HIGH ORDER EXPANSIONS IN QUASINEUTRAL LIMIT OF THE EULER-POISSON SYSTEM FOR A POTENTIAL FLOW

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Abstract : We are interested in steady-state Euler-Poisson system for a potential flow used in the mathematical modeling of plasmas and semiconductors. In the case of quasineutral limit, boundary layers can appear. We study this limit by using an asymptotic expansion. We show the existence and uniqueness of each profile and give the justification of the asymptotic expansion up to any order.

Keywords : Steady-state Euler-Poisson system, potential and subsonic flow, quasineutral limit, asymptotic expansion and justification, semiconductors and plasmas.

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1. Introduction

The hydrodynamic model is widely used in mathematical modeling and numerical simulation for plasmas [3] and semiconductors [15]. It consists in two non linear equations given by the conservation laws of momentum and density, called Euler equations, plus a Poisson equation for the electrostatic potential. Due to the hyperbolicity of the transient non linear Euler equations, the weak solution is only studied in one space dimension. In such a situation, the existence of global weak solution is shown in the set of bounded functions [14].

In this paper we only consider the unipolar steady-state case for a potential flow. Then the Euler-Poisson system reads as follows (see [5, 16, 17]) :

$$\frac{\varepsilon}{2}|\nabla\psi|^2 + h(n) = \phi + \frac{\varepsilon\psi}{\tau}, \tag{1.1}$$

$$-\mathrm{div}(n\nabla\psi) = 0, \tag{1.2}$$

$$-\lambda^2\Delta\phi = b(x) - n. \tag{1.3}$$

This system will be studied in an open and bounded domain Ω in $\mathbb{R}^d$ (d=2 or d=3 in practice) with sufficiently smooth boundary Γ. Here $n = n(x), \psi = \psi(x), \phi = \phi(x)$ represent respectively the electron density, the velocity potential and the electrostatic potential. The function $h = h(n)$ is the enthalpy for the system and is defined by :

$$h'(n) = \frac{p'(n)}{n},\ n > 0,\ \text{and } h(1) = 0,$$

where $p = p(n)$ is the pressure function, supposed to be sufficiently smooth and strictly increasing for $n > 0$. The function $b = b(x)$ represents the doping profile for a semiconductor and the ion density for a plasma. The parameters λ, ϵ, τ represent respectively the scaled Debye length, electron mass and relaxation time of the system. They are dimensionless and small compared to the caracteristic length of physical interest. Then it is important to study the limits $\lambda \to 0, \epsilon \to 0, \tau \to 0$. In [17] we used asymptotic expansions to study the zero-electron-mass limit and the zero-relaxation-time limit and to show the convergence of the Euler-Poisson system to the incompressible Euler equations. Here we study the quasineutral limit $\lambda \to 0$. In all the following we take $\tau \equiv 1$ and we keep $\varepsilon > 0$ as a small parameter independent of λ in the equations.

By eliminating ϕ of (1.1) and (1.3) and using (1.2) we have :

$$\begin{aligned}\Delta h(n) \quad &- \quad \frac{\varepsilon}{n}\sum_{i,j=1}^{d}\psi_{x_i}\psi_{x_j}n_{x_ix_j} + \frac{\varepsilon}{n^2}(\nabla\psi.\nabla n)^2 + \frac{\varepsilon}{n}(\nabla\psi.\nabla n)\\ &- \quad \frac{\varepsilon}{n}\sum_{i,j=1}^{d}\psi_{x_i}\psi_{x_ix_j}n_{x_j} - \frac{1}{\lambda^2}(n-b) + \varepsilon Q(\psi) = 0.\end{aligned} \tag{1.4}$$

where Q is given by :

$$Q(\psi) = \sum_{i,j=1}^{d}\psi^2_{x_ix_j}. \tag{1.5}$$

For $n > 0$, it is easy to see that (n, ψ, ϕ) is a smooth solution to the system (1.1)-(1.3) if and only if (n, ψ) is a smooth solution to (1.2) and (1.4). Moreover, for ψ given, the equation (1.4) is elliptic if and only if the flow is subsonic, i.e., the condition $|\nabla\psi| < \sqrt{p'(n)/\varepsilon}$ holds.

We supplement the system (1.1)-(1.3) by Dirichlet boundary conditions :

$$n = \sum_{j=0}^{m}\lambda^j n_D^j + n_{D,\lambda}^m, \;\; \psi = \sum_{j=0}^{m}\lambda^j\psi_D^j + \psi_{D,\lambda}^m, \text{ on } \Gamma = \partial\Omega, \tag{1.6}$$

where $n_{D,\lambda}^m$ and $\psi_{D,\lambda}^m$ are smooth enough and defined on $\bar{\Omega}$.

The Euler-Poisson system and its asymptotic limits have been studied by a lot of authors. In [5] it is shown the existence and uniqueness of solutions for a potential flow under an assumption on the smallness of data, which implies that the problem is in the subsonic region. In [16] the author shows that the smallness condition corresponds to the smallness of ε. Then the existence and uniqueness hold for large data provided that ε is small enough.

The quasineutral limit has been studied in several special cases. In one-dimensional steady-state Euler-Poisson system it was performed in [19] for well-prepared boundary data. The steady problem in several space variables for a potential flow without the formation of boundary layers was investigated in [16]. In [4] the authors use pseudo-differential techniques to study this limit in transient Euler-Poisson model. The quasineutral limit has also been studied in the bipolar case in the drift-diffusion equations (see [7, 8, 12]).

See also [2] for the study of this limit in a semi-linear Poisson equation in which the electron density is described by the Maxwell-Boltzmann relation. This relation is also used in [4].

The zero-electron-mass limit ($\varepsilon \to 0$) and the zero-relaxation-time limit ($\tau \to 0$) have also been studied a lot. See [1, 11, 17] for different results on these two limits. In [9], the authors study the combined relaxation-time limit and the vanishing Debye length.

This article is based on the method of asymptotic expansions presented in [18]. In [18] the justification of these asymptotic expansions is only given up to first order in the one-dimensional case. Here we will give their justification up to any order in the multidimensional case by using the Schauder fixed point Theorem. The main difficulty is to verify the assumptions in this theorem which are achieved by the Leray Schauder fixed point Theorem.

The paper is organised as follow. In Section 2, we give the formal asymptotic expansions and the systems verified by each boundary layer profiles under our assumptions. Section 3 is devoted to the justification up to any order of the asymptotic expansions.

2. Formal asymptotic expansions

In this section we study the formal asymptotic expansions of a solution to (1.1)-(1.3). We use for this the method of asymptotic expansions presented in [18]. We assume :

(H1) $b \in C^\infty(\bar{\Omega}),\ 0 < \underline{n} \le b(x) \le \bar{n},\ x \in \bar{\Omega},\ \underline{n},\ \bar{n} \in \mathbb{R}$,

(H2) $n_D^j \in C^\infty(\bar{\Omega})$ for $0 \le j \le m$,

(H3) $\psi_D^j \in C^{2,\delta}(\bar{\Omega})$ for $0 \le j \le m$,

(H4) $n_D^0(x) = b(x), n_D^1(x) = 0,\ x \in \bar{\Omega}$,

(H5) $(\lambda^{-m-1} n_{D,\lambda}^m)_{\lambda>0}$ is bounded in $W^{2,q}(\Omega),\ q > \frac{d}{1-\delta},\ \delta \in (0,1)$,

(H6) $(\lambda^{-m+1} \psi_{D,\lambda}^m)_{\lambda>0}$ is bounded in $C^{2,\delta}(\bar{\Omega})$.

The assumption (H4) is a compatibility condition for the first and second order terms. It assures that there will not appear any boundary layers in these two terms. The case without any compatibility conditions presents some difficulties which we didn't succeed in this study (see Remark 3.2).

