\epsilon_d \frac{\phi_{abs}(t) - \Phi(t, -h_d)}{h_d} \right) = -\sigma_d \frac{\phi_{abs}(t) - \Phi(t, -h_d)}{h_d} - J_{cond}. \tag{3.10}$$

- At $x = -h_c$

$$-\partial_t(\epsilon_0 \partial_x \Phi(t, h_c)) = J_{cond} - J(t, h_c). \tag{3.11}$$

Adding (3.10) and (3.11) leads to

$$\partial_t \left(\epsilon_d \frac{\phi_{abs}(t) - \Phi(t, -h_d)}{h_d} - \epsilon_0 \partial_x \Phi(t, h_c) \right) = -\sigma_d \frac{\phi_{abs}(t) - \Phi(t, -h_d)}{h_d} - J(t, h_c).$$

Combined with (3.9) it yields

$$\epsilon_0 \partial_t \partial_x \big(\Phi(t, -h_d) - \Phi(t, h_c) \big) = J(t, -h_d) - J(t, h_c) \tag{3.12}$$

Eq. (3.9) can also be recast as

$$\epsilon_0 \partial_t \Big((\partial_x \Phi'(t, -h_d) + \phi_{abs}(t) \partial_x \Phi_{ref}(-h_d) \Big) + \frac{\epsilon_d}{h_d} \partial_t \Phi'(t, -h_d) + \frac{\sigma_d}{h_d} \Phi'(t, -h_d) = J(t, -h_d). \tag{3.13}$$

The quantities $\Phi'(t, -h_d)$ and $\phi_{abs}(t)$ are entirely defined by (3.12) and (3.13).

Taking into account the scaling of the dielectric thickness $0 < h_d/h_c \ll 1$ the boundary relations become

$$J(t, -h_d) = J(t, h_c), \tag{3.14}$$

$$C_d \partial_t \big(\Phi(t, -h_d) - \phi_{abs}(t) \big) + S_d \big(\Phi(t, -h_d) - \phi_{abs}(t) \big) = J_{ext}(t, -h_d) \tag{3.15}$$

where $C_d = \epsilon_d/\epsilon_0$ and S_d are the dimensionless capacity and conductance of the dielectric respectively. Eventually, we recap the charging equations, written here in dimensionless form, as follows:

$$\begin{cases} \partial_x J = 0, \ \partial_x (J^2/n + p(n)) = -n \partial_x \Phi \\ -\lambda^2 \partial^2_{xx} \Phi = n - C, \end{cases} \tag{3.16}$$

hold on the domain $\Omega = \Omega^l \cup \Omega^r = (-L - h_d, -h_d) \cup (h_c, L + h_c)$, where λ is the ratio between the Debye length and the characteristic length, and with the boundary conditions

$$\begin{cases} n(t, -h_d) = n_0^l, \ n(t, h_c) = n_0^r \\ n(t, L + h_c) = n(t, -L - h_d) = n_\infty, \\ \Phi(t, L + h_c) = \Phi(t, -L - h_d) = 0, \end{cases} \tag{3.17}$$

together with (3.14) and(3.15). Hence we deduce that $J = J(t)$ is actually constant on the whole set Ω. Next, combining the momentum equation and the Poisson equation we get

$$\begin{aligned}
&-\partial_{xx}^2 \Phi = \frac{1}{\lambda^2}(n - C) \\
&= \partial_x \Big(\frac{1}{n} \partial_x (J^2/n + p(n)) \Big) = \partial_x \big(F_J'(n) \partial_x n \big)
\end{aligned}$$

with $F_J'(n) = -J^2/n^3 + p'(n)/n$. Therefore the density verifies the following second order equation

$$\begin{cases}
-\partial_{xx}^2 F_J(n) + \dfrac{1}{\lambda^2}\Big(n - C\Big) = 0 \qquad \text{on } \Omega \\
F_J(n) = \dfrac{J^2}{2n^2} + h(n) \\
h(n) = \displaystyle\int_1^n \frac{p'(y)}{y} dy = \frac{\gamma}{\gamma - 1} \Big(n^{\gamma-1} - 1\Big),
\end{cases} \tag{3.18}$$

endowed with Dirichlet boundary conditions.

