FIRST-ORDER ENTROPIES FOR THE DERRIDA-LEBOWITZ-SPEER-SPOHN EQUATION

ANSGAR JÜNGEL AND INGRID VIOLET

ABSTRACT. A logarithmic fourth-order parabolic equation in one space dimension with periodic boundary conditions is analyzed. Using a new semi-discrete approximation in time, a first-order entropy–entropy dissipation inequality is proved. Passing to the limit of vanishing time discretization parameter, some regularity results are deduced. Moreover, it is shown that the solution is strictly positive for large time if it does so initially.

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1. Introduction

Nonlinear fourth-order equations, whose solution signifies some nonnegative physical quantity, have attracted the attention of mathematicians since several years. The lack of a maximum principle for those equations makes necessary the development of new analytical tools in order to obtain a priori estimates and the nonnegativity or positivity of solutions. A famous example is the thin-film equation which is of degenerate parabolic type (see, e.g. [2, 3]). Another example is the nonlinear logarithmic equation

$$u_t + (u(\log u)_{xx})_{xx} = 0 \quad \text{for } x \in \mathbb{T},\ t > 0, \quad u(\cdot, 0) = u_0, \tag{1}$$

where $\mathbb{T}$ is the circle parametrized by a variable x satisfying $0 \leq x \leq 1$. This equation has been first derived by Derrida, Lebowitz, Speer, and Spohn [8, 9], and we shall therefore refer to (1) as the *Derrida-Lebowitz-Speer-Spohn equation* or simply the *DLSS equation*. Derrida et al. studied in [8, 9] interface fluctuations in a two-dimensional spin system, the so-called (time-discrete) Toom model. In a suitable scaling limit, a random variable u related to the deviation of the interface from a straight line satisfies the one-dimensional equation (1). This equation also appears in quantum semiconductor modeling as the zero-temperature, zero-field limit of the quantum drift-diffusion model [1, 7].

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The variable u describes the electron density in a microelectronic device or in a quantum plasma. In both applications, u is a nonnegative variable.

The first analytical result for (1) has been shown in [4]. There, the existence of *positive* solutions with H^1 initial data has been proved. Lacking suitable a priori estimates, the existence result holds only locally in time. The global (in time) existence of solutions was related to strict positivity: if a classical solution breaks down at a certain time t^*, then the limit profile $\lim_{t\nearrow t^*} u(x,t)$ is still an H^1 function but vanishes at some point in $\mathbb{T}$. This motivated the authors in [16] to study *nonnegative* weak solutions instead of positive classical solutions. The global existence of solutions was shown with initial data having finite generalized entropy $\int_{\mathbb{T}}(u_0 - \log u_0)dx$ and with physically motivated boundary conditions. The global existence of weak solutions to the DLSS equation (1) with periodic boundary conditions was proved in [10]. Equation (1) has been also considered with non-homogeneous boundary conditions [12], and the exponential fast decay of its solutions was shown [5, 6, 10, 12, 14, 17]. For results on the existence of solutions to the corresponding multi-dimensional DLSS equation, we refer to [11, 15].

In the paper [14], the following so-called entropy–entropy dissipation inequality has been formally derived:

$$\frac{d}{dt}\int_{\mathbb{T}}(u^{\alpha/2})_x^2 dx + \mu\int_{\mathbb{T}}\big((u^{\alpha/2})_{xxx}^2 + (u^{\alpha/6})_x^6\big)dx \le 0, \tag{2}$$

where α lies in between the two roots of $53\alpha^2 - 100\alpha + 20$, i.e. $\frac{2}{53}(25-6\sqrt{10}) < \alpha < \frac{2}{53}(25+6\sqrt{10})$, and $\mu > 0$ is some constant depending on α. The integral $\int_{\mathbb{T}}(u^{\alpha/2})_x^2 dx$ is called a first-order entropy, whereas we refer to the other integral in (2) as the corresponding entropy dissipation term. The derivation in [14] is only formal, since the manipulations require positive smooth solutions. The idea was to reformulate the necessary integration by parts leading to (2) as a decision problem for polynomial systems which can be solved by a computer algebra system.

The goal of this paper is to make the inequality (2) rigorous, to conclude some regularity properties, and to show that the solution stays positive at least for large time if it does so initially. For small time, the positivity of solutions has been already shown in [4]. We are not interested in proving regularity for the solution u itself, since it is well known that u is a classical solution to (1), at least locally in time and for positive solutions, if the initial data is strictly positive and lies in $H^1(\mathbb{T})$ [4]. Here, we are rather interested in showing that the powers of the nonnegative solution $u^{\alpha/2}$ are regular.

For $\alpha = 1$, inequality (2) has been justified in [10]. The idea was to discretize (1) in time and to consider the elliptic problem

$$\frac{1}{\tau}(u_k - u_{k-1}) + \big(u_k(\log u_k)_{xx}\big)_{xx} = 0, \tag{3}$$

where $k \in \mathbb{N}$, $\tau > 0$ is the time step and u_k is the Euler approximation of $u(\cdot, \tau k)$. The advantage of this discretization is that it is possible to show that there exists a strictly positive smooth solution u_k to (3) which allows to make the manipulations leading to a discrete version of (2) with $\alpha = 1$ rigorous. In fact, multiplying (3) by $-u_k^{\alpha/2-1}(u_k^{\alpha/2})_{xx}$ and integrating over $\mathbb{T}$ gives

$$-\frac{1}{\tau}\int_{\mathbb{T}}(u_k-u_{k-1})u_k^{\alpha/2-1}(u_k^{\alpha/2})_{xx}dx - \int_{\mathbb{T}}\big(u_k(\log u_k)_{xx}\big)_{xx}u_k^{\alpha/2-1}(u_k^{\alpha/2})_{xx}dx = 0. \tag{4}$$

Since

$$-\int_{\mathbb{T}}(u_k-u_{k-1})u_k^{\alpha/2-1}(u_k^{\alpha/2})_{xx}dx \geq \int_{\mathbb{T}}\big((u_k^{\alpha/2})_x^2-(u_{k-1}^{\alpha/2})_x^2\big)dx \quad \text{if } \alpha=1, \tag{5}$$

and the second integral in (4) can be bounded from above by $\int_{\mathbb{T}}((\sqrt{u_k})_{xxx}^2 + (\sqrt[6]{u_k})_x^6)dx$ if $\alpha=1$ (see section 3), we obtain

$$\frac{1}{\tau}\int_{\mathbb{T}}\big((\sqrt{u_k})_x^2-(\sqrt{u_{k-1}})_x^2\big)dx + \mu\int_T\big((\sqrt{u_k})_{xxx}^2+(\sqrt[6]{u_k})_x^6\big)dx \leq 0.$$

The first integral is the discrete time derivative of $(\sqrt{u})_x^2$. Then the limit $\tau\to 0$ gives (2) for $\alpha=1$.

Unfortunately, this argument does not work for general α. The problem is that (5) cannot be proved for $\alpha\neq 1$. In order to circumvent this problem, we choose another semi-discretization in time: we rewrite the time derivative u_t as $u^{1-\alpha}(u^\alpha)_t/\alpha$ and consider the semi-discretization

$$\frac{1}{\alpha\tau}\big(u_k^\alpha-u_{k-1}^\alpha\big)+u_k^{\alpha-1}\big(u_k(\log u_k)_{xx}\big)_{xx}=0. \tag{6}$$

Then it is possible to show that a discrete entropy–entropy dissipation inequality holds leading to (2) in the limit $\tau\to 0$.

The semi-discretization (6) is new in this context. It is chosen in such a way that an inequality similar to (5) can be proved. The difficulty is to deal with the expression

$$u^{\alpha-1}\big(u(\log u)_{xx}\big)_{xx},$$

since we have no control on $u^{\alpha-1}$ whose exponent may be negative. Therefore, we rewrite this expression in terms of the variable $w=u^{\alpha/4}$ for which we can derive enough regularity. Hence, we define the weak solution in terms of $u^{\alpha/4}$ instead of u as it is done in [16], for instance. More precisely, we call u a weak solution to (1) if

$$(u^\alpha)_t\in L^{3/2}(0,T;H^{-2}(\mathbb{T})),\quad u\in L^\infty(\mathbb{T}\times(0,T)), \tag{7}$$

$$u^{\alpha/2},u^{\alpha/4}\in L^2(0,T;H^2(\mathbb{T})), \tag{8}$$

is satisfied and if the following equation for $w=u^{\alpha/4}$ holds in $L^1(0,T;H^{-2}(\mathbb{T}))$:

$$(w^4)_t = -2\Big(4-\frac{4}{\alpha}\Big)\big[ww_{xx}(w^2)_{xx}-4ww_x^2w_{xx}+4\big(ww_x(ww_{xx}-w_x^2)\big)_x\big] - 4\Big(4-\frac{4}{\alpha}\Big)\Big(3-\frac{4}{\alpha}\Big)w_x^2(ww_{xx}-w_x^2)-4\big(w^2(ww_{xx}-w_x^2)\big)_{xx}. \tag{9}$$

Notice that (7)-(8) implies that $w=u^{\alpha/4}\in L^4(0,T;W^{1,4}(\mathbb{T}))$, by the Gagliardo-Nirenberg inequality (see below). Thus the right-hand side of (9) is well defined in $L^1(0,T;H^{-2}(\mathbb{T}))$. It is not difficult to see that any positive smooth solution $u=w^{4/\alpha}$ also solves (1).

