Asymptotic expansions in a steady state Euler-Poisson system and convergence to incompressible Euler equations

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Abstract : This work is concerned with a steady state Euler-Poisson system for potential flows arising in mathematical modeling for plasmas and semiconductors. We study the zero electron mass limit and the zero relaxation time limit of the system by using the method of asymptotic expansions. These two limits are expressed by the Maxwell-Boltzmann relation and the classical drift-diffusion model, respectively. For each limit, we show the existence and uniqueness of profiles and justify the asymptotic expansions up to any order. These results give also new approaches for the convergence of the Euler-Poisson system to incompressible Euler equations, which have been already obtained via the quasi-neutral limit.

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1 Introduction

The Euler-Poisson system is a hydrodynamic model widely used in the mathematical modeling and numerical simulation for plasmas [3] and semiconductors [10]. It consists of two nonlinear equations given by the conservation of density and momentum, called Euler equations, plus a Poisson equation for the electric potential. Due to the nonlinear hyperbolicity, the weak solution of the transient Euler equations is only studied in one

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space dimension. In such a situation, the existence of global weak solutions can be shown in the set of bounded functions.

In this paper, we study a steady state unipolar model for electrons. Then the corresponding Euler-Poisson system reads as follows :

$$-\mathrm{div}(nu) = 0, \tag{1.1}$$

$$\varepsilon \mathrm{div}(nu \otimes u) + \nabla p(n) = n\nabla\phi - \frac{\varepsilon n u}{\tau}, \tag{1.2}$$

$$-\lambda^2 \Delta \phi = b(x) - n. \tag{1.3}$$

This system will be studied in an open and bounded domain Ω of $\mathbb{R}^d$ ($d = 2$ or $d = 3$ in practice). Here $n = n(x)$, $u = u(x)$ and $\phi = \phi(x)$ stand for the electron density, the electron velocity and the electric potential, respectively. The given functions $b = b(x)$ is the doping profile for semiconductors and the ion density for the plasmas, and $p = p(n)$ is the pressure function. We assume that b and p are smooth functions, p is strictly increasing for $n > 0$, and there is a constant $\underline{b} > 0$ such that : $b(x) \geq \underline{b}$ for all $x \in \bar{\Omega}$. The physical parameters λ, ε and τ stand for the Debye length, the electron mass and the relaxation time of the system, respectively. They are small compared with the characteristic length of physical interest. Therefore, it is important to give the mathematical justification of these limits. In this paper, we study independently the zero electron mass limit and the zero relaxation time limit when $\lambda > 0$ is fixed. For simplicity, we assume that $\lambda = 1$.

We consider the case of a potential flow, $\mathrm{curl} u = 0$. Then by introducing a velocity potential ψ defined by $u = -\nabla\psi$ and using

$$\mathrm{div}(nu \otimes u) = \frac{n}{2}\nabla|\nabla\psi|^2 + u\,\mathrm{div}(nu),$$

we obtain from (1.1)-(1.2) :

$$\nabla\Big(\frac{1}{2}|\nabla\psi|^2 + h(n) - \phi\Big) = \nabla\psi,$$

where h is the enthalpy of the system defined by :

$$h(n) = \int_1^n \frac{p'(y)}{y}\,dy.$$

Hence, for smooth solutions, the system (1.1)-(1.3) can be written in the form :

$$-\mathrm{div}(n\nabla\psi) = 0, \tag{1.4}$$

$$\frac{\varepsilon}{2}|\nabla\psi|^2 + h(n) = \phi + \frac{\varepsilon\psi}{\tau}, \tag{1.5}$$

$$-\Delta\phi = b(x) - n. \tag{1.6}$$

Equation (1.5) is the Bernoulli law. Eliminating ϕ from (1.4) and (1.6) and using (1.5), we have

$$\begin{aligned} -\Delta h(n) &+ \frac{\varepsilon}{n}\sum_{i,j=1}^{d}\frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_j}\frac{\partial^2 n}{\partial x_i\partial x_j} - \frac{\varepsilon}{\tau n}\nabla\psi.\nabla n - \frac{\varepsilon}{n^2}(\nabla\psi.\nabla n)^2 \\ &+ \frac{\varepsilon}{n}\sum_{i,j=1}^{d}\frac{\partial\psi}{\partial x_i}\frac{\partial^2\psi}{\partial x_i\partial x_j}\frac{\partial n}{\partial x_j} + n - b(x) = Q(\psi), \end{aligned} \tag{1.7}$$

where Q is given by

$$Q(\psi) = \sum_{i,j=1}^{d}\left(\frac{\partial^2\psi}{\partial x_i\partial x_j}\right)^2. \tag{1.8}$$

For $n > 0$ it is easy to see that (n, ψ, ϕ) is a smooth solution to the system (1.4)-(1.6) if and only if (n, ψ) is a smooth solution to the system (1.4) and (1.7). Moreover, for given ψ, the equation (1.7) is elliptic if and only if the flow is subsonic, i.e., the condition $|\nabla\psi| < \sqrt{p'(n)/\varepsilon}$ holds.

For each limit of the system (1.4)-(1.6), we supplement the Dirichlet boundary conditions. In the case of the zero electron mass limit ($\varepsilon \to 0$), we associate to the system (1.4)-(1.6) (where $\tau = 1$) the following boundary conditions :

$$n = \sum_{k=0}^{m}\varepsilon^k\bar{n}_k + n_{D,\varepsilon}^{m+1}, \quad \psi = \sum_{k=0}^{m}\varepsilon^k\bar{\psi}_k + \psi_{D,\varepsilon}^{m+1} \quad \text{sur } \Gamma \stackrel{\text{def}}{=} \partial\Omega, \tag{1.9}$$

where $n_{D,\varepsilon}^{m+1}$ and $\psi_{D,\varepsilon}^{m+1}$ are smooth enough and defined in $\bar{\Omega}$ such that $n_{D,\varepsilon}^{m+1} = O(\varepsilon^{m+1})$ and $\psi_{D,\varepsilon}^{m+1} = O(\varepsilon^{m+1})$ uniformly in ε. In the case of the zero relaxation time limit ($\tau \to 0$), we first make a change of variable and then associate to the new system the following boundary conditions :

$$n = \sum_{k=0}^{m}\tau^{2k}\bar{n}_k + n_{D,\tau}^{m+1}, \quad \psi = \sum_{k=0}^{m+1}\tau^{2k}\bar{\psi}_k + \psi_{D,\tau}^{m+1} \quad \text{sur } \Gamma, \tag{1.10}$$

with similar assumptions on the data to those used in the zero electron mass limit.

For fixed ε and τ, the existence and uniqueness of solutions to the system (1.4)-(1.6) have been already shown in the space

$$B \stackrel{\text{def}}{=} \mathcal{W}^{2,q}(\Omega) \times \mathcal{C}^{2,\delta}(\bar{\Omega}) \times \mathcal{C}^{1,\delta}(\bar{\Omega})$$

for small Dirichlet data on the velocity potential (see [5]) by using the equivalent system (1.4) and (1.7). The smallness condition on the data guarantees that the problem is located in the subsonic region. In [12] it is shown that this smallness condition corresponds to the smallness of ε. Then the existence and uniqueness of solutions hold for large data provided that ε is small enough. Furthermore, there is given a result of convergence and an error estimate for an asymptotic expansion on ε up to first order of the solution in the same space.

The quasi-neutral limit $\lambda \to 0$ has been studied by lot of authors. In one-dimensional steady state Euler-Poisson system it was performed in [15] for well-prepared boundary data and in [11] for general boundary data. The steady problem in several space variables for a potential flow without the formation of boundary layers was investigated in [12]. In [4], by using pseudo-differential techniques, the quasi-neutral limit was studied for local smooth solutions of a one-dimensional and isothermal model for plasmas in which the electron density is described by the Maxwell-Boltzmann relation. This relation can be obtained in the zero electron mass limit of the Euler-Poisson equations which we will discuss below (Remark 2.1). See also [2] for the study of the quasi-neutral limit in a semi-linear Poisson equation in which the Maxwell-Boltzmann relation is also used.

