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Time-varying fractionally integrated processes with nonstationary long memory

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Abstract

Two classes $A(\mathbf{d}), B(\mathbf{d})$ of time-varying linear filters are introduced, which depend on a given sequence $\mathbf{d} = (d_t, t \in \mathbb{Z})$ of real numbers, and such that, for constant $d_t \equiv d$, $A(\mathbf{d}) = B(\mathbf{d}) = (1-L)^{-d}$ is the usual fractional differencing operator. The invertibility relations $B(-\mathbf{d})A(\mathbf{d}) = A(-\mathbf{d})B(\mathbf{d}) = I$ are established. Long memory properties and partial sums limits of the filtered white noise processes $X_t = A(\mathbf{d})\varepsilon_t$ and $Y_t = B(\mathbf{d})\varepsilon_t$ are studied, in the cases when (a) $\mathbf{d} = (d_t)$ is almost periodic with mean value $\bar{d} \in (0, 1/2)$, and (b) $\mathbf{d} = (d_t)$ admits limits $d_\pm \in (0, 1/2)$ as t tends to $\pm\infty$. We show that in the case (a), renormalized partial sums processes of X_t and Y_t converge to a fractional Brownian motion corresponding to the mean value $\bar{d}$. In the case (b), the limiting processes are generally different self-similar Gaussian processes, depending on the asymptotic constants $d_\pm$ only, which have either asymptotically stationary or asymptotically vanishing increments.

Résumé

Nous introduisons dans cet article deux classes de filtres $A(\mathbf{d}), B(\mathbf{d})$ à coefficients variant dans le temps. Ces filtres construits à partir de suites $\mathbf{d} = (d_t, t \in \mathbb{Z})$ coincident avec les filtres standard FARIMA $A(\mathbf{d}) = B(\mathbf{d}) = (1-L)^{-d}$ pour les suites constantes $d_t \equiv d$. Nous établissons les relations d'inversibilité $B(-\mathbf{d})A(\mathbf{d}) = A(-\mathbf{d})B(\mathbf{d}) = I$.

Nous étudions les propriétés de dépendance et la convergence des sommes partielles renormalisées pour les filtrés de bruits blancs $X_t = A(\mathbf{d})\varepsilon_t$ et $Y_t = B(\mathbf{d})\varepsilon_t$. Nous considérons deux types de suites : (a) les suites presque périodiques de moyenne $\bar{d} \in (0, 1/2)$, et (b) les suites convergentes vers $d_\pm \in (0, 1/2)$ en $\pm\infty$.

Nous montrons dans le cas (a), que X_t et Y_t renormalisés convergent vers le mouvement Brownien fractionnaire. Dans le cas (b), les processus limites sont gaussiens, auto similaires et leurs accroissements sont asymptotiquement soit stationnaires soit dégénérés.

1 Introduction

Fractional integration is a widely used tool to model long memory. The most important case is FARIMA(0,d,0) equation

$$(I-L)^d X_t = \varepsilon_t, \tag{1.1}$$

where $0 < d < 1/2$, $LX_t = X_{t-1}$ is the lag operator, I is the identity, and ε_t is a (weak) white noise, with zero mean and unit variance. A stationary solution to (1.1) is given by

$$X_t \;=\; (1-L)^{-d}\varepsilon_t \;=\; \sum_{j=0}^{\infty} \psi_j(d)\varepsilon_t, \tag{1.2}$$

where $\psi_0(d) := 1$ and

$$\psi_j(d) \;:=\; \Big(\frac{d}{1}\Big)\Big(\frac{d+1}{2}\Big)\cdots\Big(\frac{d-1+j}{j}\Big) \;=\; \frac{\Gamma(d+j)}{j!\Gamma(d)} \qquad (j \ge 1). \tag{1.3}$$