It is well known that it may be more convenient to formulate (1) in another variable than u. For instance, the existence of solutions to the multi-dimensional DLSS equation is shown for the variable $\sqrt{u}$ [11, 15].

Our first main result of this paper is contained in the following theorem.

Theorem 1. *Let* $0 < u_0 \in H^1(\mathbb{T})$*,* $T > 0$*, and* $\alpha \in (\alpha_0, 1]$*, where* $\alpha_0 = \frac{2}{53}(25-6\sqrt{10})$*. Then there exists a nonnegative weak solution* u *to* (1) *satisfying*

$$u^{\alpha/2} \in L^2(0,T;H^3(\mathbb{T})) \cap L^\infty(0,T;H^1(\mathbb{T})) \cap C^0([0,T];C^{0,1/2}(\mathbb{T})),$$
$$u^{\alpha/6} \in L^6(0,T;W^{1,6}(\mathbb{T})), \quad \log u \in L^2(0,T;H^2(\mathbb{T})),$$

the L^1 *norm of* u *is bounded by the* L^1 *norm of* u_0*, and there exists* $\beta_\alpha > 0$ *(independent of* u_0*) such that*

$$\|(u^{\alpha/2})_x\|_{L^2(\mathbb{T})} \le \|(u_0^{\alpha/2})_x\|_{L^2(\mathbb{T})} e^{-\beta_\alpha t}, \quad t > 0. \tag{10}$$

Notice that $\alpha_0 = 0.2274\ldots$ The lower bound on α comes from the entropy construction method of [14] which consists in reformulating the a priori estimates as a solution of a decision problem for polynomial systems. This decision problem is solved by the computer algebra system `Mathematica` which gives an exact solution. The decay estimate (10) generalizes Theorem 5.1 in [10]. Our second main result concerns the strict positivity of solutions.

Theorem 2. *Let* $0 < u_0 \in H^1(\mathbb{T})$ *and let* u *be the (continuous) solution constructed in Theorem 1 for* $\alpha = 1$*. Set* $M = \|(\sqrt{u_0})_x\|_{L^2(\mathbb{T})}$ *and* $\beta_1 = \pi^4(103 + \sqrt{214})/9 = 1273.12\ldots$ *Then the* L^1 *norm of* u *is conserved and*

$$u(x,t) > 0 \quad \text{for all } x \in \mathbb{T},\ t > (\log M)/\beta_1.$$

In particular, if $M < 1$ *then*

$$u(x,t) > 0 \quad \text{for all } x \in \mathbb{T},\ t \ge 0.$$

It is not known if Theorem 2 is optimal. In fact, there are nonnegative smooth solutions to (1) which vanish at some points in $\mathbb{T}$ if the initial data is only nonnegative [15]. Close to the points x_0 where $u(x_0,t) = 0$, the solution u seems to behave like $(x-x_0)^2$. Numerical experiments indicate that the solution stays strictly positive if the initial datum satisfies $u_0 > 0$ [4, 16]. However, we are not able to prove that $u(x,t) > 0$ for all $t > 0$ and for general initial data. Theorem 1 shows that if the solution vanishes at some point x_0, given a *positive* initial datum, then it behaves close to x_0 like $(x-x_0)^\gamma$ where $\gamma > 5/\alpha_0 = 21.9868\ldots$ This comes from the fact that $(u^{\alpha/2})^2_{xxx}$ with $u(x) \sim (x-x_0)^\gamma$ is integrable if and only if $2(\gamma\alpha/2 - 3) > -1$ or $\gamma > 5/\alpha$.

The idea of the proof of Theorem 2 is based on the decay estimate (10). Indeed, for large time t, the solution $u(x,t)$ is expected to be "close" to the homogeneous steady state $\int_{\mathbb{T}} u_0 dx$. Since this steady state is strictly positive, so does $u(x,t)$.

The paper is organized as follows. In section 2 we show the existence of classical solutions to (6). The discrete entropy inequality corresponding to (2) is proved in section 3. In order to pass to the limit $\tau \to 0$ in (6), we need further estimates on the discrete time derivative of u^α and on the spatial derivatives of $u^{\alpha/4}$ which are proved in section 4. The limit $\tau \to 0$ in (6) is performed in section 5. Finally, Theorem 2 is shown in section 6.

For convenience, we recall the Gagliardo-Nirenberg inequality [13, 19].

Lemma 3. *Let m and k be integers with $0 \le k \le m$, $0 \le \theta < 1$, $1 \le p$, q, $r \le \infty$, and let $f \in W^{m,q}(\mathbb{T}) \cap L^r(\mathbb{T})$. If both*

$$k - \frac{1}{p} \le \theta\Big(m - \frac{1}{q}\Big) + (1-\theta)\Big(-\frac{1}{r}\Big) \quad \text{and} \quad \frac{1}{p} \le \frac{\theta}{q} + \frac{1-\theta}{r}$$

holds then $f \in W^{k,p}(\mathbb{T})$, and there exists a constant $c > 0$ independent of f such that

$$\|f\|_{W^{k,p}(\mathbb{T})} \le c\|f\|^{\theta}_{W^{m,q}(\mathbb{T})}\|f\|^{1-\theta}_{L^r(\mathbb{T})}. \tag{11}$$

Moreover, if

$$k - \frac{1}{p} = \theta\Big(m - \frac{1}{q}\Big) + (1-\theta)\Big(-\frac{1}{r}\Big)$$

holds then

$$\|D^\alpha f\|_{L^p(\mathbb{T})} \le c\|f\|^{\theta}_{W^{m,q}(\mathbb{T})}\|f\|^{1-\theta}_{L^r(\mathbb{T})}, \tag{12}$$

where D^α is a differential operator of order $|\alpha| = k$, and $k/m \le \theta \le 1$ is allowed if $m - k - 1/q \notin \mathbb{N}_0$.

2. Existence of discrete solutions

In this section we solve the semi-discrete problem (6). For this, let $T > 0$ be the terminal time, $N \in \mathbb{N}$ the number of grid points, $\tau = T/N$ the time step, and let $t_k = k\tau$, $k = 0, \ldots, N$. Furthermore, let $\alpha > 0$. For $y_{k-1} \in H^1(\mathbb{T})$, we consider the following recursive sequence of equations,

$$\frac{1}{\alpha\tau} e^{(1-\alpha)y_k}\big(e^{\alpha y_k} - e^{\alpha y_{k-1}}\big) + \big(e^{y_k} y_{k,xx}\big)_{xx} = 0, \quad k \in \mathbb{N}, \tag{13}$$

and $y_0 = \log u_0$, where $u_0 \in H^1(\mathbb{T})$, $u_0 > 0$.

Lemma 4. *Given $y_{k-1} \in H^1(\mathbb{T})$, there exists a solution $y_k \in C^\infty(\mathbb{T})$ to* (13).

Proof. We employ ideas from [10]. Set $z = y_{k-1}$ and $y = y_k$. We consider first for given $\varepsilon > 0$ the elliptic fourth-order equation

$$(e^y y_{xx})_{xx} - \varepsilon y_{xx} + \varepsilon y = \frac{1}{\alpha\tau} e^{(1-\alpha)y}(e^{\alpha z} - e^{\alpha y}) \tag{14}$$

with periodic boundary conditions. In order to show the existence of solutions to this problem, we use the Leray-Schauder fixed-point theorem. Let $w \in H^1(\mathbb{T})$ and $\sigma \in [0,1]$ be given and consider

$$a(y,\phi) = F(\phi) \quad \text{for all } \phi \in H^2(\mathbb{T}), \tag{15}$$

where

$$a(y,\phi) = \int_{\mathbb{T}} (e^w y_{xx}\phi_{xx} + \varepsilon y_x \phi_x + \varepsilon y \phi)dx,$$
$$F(\phi) = \frac{\sigma}{\alpha\tau}\int_{\mathbb{T}} e^{(1-\alpha)w}(e^{\alpha z} - e^{\alpha w})\phi dx.$$

The bilinear form a is continuous and coercive on $H^2(\mathbb{T})$, i.e. $a(y,y)\geq c\|y\|^2_{H^2(\mathbb{T})}$ for some $c>0$. Furthermore, F is linear and continuous on $H^2(\mathbb{T})$. Thus, by Lax-Milgram's lemma, there exists a unique solution $y\in H^2(\mathbb{T})$ to (15). This defines a fixed-point operator $S: H^1(\mathbb{T})\times[0,1]\to H^1(\mathbb{T})$, $(w,\sigma)\mapsto y$. It holds $S(w,0)=0$ for all $w\in H^1(\mathbb{T})$. Moreover, the functional S is continuous and compact, observing that the embedding $H^2(\mathbb{T})\hookrightarrow H^1(\mathbb{T})$ is compact. We need to prove a uniform bound for all fixed points of $S(\cdot,\sigma)$. If this is shown, all conditions of Leray-Schauder's fixed-point theorem are satisfied, and the existence of a fixed point, i.e. of a weak solution, follows.