2.1. **Internal expansion.** Let :

$$n(x) = \sum_{k\ge 0} \lambda^k n_k(x);\ \psi(x) = \sum_{k\ge 0} \lambda^k \psi_k(x);\ \phi(x) = \sum_{k\ge 0} \lambda^k \phi_k(x).$$

We inject this into (1.1)-(1.3) and obtain :

$$\frac{\varepsilon}{2}|\nabla\Big(\sum_{k\ge 0}\lambda^k\psi_k(x)\Big)|^2 + h\Big(\sum_{k\ge 0}\lambda^k n_k(x)\Big) = \sum_{k\ge 0}\lambda^k\phi_k(x) + \varepsilon\sum_{k\ge 0}\lambda^k\psi_k(x), \tag{2.1}$$

$$-\mathrm{div}\Big(\sum_{k\ge 0}\lambda^k n_k(x)\nabla\Big(\sum_{k\ge 0}\lambda^k\psi_k(x)\Big)\Big) = 0, \tag{2.2}$$

$$-\lambda^2\Delta\Big(\sum_{k\ge 0}\lambda^k\phi_k(x)\Big) = b(x) - \sum_{k\ge 0}\lambda^k n_k(x),\ \text{in } \Omega. \tag{2.3}$$

Formally,

$$\begin{aligned}
&\mathrm{div}\Big(\sum_{k\geq 0}\lambda^k n_k(x)\nabla\Big(\sum_{k\geq 0}\lambda^k\psi_k(x)\Big)\Big)=\sum_{k\geq 0}\lambda^k\Big(\sum_{i=0}^{k}\mathrm{div}(n_i\nabla\psi_{k-i})\Big),\\
&|\nabla\Big(\sum_{k\geq 0}\lambda^k\psi_k(x)\Big)|^2=\sum_{k\geq 0}\lambda^k\Big(\sum_{i=0}^{k}\nabla\psi_i.\nabla\psi_{k-i}\Big),\\
&h\Big(\sum_{k\geq 0}\lambda^k n_k(x)\Big)=\sum_{k\geq 0}\lambda^k h_k(n),
\end{aligned}$$

where $n=(n_i)_{i\geq 0}$ and :

$$h_k(n)=\frac{1}{k!}\frac{d^k h\big(\sum_{k\geq 0}\lambda^k n_k\big)}{d\lambda^k}\Bigg|_{\lambda=0},\ \ k\geq 0.$$

As shown in [17] : $h_k(n)=h'(n_0)n_k+\bar{h}_k((n_i)_{0\leq i\leq k-1}),\ k\geq 1$ where $\bar{h}_k$ is smooth and $\bar{h}_1\equiv 0$. Then :

$$h(n_\lambda)=h(n_0)+\sum_{k\geq 1}\lambda^k h'(n_0)n_k+\sum_{k\geq 2}\lambda^k\bar{h}_k((n_i)_{0\leq i\leq k-1}).$$

Hence the system (2.1)-(2.3) becomes :

$$\begin{aligned}
\frac{\varepsilon}{2}\sum_{k\geq 0}\lambda^k\Big(\sum_{i=0}^{k}\nabla\psi_i.\nabla\psi_{k-i}\Big)+h(n_0)\ &+\ \sum_{k\geq 1}\lambda^k h'(n_0)n_k+\sum_{k\geq 2}\lambda^k\bar{h}_k((n_i)_{0\leq i\leq k-1})\\
&=\ \sum_{k\geq 0}\lambda^k\phi_k(x)+\varepsilon\sum_{k\geq 0}\lambda^k\psi_k(x),
\end{aligned}\tag{2.4}$$

$$-\sum_{k\geq 0}\lambda^k\Big(\sum_{i=0}^{k}\mathrm{div}(n_i\nabla\psi_{k-i})\Big)=0,\tag{2.5}$$

$$-\sum_{k\geq 2}\lambda^k\Delta\phi_{k-2}=b(x)-\sum_{k\geq 0}\lambda^k n_k.\tag{2.6}$$

We identify the order in λ and obtain the system verified by (n_k,ψ_k,ϕ_k) for all k. For $k=0$ we have :

$$\phi_0=-\frac{\varepsilon}{2}|\nabla\psi_0|^2-h(n_0)+\varepsilon\psi_0,\tag{2.7}$$

$$\mathrm{div}(n_0\nabla\psi_0)=0,\tag{2.8}$$

$$n_0=b(x).\tag{2.9}$$

For $k=1$ we have :

$$\phi_1=-\varepsilon\nabla\psi_0.\nabla\psi_1-h'(n_0)n_1+\varepsilon\psi_1,\tag{2.10}$$

$$-\mathrm{div}(n_0\nabla\psi_1)=\mathrm{div}(n_1\nabla\psi_0),\tag{2.11}$$

$$n_1=0.\tag{2.12}$$

And for $k \geq 2$:

$$\phi_k = -\frac{\varepsilon}{2}\sum_{i=0}^{k} \nabla\psi_i.\nabla\psi_{k-i} - h'(n_0)n_k - \bar{h}_k((n_i)_{0\leq i\leq k-1}) + \varepsilon\psi_k, \tag{2.13}$$

$$-\mathrm{div}(n_0\nabla\psi_k) = \sum_{i=1}^{k}\mathrm{div}(n_i\nabla\psi_{k-i}), \tag{2.14}$$

$$n_k = \Delta\phi_{k-2}. \tag{2.15}$$

All the profiles (n_k, ψ_k, ϕ_k) can be determined, uniquely and sufficiently smooth, by induction on k with boundary conditions given later. First, we obtain n_0 by (2.9), then we have ψ_0 by (2.8) and ϕ_0 by (2.7). We use the same way for determining the solutions of (2.10)-(2.12) and (2.13)-(2.15). Then the internal solution is constructed. For $m \geq 2$ let us denote :

$$n^\lambda_{I,m} = \sum_{k=0}^{m}\lambda^k n_k;\ \psi^\lambda_{I,m} = \sum_{k=0}^{m}\lambda^k\psi_k;\ \phi^\lambda_{I,m} = \sum_{k=0}^{m}\lambda^k\phi_k.$$

By construction, if (n_k, ψ_k, ϕ_k) are smooth enough, then the error equations are of order $O(\lambda^{m+1})$. Since $n_k = \Delta\phi_{k-2}$, for $k \geq 2$, and is not necessarily equal to n_D^k on Γ, then a boundary layer can appear.

2.2. **External expansion.** We follow the notations in [6]. For $x \in \Omega$, we note $t(x)$ the distance from Γ to x and $s(x)$ the point of Γ nearest from x. For $\theta > 0$, let Ω_θ be the boundary layer of size θ :

$$\Omega_\theta = \{x \in \Omega; |x - y| < \theta,\ y \in \Gamma\}.$$

If θ is small enough, $s(x)$ is defined uniquely for all $x \in \Omega_\theta$. In Ω_θ, we define the fast variable by $\xi(x, \lambda) = t(x)/\lambda$. For $x \in \Omega_\theta$, let $\nu(x) = (\nu_1, ..., \nu_d)$ the unit interior-directional normal vector of Γ passing from x. Then from :

$$t(x) = \|x - s(x)\|,\ x - s(x) = t(x)\nu(x),$$

and due to the fact that for all $i = 1, ..., d,\ \partial s(x)/\partial x_i$ is orthogonal to $\nu(x)$, it is easy to see that $\nabla_x t = \nu(x)$. Hence the partial derivative of a function $w(s(x), \xi(x, \lambda))$ may be decomposed as :

$$\frac{\partial w(s(x), \xi(x,\lambda))}{\partial x_i} = \lambda^{-1}\nu_i\frac{\partial w}{\partial\xi} + D_i w, \tag{2.16}$$

where D_i is a first order differential operator in s defined by : $D_i w = \nabla_s w.\frac{\partial s}{\partial x_i}$. Similarly :

$$\frac{\partial^2 w(s(x), \xi(x,\lambda))}{\partial x_i\partial x_j} = \lambda^{-2}\nu_i\frac{\partial^2 w}{\partial\xi^2} + \lambda^{-1}D_{ji}\frac{\partial w}{\partial\xi} + D_j D_i w + \nabla_s w.\frac{\partial^2 s}{\partial x_i\partial x_j}, \tag{2.17}$$

where $D_{ji} = \nu_i D_j + \nu_j D_i + \partial\nu_i/\partial x_j$. Note that for all $i,\ j$ we have : $D_{ji} = D_{ij}$.

For every function $w(x)$ defined in Ω_θ the equivalent function of (s, t) is designated by $\tilde{w}$ i.e. $w(x) = \tilde{w}(s(x), t(x)) = \tilde{w}(s(x), \lambda\xi(x, \lambda))$. We develop $\tilde{w}(s(x), \lambda\xi(x, \lambda))$ formally to obtain :

$$\tilde{w}(s(x), \lambda\xi(x,\lambda)) = \tilde{w}(s(x), 0) + O(\lambda).$$

Let $\bar{w}(s) = \tilde{w}(s,0)$. Then the ansatz of an approximate solution up to order m of (1.1)-(1.3) in Ω_θ are given by :

$$\begin{aligned}\tilde{n}^\lambda_{a,m}(x) &= n^\lambda_{I,m}(x) + \tilde{n}^\lambda_{B,m}(s(x),\xi(x,\lambda)),\\ \tilde{\psi}^\lambda_{a,m}(x) &= \psi^\lambda_{I,m}(x) + \tilde{\psi}^\lambda_{B,m}(s(x),\xi(x,\lambda)),\\ \tilde{\phi}^\lambda_{a,m}(x) &= \phi^\lambda_{I,m}(x) + \tilde{\phi}^\lambda_{B,m}(s(x),\xi(x,\lambda));\end{aligned}$$

where the boundary layers $(\tilde{n}^\lambda_{B,m}, \tilde{\psi}^\lambda_{B,m}, \tilde{\phi}^\lambda_{B,m})$ have the expansions :

$$\tilde{n}^\lambda_{B,m} = \sum_{k=0}^{m} \lambda^k n^b_k,\ \ \tilde{\psi}^\lambda_{B,m} = \sum_{k=0}^{m+1} \lambda^k \psi^b_k,\ \ \tilde{\phi}^\lambda_{B,m} = \sum_{k=0}^{m} \lambda^k \phi^b_k,$$