The zero relaxation time limit in one dimensional transient Euler-Poisson system has been investigated in [9] and [7, 8] by the compensated compactness arguments for global weak solutions. The limit system is governed by the classical drift-diffusion model. In multi-dimensional case and for local smooth solutions this limit has been studied in [1]. See also the references therein.

In this paper, we study the zero electron mass limit and the zero relaxation time limit in the subsonic region by the method of asymptotic expansions. For each limit, we justify the asymptotic expansions in the space B up to any order by using the elliptic properties. As applications of the asymptotic expansions up to second order, we establish in both limits the convergence of the Euler-Poisson system (1.4)-(1.6) to the incompressible Euler equations with explicit pressure expressed by the profiles. Note that the convergence of the Euler-Poisson system to the incompressible Euler equations have been already shown via the quasi-neutral limit when the Dirichlet boundary data are well prepared (see [12] and [13]).

The remainder of the paper is arranged as follows. In Sections 2-3, we consider the zero electron mass limit. We begin, in Section 2, with the asymptotic expansions of solutions to the problem by determining its all order profile. In Section 3, we justify the asymptotic expansions up to order m and establish error estimates of order ε^{m+1} for each variable. Section 4 is devoted to the zero relaxation time limit. We obtain in a same way the error estimates of order $\tau^{2(m+1)}$ for an asymptotic expansion up to order m. Finally, in the last section, we give applications of these results by showing the convergence of the system (1.4)-(1.6) to the incompressible Euler equations for each limit $\varepsilon \to 0$ and $\tau \to 0$.

2 Asymptotic expansion

2.1 Derivation of the profile equations

Let $\tau = 1$. We consider the limit $\varepsilon \to 0$ in the steady state Euler-Poisson system for the potential flow, i.e., the system (1.4)-(1.6) associated to the Dirichlet boundary conditions (1.9).

We assume that :

(A1) Ω is a bounded and convex domain of $\mathbb{R}^d$ with $\Gamma = \partial\Omega \in \mathcal{C}^{2,\delta}$, $\delta \in]0,1[$,

(A2) $p \in \mathcal{C}^{m+4}(\mathbb{R}^+)$, $m \in \mathbb{N}$, $p'(n) > 0 \ \forall\ n > 0$,

(A3) $b \in L^\infty(\Omega), 0 < \underline{b} \leq b(x)$,

(A4) $\bar{n}_k \in \mathcal{W}^{2,q}(\Omega)$ for $q > \frac{d}{1-\delta}$ and $\forall\ 0 \leq k \leq m,\ 0 < \underline{n} \leq \bar{n}_0(x) \quad \forall x \in \Gamma$,

(A5) $\bar{\psi}_k \in \mathcal{C}^{2,\delta}(\bar{\Omega}), \quad \forall\ 0 \leq k \leq m$,

(A6) the sequence $(\varepsilon^{-(m+1)} n_{D,\varepsilon}^{m+1})_{\varepsilon>0}$ is bounded in $\mathcal{W}^{2,q}(\Omega)$,

(A7) the sequence $(\varepsilon^{-(m+1)} \psi_{D,\varepsilon}^{m+1})_{\varepsilon>0}$ is bounded in $\mathcal{C}^{2,\delta}(\bar{\Omega})$.

Let $(n_{a,\varepsilon}, \psi_{a,\varepsilon}, \phi_{a,\varepsilon})$ be defined by the following ansatz :

$$n_{a,\varepsilon} = \sum_{k\geq 0} \varepsilon^k n_k,\ \ \psi_{a,\varepsilon} = \sum_{k\geq 0} \varepsilon^k \psi_k,\ \ \phi_{a,\varepsilon} = \sum_{k\geq 0} \varepsilon^k \phi_k \ \text{ in } \Omega, \tag{2.1}$$

with the boundary conditions :

$$n_{a,\varepsilon} = \sum_{k\geq 0} \varepsilon^k \bar{n}_k,\ \ \psi_{a,\varepsilon} = \sum_{k\geq 0} \varepsilon^k \bar{\psi}_k \quad \text{on } \Gamma. \tag{2.2}$$

Plugging the expression (2.1) into the system (1.4)-(1.6), we obtain formally

$$-\mathrm{div}\Big(\Big(\sum_{k\geq 0} \varepsilon^k n_k\Big)\nabla\Big(\sum_{k\geq 0} \varepsilon^k \psi_k\Big)\Big) = 0, \tag{2.3}$$

$$\frac{\varepsilon}{2}\left|\nabla\Big(\sum_{k\geq 0} \varepsilon^k \psi_k\Big)\right|^2 + h\Big(\sum_{k\geq 0} \varepsilon^k n_k\Big) = \sum_{k\geq 0} \varepsilon^k \phi_k + \varepsilon \sum_{k\geq 0} \varepsilon^k \psi_k, \tag{2.4}$$

$$-\Delta\left(\sum_{k\geq 0} \varepsilon^k \phi_k\right) = b(x) - \sum_{k\geq 0} \varepsilon^k n_k. \tag{2.5}$$

Now, we seek for the system and boundary conditions for each profile (n_k, ψ_k, ϕ_k). Obviously,

$$\mathrm{div}\Big(\Big(\sum_{k\geq 0} \varepsilon^k n_k\Big)\Big(\sum_{k\geq 0} \varepsilon^k \nabla\psi_k\Big)\Big) = \sum_{k\geq 0} \varepsilon^k \sum_{i=0}^{k} \mathrm{div}(n_i \nabla\psi_{k-i}),$$

$$\left|\nabla\Big(\sum_{k\geq 0}\varepsilon^k\psi_k\Big)\right|^2 = \sum_{k\geq 0}\varepsilon^k\Big(\sum_{i=0}^{k}\nabla\psi_i.\nabla\psi_{k-i}\Big),$$

and by the Taylor's formula,

$$h\left(\sum_{k\geq 0}\varepsilon^k n_k\right) = \sum_{k\geq 0}\varepsilon^k h_k(n),$$

where $n = (n_i)_{i\geq 0}$ and

$$h_k(n) = \frac{1}{k!}\frac{d^k h(\sum_{k\geq 0}\varepsilon^k n_k)}{d\varepsilon^k}\Big|_{\varepsilon=0}, \ k\geq 0.$$

It is immediate that

$$h_k(n) = h'(n_0)n_k + \bar{h}_k\Big((n_i)_{0\leq i\leq k-1}\Big), \ k\geq 1,$$

where h_k is of class $\mathcal{C}^{m-k+3}$ with $\bar{h}_1\equiv 0$. It follows that :

$$h(\sum_{k\geq 0}\varepsilon^k n_k) = h(n_0) + \sum_{k\geq 1}\varepsilon^k h'(n_0)n_k + \sum_{k\geq 2}\varepsilon^k\bar{h}_k\Big((n_i)_{0\leq i\leq k-1}\Big).$$

Then by identification of the order in ε in the problem (2.3)-(2.5) and (2.2), we obtain the system for each (n_k,ψ_k,ϕ_k), $k\geq 0$. More precisely, the first order (n_0,ψ_0,ϕ_0) satisfies the nonlinear problem in Ω :

$$-\mathrm{div}(n_0\nabla\psi_0) = 0, \tag{2.6}$$

$$h(n_0) = \phi_0, \tag{2.7}$$

$$-\Delta\phi_0 = b(x) - n_0, \tag{2.8}$$

with the following boundary conditions :

$$n_0 = \bar{n}_0, \quad \psi_0 = \bar{\psi}_0 \quad \text{on } \Gamma. \tag{2.9}$$

For all $k\geq 1$, (n_k,ψ_k,ϕ_k) is obtained by induction on k in the following linear problem in Ω :

$$-\mathrm{div}(n_0\nabla\psi_k) = \sum_{i=1}^{k}\mathrm{div}(n_i\nabla\psi_{k-i}), \tag{2.10}$$

$$h'(n_0)n_k - \phi_k = f_k, \tag{2.11}$$

$$-\Delta\phi_k = -n_k, \tag{2.12}$$

with the boundary conditions :

$$n_k = \bar{n}_k, \quad \psi_k = \bar{\psi}_k \quad \text{on } \Gamma, \tag{2.13}$$

where

$$f_k = \psi_{k-1} - \frac{1}{2}\sum_{i=0}^{k-1}\nabla\psi_{k-1-i}.\nabla\psi_i - \bar{h}_k((n_i)_{0\leq i\leq k-1}). \tag{2.14}$$