Let y be a fixed point of $S(\cdot,\sigma)$, i.e., $y\in H^2(\mathbb{T})$ solves for all $\phi\in H^2(\mathbb{T})$

$$\int_{\mathbb{T}}(e^y y_{xx}\phi_{xx}+\varepsilon y_x\phi_x+\varepsilon y\phi)dx=\frac{\sigma}{\alpha\tau}\int_{\mathbb{T}}e^{(1-\alpha)y}(e^{\alpha z}-e^{\alpha y})\phi dx.$$

With the test function $\phi=1-e^{-y}$ we obtain

$$\begin{aligned}\int_{\mathbb{T}}y_{xx}^2dx-&\int_{\mathbb{T}}y_{xx}y_x^2dx+\varepsilon\int_{\mathbb{T}}e^{-y}y_x^2dx+\varepsilon\int_{\mathbb{T}}y(1-e^{-y})dx\\ &=\frac{\sigma}{\alpha\tau}\int_{\mathbb{T}}e^{(1-\alpha)y}(e^{\alpha z}-e^{\alpha y})(1-e^{-y})dx.\end{aligned}$$

(Here and in the following, the expression y_x^2 means $(y_x)^2$, compared to $(y^2)_x=2yy_x$. Expressions like y_{xx}^2 etc. are understood in a similar way.) The second term on the left-hand side vanishes since $\int y_{xx}y_x^2dx=\int y_x^3dx/3=0$. The third and fourth terms on the left-hand side are nonnegative. We employ the elementary inequalities $1-e^x\leq -x$ for all $x\in\mathbb{R}$ and $\xi^\alpha-1\leq\alpha(\xi-1)$ for all $\xi\geq 0$ and $\alpha\in(0,1]$. Then, for $x=\alpha(z-y)$ and $\xi=e^{z-y}$,

$$1-e^{\alpha(z-y)}\leq-\alpha(z-y),\quad e^y(e^{\alpha(z-y)}-1)\leq\alpha(e^z-e^y),$$

and hence,

$$\begin{aligned}\frac{\sigma}{\alpha\tau}&\int_{\mathbb{T}}e^{(1-\alpha)y}(e^{\alpha z}-e^{\alpha y})(1-e^{-y})dx\\ &=\frac{\sigma}{\alpha\tau}\int_{\mathbb{T}}\big[e^y(e^{\alpha(z-y)}-1)+(1-e^{\alpha(z-y)})\big]dx\\ &\leq\frac{\sigma}{\tau}\int_{\mathbb{T}}\big[(e^z-z)-(e^y-y)\big]dx.\end{aligned}$$

Here, we need the condition $\alpha\leq 1$. Thus, we conclude that

$$\frac{\sigma}{\tau}\int_{\mathbb{T}}(e^y-y)dx+\int_{\mathbb{T}}y_{xx}^2dx\leq\frac{\sigma}{\tau}\int_{\mathbb{T}}(e^z-z)dx. \tag{16}$$

The elementary inequality $e^y-y\geq|y|$ shows that y is bounded in L^1. Then we can apply the Poincaré inequality to infer that y is uniformly bounded in $H^2(\mathbb{T})$, independently of σ. The Leray-Schauder fixed-point theorem ensures the existence of a solution $y=y_\varepsilon\in H^2(\mathbb{T})$ to (14).

Next, we perform the limit $\varepsilon\to 0$. Estimate (16) also provides a bound for y_ε in $H^2(\mathbb{T})$ independently of ε. Hence, y_ε converges, up to a subsequence which is not relabeled, to y weakly in $H^2(\mathbb{T})$ and strongly in $H^1(\mathbb{T})$. This is enough to perform the limit $\varepsilon\to 0$ in (14), showing that y is a solution to (13).

It remains to prove that $y \in C^\infty(\mathbb{T})$. For this, let $u = e^y$. Then u is strictly positive in $\mathbb{T}$ and solves, in the distributional sense,

$$u_{xxxx} = \Big(\frac{2u_{xx}u_x}{u} - \frac{u_x^3}{u^2}\Big)_x - \frac{1}{\alpha\tau}u^{1-\alpha}(u - e^{\alpha y_{k-1}}). \tag{17}$$

Since $y \in H^2(\mathbb{T}) \hookrightarrow W^{1,\infty}(\mathbb{T})$, we have $u \in H^2(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T})$ and, by the strict positivity of u, also $1/u \in L^\infty(\mathbb{T})$. This shows that $u_{xx}u_x/u \in L^2(\mathbb{T})$ and $u_x^3/u^2 \in L^\infty(\mathbb{T})$. From (17) follows that $u_{xxxx} \in H^{-1}(\mathbb{T})$ and then $u \in H^3(\mathbb{T})$. Using this regularity and taking into account (17), it follows $u_{xxxx} \in L^2(\mathbb{T})$ and thus $u \in H^4(\mathbb{T})$. By bootstrapping, we find that $u \in H^n(\mathbb{T})$ for all $n \in \mathbb{N}$, which shows that $u \in C^\infty(\mathbb{T})$. Since u is strictly positive, $y = \log u \in C^\infty(\mathbb{T})$, completing the proof of Lemma 4. □

Let $y^{(N)}(x,t) = y_k(x)$ for $x \in \mathbb{T}$ and $t \in (t_{k-1}, t_k]$. For notational convenience, we also set $u_N = \exp(y^{(N)})$. Then the proof of Lemma 4 shows the following estimate.

Lemma 5. *There exists a constant $c > 0$ independent of N such that*

$$\|y^{(N)}\|_{L^\infty(0,T;L^1(\mathbb{T}))} + \|u_N\|_{L^\infty(0,T;L^1(\mathbb{T}))} + \|y^{(N)}\|_{L^2(0,T;H^2(\mathbb{T}))} \le c. \tag{18}$$

Proof. From (16) follows that $y^{(N)}$ and u_N are uniformly bounded in $L^\infty(0,T;$ $L^1(\mathbb{T}))$ and that $y^{(N)}_{xx}$ is uniformly bounded in $L^2(0,T;L^2(\mathbb{T}))$. Then, by Poincaré's inequality, for some constant $c > 0$,

$$\Big\|y_k - \int_{\mathbb{T}} y_k dx\Big\|_{L^2(\mathbb{T})} \le c\|y_{k,x}\|_{L^2(\mathbb{T})} \le c^2\|y_{k,xx}\|_{L^2(\mathbb{T})}, \tag{19}$$

we conclude that $y^{(N)}$ is uniformly bounded in $L^2(0,T;H^2(\mathbb{T}))$. □

3. The first-order entropy estimate

In this section, we derive the first-order entropy estimate for the solution e^{y_k} to (13). Recall that $y^{(N)}(\cdot,t) = y_k$ for $t \in (t_{k-1}, t_k]$ and $u_N = \exp(y^{(N)})$.

Lemma 6. *Let u_N be a classical solution to* (6) *corresponding to $\alpha_0 < \alpha < \alpha_1$, where $\alpha_{0/1} = \frac{2}{53}(25 \pm 6\sqrt{10})$. Then there exists a constant $c > 0$ independent of N such that*

$$\|u_N^{\alpha/2}\|_{L^\infty(0,T;H^1(\mathbb{T}))} + \|u_N^{\alpha/2}\|_{L^2(0,T;H^3(\mathbb{T}))} + \|u_N^{\alpha/6}\|_{L^6(0,T;W^{1,6}(\mathbb{T}))} \le c. \tag{20}$$

We remark that we have proved the existence of a solution u_k to (6) only for $\alpha \le 1$. However, the first-order entropy estimate is valid for a larger class of values α.