We can also show that J is solution of a simple ODE. Indeed, we have

$$-\partial_x \Phi = \frac{-J^2/n^2 + p'(n)}{n} \partial_x n = \partial_x F_J(n). \tag{3.19}$$

Integrating this relation and using $\Phi(t, -L - h_d) = \Phi(t, L + h_c) = 0$, we obtain

$$\begin{cases}
\phi_{abs}(t) = F_J(n_\infty) - F_J(n_0^r), \\
\Phi(t, -h_d) = F_J(n_\infty) - F_J(n_0^l).
\end{cases}$$

Obviously if $n_0^r = n_0^l$ we get $\phi_{abs}(t) = \Phi(t, -h_d)$ for any $t \geq 0$ and (3.15) implies that there is no current at all: $J = 0$. From now on we suppose $n_0^r \neq n_0^l$. Hence the differential equation (3.15) becomes

$$\begin{aligned}
&\partial_t \Big(\frac{\gamma}{\gamma - 1} \big((n_0^r)^{\gamma-1} - (n_0^l)^{\gamma-1} \big) + \frac{J^2}{2} \Big(\frac{1}{(n_0^r)^2} - \frac{1}{(n_0^l)^2} \Big) \Big) \\
&+ \frac{S_d}{C_d} \Big(\frac{\gamma}{\gamma - 1} \big((n_0^r)^{\gamma-1} - (n_0^l)^{\gamma-1} \big) + \frac{J^2}{2} \Big(\frac{1}{(n_0^r)^2} - \frac{1}{(n_0^l)^2} \Big) \Big) = \frac{J}{C_d}
\end{aligned}$$

which, as soon as $J(t) \neq 0$, can be recast as

$$J'(t) + \frac{S_d}{C_d} \frac{J(t)}{2} = \frac{s}{J(t)} + \beta, \tag{3.20}$$

with

$$s = \frac{S_d}{C_d} \frac{\gamma}{\gamma - 1} \frac{(n_0^l)^{\gamma-1} - (n_0^r)^{\gamma-1}}{(n_0^l)^2 - (n_0^r)^2} \; (n_0^r)^2 (n_0^l)^2,$$

and

$$\beta = \frac{1}{C_d}\frac{(n_0^r)^2(n_0^l)^2}{(n_0^l)^2-(n_0^r)^2}.$$

We observe that the equation admits two stationary solutions

$$J_1 = \frac{C_d}{S_d}\big(\beta+\sqrt{\beta^2+2sS_d/C_d}\big) > 0,\ J_2 = \frac{C_d}{S_d}\big(\beta-\sqrt{\beta^2+2sS_d/C_d}\big) < 0$$

4 Analysis of the One-Dimensional Problem

According to the previous manipulations, the evolution of the current decouples from the density variations. In turn, there is no difficulty in analyzing the current equation and we obtain the following statement.

Proposition 4.1. *Let $n_0^r, n_0^l > 0$, $n_0^r \neq n_0^l$ and let J_{Init} be the initial current. Then, the equation* (3.20) *has a unique global solution. Furthermore, the solution has the following behavior*

- *if $J_{Init} > J_1$ then $J(t)$ is a positive non increasing function which converges to J_1 as t goes to ∞,*
- *if $0 < J_{Init} < J_1$ then $J(t)$ is a positive non decreasing function which converges to J_1 as t goes to ∞,*
- *if $J_{Init} < J_2$ then $J(t)$ is a negative non decreasing function which converges to J_2 as t goes to ∞,*
- *if $J_2 < J_{Init} < 0$ then $J(t)$ is a negative non increasing function which converges to J_2 as t goes to ∞.*

Therefore, the density $n(t,x)$ is determined by (3.18), which is parametrized by the time variable, via the definition of the current $J(t)$ by (3.20). Nevertheless, while J is globally defined, this is not enough to ensure the well-posedness of (3.18) due to possible change of type of the equation.

Definition 4.2. When the pair (n,J) is such that $F_J'(n) > 0$, we say that the regime is subsonic; when the pair (n,J) is such that $F_J'(n) < 0$, we say that the regime is supersonic.

We are able to justify the existence of solutions, as far as the estimates guarantees that we remain in the subsonic region, so that (3.18) is a nonlinear elliptic equation.

Theorem 4.3. *(Existence, uniqueness and regularity of subsonic solutions) Let n_0^r, n_0^l and n_∞ be positive. We set $\underline{n} = \min(n_0^r, n_0^l, n_\infty, \min C)$ and $\overline{n} = \max(n_0^r, n_0^l, n_\infty, \max C)$. We set*

$$J_{crit} = \underline{n}\ \sqrt{\gamma \underline{n}^{\gamma-1}}. \tag{4.1}$$

Then for any $|J_{Init}| \leq J_{crit}$ there exists a time $T_\star$ and a unique solution (n, Φ) of (3.18), (3.17) defined on $[0, T_\star]$. The solution lies in $C^1([0, T_\star]; C^2(\overline{\Omega}))$ and it verifies $\underline{n} \leq n(t, x) \leq \overline{n}$. If the data are such that $0 < J_{Init} < J_1 \leq J_{crit}$ or $0 < J_1 \leq J_{Init} < J_{crit}$ (resp. $-J_{crit} < J_2 \leq J_{Init} < 0$ or $-J_{crit} \leq J_{Init} < J_2 < 0$), then the solution is globally defined.