Remark 2.1 *Equation (2.7) expresses a Maxwell-Boltzmann type relation. Indeed, for the isothermal plasma, the pressure is a linear function. Then* $p(n) = a^2 n$ *with* $a > 0$. *This implies from the definition of* h *that* $h(n) = a^2 \log n$ *and hence, from (2.7)* $n_0 = \exp(\phi_0/a^2)$. *This is the classical Maxwell-Boltzmann relation which has been used in [2, 4, 14] for the study of the quasi-neutral limit.*

2.2 Existence and uniqueness of the profiles

Now we show that each problem (2.6)-(2.9) and (2.10)-(2.13) has a unique solution. We start by the problem (2.6)-(2.9). Eliminating ϕ_0 in (2.8) by (2.7), we obtain the nonlinear problem on n_0 :

$$\Delta h(n_0) - n_0 = -b(x) \quad \text{in } \Omega, \tag{2.15}$$

$$n_0 = \bar{n}_0 \quad \text{on } \Gamma. \tag{2.16}$$

Since p is smooth and strictly increasing, so is h. By the assumptions (A2)-(A4) and Lemmas 9.15 and 9.17 in [6] or Lemma 2.2 in [5], this problem admits a unique solution $n_0 \in \mathcal{W}^{2,q}(\Omega)$. Furthermore, the maximum principle gives :

$$n_0(x) \geq \min\left(\min_{x\in\Omega} b(x), \min_{x\in\Gamma} \bar{n}_0(x)\right) \geq \min(\underline{b}, \underline{n}) > 0, \quad \forall\, x \in \overline{\Omega}. \tag{2.17}$$

Then, (2.7) gives a unique $\phi_0 \in \mathcal{W}^{2,q}(\Omega)$. Finally, since the injection $\mathcal{W}^{2,q}(\Omega) \hookrightarrow \mathcal{C}^{1,\delta}(\bar{\Omega})$ is continuous, the equation (2.6) and the boundary condition in (2.9) provide a unique solution $\psi_0 \in \mathcal{C}^{2,\delta}(\bar{\Omega})$ ([6], Theorem 6.6). Hence, we have determined a unique solution (n_0, ψ_0, ϕ_0) to the problem (2.6)-(2.9) in $\mathcal{W}^{2,q}(\Omega) \times \mathcal{C}^{2,\delta}(\bar{\Omega}) \times \mathcal{W}^{2,q}(\Omega)$.

Now we consider the problem (2.10)-(2.13). Assume that for some $k \geq 1$ we know all $(n_j, \psi_j, \phi_j) \in B$ for $0 \leq j \leq k-1$, solutions of the problem (2.6)-(2.9) if $j = 0$ or (2.10)-(2.13) in which k is replaced by $j \geq 1$. Eliminating ϕ_k in (2.11)-(2.12) we obtain the following linear problem for n_k :

$$\Delta(h'(n_0)n_k) - n_k = \Delta f_k \quad \text{in } \Omega, \tag{2.18}$$

$$n_k = \bar{n}_k \quad \text{on } \Gamma. \tag{2.19}$$

Note that Δf_k contains the third order derivatives of $(\psi_i)_{0\leq i\leq k-1}$. Since $n \to h(n)$ is strictly increasing for $n > 0$, from (2.17), in order to show the existence of a unique solution $n \in \mathcal{W}^{2,q}(\Omega)$ of the linear problem (2.18)-(2.19), we have to establish the following result.

Lemma 2.1 *Let* $m \geq 1$ *and* $1 \leq k \leq m$. *Assume that* $(n_j, \psi_j) \in \mathcal{W}^{2,q}(\Omega) \times \mathcal{C}^{2,\delta}(\bar{\Omega})$ *for all* $0 \leq j \leq k-1$. *Then,* $f_k \in \mathcal{C}^{1,\delta}(\bar{\Omega})$ *and* $\Delta f_k \in L^q(\Omega)$.

Proof. By (2.14) and the continuous injection $\mathcal{W}^{2,q}(\Omega) \hookrightarrow \mathcal{C}^{1,\delta}(\bar{\Omega})$ it is clear that $f_k \in \mathcal{C}^{1,\delta}(\bar{\Omega})$. To prove $\Delta f_k \in L^q(\Omega)$ it suffices to show that Δf_k can be expressed as a function of at most second order derivatives of $(n_i)_{0\leq i\leq k-1}$ and $(\psi_i)_{0\leq i\leq k-1}$, and the second order derivative of $(n_i)_{0\leq i\leq k-1}$ in Δf_k is linear. Since

$$\Delta f_k = \Delta\psi_{k-1} - \frac{1}{2}\Delta\left(\sum_{i=0}^{k-1} \nabla\psi_{k-1-i}.\nabla\psi_i\right) - \Delta\bar{h}_k((n_i)_{0\leq i\leq k-1}),$$

and

$$\Delta\left(\nabla\psi_{k-1-i}.\nabla\psi_i\right) = \nabla(\Delta\psi_{k-1-i}).\nabla\psi_i + \nabla(\Delta\psi_i).\nabla\psi_{k-1-i} + 2\sum_{l,j=1}^{d} \frac{\partial^2\psi_{k-1-i}}{\partial x_l\partial x_j}\frac{\partial^2\psi_i}{\partial x_l\partial x_j},$$

by the assumptions and the regularity of $\bar{h}_k$, the problem is reduced to show that for all $k\geq 1$, $\Delta\psi_k$ can be expressed as a function of at most first order derivatives of $(n_i)_{0\leq i\leq k-1}$ and $(\psi_i)_{0\leq i\leq k-1}$, and the first order derivative of $(n_i)_{0\leq i\leq k-1}$ in $\Delta\psi_k$ is linear. Indeed, from (2.6) we have :

$$\Delta\psi_0 = \frac{1}{n_0}\nabla n_0.\nabla\psi_0.$$

Then the assertion is true for $k=0$. Assume it is true for all $0\leq i\leq k-1$. From (2.10) we obtain :

$$-\Delta\psi_k = \frac{1}{n_0}\sum_{i=0}^{k}(\nabla n_{k-i}.\nabla\psi_i) + \frac{1}{n_0}\sum_{i=0}^{k-1}(n_{k-i}\Delta\psi_i).$$

This shows the assertion for k and the result follows.

Consequently, for all $0\leq k\leq m$, the linear problem (2.18)-(2.19) has a unique solution $n_k \in \mathcal{W}^{2,q}(\Omega)$. Hence, (2.11) gives a unique $\phi_k \in \mathcal{C}^{1,\delta}(\bar{\Omega})$, and finally the linear equation (2.10) with the boundary condition for ψ_k in (2.13) defines a unique solution $\psi_k \in \mathcal{C}^{2,\delta}(\bar{\Omega})$. In summary, we have :

Theorem 2.1 *Let $m \in \mathbb{N}$ and the assumptions (A1)-(A5) hold. Then there exists a unique asymptotic expansion (2.1) up to order m, i.e., for all $0\leq k\leq m$, there exists a unique profile $(n_k,\psi_k,\phi_k)\in B$, solution to the problem (2.6)-(2.9) if $k=0$ or (2.10)-(2.13) if $1\leq k\leq m$.*

3 Justification of the asymptotic expansion

3.1 The main result

Let $(n_\varepsilon,\psi_\varepsilon,\phi_\varepsilon)$ be a smooth solution of (1.4)-(1.6) and (1.9) and $(n^m_{a,\varepsilon},\psi^m_{a,\varepsilon},\phi^m_{a,\varepsilon})$ be approximate solution of order m defined by :

$$n^m_{a,\varepsilon} = \sum_{k=0}^{m}\varepsilon^k n_k, \quad \psi^m_{a,\varepsilon} = \sum_{k=0}^{m}\varepsilon^k\psi_k, \quad \phi^m_{a,\varepsilon} = \sum_{k=0}^{m}\varepsilon^k\phi_k, \tag{3.1}$$

where $(n_k, \psi_k, \phi_k)_{0\leq k\leq m}$ is the unique solution of (2.6)-(2.9) for $k = 0$ and (2.10)-(2.13) for $1 \leq k \leq m$.