Proof. Let, by a slight abuse of notation, $u_k = e^{y_k}$. Then the lemma is a consequence of the following entropy–entropy dissipation inequality,

$$\int_{\mathbb{T}} (u_k^{\alpha/2})_x^2 dx + \alpha\mu \sum_{j=1}^k \tau \int_{\mathbb{T}} \big((u_j^{\alpha/2})_{xxx}^2 + (u_j^{\alpha/6})_x^6\big)dx \le \int_{\mathbb{T}} (u_0^{\alpha/2})_x^2 dx, \tag{21}$$

and Poincaré's inequality (19), where $\mu > 0$ is some constant. In order to show this inequality, we multiply (13) by a test function, integrate over $\mathbb{T}$ and integrate by parts. The integration by parts is done in a systematic way as described in [14]. Multiplication of (13) by $-u_k^{\alpha/2-1}(u_k^{\alpha/2})_{xx}$ and integration over $\mathbb{T}$ gives

$$-\frac{1}{\alpha\tau}\int_{\mathbb{T}} u_k^{1-\alpha}(u_k^\alpha - u_{k-1}^\alpha)u_k^{\alpha/2-1}(u_k^{\alpha/2})_{xx}dx - \int_{\mathbb{T}}(u_k(\log u_k)_{xx})_{xx}u_k^{\alpha/2-1}(u_k^{\alpha/2})_{xx}dx = 0. \tag{22}$$

The first integral can be rewritten as

$$\begin{aligned}
-\frac{1}{\alpha\tau}&\int_{\mathbb{T}} u_k^{1-\alpha}(u_k^\alpha - u_{k-1}^\alpha)u_k^{\alpha/2-1}(u_k^{\alpha/2})_{xx}dx \\
&= \frac{1}{\alpha\tau}\int_{\mathbb{T}}\big[(u_k^{\alpha/2})_x^2 - (u_{k-1}^{\alpha/2})_x^2\big]dx \\
&\quad + \frac{1}{\alpha\tau}\int_{\mathbb{T}}\big[(u_{k-1}^{\alpha/2})_x^2 - (u_{k-1}^\alpha u_k^{-\alpha/2})_x(u_k^{\alpha/2})_x\big]dx \\
&= \frac{1}{\alpha\tau}\int_{\mathbb{T}}\big[(u_k^{\alpha/2})_x^2 - (u_{k-1}^{\alpha/2})_x^2\big]dx + \frac{\alpha}{4\tau}\int_{\mathbb{T}} u_{k-1}^\alpha\Big(\frac{u_{k-1,x}}{u_{k-1}} - \frac{u_{k,x}}{u_k}\Big)^2 dx \\
&\geq \frac{1}{\alpha\tau}\int_{\mathbb{T}}\big[(u_k^{\alpha/2})_x^2 - (u_{k-1}^{\alpha/2})_x^2\big]dx. \qquad (23)
\end{aligned}$$

The second integral in (22) becomes, after integration by parts,

$$\begin{aligned}
I &= \int_{\mathbb{T}}(u_k(\log u_k)_{xx})_x\big(u_k^{\alpha/2-1}(u_k^{\alpha/2})_{xx}\big)_x dx \\
&= \int_{\mathbb{T}} u_k^\alpha\Big(\frac{u_{k,xxx}}{u_k} - 2\frac{u_{k,xx}u_{k,x}}{u_k^2} + \frac{u_{k,x}^3}{u_k^3}\Big) \\
&\quad \times\Big(\frac{\alpha}{2}\frac{u_{k,xxx}}{u_k} + 2\alpha\Big(\frac{\alpha}{2}-1\Big)\frac{u_{k,xx}u_{k,x}}{u_k^2} + \frac{\alpha}{2}\Big(\frac{\alpha}{2}-1\Big)(\alpha-3)\frac{u_{k,x}^3}{u_k^3}\Big)dx.
\end{aligned}$$

We wish to compare this integral with the integrals

$$\begin{aligned}
J_1 &= \int_{\mathbb{T}}(u_k^{\alpha/2})_{xxx}^2 dx \\
&= \int_T u_k^\alpha\Big(\frac{\alpha}{2}\frac{u_{k,xxx}}{u_k} + \frac{3\alpha}{2}\Big(\frac{\alpha}{2}-1\Big)\frac{u_{k,xx}u_{k,x}}{u_k^2} + \frac{\alpha}{2}\Big(\frac{\alpha}{2}-1\Big)\Big(\frac{\alpha}{2}-2\Big)\frac{u_{k,x}^3}{u_k^3}\Big)^2 dx, \\
J_2 &= \int_{\mathbb{T}}(u_k^{\alpha/6})_x^6 dx = \Big(\frac{\alpha}{6}\Big)^6\int_{\mathbb{T}} u_k^\alpha\frac{u_{k,x}^6}{u_k^6}dx.
\end{aligned}$$

Integration by parts is reformulated, according to [14], as the integral identities

$$K_1 = \int_{\mathbb{T}}\Big(\frac{u_{k,x}^5}{u_k^{5-\alpha}}\Big)_x dx = \int_{\mathbb{T}} u_k^\alpha\Big(5\frac{u_{k,xx}u_{k,x}^4}{u_k^5} - (5-\alpha)\frac{u_{k,x}^6}{u_k^6}\Big)dx = 0,$$

$$\begin{aligned} K_2 &= \int_{\mathbb{T}} \Big(\frac{u_{k,xx}u_{k,x}^3}{u_k^{4-\alpha}}\Big)_x dx \\ &= \int_{\mathbb{T}} u_k^\alpha \bigg(3\frac{u_{k,xx}^2 u_{k,x}^2}{u_k^4} + \frac{u_{k,xxx}u_{k,x}^3}{u_k^4} - (4-\alpha)\frac{u_{k,xx}u_{k,x}^4}{u_k^5}\bigg) dx = 0. \end{aligned}$$

We wish to find constants $\mu > 0$ and c_1, $c_2 \in \mathbb{R}$ such that

$$I - \mu J_1 - \mu J_2 + c_1 K_1 + c_2 K_2 \geq 0. \tag{24}$$

Since $K_1 = K_2 = 0$, this is equivalent to $I \geq \mu(J_1 + J_2)$.

The main idea now is to identify the quotients $u_{k,x}/u_k$, $u_{k,xx}/u_k$ etc. with polynomial variables ξ_1, ξ_2 etc., and to write the integrand of I (up to the factor u_k^α) as the following polynomial in the variables $\xi = (\xi_1, \xi_2, \xi_3)$,

$$\begin{aligned} P(\xi) &= (\xi_3 - 2\xi_1\xi_2 + \xi_1^3)\Big(\frac{\alpha}{2}\xi_3 + 2\alpha\Big(\frac{\alpha}{2}-1\Big)\xi_2\xi_1 + \frac{\alpha}{2}\Big(\frac{\alpha}{2}-1\Big)(\alpha-3)\xi_1^3\Big) \\ &= \frac{\alpha}{2}\Big(\frac{\alpha}{2}-1\Big)(\alpha-3)\xi_1^6 - \frac{\alpha}{2}(\alpha-2)(\alpha-5)\xi_1^4\xi_2 + \frac{\alpha}{4}(\alpha^2-5\alpha+8)\xi_1^3\xi_3 \\ &\quad - 4\alpha\Big(\frac{\alpha}{2}-1\Big)\xi_1^2\xi_2^2 + \alpha(\alpha-3)\xi_1\xi_2\xi_3 + \frac{\alpha}{2}\xi_3^2 \end{aligned}$$

the integrands of J_1 and J_2, respectively, as

$$\begin{aligned} Q_1(\xi) &= \Big(\frac{\alpha}{2}\xi_3 + \frac{3\alpha}{2}\Big(\frac{\alpha}{2}-1\Big)\xi_1\xi_2 + \frac{\alpha}{2}\Big(\frac{\alpha}{2}-1\Big)\Big(\frac{\alpha}{2}-2\Big)\xi_1^3\Big)^2 \\ &= \frac{\alpha^2}{4}\Big(\frac{\alpha}{2}-1\Big)^2\Big(\frac{\alpha}{2}-2\Big)^2\xi_1^6 + 3\frac{\alpha^2}{2}\Big(\frac{\alpha}{2}-1\Big)^2\Big(\frac{\alpha}{2}-2\Big)\xi_1^4\xi_2 \\ &\quad + \frac{\alpha^2}{2}\Big(\frac{\alpha}{2}-1\Big)\Big(\frac{\alpha}{2}-2\Big)\xi_1^3\xi_3 + \frac{9\alpha^2}{4}\Big(\frac{\alpha}{2}-1\Big)^2\xi_1^2\xi_2^2 \\ &\quad + \frac{3\alpha^2}{2}\Big(\frac{\alpha}{2}-1\Big)\xi_1\xi_2\xi_3 + \frac{\alpha^2}{4}\xi_3^2, \\ Q_2(\xi) &= \Big(\frac{\alpha}{6}\Big)^6\xi_1^6, \end{aligned}$$

and finally, the integrands of K_1 and K_2 as

$$\begin{aligned} T_1(\xi) &= -(5-\alpha)\xi_1^6 + 5\xi_1^4\xi_2, \\ T_2(\xi) &= -(4-\alpha)\xi_1^4\xi_2 + 3\xi_1^2\xi_2^2 + \xi_1^3\xi_3. \end{aligned}$$