in which each term $(n^b_k(s,\xi), \psi^b_k(s,\xi), \phi^b_k(s,\xi))$ will be chosen to decay exponentially when ξ tends to $+\infty$. They are determined by setting $(\tilde{n}^\lambda_{a,m}, \tilde{\psi}^\lambda_{a,m}, \tilde{\phi}^\lambda_{a,m})$ in (1.1)-(1.3) and by identification of the order in λ. Let $\partial/\partial\nu = \sum_{i=1}^d \nu_i \partial/\partial x_i$. After computation we obtain :

$$\psi^b_0 \equiv 0,$$

the system for $(n^b_0, \psi^b_1, \phi^b_0)$:

$$(S_0)\qquad \begin{cases} (\bar{n}_0 + n^b_0)\dfrac{\partial^2 \psi^b_1}{\partial \xi^2} + \left(\dfrac{\partial \bar{\psi}_0}{\partial \nu} + \dfrac{\partial \psi^b_1}{\partial \xi}\right)\dfrac{\partial n^b_0}{\partial \xi} = 0,\\ \dfrac{\varepsilon}{2}\left(\dfrac{\partial \psi^b_1}{\partial \xi}\right)^2 + \varepsilon \dfrac{\partial \psi^b_1}{\partial \xi}\dfrac{\partial \bar{\psi}_0}{\partial \nu} + h(\bar{n}_0 + n^b_0) - \phi^b_0 = \bar{\phi}_0 + \varepsilon\bar{\psi}_0,\\ \dfrac{\partial^2 \phi^b_0}{\partial \xi^2} = n^b_0, \end{cases}$$

and the system for $(n^b_k, \psi^b_{k+1}, \phi^b_k)$ with $k \geq 1$:

$$(S_k)\qquad \begin{cases} \varepsilon \dfrac{\partial \psi^b_{k+1}}{\partial \xi}\left(\dfrac{\partial \psi^b_1}{\partial \xi} + \dfrac{\partial \bar{\psi}_0}{\partial \nu}\right) + h'(\bar{n}_0 + n^b_0) n^b_k - \phi^b_k = F_{1,k}(n^b_l, \psi^b_{l+1}, 0 \leq l \leq k-1),\\ (\bar{n}_0 + n^b_0)\dfrac{\partial^2 \psi^b_{k+1}}{\partial \xi^2} + \left(\dfrac{\partial \bar{\psi}_0}{\partial \nu} + \dfrac{\partial \psi^b_1}{\partial \xi}\right)\dfrac{\partial n^b_k}{\partial \xi} + n^b_k \dfrac{\partial^2 \psi^b_1}{\partial \xi^2} + \dfrac{\partial n^b_0}{\partial \xi}\dfrac{\partial \psi^b_{k+1}}{\partial \xi} = \\ \qquad\qquad\qquad\qquad F_{2,k}(n^b_l, \psi^b_{l+1}, 0 \leq l \leq k-1),\\ -\dfrac{\partial^2 \phi^b_k}{\partial \xi^2} + n^b_k = F_{3,k}(\phi^b_l, k-1 \leq l \leq k-2), \end{cases}$$

where $F_{i,k},\ i = 1,2,3,$ are given functions of $(n^b_l, \psi^b_{l+1})_{0 \leq l \leq k-1}$ for $F_{1,k},\ F_{2,k}$, and of $(\phi^b_l)_{k-1 \leq l \leq k-2}$ for $F_{3,k}$.

Hence the approximate solution is constructed in Ω_θ. To complete the definition of the approximate solution in $\bar{\Omega}$, let $\sigma \in C^\infty(0,\infty)$ be a smooth function such that $\sigma(t) = 1$ for $0 \leq t \leq \theta/2$ and $\sigma \equiv 0$ for $t \geq \theta$ and we define :

$$(n^\lambda_{B,m}(x), \psi^\lambda_{B,m}(x), \phi^\lambda_{B,m}(x)) =$$
$$\begin{cases} (\tilde{n}^\lambda_{B,m}(s(x), t(x)/\lambda), \tilde{\psi}^\lambda_{B,m}(s(x)t(x)/\lambda), \tilde{\phi}^\lambda_{B,m}(s(x)t(x)/\lambda))\sigma(t(x)), & \text{for } x \in \Omega_\theta,\\ 0, \quad \text{for } x \in \Omega - \Omega_\theta. \end{cases}$$

Then, $(n^\lambda_{B,m}, \psi^\lambda_{B,m}, \phi^\lambda_{B,m})$ has the same regularity as $(\tilde{n}^\lambda_{B,m}, \tilde{\psi}^\lambda_{B,m}, \tilde{\phi}^\lambda_{B,m})$. If each $(n^b_k(s,\xi), \psi^b_k(s,\xi), \phi^b_k(s,\xi))$ decays exponentially when ξ tends to $+\infty$ it is easy to see that the difference between $(n^\lambda_{B,m}, \psi^\lambda_{B,m}, \phi^\lambda_{B,m})$ and $(\tilde{n}^\lambda_{B,m}, \tilde{\psi}^\lambda_{B,m}, \tilde{\phi}^\lambda_{B,m})$ is uniform of order of $e^{-\mu/\lambda}$ for a constant $\mu > 0$.

Finally, the boundary conditions (1.6) give for $s \in \Gamma$:

$$n_0 = n^0_D, n_1 = n^1_D, n^b_0(s,0) = n^b_1(s,0) = 0, \bar{n}_k(s) + n^b_k(s,0) = n^k_D, k \geq 2, \tag{2.18}$$

$$\psi_0 = \psi^0_D, \psi_1 = \psi^1_D, \psi_2 = \psi^2_D, \psi^b_1(s,0) = \psi^b_2(s,0) = 0, \bar{\psi}_k(s) + \psi^b_k(s,0) = \psi^k_D, k \geq 3. \tag{2.19}$$

We refer to [18] for the scheme of determination of $(n_k, \psi_k, \phi_k, n^b_k, \psi^b_{k+1}, \phi^b_k)$. We can show under the assumption (H4) :

$$n^b_0 = n^b_1 = \psi^b_1 = \psi^b_2 = 0.$$

The approximate solution up to order m is constructed in the form :

$$(n^a_\lambda, \psi^a_\lambda, \phi^a_\lambda) = (n^\lambda_{I,m} + n^\lambda_{B,m}, \psi^\lambda_{I,m} + \psi^\lambda_{B,m}, \phi^\lambda_{I,m} + \phi^\lambda_{B,m}), \text{ in } \bar{\Omega}. \tag{2.20}$$

By construction : $n^a_\lambda = \sum_{k=0}^m \lambda^k n^k_D$, $\psi^a_\lambda = \sum_{k=0}^m \lambda^k \psi^k_D$ on Γ and :

$$n^a_\lambda = n_0 + \sum_{j=2}^m \lambda^j (n_j + n^b_j), \tag{2.21}$$

$$\psi^a_\lambda = \psi_0 + \lambda\psi_1 + \lambda^2\psi_2 + \sum_{j=3}^m \lambda^j(\psi_j + \psi^b_j) + \lambda^{m+1}\psi^b_{m+1}. \tag{2.22}$$

The existence and uniqueness of boundary layers (n^b_k, ψ^b_{k+1}) with exponential decay have been shown in [18] for each $k \geq 0$. Thus we obtain,

Theorem 2.1. *Under the assumptions (H1)-(H6), there exists a unique asymptotic expansion (2.20) up to order m, sufficiently smooth, satisfying (2.21)-(2.22).*

3. Justification of the quasineutral limit

We have seen in the introduction that (1.1)-(1.3), (1.6) is equivalent to (1.1)-(1.2), (1.4), (1.6). The main result of this paper is :

Theorem 3.1. *Under the assumption (H1)-(H6), for λ small enough, there is an $\varepsilon_0 > 0$ independent of λ such that for all $\varepsilon \in [0, \varepsilon_0]$, the problem (1.2), (1.4), (1.6) has a solution $(n_\lambda, \psi_\lambda) \in W^{2,q}(\Omega) \times C^{2,\delta}(\bar{\Omega})$ which satisfies :*

$$\|n_\lambda - n^a_\lambda\|_{W^{2,q}(\Omega)} \leq A\lambda^{m-1}, \ \|\psi_\lambda - \psi^a_\lambda\|_{C^{2,\delta}(\bar{\Omega})} \leq A\lambda^{m-1}, \tag{3.1}$$

where A is a constant independent of λ.

Remark 3.2. *Using equation (1.1), the continuity of h and the estimates from Theorem 3.1, we can easily show, for λ small enough :*

$$\|\phi_\lambda - \phi^a_\lambda\|_{C^{1,\delta}(\bar{\Omega})} \leq A\lambda^{m-1},$$

where A is a constant independent of λ.