We plot on figure 4.1 the phase portrait of the current J which summarizes the different situations described in Proposition 4.1 and Theorem 4.3. We depict the subsonic and supersonic regions respectively by white and grey colored areas. Values of J_1 and J_2 are 0.8035 and -4.0250, whereas $J_{crit} = 1.1832$. The trajectories converge very fast to J_1 for positive current, and the contrary is observed for J_2. This situation is exchanged if we switch n_0^l and n_0^r.

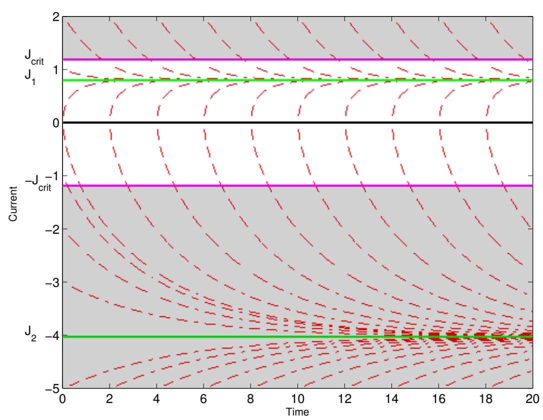

Figure 4.1: Phase portrait of current J for $n_0^l = 1.1$, $n_0^r = 1.9$, $n_\infty = 1$, $C = 1$, $\gamma = 1.4$, $S_d = 1.13$ and $C_d = 3$

The proof of Theorem 4.3 follows the lines of [14] and it is based on a suitable fixed point method. Indeed, we show that the mapping $\mathcal{T} : n \mapsto \tilde{n}$ defined by

$$\tilde{n} - \lambda^2 \partial_x (F_J'(n) \partial_x \tilde{n}) = C$$

endowed with the Dirichlet boundary conditions (3.17) has a unique fixed point. The proof uses the regularizing effect of elliptic equations. Hence it works as soon as the regime is subsonic, which leads to the condition (4.1) on the current, see [14]. Coming back to (3.20), we can exhibit conditions on the data (that is on n_0^l, n_0^r, n_∞) such that the current $J(t)$ remains in the interval $0 < J(t) < J_1 < J_{crit}$ for any time $t \geq 0$, and therefore the solution of the whole problem is globally defined.

According to [23], we guess that we can exhibit some $J^{crit} > J_{crit}$ such that if the initial current is large enough $|J_{Init}| \geq J^{crit}$ then, we remain in a supersonic case and we can also show the existence-uniqueness of a smooth solution. The proof is much more delicate since we do not have in the supersonic case a so nice elliptic structure and helpful estimates (like in particular the maximum principle) are not easily available. The analysis of the possible change of type and transonic regimes would be very interesting and challenging; we refer to [15, 16] for results in this direction.

5 Numerical Simulation of the One-Dimensional Problem

We investigate numerically the following system

$$J'(t) + \frac{S_d}{C_d}\frac{J(t)}{2} = \frac{s}{J(t)} + \beta, \qquad t \in [0,T], \tag{5.1}$$

$$-\partial^2_{xx} F_J(n) + \frac{1}{\lambda^2}\Big(n - C\Big) = 0,\ (x,t) \in \Omega \times [0,T] \tag{5.2}$$

$$-\partial^2_{xx}\Phi = \frac{1}{\lambda^2}\Big(n - C\Big),\ (x,t) \in \Omega \times [0,T] \tag{5.3}$$

$$n(t,-h_d) = n_0^l, \quad n(t,h_c) = n_0^r, \qquad t \in [0,T], \tag{5.4}$$

$$n(t, L + h_c) = n(t, -L - h_d) = n_\infty, \qquad t \in [0,T], \tag{5.5}$$

$$\Phi(t,h_c) = \phi_{abs}(t) = F_J(n_\infty) - F_J(n_0^r), \qquad t \in [0,T], \tag{5.6}$$

$$\Phi(t,-h_d) = F_J(n_\infty) - F_J(n_0^l), \qquad t \in [0,T], \tag{5.7}$$

$$\Phi(t, L + h_c) = \Phi(t, -L - h_d) = 0, \qquad t \in [0,T]. \tag{5.8}$$

with $F_J(n) = \dfrac{J^2}{2n^2} + h(n)$ and $h(n) = \displaystyle\int_1^n \frac{p'(y)}{y}dy = \frac{\gamma}{\gamma-1}\Big(n^{\gamma-1} - 1\Big)$.