Thus, we can reformulate the task of determining constants μ, c_1, and c_2 such that (24) holds as the following decision problem: Find $\mu > 0$, c_1, $c_2 \in \mathbb{R}$ such that for all $\xi \in \mathbb{R}^3$,

$$P(\xi) - \mu Q_1(\xi) - \mu Q_2(\xi) + c_1 T_1(\xi) + c_2 T_2(\xi) \geq 0. \tag{25}$$

If this problem is solved, we obtain the inequality

$$\frac{1}{\alpha\tau}\int_{\mathbb{T}} \big((u_k^{\alpha/2})_x^2 - (u_{k-1}^{\alpha/2})_x^2\big)dx + \mu\int_{\mathbb{T}} \big((u_k^{\alpha/2})_{xxx}^2 + (u_k^{\alpha/6})_x^6\big)dx \leq 0,$$

from which (21) follows by taking the sum over all k.

Actually, the decision problem (25) can be solved by quantifier elimination. For this, we write (25) in the following way:

$$\begin{aligned} 0 &\le (P - \mu Q_1 - \mu Q_2 + c_1 T_1 + c_2 T_2)(\xi) \\ &= a_1\xi_1^6 + a_2\xi_1^4\xi_2 + a_3\xi_1^3\xi_3 + a_4\xi_1^2\xi_2^2 + a_5\xi_1\xi_2\xi_3 + a_6\xi_3^2, \end{aligned} \tag{26}$$

where

$$\begin{aligned} a_1 &= \frac{\alpha}{2}\Big(\frac{\alpha}{2}-1\Big)(\alpha-3) - \mu\frac{\alpha^2}{4}\Big(\frac{\alpha}{2}-1\Big)^2\Big(\frac{\alpha}{2}-2\Big)^2 - \mu\Big(\frac{\alpha}{6}\Big)^6 - c_1(5-\alpha), \\ a_2 &= -\frac{\alpha}{2}(\alpha-2)(\alpha-5) - \mu\frac{3\alpha^2}{2}\Big(\frac{\alpha}{2}-1\Big)^2\Big(\frac{\alpha}{2}-2\Big) + 5c_1 - c_2(4-\alpha), \\ a_3 &= \frac{\alpha}{4}(\alpha^2-5\alpha+8) - \mu\frac{\alpha^2}{2}\Big(\frac{\alpha}{2}-1\Big)\Big(\frac{\alpha}{2}-2\Big) + c_2, \\ a_4 &= -4\alpha\Big(\frac{\alpha}{2}-1\Big) - \mu\frac{9\alpha^2}{4}\Big(\frac{\alpha}{2}-1\Big)^2 + 3c_2, \\ a_5 &= \alpha(\alpha-3) - \mu\frac{3\alpha^2}{2}\Big(\frac{\alpha}{2}-1\Big), \\ a_6 &= \frac{\alpha}{2} - \mu\frac{\alpha^2}{4}. \end{aligned}$$

By Lemma 12 of [14], (26) holds if the coefficients satisfy either $a_6 > 0$, $4a_4a_6 - a_5^2 = 2a_2a_6 - a_3a_5 = 0$, and $4a_1a_6 - a_3^2 \ge 0$ or $a_6 > 0$, $4a_4a_6 - a_5^2 > 0$, and $4a_1a_4a_6 - a_1a_5^2 - a_2^2a_6 - a_3^2a_4 + a_2a_3a_5 \ge 0$. In the case $\alpha = 1$ this system of nonlinear equations can be solved explicitly [10]. The general case is much more involved. However, the solution of this system can be obtained by the computer algebra system `Mathematica` which provides an exact solution. The computer program ensures the existence of constants $\mu > 0$, c_1, and c_2 (depending on α) such that (26) holds for all $\xi \in \mathbb{R}^3$ under the condition $\frac{2}{53}(25 - 6\sqrt{10}) < \alpha < \frac{2}{53}(25 + 6\sqrt{10})$. Moreover, if $\alpha = 1$, we can choose $\mu = (103 + \sqrt{214})/72$ [10]. □

4. Further a priori estimates

From Lemmas 5 and 6 we are able to derive more a priori estimates. As in the previous section, we assume that $y_k \in C^\infty(\mathbb{T})$ is a solution to (13) and we set $u_N = \exp(y^{(N)}) > 0$.

We will need the following "parabolic" Gagliardo-Nirenberg-type inequality. Let $f \in L^2(0,T;H^3(\mathbb{T})) \cap L^\infty(0,T;H^1(\mathbb{T}))$. Then

$$\begin{aligned} \|f_{xx}\|_{L^2(\mathbb{T})}^2 &= (f_{xx}, f_{xx})_{L^2(\mathbb{T})} = -(f_{xxx}, f_x)_{L^2(\mathbb{T})} \le \|f\|_{H^3(\mathbb{T})}\|f\|_{H^1(\mathbb{T})}, \\ \|f\|_{H^1(\mathbb{T})}^2 &\le \|f\|_{H^2(\mathbb{T})}\|f\|_{H^1(\mathbb{T})} \end{aligned}$$

and hence,

$$\begin{aligned} \|f\|_{H^2(\mathbb{T})}^2 &= \|f\|_{H^1(\mathbb{T})}^2 + \|f_{xx}\|_{L^2(\mathbb{T})}^2 \le (\|f\|_{H^2(\mathbb{T})} + \|f\|_{H^3(\mathbb{T})})\|f\|_{H^1(\mathbb{T})} \\ &\le 2\|f\|_{H^3(\mathbb{T})}\|f\|_{H^1(\mathbb{T})} \end{aligned}$$

or

$$\|f\|_{H^2(\mathbb{T})} \le \sqrt{2}\|f\|_{H^3(\mathbb{T})}^{1/2}\|f\|_{H^1(\mathbb{T})}^{1/2}.$$

This gives

$$\|f\|^4_{L^4(0,T;H^2(\mathbb{T}))} \le 4\int_0^T \|f\|^2_{H^3(\mathbb{T})}\|f\|^2_{H^1(\mathbb{T})}dt$$
$$\le 4\|f\|^2_{L^2(0,T;H^3(\mathbb{T}))}\|f\|^2_{L^\infty(0,T;H^1(\mathbb{T}))}. \tag{27}$$

Lemma 7. *There exists a constant $c > 0$ independent of N such that*

$$\|u_N^\alpha\|_{L^2(0,T;H^3(\mathbb{T}))} \le c.$$

Proof. Set $v = u_N^{\alpha/2}$. We have to show that v^2 is uniformly bounded in $L^2(0,T;H^3(\mathbb{T}))$. Since $H^1(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$, the estimate (20) shows that v is uniformly bounded in $L^\infty(0,T;L^\infty(\mathbb{T}))$. In particular, v^2 is bounded in $L^2(0,T;L^2(\mathbb{T}))$. Then, by (20),

$$\|(v^2)_x\|_{L^2(0,T;L^2(\mathbb{T}))} \le 2\|v\|_{L^\infty(0,T;L^\infty(\mathbb{T}))}\|v_x\|_{L^2(0,T;L^2(\mathbb{T}))} \le c$$

and

$$\|(v^2)_{xx}\|_{L^2(0,T;L^2(\mathbb{T}))} \le 2\|v_x\|^2_{L^4(0,T;L^4(\mathbb{T}))} + 2\|v\|_{L^\infty(0,T;L^\infty(\mathbb{T}))}\|v_{xx}\|_{L^2(0,T;L^2(\mathbb{T}))} \le c.$$

Here and in the following, $c > 0$ denotes a generic constant. We have used the fact that, by (27), $v = u_N^{\alpha/2}$ is uniformly bounded in $L^4(0,T;H^2(\mathbb{T}))$ and hence also in $L^4(0,T;W^{1,4}(\mathbb{T}))$. Furthermore, since $H^3(\mathbb{T})$ embeddes continuously into $W^{2,\infty}(\mathbb{T})$,

$$\|(v^2)_{xxx}\|_{L^2(0,T;L^2(\mathbb{T}))} \le 6\|v_x\|_{L^\infty(0,T;L^2(\mathbb{T}))}\|v_{xx}\|_{L^2(0,T;L^\infty(\mathbb{T}))} + 2\|v\|_{L^\infty(0,T;L^\infty(\mathbb{T}))}\|v_{xxx}\|_{L^2(0,T;L^2(\mathbb{T}))} \le c.$$