One of the goals of this paper is to show the following result.

Theorem 3.1 *Let $(n_\varepsilon, \psi_\varepsilon, \phi_\varepsilon)$ be the solution of the problem (1.4)-(1.6) and (1.9) and $(n^m_{a,\varepsilon}, \psi^m_{a,\varepsilon}, \phi^m_{a,\varepsilon})$ be the approximate solution given by the asymptotic expansion (3.1). Let the assumptions (A1)-(A7) hold. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$, we have the following estimates :*

$$\|n_\varepsilon - n^m_{a,\varepsilon}\|_{\mathcal{W}^{2,q}(\Omega)} \leq A_1\varepsilon^{m+1}, \|\psi_\varepsilon - \psi^m_{a,\varepsilon}\|_{\mathcal{C}^{2,\delta}(\bar{\Omega})} \leq A_1\varepsilon^{m+1}, \|\phi_\varepsilon - \phi^m_{a,\varepsilon}\|_{\mathcal{C}^{1,\delta}(\bar{\Omega})} \leq A_1\varepsilon^{m+1},$$

where $A_1 > 0$ is a constant independent of ε.

3.2 Derivation of the problem on $(n_\varepsilon - n^m_{a,\varepsilon}, \psi_\varepsilon - \psi^m_{a,\varepsilon}, \phi_\varepsilon - \phi^m_{a,\varepsilon})$

First, since $(n_\varepsilon, \psi_\varepsilon, \phi_\varepsilon)$ is the solution of (1.4)-(1.6) and (1.9), we have in Ω :

$$-\mathrm{div}(n_\varepsilon \nabla \psi_\varepsilon) = 0, \tag{3.2}$$

$$\frac{\varepsilon}{2}|\nabla \psi_\varepsilon|^2 + h(n_\varepsilon) = \phi_\varepsilon + \varepsilon\psi_\varepsilon, \tag{3.3}$$

$$-\Delta\phi_\varepsilon = b(x) - n_\varepsilon, \tag{3.4}$$

and

$$n_\varepsilon = \sum_{k=0}^{m} \varepsilon^k \bar{n}_k + n^{m+1}_{D,\varepsilon}, \quad \psi_\varepsilon = \sum_{k=0}^{m} \varepsilon^k \bar{\psi}_k + \psi^{m+1}_{D,\varepsilon} \quad \text{on } \Gamma. \tag{3.5}$$

Now we determine the system verified by the approximate solution $(n^m_{a,\varepsilon}, \psi^m_{a,\varepsilon}, \phi^m_{a,\varepsilon})$. Since

$$\mathrm{div}(n^m_{a,\varepsilon}\nabla\psi^m_{a,\varepsilon}) = \sum_{k=0}^{m} \varepsilon^k \Big(\sum_{i=0}^{k} \mathrm{div}(n_i \nabla\psi_{k-i})\Big) + \varepsilon^{m+1} D^\varepsilon_1,$$

where

$$D^\varepsilon_1 = \sum_{k=m+1}^{2m} \Big(\varepsilon^{k-m-1} \sum_{i=k-m}^{m} \mathrm{div}(n_i \nabla\psi_{k-i})\Big), \tag{3.6}$$

from (2.6) and (2.10), we have :

$$-\mathrm{div}(n^m_{a,\varepsilon}\nabla\psi^m_{a,\varepsilon}) = -\varepsilon^{m+1} D^\varepsilon_1. \tag{3.7}$$

Similarly,

$$\frac{\varepsilon}{2}|\nabla\psi^m_{a,\varepsilon}|^2 + h(n^m_{a,\varepsilon}) = \phi^m_{a,\varepsilon} + \varepsilon\psi^m_{a,\varepsilon} - \varepsilon^{m+1} D^\varepsilon_2, \tag{3.8}$$

$$-\Delta\phi^m_{a,\varepsilon} = b(x) - n^m_{a,\varepsilon}, \tag{3.9}$$

$$n_{a,\varepsilon}^m = \sum_{k=0}^m \varepsilon^k \bar{n}_k, \quad \psi_{a,\varepsilon}^m = \sum_{k=0}^m \varepsilon^k \bar{\psi}_k \quad \text{sur } \Gamma, \tag{3.10}$$

where

$$D_2^\varepsilon = -\frac{1}{2} \sum_{k=m}^{2m} \Big(\varepsilon^{k-m} \sum_{i=k-m}^{m} \nabla\psi_i . \nabla\psi_{k-i} \Big) - r_\varepsilon(n) + \psi_m, \tag{3.11}$$

and

$$r_\varepsilon(n) = \frac{1}{(m+1)!} \frac{d^{m+1} h(n_{a,\xi}^m)}{d\varepsilon^{m+1}} \quad \text{with } \xi \in [0, \varepsilon]. \tag{3.12}$$

By subtraction of the systems (3.2)-(3.5) and (3.7)-(3.10), we obtain in Ω :

$$-\text{div}(n_\varepsilon \nabla\psi_\varepsilon) + \text{div}(n_{a,\varepsilon}^m \nabla\psi_{a,\varepsilon}^m) = \varepsilon^{m+1} D_1^\varepsilon, \tag{3.13}$$

$$\frac{\varepsilon}{2}(|\nabla\psi_\varepsilon|^2 - |\nabla\psi_{a,\varepsilon}^m|^2) + h(n_\varepsilon) - h(n_{a,\varepsilon}^m) = \phi_\varepsilon - \phi_{a,\varepsilon}^m + \varepsilon(\psi_\varepsilon - \psi_{a,\varepsilon}^m) + \varepsilon^{m+1} D_2^\varepsilon, \tag{3.14}$$

$$-\Delta(\phi_\varepsilon - \phi_{a,\varepsilon}^m) = -(n_\varepsilon - n_{a,\varepsilon}^m), \tag{3.15}$$

and

$$n_\varepsilon - n_{a,\varepsilon}^m = n_{D,\varepsilon}^{m+1}, \quad \psi_\varepsilon - \psi_{a,\varepsilon}^m = \psi_{D,\varepsilon}^{m+1} \quad \text{on } \Gamma. \tag{3.16}$$

Eliminating $\phi_\varepsilon - \phi_{a,\varepsilon}^m$ in (3.14)-(3.15), we obtain finally the following system in Ω :

$$\frac{\varepsilon}{2}\Delta(|\nabla\psi_\varepsilon|^2 - |\nabla\psi_{a,\varepsilon}^m|^2) + \Delta(h(n_\varepsilon) - h(n_{a,\varepsilon}^m)) = n_\varepsilon - n_{a,\varepsilon}^m + \varepsilon\Delta(\psi_\varepsilon - \psi_{a,\varepsilon}^m) + \varepsilon^{m+1}\Delta D_2^\varepsilon, \tag{3.17}$$

$$-\text{div}((n_\varepsilon - n_{a,\varepsilon}^m)\nabla\psi_\varepsilon + n_{a,\varepsilon}^m \nabla(\psi_\varepsilon - \psi_{a,\varepsilon}^m)) = \varepsilon^{m+1} D_1^\varepsilon. \tag{3.18}$$

3.3 Some preliminary results

In what follows, $C_i > 0$ ($i \geq 1$) denote different constants independent of ε. We give some preliminary results for the proof of Theorem 3.1. Lemma 3.1 is a direct consequence of the existence and uniqueness of solutions to the system (3.2)-(3.4) with Dirichlet data bounded in $\mathcal{W}^{2,q}(\Omega) \times \mathcal{C}^{2,\delta}(\bar{\Omega})$. Its proof can be found in [12]. The system (3.17)-(3.18) contains also the third order derivatives which can be treated in a similar way as that for Δf_k (Lemma 3.2). The key estimate is given in Lemma 3.3. These results allow us to justify rigorously the asymptotic expansion (2.1) in B.