The main purpose of this paper is to introduce time-varying fractionally integrated filters for modeling *nonstationary long memory* and *multifractality.* Ayache and Lévy Véhel (1999) define multifractional BM (Brownian motion) $W_{H(t)}(t)$, by simply substituting the Hurst parameter H in the usual fractional Brownian motion $W_H(t)$ by a function $H(t)$ taking values in the interval $(0,1)$. In a similar way, one could define "multifractional FARIMA" as

$$X_t = \sum_{j=0}^{\infty} \psi_j(d_t)\varepsilon_{t-j}, \tag{1.4}$$

for a suitable sequence $(d_t, t \in \mathbb{Z})$. However, the filter (1.4) is only formally related to fractional integration because the inverse of (1.4) ("differentiation") is unknown and is certainly not of the form $\sum_{j=0}^{\infty} \psi_j(-d_t)X_{t-j}$. Moreover, the intensity of long memory in the process (1.4) changes abruptly together with with d_t, and partial sums of (1.4) are more likely to converge to a usual fractional Brownian motion with the Hurst parameter $H = \limsup_{t\in\mathbb{Z}} d_t + (1/2)$ than to a multifractional Brownian motion.

In this paper we study time-varying fractionally integrated filters given by

$$A(\mathbf{d})x_t \;=\; \sum_{j=0}^{\infty} a_j(t)x_{t-j}, \qquad B(\mathbf{d})x_t \;=\; \sum_{j=0}^{\infty} b_j(t)x_{t-j}, \tag{1.5}$$

where $\mathbf{d} = (d_t, t \in \mathbb{Z})$ is a given function of $t \in \mathbb{Z}$, and where $a_0(t) = b_0(t) := 1$,

$$\begin{aligned} a_j(t) \;&:=\; \Big(\frac{d_{t-1}}{1}\Big)\Big(\frac{d_{t-2}+1}{2}\Big)\Big(\frac{d_{t-3}+2}{3}\Big)\cdots\Big(\frac{d_{t-j}+j-1}{j}\Big) \\ &=\; \prod_{k=1}^{j} \frac{k-1+d_{t-k}}{k}, \end{aligned} \tag{1.6}$$

$$\begin{aligned} b_j(t) \;&:=\; \Big(\frac{d_{t-1}}{1}\Big)\Big(\frac{d_{t-j}+1}{2}\Big)\Big(\frac{d_{t-j+1}+2}{3}\Big)\cdots\Big(\frac{d_{t-2}+j-1}{j}\Big) \\ &=\; \Big(\frac{d_{t-1}}{1}\Big)\prod_{k=2}^{j} \frac{k-1+d_{t-j+k-2}}{k}, \qquad j \ge 1. \end{aligned} \tag{1.7}$$

If $d_t \equiv d$ is a constant, then $A(\mathbf{d}) = B(\mathbf{d}) = (I-L)^{-d}$ is the usual fractional integration operator. In Section 2 we give general conditions on $\mathbf{d}$ under which the sequences $(a_j(t), j \ge 1)$ and $(b_j(t), j \ge 1)$ are bounded in $L^1(\mathbb{Z})$ or $L^2(\mathbb{Z})$ and are dominated in absolute value by some FARIMA coefficients. As shown in Section 3, the operators $A(\mathbf{d})$ and $B(\mathbf{d})$ are related by

$$B(-\mathbf{d})A(\mathbf{d}) \;=\; A(-\mathbf{d})B(\mathbf{d}) \;=\; I, \tag{1.8}$$

where $-\mathbf{d} := (-d_t, t \in \mathbb{Z})$; in other words, $B(-\mathbf{d})$ is the inverse of $A(\mathbf{d})$ and vice versa. Note both $a_j(t)$ and $b_j(t)$ depend on $(d_{t-1}, \dots, d_{t-j})$ and $b_j(t)$ is obtained from $a_j(t)$ by permutation ("reflection") $(d_{t-2}, d_{t-3}, \dots, d_{t-j+1}, d_{t-j}) \to (d_{t-j}, d_{t-j+1}, \dots, d_{t-3}, d_{t-2})$. Although the form of $a_j(t)$ seems "more natural" than that of $b_j(t)$, there are no other reasons to restrict the discussion to the filter $A(\mathbf{d})$ only. In this paper we attempt a parallel study of $A(\mathbf{d})$ and $B(\mathbf{d})$, trying to expose the differences between the two filters and between non-stationary long memory properties of the corresponding filtered white noise processes.