Remark 3.3. *Without the assumption (H4), some boundary layers of order 0 and 1 appear, and we are not able to estimate $n_\lambda - n^a_\lambda$ and $\psi_\lambda - \psi^a_\lambda$ in the same spaces as in (3.1).*

Proof of Theorem 3.1 : We search a solution of the problem (1.2), (1.4), (1.6) of the form :

$$n_\lambda = n_\lambda^a + \lambda^{m-1} r_\lambda, \ \psi_\lambda = \psi_\lambda^a + \lambda^{m-1} p_\lambda,$$

where n_λ^a and ψ_λ^a are the expansions defined in (2.21)-(2.22). To this end we define the two following operators :

$$\begin{aligned} N(n,\psi) &:= L(n,\psi) - \frac{1}{\lambda^2}(n-b) + \varepsilon Q(\psi), \\ M(n,\psi) &:= -\mathrm{div}(n\nabla\psi), \end{aligned}$$

where :

$$\begin{aligned} L(n,\psi) := \Delta h(n) \ &- \ \frac{\varepsilon}{n}\sum_{i,j=1}^{3} \psi_{x_i}\psi_{x_j} n_{x_i x_j} + \frac{\varepsilon}{n}\nabla\psi.\nabla n + \frac{\varepsilon}{n^2}(\nabla\psi.\nabla n)^2 \\ &- \ \frac{\varepsilon}{n}\sum_{i,j=1}^{3} \psi_{x_i}\psi_{x_i x_j} n_{x_j}. \end{aligned}$$

Then the system (1.2),(1.4),(1.6) can be written as :

$$N(n_\lambda,\psi_\lambda)(x,\lambda) = 0, \tag{3.2}$$

$$M(n_\lambda,\psi_\lambda)(x,\lambda) = 0, \ \text{in } \Omega, \tag{3.3}$$

$$n_\lambda = \sum_{k=0}^{m} \lambda^k n_D^k + n_{D,\lambda}^m, \ \psi_\lambda = \sum_{k=0}^{m} \lambda^k \psi_D^k + \psi_{D,\lambda}^m, \ \text{on } \Gamma. \tag{3.4}$$

By construction it is easy to check that (see [18]) :

$$N(n_\lambda^a,\psi_\lambda^a) = O(\lambda^{m-1}), \ M(n_\lambda^a,\psi_\lambda^a) = O(\lambda^m), \ \text{in } L^\infty(\Omega), \tag{3.5}$$

$$N(n_\lambda^a,\psi_\lambda^a) = O(\lambda^{m-2}), \ M(n_\lambda^a,\psi_\lambda^a) = O(\lambda^{m-1}), \ \text{in } C^1(\bar\Omega), \tag{3.6}$$

uniformly with respect to λ.

We search now the system verified by (r_λ, p_λ) by replacing formally n_λ by $n_\lambda^a + \lambda^{m-1} r_\lambda$ and ψ_λ by $\psi_\lambda^a + \lambda^{m-1} p_\lambda$ in (3.2)-(3.4). From (3.3) we have :

$$-\mathrm{div}[(n_\lambda^a + \lambda^{m-1} r_\lambda)\nabla p_\lambda] = \lambda^{-m+1}[-M(n_\lambda^a,\psi_\lambda^a) + \lambda^{m-1}\mathrm{div}(r_\lambda\nabla\psi_\lambda^a)]. \tag{3.7}$$

Similarly, (3.2) gives :

$$\begin{aligned} &\Delta H(r_\lambda,x,\lambda) - \frac{\varepsilon}{n_\lambda^a + \lambda^{m-1} r_\lambda}\sum_{i,j=1}^{3}\left(\frac{\partial \psi_\lambda^a}{\partial x_i} + \lambda^{m-1}\frac{\partial p_\lambda}{\partial x_i}\right)\left(\frac{\partial \psi_\lambda^a}{\partial x_j} + \lambda^{m-1}\frac{\partial p_\lambda}{\partial x_j}\right)\frac{\partial^2 r_\lambda}{\partial x_i \partial x_j} \\ &+\frac{\varepsilon}{(n_\lambda^a + \lambda^{m-1} r_\lambda)^2}\Big[\lambda^{m-1}(\nabla(\psi_\lambda^a + \lambda^{m-1}p_\lambda).\nabla r_\lambda)^2 + 2\nabla(\psi_\lambda^a + \lambda^{m-1}p_\lambda).\nabla r_\lambda \times \\ &(\nabla\psi_\lambda^a.\nabla n_\lambda^a + \lambda^{m-1}\nabla p_\lambda.\nabla n_\lambda^a)\Big] + \frac{\varepsilon}{n_\lambda^a + \lambda^{m-1} r_\lambda}\nabla(\psi_\lambda^a + \lambda^{m-1}p_\lambda).\nabla r_\lambda \\ &-\frac{\varepsilon}{n_\lambda^a + \lambda^{m-1} r_\lambda}\sum_{i,j=1}^{3}\left(\frac{\partial \psi_\lambda^a}{\partial x_i} + \lambda^{m-1}\frac{\partial p_\lambda}{\partial x_i}\right)\left(\frac{\partial^2 \psi_\lambda^a}{\partial x_i \partial x_j} + \lambda^{m-1}\frac{\partial^2 p_\lambda}{\partial x_i \partial x_j}\right)\frac{\partial r_\lambda}{\partial x_j} \\ &-\frac{1}{\lambda^2} r_\lambda(1-\lambda^2\beta(r_\lambda,x)) = \frac{-\lambda^{-m+1} n_\lambda^a}{n_\lambda^a + \lambda^{m-1} r_\lambda} N(n_\lambda^a,\psi_\lambda^a) - g_1(n_\lambda^a,\psi_\lambda^a,r_\lambda,p_\lambda) - g_2(n_\lambda^a,\psi_\lambda^a,r_\lambda) \end{aligned} \tag{3.8}$$

with :

$$H(r_\lambda, x, \lambda) := r_\lambda \int_0^1 h'(n^a_\lambda(x) + \lambda^{m-1} r_\lambda t)dt,$$

$$\begin{aligned} g_1(n^a_\lambda, \psi^a_\lambda, r_\lambda, p_\lambda) \quad &:= \quad -\frac{\varepsilon}{n^a_\lambda + \lambda^{m-1} r_\lambda} \sum_{i,j=1}^3 \left[\frac{\partial \psi^a_\lambda}{\partial x_i}\frac{\partial p_\lambda}{\partial x_j} + \frac{\partial p_\lambda}{\partial x_i}\left(\frac{\partial \psi^a_\lambda}{\partial x_j} + \lambda^{m-1}\frac{\partial p_\lambda}{\partial x_j}\right)\right]\frac{\partial^2 n^a_\lambda}{\partial x_i \partial x_j} \\ &+ \quad \frac{\varepsilon}{(n^a_\lambda + \lambda^{m-1} r_\lambda)^2}\left[\lambda^{m-1}(\nabla p_\lambda . \nabla n^a_\lambda)^2 + 2(\nabla \psi^a_\lambda . \nabla n^a_\lambda)(\nabla p_\lambda . \nabla n^a_\lambda)\right] \\ &+ \quad \frac{\varepsilon}{n^a_\lambda + \lambda^{m-1} r_\lambda}\nabla p_\lambda . \nabla n^a_\lambda - \frac{\varepsilon}{n^a_\lambda + \lambda^{m-1} r_\lambda}\left[\sum_{i,j=1}^3 \frac{\partial \psi^a_\lambda}{\partial x_i}\frac{\partial n^a_\lambda}{\partial x_j}\frac{\partial^2 p_\lambda}{\partial x_i \partial x_j}\right. \\ &+ \quad \left.\sum_{i,j=1}^3 \frac{\partial p_\lambda}{\partial x_i}\frac{\partial n^a_\lambda}{\partial x_j}\left(\frac{\partial^2 \psi^a_\lambda}{\partial x_i \partial x_j} + \lambda^{m-1}\frac{\partial^2 p_\lambda}{\partial x_i \partial x_j}\right)\right] \\ &+ \quad \varepsilon\left[\lambda^{m-1} Q(p_\lambda) + 2\sum_{i,j=1}^3 \frac{\partial^2 \psi^a_\lambda}{\partial x_i \partial x_j}\frac{\partial^2 p_\lambda}{\partial x_i \partial x_j}\right], \end{aligned}$$

$$g_2(n^a_\lambda, \psi^a_\lambda, r_\lambda) := -\frac{\varepsilon r_\lambda}{n^a_\lambda(n^a_\lambda + \lambda^{m-1} r_\lambda)^2}(\nabla \psi^a_\lambda . \nabla n^a_\lambda)^2 + \frac{\varepsilon r_\lambda}{n^a_\lambda + \lambda^{m-1} r_\lambda} Q(\psi^a_\lambda),$$

and,

$$\beta(r_\lambda, x) = \frac{1}{n^a_\lambda(x) + \lambda^{m-1} r_\lambda}\Delta h(n^a_\lambda(x)) - \frac{1}{\lambda^2(n^a_\lambda(x) + \lambda^{m-1} r_\lambda)}(n^a_\lambda(x) - b(x)).$$

From (1.6) the boundary conditions associated to (3.7)-(3.8) are :

$$p_\lambda = \lambda^{-m+1}\psi^m_{D,\lambda}, \ r_\lambda = \lambda^{-m+1} n^m_{D,\lambda}, \text{ on } \Gamma. \tag{3.9}$$

Note that by definition, using the assumptions, the application $H(.,x,\lambda): \ r_\lambda \mapsto H(r_\lambda, x, \lambda)$ is continuous and strictly increasing and then invertible.