This proves the lemma. □

Lemma 8. *There exists a constant $c > 0$ independent of N such that*

$$\|u_N^{\alpha/4}\|_{L^2(0,T;H^2(\mathbb{T}))} \le c.$$

Proof. It is clear from the L^∞ bound on u_N that $u_N^{\alpha/4}$ is uniformly bounded in $L^2(0,T;L^2(\mathbb{T}))$. Furthermore, since $u_N^{\alpha/6}$ is bounded in $L^6(0,T;W^{1,6}(\mathbb{T}))$ (see Lemma 6),

$$\|(u_N^{\alpha/4})_x\|_{L^2(0,T;L^2(\mathbb{T}))} \le c\|u_N^{\alpha/12}(u_N^{\alpha/6})_x\|_{L^2(0,T;L^2(\mathbb{T}))}$$
$$\le c\|u_N^{\alpha/2}\|^{1/6}_{L^\infty(0,T;L^\infty(\mathbb{T}))}\|u_N^{\alpha/6}\|_{L^2(0,T;H^1(\mathbb{T}))} \le c.$$

For the second derivative of $u_N^{\alpha/4}$ we obtain

$$\|(u_N^{\alpha/4})_{xx}\|^2_{L^2(0,T;L^2(\mathbb{T}))} \le \int_0^T\int_{\mathbb{T}}\left(\frac{\alpha}{4}e^{\alpha y^{(N)}/4}y^{(N)}_{xx} + \left(\frac{\alpha}{4}\right)^2 e^{\alpha y^{(N)}/4}(y^{(N)}_x)^2\right)^2 dxdt$$
$$\le c\|u_N\|^{\alpha/2}_{L^\infty(0,T;L^\infty(\mathbb{T}))}\|y^{(N)}_{xx}\|^2_{L^2(0,T;L^2(\mathbb{T}))} + c\int_0^T\int_{\mathbb{T}} e^{\alpha y^{(N)}/2}(y^{(N)}_x)^4 dxdt.$$

The first term is uniformly bounded by Lemmas (5) and 6. For the second term we apply Hölder's inequality,

$$\begin{aligned}\int_0^T\int_{\mathbb{T}} e^{\alpha y^{(N)}/2}(y_x^{(N)})^4dxdt &\le c\int_0^T\int_{\mathbb{T}} y_x^{(N)}(e^{\alpha y^{(N)}/6})_x^3dxdt \\ &\le c\int_0^T \|y_x^{(N)}\|_{L^2(\mathbb{T})}\|(e^{\alpha y^{(N)}/6})_x\|_{L^6(\mathbb{T})}^3dt \\ &\le c\|y^{(N)}\|_{L^2(0,T;H^1(\mathbb{T}))}\|e^{\alpha y^{(N)}/6}\|^3_{L^6(0,T;W^{1,6}(\mathbb{T}))} \le c.\end{aligned}$$

The above estimates show that $u_N^{\alpha/4} = e^{\alpha y^{(N)}/4}$ is uniformly bounded in $L^2(0,T;$ $H^2(\mathbb{T}))$. □

For the compactness argument, we also need a priori estimates for the discrete time derivative of u_N^α. For this we introduce the shift operator $(\sigma_N(u_N^\alpha))(x,t) = (\sigma_N\exp(\alpha y^{(N)}))(x,t) = \exp(\alpha y_{k-1}(x))$ for $x\in\mathbb{T}$ and $t\in(t_{k-1},t_k]$.

Lemma 9. *There exists a constant $c>0$ independent of N and τ such that*

$$\|u_N^\alpha - \sigma_N(u_N^\alpha)\|_{L^{3/2}(0,T;H^{-2}(\mathbb{T}))} \le c\tau.$$

Proof. The function $u_N = \exp(y^{(N)})$ satisfies

$$\frac{1}{\tau}(u_N^\alpha - \sigma_N(u_N^\alpha)) = -\alpha u_N^{\alpha-1}(u_N(\log u_N)_{xx})_{xx}.$$

We need to estimate the right-hand side of this equation. Since we have uniform bounds for u_N^α and $u_N^{\alpha/2}$ rather than for u_N, we formulate the right-hand side in terms of $u_N^{\alpha/k}$ where $k=1,2,3,4$:

$$\begin{aligned}\alpha u_N^{\alpha-1}(u_N(\log u_N)_{xx})_{xx} &= 2(u_N^\alpha(\log u_N^\alpha)_{xx})_{xx} - (u_N^\alpha)_{xxxx} + \frac{4}{\alpha}((u_N^{\alpha/2})_x^2)_{xx} \\ &\quad - c_1(\alpha)(u_N^{\alpha/2})^2_{xx} - c_2(\alpha)((u_N^{\alpha/3})_x^3)_x + c_3(\alpha)(u_N^{\alpha/4})_x^4,\end{aligned}$$

where

$$\begin{aligned}c_1(\alpha) &= \frac{4}{\alpha}(1-\alpha), \\ c_2(\alpha) &= \frac{\alpha}{3}(1-\alpha)(3\alpha-1), \\ c_3(\alpha) &= \frac{64}{3\alpha^2}(-3\alpha^2+7\alpha-4).\end{aligned}$$

This gives

$$\begin{aligned}&\tau^{-1}\|u_N^\alpha - \sigma_N(u_N^\alpha)\|_{L^{3/2}(0,T;H^{-2}(\mathbb{T}))} \\ &\quad\le 2\alpha\|u_N^\alpha y_{xx}^{(N)}\|_{L^{3/2}(0,T;L^2(\mathbb{T}))} + \|(u_N^\alpha)_{xx}\|_{L^{3/2}(0,T;L^2(\mathbb{T}))} \\ &\qquad + \frac{4}{\alpha}\|(u_N^{\alpha/2})_x^2\|_{L^{3/2}(0,T;L^2(\mathbb{T}))} + |c_1(\alpha)|\,\|(u_N^{\alpha/2})^2_{xx}\|_{L^{3/2}(0,T;H^{-2}(\mathbb{T}))} \\ &\qquad + |c_2(\alpha)|\,\|(u_N^{\alpha/3})_x^3\|_{L^{3/2}(0,T;H^{-1}(\mathbb{T}))} \\ &\qquad + |c_3(\alpha)|\,\|(u_N^{\alpha/4})_x^4\|_{L^{3/2}(0,T;H^{-2}(\mathbb{T}))} \\ &\quad = I_1 + \cdots + I_6.\end{aligned}$$

By (27), applied to $f = u_N^{\alpha/2}$, and Lemmas 5 and 6, we obtain

$$
\begin{aligned}
I_1 &\le \|u_N\|^\alpha_{L^\infty(0,T;L^\infty(\mathbb{T}))}\|y^{(N)}_{xx}\|_{L^{3/2}(0,T;L^2(\mathbb{T}))} \le c,\\
I_2 &= 2\|(u_N^{\alpha/2})_x^2 + u_N^{\alpha/2}(u_N^{\alpha/2})_{xx}\|_{L^{3/2}(0,T;L^2(\mathbb{T}))}\\
&\le 2\|u_N^{\alpha/2}\|^2_{L^3(0,T;W^{1,4}(\mathbb{T}))} + 2\|u_N^{\alpha/2}\|_{L^\infty(0,T;L^\infty(\mathbb{T}))}\|u_N^{\alpha/2}\|_{L^{3/2}(0,T;H^2(\mathbb{T}))} \le c,\\
I_3 &= \|u_N^{\alpha/2}\|^2_{L^3(0,T;W^{1,4}(\mathbb{T}))} \le c\|u_N^{\alpha/2}\|^2_{L^4(0,T;H^2(\mathbb{T}))} \le c,\\
I_4 &\le c\|(u_N^{\alpha/2})^2_{xx}\|_{L^{3/2}(0,T;L^1(\mathbb{T}))} = c\|(u_N^{\alpha/2})_{xx}\|^2_{L^3(0,T;L^2(\mathbb{T}))}\\
&\le c\|u_N^{\alpha/2}\|^2_{L^3(0,T;H^2(\mathbb{T}))} \le c,\\
I_5 &\le c\|(u_N^{\alpha/3})^3_x\|_{L^2(0,T;L^2(\mathbb{T}))} \le c\|u_N^{\alpha/6}(u_N^{\alpha/6})_x\|^3_{L^6(0,T;L^6(\mathbb{T}))}\\
&\le c\|u_N^{\alpha/2}\|_{L^\infty(0,T;L^\infty(\mathbb{T}))}\|u_N^{\alpha/6}\|^3_{L^6(0,T;W^{1,6}(\mathbb{T}))} \le c,\\
I_6 &\le c\|(u_N^{\alpha/4})^4_x\|_{L^{3/2}(0,T;L^{3/2}(\mathbb{T}))} \le c\|u_N^{\alpha/3}(u_N^{\alpha/6})^4_x\|_{L^{3/2}(0,T;L^{3/2}(\mathbb{T}))}\\
&\le c\|u_N^{\alpha/2}\|^{2/3}_{L^\infty(0,T;L^\infty(\mathbb{T}))}\|u_N^{\alpha/6}\|^4_{L^6(0,T;W^{1,6}(\mathbb{T}))} \le c.
\end{aligned}
$$