Below we call $\mathbf{d} = (d_t, t \in \mathbb{Z})$ the *Hurst sequence* of filters $A(\mathbf{d})$ and $B(\mathbf{d})$ given by (1.5)-(1.7). In order to avoid "pathological" situations, we shall assume the sequence $\mathbf{d} = (d_t, t \in \mathbb{Z})$ bounded, i.e. $\mathbf{d} \in L^\infty(\mathbb{Z})$. Moreover, we shall often assume that $d_t \notin \mathbb{Z}_-$ for any $t \in \mathbb{Z}$, where $\mathbb{Z}_- := \{0, -1, -2, \dots\}$, which guarantees that $a_j(t), b_j(t)$ do not vanish for any t, j. We do not assume, however, that the values d_t are restricted to the interval $(0, 1/2)$ or any other compact subset of the real line.

Let us describe our main results. Recall that a (stationary) FARIMA(0,d,0) process of (1.1) exibits long memory in the sense that its covariance function EX_sX_t decays as $|t-s|^{2d-1}$ as $|t-s| \to \infty$ in an *arbitrary* way. It is also well-known that partial sums process $\sum_{t=1}^{[N\tau]} X_t, \tau \ge 0$ of the process X_t in (1.1), normalized by $N^{d+(1/2)}$, tends to a fractional BM $W_H(\tau)$ with Hurst parameter $H = d + (1/2)$ (Davydov (1970)). On the other hand, the autocovariance functions and partial sums processes of of nonstationary processes $A(\mathbf{d})\varepsilon_t$ and $B(\mathbf{d})\varepsilon_t$ may exhibit a variety of new behaviors. In the present paper, we focus on the two opposite classes of sequences $\mathbf{d}$, namely:

(a) *Almost periodic* sequences. An element $\mathbf{d} = (d_t, t \in \mathbb{Z}) \in L^\infty(\mathbb{Z})$ is *almost periodic* if and only if it is the limit in $L^\infty(\mathbb{Z})$ of periodic elements. Recall that for such a sequence the limit

$$\bar{d} \;:=\; \lim_{t-s\to\infty} \frac{1}{t-s} \sum_{u=s}^{t} d_u \tag{1.9}$$

exists as $t - s \to \infty$ in arbitrary way. The number $\bar{d}$ in (1.9) is called the *mean value* of $\mathbf{d}$.

(b) *Asymptotic* sequences, in the sense that there exist limits at $t = \pm\infty$:

$$d_\pm := \lim_{t\to\pm\infty} d_t. \tag{1.10}$$

As shown in Section 5, in case (a), or almost periodic $\mathbf{d}$ with mean value $\bar{d} \in (0, 1/2)$, the autocovariances of $X_t = A(\mathbf{d})\varepsilon_t$ and $Y_t = B(\mathbf{d})\varepsilon_t$ decay as $O(|t-s|^{2\bar{d}-1})$ when $|t-s| \to \infty$ in arbitrary way, and the corresponding partial sum processes tend to a multiple of a fractional BM $W_H(\tau)$ with Hurst parameter $H = \bar{d} + (1/2)$ (the multiplicative constants depend on the type of the filter and the sequence $\mathbf{d}$). One may say that in case (a), "averaging of nonstationary long memory" of time-varying filters X_t and Y_t occurs and their long memory properties are similar to the properties of FARIMA filter in (1.1) with parameter $d = \bar{d}$ equal to the mean value of the almost periodic sequence $\mathbf{d}$.

In case (b), the filtered processes $X_t = A(\mathbf{d})\varepsilon_t$ and $Y_t = B(\mathbf{d})\varepsilon_t$ may exhibit a more interesting nonstationary long memory behavior. We say that a random process $\{W_\tau, \tau \ge 0\}$ has *asymptotically stationary increments* if

$$\{W_{T+\tau} - W_T, \tau \ge 0\} \ \to_{\text{f.d.d.}} \ \{\tilde{W}_\tau, \tau \ge 0\} \qquad (T \to \infty), \tag{1.11}$$