To prove Theorem 3.1, it suffices to prove the following Lemma.

Lemma 3.4. *Under the assumptions (H1)-(H6), for λ small enough, there is an $\varepsilon_0 > 0$ independent of λ such that for all $\varepsilon \in [0, \varepsilon_0]$, the problem (3.7)-(3.9) has a solution $(r_\lambda, p_\lambda) \in W^{2,q}(\Omega) \times C^{2,\delta}(\bar{\Omega})$ which satisfies :*

$$\|r_\lambda\|_{W^{2,q}(\Omega)} \le A_1, \ \|p_\lambda\|_{C^{2,\delta}(\bar{\Omega})} \le A_1,$$

where A_1 is a constant independent of λ.

Proof : Let $\sigma_\lambda \in S$ with :

$$S = \{\rho \in C^{1,\delta}(\bar{\Omega}); \ \|\rho\|_{C^{1,\delta}(\bar{\Omega})} \le K\},$$

where K is a constant independent of λ which will be fixed later. Here S is a closed and convex set. We define the mapping $T : \sigma_\lambda \mapsto p_\lambda \mapsto r_\lambda$ by :
(A) the solution of :

$$\begin{aligned} &-\text{div}[(n^a_\lambda + \lambda^{m-1}\sigma_\lambda)\nabla p_\lambda] = f_1(\sigma_\lambda, \psi^a_\lambda, n^a_\lambda), \text{ in } \Omega, \\ &p_\lambda = \lambda^{-m+1}\psi^m_{D,\lambda}, \text{ on } \Gamma \end{aligned}$$

with

$$f_1(\sigma_\lambda, \psi_\lambda^a, n_\lambda^a) := \lambda^{-m+1}\Big[-M(n_\lambda^a, \psi_\lambda^a) + \lambda^{m-1}\mathrm{div}(\sigma_\lambda \nabla \psi_\lambda^a)\Big];$$

(B) let $r_\lambda = G(v_\lambda, x, \lambda)$ where $G(., x, \lambda) = H^{-1}(., x, \lambda)$ with v_λ being solution of :

$$\begin{aligned}
&\Delta v_\lambda - \frac{\varepsilon G'(H(\sigma_\lambda, x, \lambda), x, \lambda)}{n_\lambda^a + \lambda^{m-1}\sigma_\lambda} \sum_{i,j=1}^{3} \frac{\partial \psi_\lambda}{\partial x_i} \frac{\partial \psi_\lambda}{\partial x_j} \frac{\partial^2 v_\lambda}{\partial x_i \partial x_j} \\
&- \frac{\varepsilon G''(H(\sigma_\lambda, x, \lambda), x, \lambda)}{(n_\lambda^a + \lambda^{m-1}\sigma_\lambda) G'(H(\sigma_\lambda, x, \lambda), x, \lambda)} \sum_{i,j=1}^{3} \frac{\partial \psi_\lambda}{\partial x_i} \frac{\partial \psi_\lambda}{\partial x_j} \frac{\partial \sigma_\lambda}{\partial x_i} \frac{\partial v_\lambda}{\partial x_j} \\
&- \frac{\varepsilon}{n_\lambda^a + \lambda^{m-1}\sigma_\lambda} \sum_{i,j=1}^{3} \frac{\partial \psi_\lambda}{\partial x_i} \frac{\partial \psi_\lambda}{\partial x_j} \left(\frac{\partial G'(H(\sigma_\lambda, x, \lambda), x, \lambda)}{\partial x_j} \frac{\partial v_\lambda}{\partial x_i} + \frac{\partial G'(H(\sigma_\lambda, x, \lambda), x, \lambda)}{\partial x_i} \frac{\partial v_\lambda}{\partial x_j} \right) \\
&- \frac{\varepsilon G''(H(\sigma_\lambda, x, \lambda), x, \lambda)}{n_\lambda^a + \lambda^{m-1}\sigma_\lambda} \sum_{i,j=1}^{3} \frac{\partial \psi_\lambda}{\partial x_i} \frac{\partial \psi_\lambda}{\partial x_j} \frac{\partial H(\sigma_\lambda, x, \lambda)}{\partial x_i} \frac{\partial v_\lambda}{\partial x_j} \\
&+ \frac{\varepsilon G'(H(\sigma_\lambda, x, \lambda), x, \lambda)}{(n_\lambda^a + \lambda^{m-1}\sigma_\lambda)^2} \Big[\lambda^{m-1}(\nabla \psi_\lambda . \nabla \sigma_\lambda) \nabla \psi_\lambda + 2(\nabla \psi_\lambda^a . \nabla n_\lambda^a + \lambda^{m-1} \nabla p_\lambda . \nabla n_\lambda^a) \nabla \psi_\lambda\Big] . \nabla v_\lambda \\
&+ \frac{\varepsilon G'(H(\sigma_\lambda, x, \lambda), x, \lambda)}{n_\lambda^a + \lambda^{m-1}\sigma_\lambda} \nabla \psi_\lambda . \nabla v_\lambda - \frac{\varepsilon G'(H(\sigma_\lambda, x, \lambda), x, \lambda)}{n_\lambda^a + \lambda^{m-1}\sigma_\lambda} \sum_{i,j=1}^{3} \frac{\partial \psi_\lambda}{\partial x_i} \frac{\partial^2 \psi_\lambda}{\partial x_i \partial x_j} \frac{\partial v_\lambda}{\partial x_j} \\
&- \frac{1}{\lambda^2} G(v_\lambda, x, \lambda)(1 - \lambda^2 \beta(\sigma_\lambda, x)) = f_2(\sigma_\lambda, x, \lambda), \; x \in \Omega, \\
&v_\lambda = H(\lambda^{-m+1} n_{D,\lambda}^m, x, \lambda), \; \text{on } \Gamma,
\end{aligned}$$

where we note :

$$G'(v_\lambda, x, \lambda) = \frac{\partial G}{\partial v_\lambda}(v_\lambda, x, \lambda), \; G''(v_\lambda, x, \lambda) = \frac{\partial^2 G}{\partial v_\lambda^2}(v_\lambda, x, \lambda),$$

$$\begin{aligned}
f_2(\sigma_\lambda, x, \lambda) \;\; = & \;\; -\lambda^{-m+1} \frac{n_\lambda^a}{n_\lambda^a + \lambda^{m-1}\sigma_\lambda} N(n_\lambda^a, \psi_\lambda^a) - g_1(n_\lambda^a, \psi_\lambda^a, \sigma_\lambda, p_\lambda) \\
- & \;\; g_2(n_\lambda^a, \psi_\lambda^a, \sigma_\lambda) + \frac{\varepsilon}{n_\lambda^a + \lambda^{m-1}\sigma_\lambda} \sum_{i,j=1}^{3} \frac{\partial \psi_\lambda}{\partial x_i} \frac{\partial \psi_\lambda}{\partial x_j} \frac{\partial^2 G}{\partial x_i \partial x_j}(H(\sigma_\lambda, x, \lambda), x, \lambda) \\
- & \;\; \frac{\varepsilon}{(n_\lambda^a + \lambda^{m-1}\sigma_\lambda)^2} \Big[\lambda^{m-1}(\nabla \psi_\lambda . \nabla \sigma_\lambda) \nabla \psi_\lambda + 2(\nabla \psi_\lambda^a . \nabla n_\lambda^a + \lambda^{m-1} \nabla p_\lambda . \nabla n_\lambda^a) \nabla \psi_\lambda\Big] \\
& \;\; . \nabla_x G(H(\sigma_\lambda, x, \lambda), x, \lambda) \\
- & \;\; \frac{\varepsilon}{n_\lambda^a + \lambda^{m-1}\sigma_\lambda} \nabla \psi_\lambda . \nabla_x G(H(\sigma_\lambda, x, \lambda), x, \lambda) \\
+ & \;\; \frac{\varepsilon}{n_\lambda^a + \lambda^{m-1}\sigma_\lambda} \sum_{i,j=1}^{3} \frac{\partial \psi_\lambda}{\partial x_i} \frac{\partial^2 \psi_\lambda}{\partial x_i \partial x_j} \frac{\partial G}{\partial x_j}(H(\sigma_\lambda, x, \lambda), x, \lambda).
\end{aligned}$$

By construction n_λ^a and ψ_λ^a being sufficiently smooth, $f_1 \in C^{0,\delta}(\bar{\Omega})$ and is bounded in $C^{0,\delta}(\bar{\Omega})$ uniformly in λ. Moreover by assumption $n_0 \geq \underline{n} > 0$, hence for λ small enough we have : $n^a_\lambda + \lambda^{m-1}\sigma_\lambda \geq \frac{1}{2}\underline{n} > 0$. Finally, using the Theorem 6.6 in [10] we obtain that the problem (A) has a unique solution $p_\lambda \in C^{2,\delta}(\bar{\Omega})$ and :

$$\|p_\lambda\|_{C^{2,\delta}(\bar{\Omega})} \leq C(K),$$

with $C(K)$ a constant independent of λ. Therefore, there exists $\varepsilon_1 > 0$, independent of λ, such that for all $\varepsilon \in [0, \varepsilon_1]$, (B) is an elliptic problem.