The expressions in $I_1, \ldots, I_5$ could be bounded also in L^2 but the expression in I_6 can be only bounded in $L^{3/2}$. Putting together the above inequalities, we conclude the proof. □

5. Proof of Theorem 1

Lemma 7 gives a uniform bound on u_N^α in $L^2(0,T;H^3(\Omega))$. Lemma 6 and the continuous embedding $H^1(\mathbb{T}) \subset L^\infty(\mathbb{T})$ show that $u_N^\alpha = (u_N^{\alpha/2})^2$ is uniformly bounded in $L^\infty(0,T;H^1(\mathbb{T}))$. These bounds and the bound on the discrete time derivative of u_N^α in $L^{3/2}(0,T;\, H^{-2}(\mathbb{T}))$ (Lemma 9) allow to apply the Aubin lemma [18], providing the existence of subsequences of u_N^α and u_N, which are not relabeled, such that

$$u_N^\alpha \to v \quad \text{strongly in } L^2(0,T;W^{2,q}(\mathbb{T})), \tag{28}$$

$$\tau^{-1}(u_N^\alpha - \sigma_N(u_N^\alpha)) \rightharpoonup v_t \quad \text{weakly in } L^{3/2}(0,T;H^{-2}(\mathbb{T})), \tag{29}$$

$$u_N \to u \quad \text{strongly in } L^\infty(0,T;L^\infty(\mathbb{T})) \text{ as } N \to \infty, \tag{30}$$

and the limit function u lies in $C^0([0,T];C^{0,1/2}(\mathbb{T}))$. Here, we have used the fact that $H^3(\mathbb{T})$ embeddes compactly into $W^{2,q}(\mathbb{T})$ for all $q < \infty$ and that $H^1(\mathbb{T})$ embeddes compactly into $L^\infty(\mathbb{T})$ and continuously into $C^{0,1/2}(\mathbb{T})$. Since we have in particular pointwise convergence of the sequence (u_N^α), we obtain $u_N^\alpha \to v$ in $L^1(0,T;L^1(\mathbb{T}))$, and $v = u^\alpha$. Moreover, we conclude from Lemma 6 for a subsequence, that

$$y^{(N)} \rightharpoonup y \quad \text{weakly in } L^2(0,T;H^2(\mathbb{T})).$$

We claim that $u = e^y$ which shows $u \ge 0$ a.e. in Q_T. Let χ be a smooth function. The monotonicity of the exponential function implies that

$$\int_0^T \int_{\mathbb{T}} (u_N^\alpha - e^{\alpha\chi})(y^{(N)} - \chi)\,dxdt = \int_0^T \int_{\mathbb{T}} (e^{\alpha y^{(N)}} - e^{\alpha\chi})(y^{(N)} - \chi)\,dxdt \ge 0.$$

Passing to the limit $N \to \infty$ yields

$$\int_0^T \int_{\mathbb{T}} (v - e^{\alpha\chi})(y-\chi)dxdt \geq 0 \quad \text{for all smooth } \chi.$$

Thus, by the monotonicity of $x \mapsto e^x$, $v = e^{\alpha y}$. Since $v = u^\alpha$, this implies $u = e^y$.

We perform the limit $\tau \to 0$ (or $N \to \infty$) in

$$\frac{1}{\alpha\tau}\big(u_N^\alpha - \sigma_N(u_N^\alpha)\big) + u_N^{\alpha-1}\big(u_N y_{xx}^{(N)}\big)_{xx} = 0.$$

Let $\phi \in L^\infty(0,T;H^2(\mathbb{T}))$. Then the weak formulation of this equation reads as

$$\frac{1}{\alpha\tau}\int_{Q_T} \big(u_N^\alpha - \sigma_N(u_N^\alpha)\big)\phi dxdt = -\int_{Q_T} u_N y_{xx}^{(N)}\big(u_N^{\alpha-1}\phi\big)_{xx}dxdt,$$

where we recall that $Q_T = \mathbb{T} \times (0,T)$. In the following, we set $w_N = u_N^{\alpha/4}$. We wish to write the right-hand side of the above equation in terms of w_N, $w_{N,x}$ and $w_{N,xx}$. A calculation shows that

$$\begin{aligned}
&\frac{1}{\alpha\tau}\int_{Q_T} \big(u_N^\alpha - \sigma_N(u_N^\alpha)\big)\phi dxdt \\
&\quad = -\frac{2}{\alpha}\Big(4 - \frac{4}{\alpha}\Big)\Big(\int_{Q_T} w_{N,xx}(w_N^2)_{xx}w_N\phi dxdt \\
&\qquad - 4\int_{Q_T} w_{N,xx}w_{N,x}^2 w_N\phi dxdt \\
&\qquad + 4\int_{Q_T} w_{N,xx}w_{N,x}w_N^2\phi_x dxdt - 4\int_{Q_T} w_{N,x}^3 w_N\phi_x dxdt\Big) \\
&\qquad - \frac{4}{\alpha}\Big(4 - \frac{4}{\alpha}\Big)\Big(3 - \frac{4}{\alpha}\Big)\Big(\int_{Q_T} w_{N,xx}w_{N,x}^2 w_N\phi dxdt - \int_{Q_T} w_{N,x}^4\phi dxdt\Big) \\
&\qquad - \frac{4}{\alpha}\Big(\int_{Q_T} w_{N,xx}w_N^3\phi_{xx}dxdt - \int_{Q_T} w_{N,x}^2 w_N^2\phi_{xx}dxdt\Big).
\end{aligned} \tag{31}$$

We collect now some convergence results for the sequence (w_N) which allow to pass to the limit $N \to \infty$ in (31). Lemma 8 shows that, for a subsequence,

$$w_N \rightharpoonup w \quad \text{weakly in } L^2(0,T;H^2(\mathbb{T})). \tag{32}$$

We claim that (maybe for a subsequence)

$$w_N^2 \to w^2 \quad \text{strongly in } L^2(0,T;H^2(\mathbb{T})). \tag{33}$$

By (30), we clearly have $w_N^2 \to w^2$ in $L^\infty(0,T;L^\infty(\mathbb{T}))$. Then the Gagliardo-Nirenberg inequality (12) shows that

$$\begin{aligned}
\|w_N^2 - w^2\|_{L^2(0,T;H^2(\mathbb{T}))}^2 &\leq c\int_0^T \|w_N^2 - w^2\|_{H^3(\mathbb{T})}^{6/5}\|w_N^2 - w^2\|_{L^\infty(\mathbb{T})}^{4/5}dt \\
&\leq c\|w_N^2 - w^2\|_{L^\infty(0,T;L^\infty(\mathbb{T}))}^{4/5}\|u_N^{\alpha/2} - u^{\alpha/2}\|_{L^2(0,T;H^3(\mathbb{T}))}^{6/5} \\
&\to 0,
\end{aligned}$$

since the second factor is bounded and the first factor converges to zero. Since $u_N = w_N^{4/\alpha}$ converges pointwise to u, we have $u = w^{4/\alpha}$.