Lemma 3.1 *Under the assumptions (A1)-(A7), there exists $\varepsilon_1 > 0$ such that the problem (3.2)-(3.5) has a unique solution $(n_\varepsilon, \psi_\varepsilon, \phi_\varepsilon) \in B$ for all $\varepsilon \in [0, \varepsilon_1]$. Furthermore, the sequence of solutions $(n_\varepsilon, \psi_\varepsilon, \phi_\varepsilon)_{\varepsilon>0}$ is bounded in B.*

Lemma 3.2 *Assume that $(n_i,\psi_i)\in\mathcal{W}^{2,q}(\Omega)\times\mathcal{C}^{2,\delta}(\bar\Omega)$ for all $0\le i\le m$. Then the sequences $(\nabla D_1^\varepsilon)_{\varepsilon>0}$ and $(\Delta D_2^\varepsilon)_{\varepsilon>0}$ are bounded in $L^q(\Omega)$.*

Proof. We only prove Lemma 3.2 for the sequence $(\nabla D_1^\varepsilon)_{\varepsilon>0}$ since the proof is similar for the sequence $(\Delta D_2^\varepsilon)_{\varepsilon>0}$. By the definition of D_1^ε in (3.6), we have :

$$\nabla D_1^\varepsilon=\sum_{k=m+1}^{2m}\varepsilon^{k-m-1}\sum_{i=k-m}^{m}(\nabla n_i\Delta\psi_{k-i}+\nabla(\nabla n_i.\nabla\psi_{k-i})+n_i\nabla(\Delta\psi_{k-i})),$$

in which the third order derivatives appear only in the terms $\nabla(\Delta\psi_{k-i})$ for $m+1\le k\le 2m$ and $k-m\le i\le m$, i.e., in the terms of the form $\nabla(\Delta\psi_k)$ for $1\le k\le m$. Furthermore, it is easy to check that the second order derivative of $(n_i)_{0\le i\le m}$ in ∇D_1^ε is linear. Since $(n_i,\psi_i)\in\mathcal{W}^{2,q}(\Omega)\times\mathcal{C}^{2,\delta}(\bar\Omega)$ for all $0\le i\le m$, from the proof of Lemma 2.1 we know that $\Delta\psi_k$ can be expressed as a function of at most first order derivatives of $(n_i)_{0\le i\le k-1}$ and $(\psi_i)_{0\le i\le k-1}$, and the first order derivative of $(n_i)_{0\le i\le k-1}$ in $\Delta\psi_k$ is linear. This shows that the sequence $(\nabla D_1^\varepsilon)_{\varepsilon>0}$ is bounded in $L^q(\Omega)$.

Lemma 3.3 *Let $I_\varepsilon=\frac12\Delta(|\nabla\psi_\varepsilon|^2-|\nabla\psi_{a,\varepsilon}^m|^2)$. Under the assumption $(n_i,\psi_i)\in\mathcal{W}^{2,q}(\Omega)\times\mathcal{C}^{2,\delta}(\bar\Omega)$ for all $0\le i\le m$, there exists a constant $C>0$ independent of ε such that :*

$$\|I_\varepsilon\|_{L^q(\Omega)}\le C\Big(\|\psi_\varepsilon-\psi_{a,\varepsilon}^m\|_{\mathcal{C}^{2,\delta}(\bar\Omega)}+\|n_\varepsilon-n_{a,\varepsilon}^m\|_{\mathcal{W}^{2,q}(\Omega)}+\varepsilon^{m+1}\Big). \tag{3.19}$$

Proof. From the relation :

$$\frac12\Delta(|\nabla\psi|^2)=\nabla\psi.\nabla(\Delta\psi)+Q(\psi),$$

where $Q(\psi)$ is defined in (1.8), we obtain :

$$\begin{aligned}I_\varepsilon&=\nabla\psi_\varepsilon.\nabla(\Delta\psi_\varepsilon)-\nabla\psi_{a,\varepsilon}^m.\nabla(\Delta\psi_{a,\varepsilon}^m)+Q(\psi_\varepsilon)-Q(\psi_{a,\varepsilon}^m)\\&=\nabla\psi_{a,\varepsilon}^m.\nabla(\Delta\psi_\varepsilon-\Delta\psi_{a,\varepsilon}^m)+\nabla(\psi_\varepsilon-\psi_{a,\varepsilon}^m).\nabla(\Delta\psi_\varepsilon)+Q(\psi_\varepsilon)-Q(\psi_{a,\varepsilon}^m).\end{aligned}$$

It follows from (3.2) and (3.7) that :

$$\begin{aligned}\Delta\psi_\varepsilon&=-\frac{\nabla n_\varepsilon}{n_\varepsilon}\nabla\psi_\varepsilon,\\\Delta\psi_{a,\varepsilon}^m&=-\frac{\nabla n_{a,\varepsilon}^m}{n_{a,\varepsilon}^m}\nabla\psi_{a,\varepsilon}^m+\varepsilon^{m+1}D_1^\varepsilon.\end{aligned}$$

Then,

$$\begin{aligned}\Delta\psi_\varepsilon-\Delta\psi_{a,\varepsilon}^m&=\frac{\nabla n_{a,\varepsilon}^m}{n_{a,\varepsilon}^m}.\nabla\psi_{a,\varepsilon}^m-\frac{\nabla n_\varepsilon}{n_\varepsilon}.\nabla\psi_\varepsilon-\varepsilon^{m+1}D_1^\varepsilon\\&=\frac{\nabla n_{a,\varepsilon}^m}{n_{a,\varepsilon}^m}.\nabla(\psi_{a,\varepsilon}^m-\psi_\varepsilon)+\Big(\frac{\nabla n_{a,\varepsilon}^m}{n_{a,\varepsilon}^m}-\frac{\nabla n_\varepsilon}{n_\varepsilon}\Big).\nabla\psi_\varepsilon-\varepsilon^{m+1}D_1^\varepsilon\\&=\frac{\nabla n_{a,\varepsilon}^m}{n_{a,\varepsilon}^m}.\nabla(\psi_{a,\varepsilon}^m-\psi_\varepsilon)+\nabla(\ln n_{a,\varepsilon}^m-\ln n_\varepsilon).\nabla\psi_\varepsilon-\varepsilon^{m+1}D_1^\varepsilon.\end{aligned}$$

This gives :

$$
\begin{aligned}
I_\varepsilon &= \nabla\psi^m_{a,\varepsilon}.\nabla\Big[\frac{\nabla n^m_{a,\varepsilon}}{n^m_{a,\varepsilon}}.\nabla(\psi^m_{a,\varepsilon}-\psi_\varepsilon)+\nabla(\ln n^m_{a,\varepsilon}-\ln n_\varepsilon).\nabla\psi_\varepsilon-\varepsilon^{m+1}D_1^\varepsilon\Big]\\
&\quad +\nabla(\psi^m_{a,\varepsilon}-\psi_\varepsilon).\nabla\Big(\frac{\nabla n_\varepsilon}{n_\varepsilon}.\nabla\psi_\varepsilon\Big)+Q(\psi_\varepsilon)-Q(\psi^m_{a,\varepsilon}).
\end{aligned}
$$

By Lemma 3.1 and Theorem 2.1, since $(n_\varepsilon)_{\varepsilon>0}$ and $(n^m_{a,\varepsilon})_{\varepsilon>0}$ are bounded in $\mathcal{W}^{2,q}(\Omega)$ and $(\psi_\varepsilon)_{\varepsilon>0}$ and $(\psi^m_{a,\varepsilon})_{\varepsilon>0}$ are bounded in $\mathcal{C}^{2,\delta}(\bar\Omega)$, we deduce from the continuous injection from $\mathcal{W}^{2,q}(\Omega)$ to $\mathcal{C}^{1,\delta}(\bar\Omega)$ that :

$$
\|Q(\psi_\varepsilon)-Q(\psi^m_{a,\varepsilon})\|_{L^q(\Omega)}\le C_1\|\psi_\varepsilon-\psi^m_{a,\varepsilon}\|_{\mathcal{C}^{2,\delta}(\bar\Omega)}
$$

and

$$
\|\ln n_\varepsilon-\ln n^m_{a,\varepsilon}\|_{\mathcal{W}^{2,q}(\Omega)}\le C_2\|n_\varepsilon-n^m_{a,\varepsilon}\|_{\mathcal{W}^{2,q}(\Omega)}.
$$

Hence,

$$
\|I_\varepsilon\|_{L^q(\Omega)}\le C_3\Big(\|\psi_\varepsilon-\psi^m_{a,\varepsilon}\|_{\mathcal{C}^{2,\delta}(\bar\Omega)}+\|n_\varepsilon-n^m_{a,\varepsilon}\|_{\mathcal{W}^{2,q}(\Omega)}+\varepsilon^{m+1}\|\nabla D_1^\varepsilon\|_{L^q(\Omega)}\Big).
$$

This ends the proof of Lemma 3.3, by using Lemma 3.2.