Here, we cannot use the classical result used in [5] due to the fact that the function G depends not only on v_λ, but also on the variable x. The idea here is to use the Leray-Schauder fixed point Theorem to show first the existence and boundedness of solutions v_λ to (B). This implies the existence and boundedness of $r_\lambda = G(v_\lambda, x, \lambda)$ which are needed to prove that T is an application from S to S, and then to apply the Schauder fixed point Theorem. More precisely, we use the Leray-Schauder fixed point Theorem to prove the following Lemma.

Lemma 3.5. *Under the assumptions (H1)-(H6), for λ small enough, there is an $\varepsilon_0 > 0$, independent of λ such that for all $\varepsilon \in [0, \varepsilon_0]$, (B) has a unique solution $v_\lambda \in W^{2,q}(\Omega)$ which satisfies :*

$$\|v_\lambda\|_{W^{2,q}(\Omega)} \leq A_2, \text{ with } A_2 \text{ independent of } \lambda, K.$$

Proof : As mention previously, to prove this lemma, we will use the Leray-Schauder fixed point theorem. Let $\tau \in [0,1]$ and $w \in W^{2,q}(\Omega)$. We define the application $\tilde{T} : W^{2,q}(\Omega) \times [0,1] \to C^{1,\delta}(\bar{\Omega})$ by $(w, \tau) \mapsto v$ where v solves the problem :

$$\begin{aligned}
&\Delta v - \frac{\varepsilon G'(H(\sigma_\lambda, x, \lambda), x, \lambda)}{n^a_\lambda + \lambda^{m-1}\sigma_\lambda} \sum_{i,j=1}^{3} \frac{\partial \psi_\lambda}{\partial x_i} \frac{\partial \psi_\lambda}{\partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} \\
&- \frac{\tau\varepsilon G''(H(\sigma_\lambda, x, \lambda), x, \lambda)}{(n^a_\lambda + \lambda^{m-1}\sigma_\lambda) G'(H(\sigma_\lambda, x, \lambda), x, \lambda)} \sum_{i,j=1}^{3} \frac{\partial \psi_\lambda}{\partial x_i} \frac{\partial \psi_\lambda}{\partial x_j} \frac{\partial \sigma_\lambda}{\partial x_i} \frac{\partial w}{\partial x_j} \\
&- \frac{\tau\varepsilon}{n^a_\lambda + \lambda^{m-1}\sigma_\lambda} \sum_{i,j=1}^{3} \frac{\partial \psi_\lambda}{\partial x_i} \frac{\partial \psi_\lambda}{\partial x_j} \left(\frac{\partial G'(H(\sigma_\lambda, x, \lambda), x, \lambda)}{\partial x_j} \frac{\partial w}{\partial x_i} + \frac{\partial G'(H(\sigma_\lambda, x, \lambda), x, \lambda)}{\partial x_i} \frac{\partial w}{\partial x_j} \right) \\
&- \frac{\tau\varepsilon G''(H(\sigma_\lambda, x, \lambda), x, \lambda)}{n^a_\lambda + \lambda^{m-1}\sigma_\lambda} \sum_{i,j=1}^{3} \frac{\partial \psi_\lambda}{\partial x_i} \frac{\partial \psi_\lambda}{\partial x_j} \frac{\partial H(\sigma_\lambda, x, \lambda)}{\partial x_i} \frac{\partial w}{\partial x_j} \\
&+ \frac{\tau\varepsilon G'(H(\sigma_\lambda, x, \lambda), x, \lambda)}{(n^a_\lambda + \lambda^{m-1}\sigma_\lambda)^2} \Big[\lambda^{m-1} (\nabla\psi_\lambda . \nabla\sigma_\lambda) \nabla\psi_\lambda + 2(\nabla\psi^a_\lambda . \nabla n^a_\lambda + \lambda^{m-1} \nabla p_\lambda . \nabla n^a_\lambda) \nabla\psi_\lambda \Big] . \nabla w \\
&+ \frac{\tau\varepsilon G'(H(\sigma_\lambda, x, \lambda), x, \lambda)}{n^a_\lambda + \lambda^{m-1}\sigma_\lambda} \nabla\psi_\lambda . \nabla w - \frac{\tau\varepsilon G'(H(\sigma_\lambda, x, \lambda), x, \lambda)}{n^a_\lambda + \lambda^{m-1}\sigma_\lambda} \sum_{i,j=1}^{3} \frac{\partial \psi_\lambda}{\partial x_i} \frac{\partial^2 \psi_\lambda}{\partial x_i \partial x_j} \frac{\partial w}{\partial x_j} \\
&- \frac{\tau}{\lambda^2} G(w, x, \lambda)(1 - \lambda^2 \beta(\sigma_\lambda, x)) = \tau f_2(\sigma_\lambda, x, \lambda), \; x \in \Omega, \\
&v = \tau H(\lambda^{-m+1} n^m_{D,\lambda}, x, \lambda), \text{ on } \Gamma.
\end{aligned}$$

This problem is elliptic and admits a unique solution for ε small enough, hence $\tilde{T}$ is well-defined. We have : $\tilde{T}(w,0) = v_1$ where v_1 is a solution of the linear problem :

$$\Delta v_1 - \frac{\varepsilon G'(H(\sigma_\lambda, x, \lambda), x, \lambda)}{n_\lambda^a + \lambda^{m-1}\sigma_\lambda} \sum_{i,j=1}^{3} \frac{\partial \psi_\lambda}{\partial x_i} \frac{\partial \psi_\lambda}{\partial x_j} \frac{\partial^2 v_1}{\partial x_i \partial x_j} = 0$$
$$v_1 = 0, \text{ on } \Gamma.$$

Then $v_1 = 0$ and hence $\tilde{T}(w,0) = 0$. Furthermore, it is not difficult to check that $\tilde{T}$ is continuous and also compact (due to the compact injection from $W^{2,q}(\Omega)$ to $C^{1,\delta}(\bar{\Omega})$).

To achieve the proof of Lemma 3.5, we need to prove that if v_2 is a fixed point of $\tilde{T}$ then we have

$$\|v_2\|_{W^{2,q}(\Omega)} \leq A_3,$$

with A_3 being independent of λ, K. This is the statement of Lemma 3.6. Since its proof being technical, it is moved to Appendix.

Lemma 3.6. *Assuming (H1)-(H6), let v_2 be a fixed point of $\tilde{T}$. Then for λ small enough there is an $\varepsilon_0 > 0$, independent of λ such that for all $\varepsilon \in [0, \varepsilon_0]$:*

$$\|v_2\|_{W^{2,q}(\Omega)} \leq A_3, \ \forall \tau \in [0,1],$$

where A_3 is a constant independent of λ and K.

Then by the Leray-Schauder fixed point theorem, for λ small enough, and $\varepsilon \leq \varepsilon_0$, ε_0 independent of λ, $\tilde{T}_1 = \tilde{T}(w,1)$ has a fixed point $v_\lambda \in W^{2,q}(\Omega)$ such that :

$$\|v_\lambda\|_{W^{2,q}(\Omega)} \leq A_2, \text{ with } A_2 \text{ independent of } \lambda, K.$$

By definition of $\tilde{T}, v_\lambda$ is a solution to the problem (B). For the uniqueness of solution, we assume that there exist two solutions $v_{\lambda,1}, v_{\lambda,2}$, we substract the two systems. Using Theorem 9.15 in [10], we show that the obtained system has the unique solution zero. This gives $v_{\lambda,1} = v_{\lambda,2}$, and then we obtain the uniqueness of solutions for (B). This completes the proof of Lemma 3.5.

Now using Lemma 3.5 and the injection $W^{2,q}(\Omega) \hookrightarrow C^{1,\delta}(\bar{\Omega})$, with $K = cA_2$, where c is the injection constant, we obtain $v_\lambda \in S$ which implies $r_\lambda \in S$ and then $T : S \to S$. To complete the verification of the assumptions in the Schauder fixed point Theorem for T we need to show that T is continuous and compact. This is the statement of the following Lemma.