We solve the current equation (5.1) for the variable $y(t) = J(t)^2$, the equation being transformed as

$$y'(t) + \frac{S_d}{C_d}y(t) = 2\Big(s \pm \beta\sqrt{y(t)}\Big). \tag{5.9}$$

The choice of the sign in the r.h.s of (5.9) is determined by the sign of the initial datum J_{Init} since the sign of J remains constant in time. Therefore, $J(t) = \pm\sqrt{y(t)}$. The equation (5.9) is solved once for all by a standard Runge-Kutta scheme.

Then, knowing the current J^k, approximation of $J(k\Delta t)$, we approach (5.2) with a basic finite difference scheme

$$n_j^k - \frac{\lambda^2}{\Delta x}\left(F'_{J^k}(n_{j+1}^k)\frac{n_{j+1}^k - n_j^k}{\Delta x} - F'_{J^k}(n_j^k)\frac{n_j^k - n_{j-1}^k}{\Delta x}\right) = C_j. \quad (5.10)$$

The nonlinear equation (5.10) is solved by a Newton algorithm. The elliptic Poisson equation (5.3) is also solved by classical finite difference scheme. Although seeming stationnary, those equations depends on time by their boundary condition (5.4)-(5.8).

The simulation reveals the threshold effect in the choice of the initial current: for a small enough J_{Init} the scheme works well and reproduce a smooth density profile, as expected. But, starting with a larger initial current, singularity might appear characterized by the non invertibility of the linear systems involved in the resolution of (5.10). To emphasize this point we make the following experiment with $-h_d = h_c = 0$ and $L = 1$ for the domain Ω. We consider $\gamma = 1.4$, $S_d = 1.13$, $C_d = 3$, $n_\infty = 1$, $n_0^l = 1.1$, $n_0^r = 1.9$ and $C = 1$. In this case we recall that the critical current is $J_{crit} = 1.1832$. In figures 5.1, 5.2, 5.3 we take $J_{Init} = 1.15$ such that $J_{Init} < J_{crit}$ and we are in the subsonic case. Here the current is a smooth decreasing function of time. With the same values of parameters, taking $J_{Init} = 1.2$, singularities appear directly from the begining. If $J_{Init} = -0.5$, we also observe a problem when $J(t)$ crosses the value of $-J_{crit}$ and singularities appear.

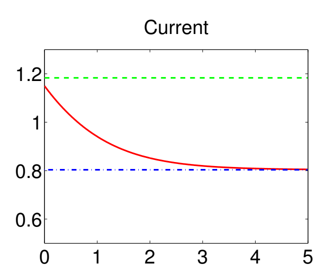

Figure 5.1: Evolution of current (line at top corresponds to the value of J_{crit})