3.4 Proof of Theorem 3.1

Let

$$
F_\varepsilon=I_\varepsilon-\Delta(\psi_\varepsilon-\psi^m_{a,\varepsilon})-\varepsilon^m\Delta D_2^\varepsilon
$$

Then the equation (3.17) can be written as :

$$
-\Delta(h(n_\varepsilon)-h(n^m_{a,\varepsilon}))+(n_\varepsilon-n^m_{a,\varepsilon})=\varepsilon F_\varepsilon. \tag{3.20}
$$

By Lemmas 3.2 and 3.3, we have :

$$
\|F_\varepsilon\|_{L^q(\Omega)}\le C_4\Big(\|\psi_\varepsilon-\psi^m_{a,\varepsilon}\|_{\mathcal{C}^{2,\delta}(\bar\Omega)}+\|n_\varepsilon-n^m_{a,\varepsilon}\|_{\mathcal{W}^{2,q}(\Omega)}+\varepsilon^m\Big).
$$

Moreover, under the assumption (A6), the Lemmas 9.15 and 9.17 in [6] applied to the equation (3.20) and the boundary condition (3.16) give :

$$
\|h(n_\varepsilon)-h(n^m_{a,\varepsilon})\|_{\mathcal{W}^{2,q}(\Omega)}\le C_5\Big(\varepsilon\|\psi_\varepsilon-\psi^m_{a,\varepsilon}\|_{\mathcal{C}^{2,\delta}(\bar\Omega)}+\varepsilon\|n_\varepsilon-n^m_{a,\varepsilon}\|_{\mathcal{W}^{2,q}(\Omega)}+\varepsilon^{m+1}\Big).
$$

Noting that h is smooth and strictly increasing, we have :

$$
C_6\|n_\varepsilon-n^m_{a,\varepsilon}\|_{\mathcal{W}^{2,q}(\Omega)}\le\|h(n_\varepsilon)-h(n^m_{a,\varepsilon})\|_{\mathcal{W}^{2,q}(\Omega)}.
$$

We conclude that for ε small enough :

$$
\|n_\varepsilon-n^m_{a,\varepsilon}\|_{\mathcal{W}^{2,q}(\Omega)}\le C_7\Big(\varepsilon\|\psi_\varepsilon-\psi^m_{a,\varepsilon}\|_{\mathcal{C}^{2,\delta}(\bar\Omega)}+\varepsilon^{m+1}\Big). \tag{3.21}
$$

Next, we write the equation (3.17) under the form :

$$-\Delta(\psi_\varepsilon - \psi^m_{a,\varepsilon}) - \frac{\nabla n^m_{a,\varepsilon}}{n^m_{a,\varepsilon}}.\nabla(\psi_\varepsilon - \psi^m_{a,\varepsilon}) = g_\varepsilon, \tag{3.22}$$

where

$$g_\varepsilon = \frac{1}{n^m_{a,\varepsilon}}\mathrm{div}\Big((n_\varepsilon - n^m_{a,\varepsilon})\nabla\psi_\varepsilon\Big) + \frac{\varepsilon^{m+1}}{n^m_{a,\varepsilon}}D^\varepsilon_1.$$

From (3.21) and the continuous injection from $\mathcal{W}^{2,q}(\Omega)$ to $\mathcal{C}^{1,\delta}(\bar\Omega)$, we have :

$$\|g_\varepsilon\|_{C^{0,\delta}(\bar\Omega)} \le C_8\Big(\varepsilon\|\psi_\varepsilon - \psi^m_{a,\varepsilon}\|_{\mathcal{C}^{2,\delta}(\bar\Omega)} + \varepsilon^{m+1}\Big).$$

Hence, from the assumption (A7), Theorem 6.6 in [6] shows that the solution ψ_ε of the equation (3.22) associated to the boundary condition given in (3.16) satisfies :

$$\|\psi_\varepsilon - \psi^m_{a,\varepsilon}\|_{\mathcal{C}^{2,\delta}(\bar\Omega)} \le C_9\Big(\varepsilon\|\psi_\varepsilon - \psi^m_{a,\varepsilon}\|_{\mathcal{C}^{2,\delta}(\bar\Omega)} + \varepsilon^{m+1}\Big).$$

We deduce that, for all ε small enough, (for example $\varepsilon \le 1/2C_9$) :

$$\|\psi_\varepsilon - \psi^m_{a,\varepsilon}\|_{\mathcal{C}^{2,\delta}(\bar\Omega)} \le C_{10}\varepsilon^{m+1},$$

which yields from (3.21) :

$$\|n_\varepsilon - n^m_{a,\varepsilon}\|_{\mathcal{W}^{2,q}(\Omega)} \le C_{11}\varepsilon^{m+1}.$$

Finally, (3.14) gives :

$$\|\phi_\varepsilon - \phi^m_{a,\varepsilon}\|_{\mathcal{C}^{1,\delta}(\bar\Omega)} \le C_{12}\varepsilon^{m+1}.$$

This completes the proof of Theorem 3.1.

4 Zero relaxation time limit

In this section we deal with the zero relaxation time limit $\tau \to 0$ in the system (1.4)-(1.6). We present the results and omit the proofs since they are similar to those of Sections 2-3. To simplify the presentation, we make the following change of variable :

$$n_\tau = n,\ \psi_\tau = \frac{\psi}{\tau},\ \phi_\tau = \phi.$$

Then from (1.4)-(1.6), $(n_\tau, \psi_\tau, \phi_\tau)$ satisfies :

$$-\mathrm{div}(n_\tau\nabla\psi_\tau) = 0, \tag{4.1}$$

$$\frac{\varepsilon}{2}\tau^2|\nabla\psi_\tau|^2 + h(n_\tau) = \phi_\tau + \varepsilon\psi_\tau, \tag{4.2}$$

$$-\Delta\phi_\tau = b(x) - n_\tau. \tag{4.3}$$

We associate to this system the following Dirichlet boundary conditions :

$$n_\tau = \sum_{k=0}^{m} \tau^{2k}\bar{n}_k + n_{D,\tau}^{m+1}, \quad \psi_\tau = \sum_{k=0}^{m} \tau^{2k}\bar{\psi}_k + \psi_{D,\tau}^{m+1} \quad \text{on } \Gamma, \tag{4.4}$$

where $n_{D,\tau}^{m+1}$ and $\psi_{D,\tau}^{m+1}$ satisfy the following conditions :

(A6)' the sequence $(\tau^{-2(m+1)} n_{D,\tau}^{m+1})_{\tau>0}$ is bounded in $\mathcal{W}^{2,q}(\Omega)$.

(A7)' the sequence $(\tau^{-2(m+1)} \psi_{D,\tau}^{m+1})_{\tau>0}$ is bounded in $\mathcal{C}^{2,\delta}(\bar{\Omega})$.

Similar to Lemma 3.1, if the assumptions (A1)-(A5) and (A6)'-(A7)' hold, we can prove that there is a $\varepsilon_2 > 0$ independent of τ such that for all $\varepsilon \in (0, \varepsilon_2]$, the problem (4.1)-(4.4) has a unique solution $(n_\tau, \psi_\tau, \phi_\tau)$ in B. Moreover, the sequences $(n_\tau)_{\tau>0}$, $(\psi_\tau)_{\tau>0}$ and $(\phi_\tau)_{\tau>0}$ are bounded in $\mathcal{W}^{2,q}(\Omega)$, $\mathcal{C}^{2,\delta}(\bar{\Omega})$ and $\mathcal{C}^{1,\delta}(\bar{\Omega})$, respectively. We refer to [12] for details.