Lemma 3.7. *The application T, as an application from $C^{1,\delta}(\bar{\Omega})$ to $C^{1,\delta}(\bar{\Omega})$, is continuous and compact.*

Proof :

We define $T = I \circ G(.,x,\lambda) \circ \chi \circ \gamma$ by :

$$I : W^{2,q}(\Omega) \hookrightarrow C^{1,\delta}(\bar{\Omega}), \text{ which is continuous and compact,}$$
$$G(.,x,\lambda) : v_\lambda \mapsto r_\lambda, \text{ which is continuous by assumption,}$$
$$\gamma : C^{1,\delta}(\bar{\Omega}) \longrightarrow C^{2,\delta}(\bar{\Omega}), \ \sigma_\lambda \mapsto p_\lambda, \text{ where } p_\lambda \text{ is solution of (A),}$$
$$\chi : C^{2,\delta}(\bar{\Omega}) \longrightarrow W^{2,q}(\Omega), \ p_\lambda \mapsto v_\lambda, \text{ where } v_\lambda \text{ is solution of (B).}$$

We have to show that χ and γ are continuous.

Let $p_{\lambda,1}$ and $p_{\lambda,2}$ be two solutions of (A). Then by substraction of the two systems we obtain the following one :

$$\begin{aligned}&-\mathrm{div}[(n_\lambda^a+\lambda^{m-1}\sigma_{\lambda,1})\nabla(p_{\lambda,1}-p_{\lambda,2})]=f,\ \text{ in }\Omega,\\&p_{\lambda,1}-p_{\lambda,2}=0,\ \text{ on }\Gamma,\end{aligned}$$

with :

$$f=f_1(\sigma_{\lambda,1},\psi_\lambda^a,n_\lambda^a)-f_1(\sigma_{\lambda,2},\psi_\lambda^a,n_\lambda^a)-\lambda^{m-1}\mathrm{div}[(\sigma_{\lambda,1}-\sigma_{\lambda,2})\nabla p_{\lambda,2}].$$

It is clear that :

$$\|\lambda^{m-1}\mathrm{div}[(\sigma_{\lambda,1}-\sigma_{\lambda,2})\nabla p_{\lambda,2}]\|_{C^{0,\delta}(\bar\Omega)}\le\tilde C\|\sigma_{\lambda,1}-\sigma_{\lambda,2}\|_{C^{1,\delta}(\bar\Omega)},$$

since $\|p_{\lambda,2}\|_{C^{2,\delta}(\bar\Omega)}\le\tilde C$.
Moreover, by definition we have :

$$f_1(\sigma_{\lambda,1},\psi_\lambda^a,n_\lambda^a)-f_1(\sigma_{\lambda,2},\psi_\lambda^a,n_\lambda^a)=\mathrm{div}[(\sigma_{\lambda,1}-\sigma_{\lambda,2})\nabla\psi_\lambda^a],$$

and by construction : $\|\psi_\lambda^a\|_{C^{2,\delta}(\bar\Omega)}\le\tilde C$. Then

$$\|f_1(\sigma_{\lambda,1},\psi_\lambda^a,n_\lambda^a)-f_1(\sigma_{\lambda,2},\psi_\lambda^a,n_\lambda^a)\|_{C^{0,\delta}(\bar\Omega)}\le\tilde C\|\sigma_{\lambda,1}-\sigma_{\lambda,2}\|_{C^{1,\delta}(\bar\Omega)}.$$

Hence we obtain that :

$$\|f\|_{C^{0,\delta}(\bar\Omega)}\le\tilde C\|\sigma_{\lambda,1}-\sigma_{\lambda,2}\|_{C^{1,\delta}(\bar\Omega)}.$$

Using Theorem 6.6 in [10]

$$\|p_{\lambda,1}-p_{\lambda,2}\|_{C^{2,\delta}(\bar\Omega)}\le\|f\|_{C^{0,\delta}(\bar\Omega)}.$$

Hence :

$$\|p_{\lambda,1}-p_{\lambda,2}\|_{C^{2,\delta}(\bar\Omega)}\le\tilde C\|\sigma_{\lambda,1}-\sigma_{\lambda,2}\|_{C^{1,\delta}(\bar\Omega)},$$

and the application γ is continuous from $C^{1,\delta}(\bar\Omega)$ to $C^{2,\delta}(\bar\Omega)$.

In a same way, with the pressure function p_λ smooth enough we obtain that χ is a continuous application from $C^{2,\delta}(\bar\Omega)$ to $W^{2,q}(\Omega)$. Therefore, T is continuous. The application I being compact, we have that T is also compact. This completes the proof of Lemma 3.7.

Finally all the assumptions in the Schauder fixed point Theorem are satisfied. As a consequence, for λ small enough and $\varepsilon\le\varepsilon_0$, ε_0 being independent of λ, T has a fixed point. This completes the proof of Lemma 3.4.

Theorem 3.8. *Assume (H1)-(H6). If in addition, $(\lambda^{-m-1}n_{D,\lambda}^m)_{\lambda>0}$ is bounded in $W^{2,\infty}(\Omega)$, and $(\lambda^{-m-1}\psi_{D,\lambda}^m)_{\lambda>0}$ is bounded in $W^{1,q}(\Omega)$. Then,*

$$\|n_\lambda-n_\lambda^a\|_{L^\infty(\Omega)}\le A_4\lambda^{m+1},\ \ \|\psi_\lambda-\psi_\lambda^a\|_{W^{1,q}(\Omega)}\le A_4\lambda^{m+1},$$

where A_4 is a constant independent of λ.

Since the proof uses notations defined in the proof of Lemma 3.6, it is also moved in Appendix.

Appendix :

Proof of Lemma 3.6.
Let v_2 be a fixed point of $\tilde{T}$. Let :

$$\begin{aligned}
&\tilde{L}(\tau,\sigma_\lambda,n_\lambda^a,\psi_\lambda^a,p_\lambda)v_2 := -\Delta v_2 + \frac{\varepsilon G'(H(\sigma_\lambda,x,\lambda),x,\lambda)}{n_\lambda^a+\lambda^{m-1}\sigma_\lambda}\sum_{i,j=1}^{3}\frac{\partial\psi_\lambda}{\partial x_i}\frac{\partial\psi_\lambda}{\partial x_j}\frac{\partial^2 v_2}{\partial x_i\partial x_j}\\
&+\frac{\tau\varepsilon G''(H(\sigma_\lambda,x,\lambda),x,\lambda)}{(n_\lambda^a+\lambda^{m-1}\sigma_\lambda)G'(H(\sigma_\lambda,x,\lambda),x,\lambda)}\sum_{i,j=1}^{3}\frac{\partial\psi_\lambda}{\partial x_i}\frac{\partial\psi_\lambda}{\partial x_j}\frac{\partial\sigma_\lambda}{\partial x_i}\frac{\partial v_2}{\partial x_j}\\
&+\frac{\tau\varepsilon}{n_\lambda^a+\lambda^{m-1}\sigma_\lambda}\sum_{i,j=1}^{3}\frac{\partial\psi_\lambda}{\partial x_i}\frac{\partial\psi_\lambda}{\partial x_j}\Big(\frac{\partial G'(H(\sigma_\lambda,x,\lambda),x,\lambda)}{\partial x_j}\frac{\partial v_2}{\partial x_i}\\
&\qquad\qquad+\frac{\partial G'(H(\sigma_\lambda,x,\lambda),x,\lambda)}{\partial x_i}\frac{\partial v_2}{\partial x_j}\Big)\\
&+\frac{\tau\varepsilon G''(H(\sigma_\lambda,x,\lambda),x,\lambda)}{n_\lambda^a+\lambda^{m-1}\sigma_\lambda}\sum_{i,j=1}^{3}\frac{\partial\psi_\lambda}{\partial x_i}\frac{\partial\psi_\lambda}{\partial x_j}\frac{\partial H(\sigma_\lambda,x,\lambda)}{\partial x_i}\frac{\partial v_2}{\partial x_j}\\
&-\frac{\tau\varepsilon G'(H(\sigma_\lambda,x,\lambda),x,\lambda)}{(n_\lambda^a+\lambda^{m-1}\sigma_\lambda)^2}\Big[\lambda^{m-1}(\nabla\psi_\lambda.\nabla\sigma_\lambda)\nabla\psi_\lambda+2(\nabla\psi_\lambda^a.\nabla n_\lambda^a\\
&\qquad\qquad+\lambda^{m-1}\nabla p_\lambda.\nabla n_\lambda^a)\nabla\psi_\lambda\Big].\nabla v_2\\
&-\frac{\tau\varepsilon G'(H(\sigma_\lambda,x,\lambda),x,\lambda)}{n_\lambda^a+\lambda^{m-1}\sigma_\lambda}\nabla\psi_\lambda.\nabla v_2+\frac{\tau\varepsilon G'(H(\sigma_\lambda,x,\lambda),x,\lambda)}{n_\lambda^a+\lambda^{m-1}\sigma_\lambda}\sum_{i,j=1}^{3}\frac{\partial\psi_\lambda}{\partial x_i}\frac{\partial^2\psi_\lambda}{\partial x_i\partial x_j}\frac{\partial v_2}{\partial x_j}.
\end{aligned}$$