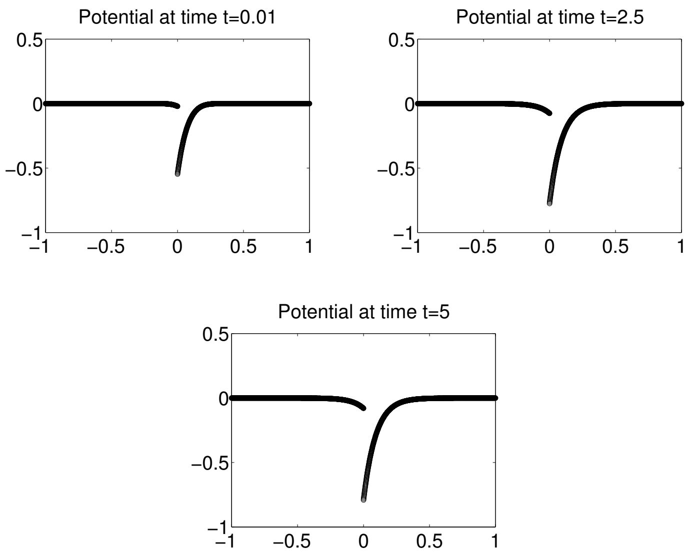

Figure 5.2: Evolution of potential

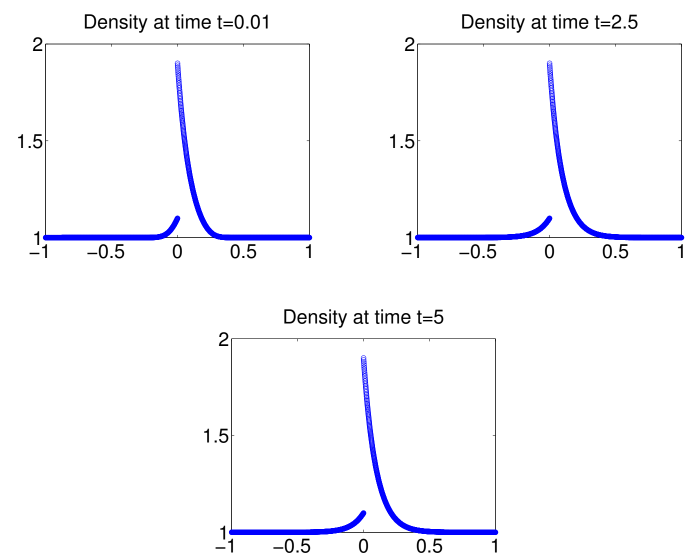

Figure 5.3: Evolution of density

As a final comment, it is worth having in mind that in the rescaled problem (3.16)-(3.17) the Debye length might be small compared to the characteristic length scale. Hence, in LEO environment we usually have $0 < \lambda \ll 1$. It leads to the formation of boundary layers. Indeed, let us set $C = n_\infty = 1$. Writing the equation for $\lambda = 0$ we simply get

$$\partial_x j = 0, \quad F_j'(n)\partial_x n = \partial_x \Phi, \quad n = 1.$$

Taking into account the condition at infinity (or at the far end boundary $x = L + h_c$ or $-L - h_d$), the solution reads

$$\overline{n} = 1, \quad \overline{j} = j(t), \quad \overline{\Phi} = 0.$$

However, this solution does not verify the boundary condition at $x = h_c$ nor $x = -h_d$. Let us expand the solution of (3.16)-(3.17) as follows

$$j = \overline{j} + \widetilde{j}(x/\lambda) + \lambda\widehat{j}, \quad \Phi = \overline{\Phi} + \widetilde{\Phi}(x/\lambda) + \lambda\widehat{\Phi}, \quad n = 1 + \widetilde{n}(x/\lambda) + \lambda\widehat{n}.$$

At leading order we obtain the following relations satisfied by the boundary correctors:

$$\begin{cases} \dfrac{1}{\lambda}\partial_y \widetilde{j}(x/\lambda) = 0, \\ \dfrac{\gamma - \overline{j}^2}{\lambda}\partial_y \widetilde{n} = \dfrac{1}{\lambda}\partial_y \widetilde{\Phi}, \\ -\partial_{yy}^2 \widetilde{\Phi} = \widetilde{n}. \end{cases}$$

The equation is completed by the boundary condition matching the data to the solution corresponding to $\lambda = 0$, that is

$$\begin{aligned} &\widetilde{\Phi}(y = 0) = \phi_{abs}(t) = F_{J(t)}(n_\infty) - F_{J(t)}(n_0^r) \\ &\qquad\qquad \text{or } \Phi(t, -h_d) = F_{J(t)}(n_\infty) - F_{J(t)}(n_0^l), \\ &\widetilde{\Phi}(y \to \infty) = 0, \\ &\widetilde{n}(y = 0) = n_0^r \text{ or } n_0^l, \\ &\widetilde{n}(y \to \infty) = 1. \end{aligned}$$

The numerical treatment of this kind of asymptotic problem leads to severe stiff problems, which require a specific treatment. A deep understanding of the boundary layer formation and of the scale separation helps in designing an efficient numerical scheme, as in [26].

6 Conclusions

Considering LEO environment instead of GEO, it can be tempting to describe spacecraft charge phenomena by using hydrodynamic models, at least as a first approximation. Such models are indeed less complicated than a full kinetic description of the plasma and can be treated for a reduced numerical cost. The underlying Euler equations are thus coupled to the Poisson equation for the electric potential, with complex and non standard boundary conditions. These boundary conditions for the potential, which take into account that different places on the spacecraft surface can have a different electrical behavior, are at the origin of the charging phenomena. We point out several difficulties related to the hydrodynamic modeling:

- A crucial issue concerns the boundary condition to be satisfied by the hydrodynamic unknowns. A convincing derivation should certainly goes back to the kinetic model and the hydrodynamic limit through a fine analysis of the kinetic boundary layer.
- Due to the time evolution through the boundary condition, change of type of the flow can occur. Such passage from subsonic to supersonic regimes make the mathematical analysis difficult and might lead to breakdown of the numerical methods. This is illustrated on a simple one dimensional caricature model.
- Eventually, a careful discussion of the various scales involved in the equations is necessary. The multiscale features of the problem definitely make it challenging for numerical simulations which require the design of refined and dedicated schemes.

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