where $\{\tilde{W}_\tau, \tau \ge 0\}$ is a nontrivial stochastic process, and $\to_{\text{f.d.d.}}$ denotes the weak convergence of finite dimensional distributions. Similarly, we say that $\{W_\tau, \tau \ge 0\}$ has *asymptotically vanishing increments* if (1.11) holds with $\tilde{W}_\tau \equiv 0$. According to Theorem 4.1 below, in case (b) (assuming $d_\pm \in (0, 1/2)$ and some additional conditions), the partial sums process of X_t converges, under normalization $N^{d_+ + (1/2)}$, to a self-similar process $C^+_{d_+,0,0}(\tau) + C^-_{d_-,d_+-d_-,0}(\tau)$ with asymptotically stationary increments. The limit process is a particular linear combination from the three-parametric family of self-similar Gaussian processes

$$C^+_{h_1,h_2,h_3}(\tau) := \int_0^\tau Z(\mathrm{d}s) s^{h_3} \int_s^\tau t^{h_2}(t-s)^{h_1-1}\mathrm{d}t, \tag{1.12}$$

$$C^-_{h_1,h_2,h_3}(\tau) := \int_{-\infty}^0 Z(\mathrm{d}s)(-s)^{h_3} \int_0^\tau t^{h_2}(t-s)^{h_1-1}\mathrm{d}t, \tag{1.13}$$

written as stochastic integrals w.r.t. a standard Gaussian white noise $Z(\mathrm{d}s)$. The process $C^+_{d_+,0,0}(\tau)$ (called also type II fractional BM, see Marinucci and Robinson (1998)) has asymptotically stationary increments tending (in the sense of (1.11)) to a (usual) fractional BM, while $C^-_{d_-,d_+-d_-,0}(\tau)$ has asymptotically vanishing increments and trajectories a.s. infinitely differentiable on $(0, \infty)$, see Proposition 6.3 below.

On the other hand, according to Theorem 4.2 the partial sums process of $Y_t = B(\mathbf{d})\varepsilon_t$ converges to different limits depending on whether $d_+ > d_-$ or $d_+ < d_-$ ($d_\pm \in (0, 1/2)$) holds: in the first case, the limit process is a type II fractional BM $C^+_{d_+,0,0}(\tau)$ with asymptotically stationary increments, while in the second case, it is a process $C^-_{d_+,0,d_--d_+}(\tau)$ with smooth paths and asymptotically vanishing increments.

The above differences in partial sums behavior of the filtered processes $X_t = A(\mathbf{d})\varepsilon_t$ and $Y_t = B(\mathbf{d})\varepsilon_t$ are quite interesting and suggest different memory transmission mechanisms between distant past and distant future for filters $A(\mathbf{d})$ and $B(\mathbf{d})$. In view of the form of the filter coefficients $a_j(t)$ of (1.6), the limit $d_+ = \lim_{t\to\infty} d_t$ can be interpreted as the intensity of long memory of X_t in the distant future and $d_- = \lim_{t\to-\infty} d_t$ as the corresponding intensity in the distant past. A similar interpretation of $d_\pm$ seems possible also in the case of Y_t. The differences between the two filters are most clearly seen in the case of $\mathbf{d} = (d_t, t \in \mathbb{Z})$ taking two values $d_+, d_- \in (0, 1/2)$:

$$d_t = \begin{cases} d_+, & t \ge 0, \\ d_-, & t < 0. \end{cases} \tag{1.14}$$

For such $\mathbf{d}$, from (1.6)-(1.7) we obtain

$$a_{t-s}(t) = \begin{cases} \psi_{t-s}(d_+), & 0 \le s \le t, \\ \psi_{t-s}(d_-)\frac{\psi_t(d_+)}{\psi_t(d_-)}, & s < 0 \le t, \end{cases} \tag{1.15}$$

and

$$b_{t-s}(t) \quad = \quad \begin{cases} \psi_{t-s}(d_+), & 0 \le s \le t, \\ \psi_{t-s}(d_+)\frac{d_+\psi_{-s}(d_-)}{d_-\psi_{-s}(d_+)}, & s \le 0 < t, \end{cases} \tag{1.16}$$