For $\varepsilon\in[0,\varepsilon_1]$, the differential operator $\tilde{L}$ is elliptic. By definition v_2 is solution of :

$$\text{(A0)}\quad \begin{aligned}
&-\tilde{L}(\tau,\sigma_\lambda,n_\lambda^a,\psi_\lambda^a,p_\lambda)v_2-\frac{\tau}{\lambda^2}G(v_2,x,\lambda)(1-\lambda^2\beta(\sigma_\lambda,x))=\tau f_2(\sigma_\lambda,x,\lambda),\ x\in\Omega,\\
&v_2=\tau H(\lambda^{-m+1}n_{D,\lambda}^m,x,\lambda),\ \text{on }\Gamma.
\end{aligned}$$

Let

$$\begin{aligned}
&f_3(\tau,\sigma_\lambda,x,\lambda):=f_2(\sigma_\lambda,x,\lambda)+\frac{1}{\lambda^2}G(B_\tau,x,\lambda)(1-\lambda^2\beta(\sigma_\lambda,x)),\\
&B_\tau:=\tau H(\lambda^{-m+1}n_{D,\lambda}^m,x,\lambda).
\end{aligned}$$

Then (A0) can be rewritten as:

$$\begin{aligned}
&\tilde{L}(\tau,\sigma_\lambda,n_\lambda^a,\psi_\lambda^a,p_\lambda)v_2+\frac{\tau}{\lambda^2}\big(G(v_2,x,\lambda)-G(B_\tau,x,\lambda)\big)(1-\lambda^2\beta(\sigma_\lambda,x))\\
&\qquad\qquad=-\tau f_3(\tau,\sigma_\lambda,x,\lambda),\ \text{in }\Omega,\\
&v_2=B_\tau,\ \text{on }\Gamma.
\end{aligned}$$

We can show that for λ small enough,

$$\text{(A1)}\qquad \frac{1}{2}\le 1-\lambda^2\beta(\sigma_\lambda,x)\le C_1,\ \forall x\in\Omega,$$

with C_1 a constant independent of λ and K. Using the result of [16] we show that, there exists $\varepsilon_2 > 0$, independent of λ, such that for all $\varepsilon \in [0, \varepsilon_2]$:

$$\int_\Omega z|z|^{q-2}\tilde{L}(\tau, \sigma_\lambda, n^a_\lambda, \psi^a_\lambda, p_\lambda)z dx \geq 0,\ \forall z \in W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega).$$

Let $u = v_2 - B_\tau$. We have $u \in W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega)$ and u is a solution of :

$$\text{(A2)}\quad \begin{aligned} \tilde{L}(\tau, \sigma_\lambda, n^a_\lambda, \psi^a_\lambda, p_\lambda)u &+ \frac{\tau}{\lambda^2}\big(G(u + B_\tau, x, \lambda) - G(B_\tau, x, \lambda)\big)(1 - \lambda^2\beta(\sigma_\lambda, x)) \\ &= -\tau f_3(\tau, \sigma_\lambda, x, \lambda) - \tilde{L}(\tau, \sigma_\lambda, n^a_\lambda, \psi^a_\lambda, p_\lambda)B_\tau,\ \text{in } \Omega \\ u &= 0,\ \text{on } \Gamma. \end{aligned}$$

For λ small enough, the function $H(., x, \lambda)$ is strictly increasing and so is $G(., x, \lambda)$ by definition. Then there exists two constants C_2 and C_3 independent of λ, τ and K such that :

$$\text{(A3)}\qquad C_2u^2 \leq \big(G(u + B_\tau, x, \lambda) - G(B_\tau, x, \lambda)\big)u = u^2\int_0^1 G'(u + tB_\tau, x, \lambda)dt \leq C_3u^2.$$

We multiply (A2) by $u|u|^{q-2}$ and we integrate on Ω. Then using (A1), (A3), for λ small enough, by Hölder inequality,

$$\begin{aligned} \frac{C_2\tau}{2\lambda^2}\int_\Omega |u|^q dx &\leq -\int_\Omega \big(\tau f_3(\tau, \sigma_\lambda, x, \lambda) + \tilde{L}(\tau, \sigma_\lambda, n^a_\lambda, \psi^a_\lambda, p_\lambda)B_\tau\big)u|u|^{q-2}dx \\ &\leq \big(\tau\|f_3(\tau, \sigma_\lambda, x, \lambda)\|_{L^q(\Omega)} + \|\tilde{L}(\tau, \sigma_\lambda, n^a_\lambda, \psi^a_\lambda, p_\lambda)B_\tau\|_{L^q(\Omega)}\big)\|u\|^{q-1}_{L^q(\Omega)}. \end{aligned}$$

Then :

$$\|u\|_{L^q(\Omega)} \leq \frac{2\lambda^2}{C_2}\bigg(\|f_3(\tau, \sigma_\lambda, x, \lambda)\|_{L^q(\Omega)} + \frac{1}{\tau}\|\tilde{L}(\tau, \sigma_\lambda, n^a_\lambda, \psi^a_\lambda, p_\lambda)B_\tau\|_{L^q(\Omega)}\bigg).$$

For λ small enough, there exists $\varepsilon_3 > 0$ independent of λ such that for all $\varepsilon \in [0, \varepsilon_3]$:

$$\|f_3(\tau, \sigma_\lambda, x, \lambda)\|_{L^q(\Omega)} \leq C_4,\ \frac{1}{\tau}\|\tilde{L}(\tau, \sigma_\lambda, n^a_\lambda, \psi^a_\lambda, p_\lambda)B_\tau\|_{L^q(\Omega)} \leq C_5,$$

with C_4 and C_5 constants independent of λ, τ, K. Hence :

$$\|u\|_{L^q(\Omega)} \leq C_6\lambda^2,$$

with C_6 independent of λ, τ, K. Like in [16], we obtain, for λ small enough,

$$\|u\|_{W^{2,q}(\Omega)} \leq C_7,$$

with C_7 independent of λ, τ, K. Finally :

$$\|v_2\|_{W^{2,q}(\Omega)} \leq \|u\|_{W^{2,q}(\Omega)} + \|B\|_{W^{2,q}(\Omega)}.$$

This completes the proof of Lemma 3.6 with $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2, \varepsilon_3)$.

Proof of Theorem 3.8. In all the following, the constants C_i are independent of λ. It is clear that $u_\lambda = v_\lambda - B_1$, where $B_1 = B_\tau$ for $\tau = 1$, is solution of :

$$\text{(A4)}\quad \begin{aligned} \tilde{L}(1, r_\lambda, n^a_\lambda, \psi^a_\lambda, p_\lambda)u_\lambda &+ \frac{1}{\lambda^2}\big(G(u_\lambda + B_1, x, \lambda) - G(B_1, x, \lambda)\big)(1 - \lambda^2\beta(r_\lambda, x)) \\ &= -f_3(1, r_\lambda, x, \lambda) - \tilde{L}(1, r_\lambda, n^a_\lambda, \psi^a_\lambda, p_\lambda)B_1,\ \text{in } \Omega \\ u_\lambda &= 0,\ \text{on } \Gamma. \end{aligned}$$

Since by assumption $(\lambda^{-m-1}n^m_{D,\lambda})_{\lambda>0}$ is bounded in $W^{2,\infty}(\Omega)$, we have :

$$\|f_3(1,r_\lambda,x,\lambda)+\tilde{L}(1,r_\lambda,n^a_\lambda,\psi^a_\lambda,p_\lambda)B_1\|_{L^\infty(\Omega)} \leq C_8.$$

Let :

$$\bar{u}=\frac{2\lambda^2 C_8}{C_2}, \text{ and } \underline{u}=-\frac{2\lambda^2 C_8}{C_3}.$$

We can show that $\bar{u}$ (resp. $\underline{u}$) is an upper-solution (resp. lower-solution) of (A4). Hence,

$$\underline{u}\leq u_\lambda \leq \bar{u}, \text{ and, } \|u_\lambda\|_{L^\infty(\Omega)} \leq C_9\lambda^2.$$

Using the assumption on $n^m_{D,\lambda}$,

$$\|B_1\|_{L^\infty(\Omega)} \leq C_{10}\lambda^2.$$

Hence :

$$\|v_\lambda\|_{L^\infty(\Omega)} \leq \|u_\lambda\|_{L^\infty(\Omega)} + \|B_1\|_{L^\infty(\Omega)} \leq C_{11}\lambda^2.$$

Then, using the continuity of $H(.,x,\lambda)$, we obtain

$$\|r_\lambda\|_{L^\infty(\Omega)} \leq C_{12}\lambda^2,$$

which gives the first estimate in Theorem 3.8. Using the equations satisfied by the boundary layers profiles, we can show that

$$M(n^a_\lambda,\psi^a_\lambda)=O(\lambda^{m+1}) \text{ in } W^{-1,q}(\Omega).$$

Then in a same way than in [16], using the assumption on $\psi^m_{D,\lambda}$, the boundedness of r_λ in $L^\infty(\Omega)$, $\psi^a_\lambda \in C^{2,\delta}(\bar{\Omega})$, (3.5), and the strict positivity of n_λ, for λ small enough, we have :

$$\|p_\lambda\|_{W^{1,q}(\Omega)} \leq C_{13}\lambda^2.$$

This completes the proof of Theorem 3.8.

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