4.1 Asymptotic expansion

In view of the system (4.1)-(4.3), it is natural to consider the ansatz defined by :

$$n_{a,\tau} = \sum_{k\geq 0} \tau^{2k} n_k, \quad \psi_{a,\tau} = \sum_{k\geq 0} \tau^{2k}\psi_k, \quad \phi_{a,\tau} = \sum_{k\geq 0} \tau^{2k}\phi_k. \tag{4.5}$$

Plugging (4.5) into the system (4.1)-(4.3) and comparing the orders in τ^2, we have :

(1) The first order profile (n_0, ψ_0, ϕ_0) satisfies the nonlinear drift-diffusion system in Ω :

$$-\mathrm{div}(n_0 \nabla\psi_0) = 0, \tag{4.6}$$

$$h(n_0) = \phi_0 + \varepsilon\psi_0, \tag{4.7}$$

$$-\Delta\phi_0 = b(x) - n_0, \tag{4.8}$$

with the following boundary conditions :

$$n_0 = \bar{n}_0, \quad \psi_0 = \bar{\psi}_0 \quad \text{on } \Gamma. \tag{4.9}$$

(2) For all $k \geq 1$, (n_k, ψ_k, ϕ_k) satisfies the linearized drift-diffusion system in Ω :

$$-\mathrm{div}(n_0\nabla\psi_k) = \sum_{i=1}^{k} \mathrm{div}(n_i \nabla\psi_{k-i}), \tag{4.10}$$

$$h'(n_0)n_k - \phi_k - \varepsilon\psi_k = -\frac{\varepsilon}{2}\sum_{i=0}^{k-1} \nabla\psi_{k-1-i}.\nabla\psi_i - \bar{h}_k((n_i)_{0\le i\le k-1}), \tag{4.11}$$

$$-\Delta\phi_k = -n_k, \tag{4.12}$$

with the boundary conditions :

$$n_k = \bar{n}_k, \quad \psi_k = \bar{\psi}_k \quad \text{on } \Gamma. \tag{4.13}$$

Now we show that each problem (4.6)-(4.9) and (4.10)-(4.13) for all $k \geq 1$ has a unique solution in B when the parameter $\varepsilon > 0$ is small enough. First, eliminating ϕ_0 in (4.7)-(4.8), we have :

$$\Delta h(n_0) - n_0 - \varepsilon\Delta\psi_0 = -b(x).$$

It follows from (4.6) that :

$$\Delta h(n_0) - n_0 - \frac{\varepsilon\nabla n_0}{n_0}.\nabla\psi_0 = -b(x), \tag{4.14}$$

The system (4.6) and (4.14) is a simplified version of (1.4) and (1.7). Then applying the Schauder's fixed point Theorem, this system associated to the boundary condition given in (4.9) admits a unique solution $(n_0, \psi_0) \in \mathcal{W}^{2,q}(\Omega) \times \mathcal{C}^{2,\delta}(\bar{\Omega})$ provided that $\varepsilon > 0$ is small enough (see [12]). Furthermore, the maximum principle implies :

$$n_0(x) \geq \min(\underline{b}, \underline{n}) > 0, \quad \forall\, x \in \overline{\Omega}.$$

Next, we determine a unique $\phi_0 \in \mathcal{W}^{2,q}(\Omega)$ from (4.7). This shows the existence and uniqueness of (n_0, ψ_0, ϕ_0). By induction and an analogous method used above, we show that the linear problem (4.10)-(4.13), for all $k \geq 1$, has a unique solution $(n_k, \psi_k, \phi_k) \in B$.

Hence, we have proved the following theorem.

Theorem 4.1 *Let $m \in \mathbb{N}$. Assume that the assumptions (A1)-(A5) hold. Then, there exists $\varepsilon_3 > 0$ and a unique asymptotic expansion (4.5) up to order m for all $\varepsilon \in (0, \varepsilon_3]$, i.e., for all $0 \leq k \leq m$, there exists a unique profile $(n_k, \psi_k, \phi_k) \in B$, solution of the problem (4.6)-(4.9) if $k = 0$ or (4.10)-(4.13) if $1 \leq k \leq m$.*

4.2 Justification of the asymptotic expansion

Let us define $(n^m_{a,\tau}, \psi^m_{a,\tau}, \phi^m_{a,\tau})$ by :

$$n^m_{a,\tau} = \sum_{k=0}^{m} \tau^{2k} n_k, \quad \psi^m_{a,\tau} = \sum_{k=0}^{m} \tau^{2k}\psi_k, \quad \phi^m_{a,\tau} = \sum_{k=0}^{m} \tau^{2k}\phi_k, \tag{4.15}$$

where (n_0, ψ_0, ϕ_0) is the unique solution of the problem (4.6)-(4.9) and (n_k, ψ_k, ϕ_k) is the unique solution of (4.10)-(4.13) for all $1 \le k \le m$. Similar to the arguments used Section 3, we obtain :

$$-\mathrm{div}(n^m_{a,\tau}\nabla\psi^m_{a,\tau}) = -\tau^{2(m+1)}E^\tau_1, \tag{4.16}$$

$$\frac{\varepsilon}{2}\tau^2|\nabla\psi^m_{a,\tau}|^2 + h(n^m_{a,\tau}) = \phi^m_{a,\tau} + \varepsilon\psi^m_{a,\tau} - \tau^{2(m+1)}E^\tau_2, \tag{4.17}$$

$$-\Delta\phi^m_{a,\tau} = b(x) - n^m_{a,\tau}, \tag{4.18}$$

where

$$E^\tau_1 = \sum_{k=m+1}^{2m}\Big(\tau^{2(k-m-1)}\sum_{i=k-m}^{m}\mathrm{div}(n_i\nabla\psi_{k-i})\Big),$$

$$E^\tau_2 = -\frac{\varepsilon}{2}\sum_{k=m}^{2m}\Big(\tau^{2(k-m)}\sum_{i=k-m}^{m}\nabla\psi_i.\nabla\psi_{k-i}\Big) - s_\tau(n),$$

$$s_\tau(n) = \frac{1}{(m+1)!}\frac{d^{m+1}h(n^m_{a,\xi})}{d(\tau^2)^{m+1}} \quad \text{with } \xi \in [0,\tau].$$

It follows that

$$-\mathrm{div}(n_\tau\nabla\psi_\tau) + \mathrm{div}(n^m_{a,\tau}\nabla\psi^m_{a,\tau}) = \tau^{m+1}E^\tau_1,$$

$$\frac{\varepsilon}{2}\tau^2(|\nabla\psi_\tau|^2 - |\nabla\psi^m_{a,\tau}|^2) + h(n_\tau) - h(n^m_{a,\tau}) = \phi_\tau - \phi^m_{a,\tau} + \varepsilon(\psi_\tau - \psi^m_{a,\tau}) + \tau^{m+1}E^\tau_2,$$

$$-\Delta(\phi_\tau - \phi^m_{a,\tau}) = -(n_\tau - n^m_{a,\tau}),$$

in Ω with the boundary conditions

$$n_\tau - n^m_{a,\tau} = n^{m+1}_{D,\tau}, \quad \psi_\tau - \psi^m_{a,\varepsilon} = \psi^{m+1}_{D,\tau} \quad \text{on } \Gamma.$$

These equations are similar to (3.13)-(3.16). Like Lemma 3.2, we can show that the sequences $(\nabla E^\tau_1)_{\tau>0}$ and $(\Delta E^\tau_2)_{\tau>0}$ are bounded in $L^q(\Omega)$. Then from the assumptions (A6)' and (A7)', we obtain the following convergence result similar to Theorem 3.1.

Theorem 4.2 *Let $(n_\tau, \psi_\tau, \phi_\tau)$ be the solution of the problem (4.1)-(4.4) and $(n^m_{a,\tau}, \psi^m_{a,\tau}, \phi^m_{a,\tau})$ be the approximate solution given by (4.15). Under the assumptions (A1)-(A5) and (A6)'-(A7)', there exists $\varepsilon_4 > 0$ independent of $\tau \in (0,1)$ such that for all $\varepsilon \in (0, \varepsilon_4]$ we have the following error estimates :*

$$\|n_\tau - n^m_{a,\tau}\|_{\mathcal{W}^{2,q}(\Omega)} \le A_2\tau^{2(m+1)}, \|\psi_\tau - \psi^m_{a,\tau}\|_{\mathcal{C}^{2,\delta}(\bar\Omega)} \le A_2\tau^{2(m+1)}, \|\phi_\tau - \phi^m_{a,\tau}\|_{\mathcal{C}^{1,\delta}(\bar\Omega)} \le A_2\tau^{2(m+1)},$$

where $A_2 > 0$ is a constant independent of ε and τ.