In a similar way, we can prove that, again for a subsequence,

$$w_N \to w \quad \text{strongly in } L^4(0,T;W^{1,4}(\mathbb{T})). \tag{34}$$

Indeed, employing the Gagliardo-Nirenberg inequality (11), we obtain

$$\begin{aligned}\|w_N - w\|^4_{L^4(0,T;W^{1,4}(\mathbb{T}))} &\leq c\int_0^T \|w_N - w\|^2_{H^2(\mathbb{T})}\|w_N - w\|^2_{L^\infty(\mathbb{T})}dt \\ &\leq c\|w_N - w\|^2_{L^\infty(0,T;L^\infty(\mathbb{T}))}\|w_N - w\|^2_{L^2(0,T;H^2(\mathbb{T}))} \to 0,\end{aligned}$$

since the second factor is bounded, by Lemma 8, and the first factor converges to zero. The convergence (34) implies that

$$w^2_{N,x} \to w^2_x \quad \text{strongly in } L^2(0,T;L^2(\mathbb{T})), \tag{35}$$

$$w^3_{N,x} \to w^3_x \quad \text{strongly in } L^{4/3}(0,T;L^{4/3}(\mathbb{T})). \tag{36}$$

The above convergence results (32)-(36) are sufficient to pass to the limit in (31). Then the function w satisfies

$$\begin{aligned}&\frac{1}{\alpha}\int_0^T \langle \partial_t(u^\alpha), \phi\rangle_{H^{-2},H^2}dt \\ &\quad = -\frac{2}{\alpha}\Big(4 - \frac{4}{\alpha}\Big)\Big[\int_{Q_T} w_{xx}(w^2)_{xx}w\phi dxdt - 4\int_{Q_T} w_{xx}w_x^2 w\phi dxdt \\ &\qquad + 4\int_{Q_T} w_x w(w_{xx}w - w_x^2)\phi_x dxdt\Big] \\ &\qquad - \frac{4}{\alpha}\Big(4 - \frac{4}{\alpha}\Big)\Big(3 - \frac{4}{\alpha}\Big)\int_{Q_T} w_x^2(w_{xx}w - w_x^2)\phi dxdt \\ &\qquad - \frac{4}{\alpha}\int_{Q_T} w^2(w_{xx}w - w_x^2)\phi_{xx}dxdt.\end{aligned}$$

Next, we show that the L^1 norm of the weak solution u to (1) is bounded,

$$\int_{\mathbb{T}} u(x,t)dx \leq \int_{\mathbb{T}} u_0(x)dx. \tag{37}$$

The function $f(x) = x^{1/(1-\alpha)}$ is convex for $\alpha < 1$, i.e., for all x, $y > 0$,

$$f(x) - f(y) \leq f'(x)(x-y) = \frac{1}{1-\alpha}x^{\alpha/(1-\alpha)}(x-y).$$

Hence, for $x = u_{k-1}^{1-\alpha}$ and $y = u_k^{1-\alpha}$, we have

$$\int_{\mathbb{T}}(u_{k-1} - u_k)dx \leq \frac{1}{1-\alpha}\int_{\mathbb{T}} u_{k-1}^\alpha(u_{k-1}^{1-\alpha} - u_k^{1-\alpha})dx$$

and, multiplying (6) by $u_k^{1-\alpha}$ and integrating over $\mathbb{T}$,

$$\begin{aligned}
0 &= \int_{\mathbb{T}} u_k^{1-\alpha}(u_k^\alpha - u_{k-1}^\alpha)dx = \int_{\mathbb{T}}(u_k - u_{k-1})dx + \int_{\mathbb{T}} u_{k-1}^\alpha(u_{k-1}^{1-\alpha} - u_k^{1-\alpha})dx \\
&\geq \int_{\mathbb{T}}(u_k - u_{k-1})dx + (1-\alpha)\int_{\mathbb{T}}(u_{k-1} - u_k)dx \\
&= \alpha\int_{\mathbb{T}}(u_k - u_{k-1})dx.
\end{aligned}$$

This gives

$$\int_{\mathbb{T}} u_k dx \leq \int_{\mathbb{T}} u_{k-1}dx \leq \cdots \leq \int_{\mathbb{T}} u_0 dx$$

and

$$\int_{\mathbb{T}} u_N(\cdot,t)dx \leq \int_{\mathbb{T}} u_0 dx,$$

The limit $N \to \infty$ leads to (37). If $\alpha = 1$, the L^1 norm of u is conserved, see the proof of Theorem 2 below.

Finally, we need to verify (10). By the Poincaré inequality

$$\int_{\mathbb{T}}(u^{\alpha/2})_x^2 dx \leq \frac{1}{4\pi^2}\int_{\mathbb{T}}(u^{\alpha/2})_{xx}^2 dx \leq \frac{1}{16\pi^4}\int_{\mathbb{T}}(u^{\alpha/2})_{xxx}^2 dx$$

with optimal constant, we obtain from (21)

$$\int_{\mathbb{T}}(u_k^{\alpha/2})_x^2 dx + 16\pi^4\alpha\mu\tau\int_{\mathbb{T}}(u_k^{\alpha/2})_x^2 dx \leq \int_{\mathbb{T}}(u_{k-1}^{\alpha/2})_x^2 dx.$$

Solving this recursive inequality gives, for $t \in (t_{k-1}, t_k]$, $t_k = k\tau$,

$$\begin{aligned}
\int_{\mathbb{T}}(u_k^{\alpha/2})_x^2 dx &\leq \frac{1}{(1+16\pi^4\alpha\mu\tau)^k}\int_{\mathbb{T}}(u_0^{\alpha/2})_x^2 dx \\
&= \frac{1}{(1+16\pi^4\alpha\mu\tau)^{t_k/\tau}}\int_{\mathbb{T}}(u_0^{\alpha/2})_x^2 dx \\
&\leq \frac{1}{(1+16\pi^4\alpha\mu\tau)^{t/\tau}}\int_{\mathbb{T}}(u_0^{\alpha/2})_x^2 dx.
\end{aligned}$$

The limit $\tau \to 0$ or $N \to \infty$ yields

$$\int_{\mathbb{T}}(u^{\alpha/2})_x^2(x,t)dx \leq e^{-16\pi^4\alpha\mu t}\int_{\mathbb{T}}(u_0^{\alpha/2})_x^2 dx.$$

This is equal to (10) with $\beta_\alpha = 8\pi^4\alpha\mu$ (we remark that also μ depends on α), finishing the proof of Theorem 1.

6. Proof of Theorem 2

First, we show that the L^1 norm of u is conserved:

$$\int_{\mathbb{T}} u(x,t)dx = \int_{\mathbb{T}} u_0 dx. \tag{38}$$

The discrete solution $w_N = u_N^{1/4}$ satisfies (31) for $\alpha = 1$ which is in $(t_{k-1}, t_k]$ equal to

$$\frac{1}{\tau}\int_{\mathbb{T}}(u_k - u_{k-1})\phi dx = -4\int_{\mathbb{T}}\big(u_{k,xx}^{1/4}u_k^{3/4} - u_{k,x}^{1/2}u_k^{1/2}\big)\phi_{xx}dx,$$

where $\phi \in H^2(\mathbb{T})$ is a test function. Choosing $\phi = 1$ gives

$$\int_{\mathbb{T}}(u_k - u_{k-1})dx = 0,$$

and hence

$$\int_{\mathbb{T}} u_k dx = \int_{\mathbb{T}} u_{k-1}dx = \cdots = \int_{\mathbb{T}} u_0 dx.$$

Then, after passing to the limit $N \to \infty$, we conclude (38).

Next, we employ the decay estimate (10) to prove the strict positivity of u. For this, we remark that in the case $\alpha = 1$, we have $\mu = (103 + \sqrt{214})/72$ [10]. This gives $\beta_1 = \pi^4(103 + \sqrt{214})/9$. Let $\bar{u} = \int u_0 dx$ and $M = \|(\sqrt{u_0})_x\|_{L^2(\mathbb{T})}$. Employing the Poincaré-type inequality with optimal constant,

$$\|f - \bar{f}\|_{L^\infty(\mathbb{T})} \le \frac{1}{2}\|f_x\|_{L^1(\mathbb{T})}$$

for $f \in W^{1,1}(\mathbb{T})$, where $\bar{f} = \int_{\mathbb{T}} f dx$, and inequality (10), we obtain

$$\begin{aligned}\|u(\cdot,t) - \bar{u}\|_{L^\infty(\mathbb{T})} &\le \frac{1}{2}\|u_x\|_{L^1(\mathbb{T})} = \int_{\mathbb{T}} |\sqrt{u}(\sqrt{u})_x| dx \\ &\le \|\sqrt{u}\|_{L^2(\mathbb{T})}\|(\sqrt{u})_x\|_{L^2(\mathbb{T})} = \|u\|_{L^1(\mathbb{T})}\|(\sqrt{u})_x\|_{L^2(\mathbb{T})} \\ &= \bar{u}Me^{-\beta_1 t},\end{aligned}$$

observing (38) in the last equality, and thus

$$u(\cdot,t) \ge \bar{u} - \bar{u}Me^{-\beta_1 t} = \bar{u}(1 - Me^{-\beta_1 t}), \quad t > 0.$$

Hence, $u(\cdot,t)$ is strictly positive if $t > (\log M)/\beta_1$. Furthermore, if $M < 1$ then $u(\cdot,t)$ is strictly positive for all $t \ge 0$.

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A.J.: Institut für Analysis und Scientific Computing, Technische Universität Wien, Wiedner Hauptstr. 8, 1040 Wien, Austria

E-mail address: juengel@mathematik.uni-mainz.de.

I.V.: Laboratoire de Mathématiques Appliquées, CNRS UMR 6620, Université Blaise Pascal (Clermont-Ferrand 2), 63177 Aubière, France

E-mail address: violet@math.univ-bpclermont.fr.