5 Convergence to the incompressible Euler equations

As applications of Theorems 3.1 and 4.2, we show in this section that when the boundary data are compatible with the function b, the velocity $u = -\nabla\psi$ in each limit satisfies the incompressible Euler equations, in which the pressures are determined as functions of the profiles (n_1, ψ_1, ϕ_1) in both cases. To see this property, we assume in what follows $b(x) \equiv 1$.

5.1 Via the zero electron mass limit

Let $(n_\varepsilon, u_\varepsilon, \phi_\varepsilon)$ be a smooth solution of the steady state Euler-Poisson system (1.1)-(1.3) with $\tau = \lambda = 1$. Then :

$$-\mathrm{div}(n_\varepsilon u_\varepsilon) = 0, \tag{5.1}$$

$$\varepsilon \mathrm{div}(n_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \nabla p(n_\varepsilon) = n_\varepsilon \nabla\phi_\varepsilon - \varepsilon n_\varepsilon u_\varepsilon, \tag{5.2}$$

$$-\Delta\phi_\varepsilon = 1 - n_\varepsilon. \tag{5.3}$$

For $n_\varepsilon > 0$, the equation (5.2) is equivalent to :

$$(u_\varepsilon.\nabla)u_\varepsilon + \frac{1}{\varepsilon}\nabla(h(n_\varepsilon) - \phi_\varepsilon) + u_\varepsilon = 0.$$

If we take the following ansatz :

$$\begin{aligned} n_\varepsilon &= 1 + \varepsilon n_1 + O(\varepsilon^2), \\ u_\varepsilon &= u_0 + \varepsilon u_1 + O(\varepsilon^2), \\ \phi_\varepsilon &= \phi_0 + \varepsilon\phi_1 + O(\varepsilon^2), \end{aligned}$$

it is easy to see that $\phi_0 = h(1)$ and u_0 satisfies the incompressible Euler equations :

$$(u_0.\nabla)u_0 + u_0 + \nabla P = 0, \quad \mathrm{div} u_0 = 0, \tag{5.4}$$

where the pressure P is defined by :

$$P = h'(1)n_1 - \phi_1. \tag{5.5}$$

This formal analysis can be easily extended to the transient Euler-Poisson system.

For the potential flow, if we take $\bar{n}_0 = 1$, then by Theorem 2.1 the problem (2.6)-(2.9) has a unique solution (n_0, ψ_0, ϕ_0) given by :

$$n_0 = 1, \ \phi_0 = h(1) \tag{5.6}$$

and

$$-\mathrm{div}(\nabla\psi_0) = 0 \ \text{ in } \Omega, \quad \psi_0 = \bar{\psi}_0 \text{ on } \Gamma. \tag{5.7}$$

Hence, $u_0 = -\nabla\psi_0$ satisfies the incompressible Euler equations (5.4) with $P = h'(1)n_1 - \phi_1$, where (n_1, ψ_1, ϕ_1) is the unique solution of the problem (2.10)-(2.13) for $k = 1$. By Theorem 3.1, we have :

$$\|\psi_\varepsilon - \psi_0\|_{\mathcal{C}^{2,\delta}(\bar{\Omega})} \le A_1\varepsilon.$$

Then, the velocity $u_\varepsilon = -\nabla\psi_\varepsilon$ satisfies :

$$\|u_\varepsilon - u_0\|_{\mathcal{C}^{1,\delta}(\bar{\Omega})} \le A_1\varepsilon. \tag{5.8}$$

In summary, we have obtained :

Corollary 5.1 *Let $b(x) \equiv 1$, $\bar{n}_0(x) \equiv 1$ and the assumptions (A1)-(A7) hold for $m = 1$. Then the sequence of solutions $(n_\varepsilon, \psi_\varepsilon, \phi_\varepsilon)_{\varepsilon>0}$ of the problem (1.4)-(1.6) and (1.9) converges, as ε tends to 0, to (n_0, ψ_0, ϕ_0) in B, where (n_0, ψ_0, ϕ_0) is the unique solution of the problem (5.6)-(5.7). Moreover, $u_0 = -\nabla\psi_0$ satisfies the incompressible Euler equations (5.4) in which P is defined by (5.5) and (n_1, ψ_1, ϕ_1) is the unique solution of the problem (2.10)-(2.13) for $k = 1$. Furthermore, the estimate (5.8) holds.*

5.2 Via the zero relaxation time limit

Let (n, u, ϕ) be a smooth solution of the steady state Euler-Poisson system (1.1)-(1.3) with $\lambda = 1$. As above, we make the following change of variable :

$$n_\tau = n, \;\; u_\tau = \frac{u}{\tau}, \;\; \phi_\tau = \phi.$$

If $n_\tau > 0$, then $(n_\tau, u_\tau, \phi_\tau)$ satisfies :

$$-\mathrm{div}(n_\tau u_\tau) = 0, \tag{5.9}$$

$$\varepsilon\tau^2(u_\tau.\nabla)u_\tau + \nabla(h(n_\tau) - \phi_\tau) + \varepsilon u_\tau = 0, \tag{5.10}$$

$$-\Delta\phi_\tau = 1 - n_\tau. \tag{5.11}$$

If we take the following ansatz :

$$\begin{aligned} n_\tau &= 1 + \tau^2 n_1 + O(\tau^4), \\ u_\tau &= u_0 + \tau^2 u_1 + O(\tau^4), \\ \phi_\tau &= \phi_0 + \tau^2\phi_1 + O(\tau^4), \end{aligned}$$

it is easy to see that $\Delta\phi_0 = 0$ and u_0 satisfies the incompressible Euler equations :

$$(u_0.\nabla)u_0 + \nabla P = 0, \quad \mathrm{div}u_0 = 0, \tag{5.12}$$

where

$$\nabla P = \frac{1}{\varepsilon}\nabla(h'(1)n_1 - \phi_1) + u_1.$$

This formal analysis can also be extended to the transient Euler-Poisson equations.

For the potential flow, if we take $\bar{n}_0 = 1$, then the problem (4.6)-(4.9) has a unique solution (n_0, ψ_0, ϕ_0) given by :

$$n_0 = 1, \ \phi_0 = h(1) - \varepsilon\psi_0, \tag{5.13}$$

and

$$-\mathrm{div}(\nabla\psi_0) = 0 \ \text{ in } \Omega, \ \psi_0 = \bar{\psi}_0 \ \text{ on } \Gamma. \tag{5.14}$$

Then, $u_0 = -\nabla\psi_0$ satisfies the incompressible Euler equations (5.12) with

$$P = \frac{1}{\varepsilon}(h'(1)n_1 - \phi_1) - \psi_1, \tag{5.15}$$

where (n_1, ψ_1, ϕ_1) is the unique solution of the problem (4.10)-(4.13) for $k = 1$. By Theorem 4.2, we have :

$$\|\psi_\tau - \psi_0\|_{\mathcal{C}^{2,\delta}(\bar{\Omega})} \leq A_2\tau^2.$$

Therefore, the velocity $u_\tau = -\nabla\psi_\tau$ satisfies :

$$\|u_\tau - u_0\|_{\mathcal{C}^{1,\delta}(\bar{\Omega})} \leq A_2\tau^2. \tag{5.16}$$

In summary, we have obtained :

Corollary 5.2 *Let $b(x) \equiv 1$, $\bar{n}_0(x) \equiv 1$ and the assumptions (A1)-(A5) and (A6)'-(A7)' hold for $m = 1$. Then for all $\varepsilon \in (0, \varepsilon_4]$, the sequence of solutions $(n_\tau, \psi_\tau, \phi_\tau)_{\tau>0}$ of the problem (4.1)-(4.4) converges, as τ tends to 0, to (n_0, ψ_0, ϕ_0) in B, where (n_0, ψ_0, ϕ_0) is the unique solution of the problem (5.13)-(5.14). Moreover, $u_0 = -\nabla\psi_0$ satisfies the incompressible Euler equations (5.12) with P being defined by (5.15) and (n_1, ψ_1, ϕ_1) being the unique solution of the problem (4.10)-(4.13) for $k = 1$. Furthermore, the estimate (5.16) holds.*

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