so that for "typically" large values $t \asymp N$, $-s \asymp N$ we have $a_{t-s}(t) \asymp N^{d_+-1}$ and $b_{t-s}(t) \asymp N^{d_--1}$ decaying at different rates. In particular, in the case $d_- < d_+$ ("positive memory jump at $t = 0$") the process X_t has much stronger memory of its "distant negative past" (as compared with the process Y_t) whose intensity is the same as the intensity d_+ of the "future memory". On the other hand, if $d_- > d_+$ ("negative memory jump at $t = 0$") the process Y_t has much stronger "negative past memory" which dominates its "future memory" so that its limit partial sums process $C^-_{d_+,0,d_--d_+}(\tau)$ is essentially the "echo from the distant past".

The paper is organized as follows. In Section 2 we study general properties of time-varying filters of (1.5). Section 3 is given to the proof of identities (1.8). Section 4 and 5 discuss the covariance decay and the convergence of partial sums processes of the filtered processes $X_t = A(\mathbf{d})\varepsilon_t$ and $Y_t = B(\mathbf{d})\varepsilon_t$ for asymptotic and almost periodic Hurst sequences $\mathbf{d}$, respectively. Section 6 discusses some properties of self-similar processes $C^+_{h_1,h_2,h_3}(\tau)$ and $C^-_{h_1,h_2,h_3}(\tau)$. Technical lemmas are proved in the Appendix.

2 General properties of time-varying fractional filters

Let $\mathcal{L}^p$ ($p \ge 1$) denote the class of all real-valued random processes $\{x_t, t \in \mathbb{Z}\}$ such that $\sup_t E|x_t|^p < \infty$. Let $\mathcal{L}^2_o \subset \mathcal{L}^2$ denote the subclass of all *orthogonal* sequences $\{x_t, t \in \mathbb{Z}\}$ with zero mean, i.e. such that $Ex_t = 0$, $Ex_tx_s = 0$ for any $t, s \in \mathbb{Z}, t \ne s$. Everywhere below we assume that the sequence $\mathbf{d} = (d_t, t \in \mathbb{Z})$ is bounded, i.e.

$$\sup_{t\in\mathbb{Z}} |d_t| < \infty.$$

Definition 2.1. *Let D be a real number. We say that a Hurst sequence $\mathbf{d} = (d_t, t \in \mathbb{Z})$ satisfies condition $A(D)$ if there exists $K < \infty$ such that for all integers $s < t$ such that $t - s > K$*

$$\sum_{s<u<t} \frac{d_u - D}{t-u} \quad \le \quad 0.$$

Similarly, we say that $\mathbf{d} = (d_t, t \in \mathbb{Z})$ satisfies condition $B(D)$ if there exists $K < \infty$ such that for all integers $s < t$ such that $t - s| > K$

$$\sum_{s<u<t} \frac{d_u - D}{u-s} \quad \le \quad 0.$$

Clearly, if $(d_t, t \in \mathbb{Z})$ satisfies condition $A(D)$ (respectively, condition $B(D)$), then the sequence $(d_{t+\tau}, t \in \mathbb{Z})$ also satisfies condition $A(D)$ (respectively, condition $B(D)$), for any $\tau \in \mathbb{Z}$; in other words, conditions $A(D)$ and $B(D)$ are invariant w.r.t. arbitrary translations of Hurst sequences. It is also obvious that if $(d_t, t \in \mathbb{Z})$ satisfies condition $A(D)$ (respectively, $B(D)$), then the time reversed sequence $(d_{-t}, t \in \mathbb{Z})$ satisfies condition $B(D)$ (respectively, condition $A(D)$). The following example shows that conditions $A(D)$ and $B(D)$ are not equivalent.

Example 2.1. Let

$$d_t = \begin{cases} D - 2\Delta, & t = 0 \\ D + \Delta, & t = 1 \\ D - \Delta, & t = 2 \\ D, & t \neq 0, 1, 2, \end{cases}$$

where $\Delta > 0$. Then the sequence $(d_t, t \in \mathbb{Z})$ satisfies condition $A(D)$ and does not satisfy condition $B(D)$. Indeed, let $D = 0, \Delta = 1$, $a_{s,t} := \sum_{s<u<t} d_u/(t-u), b_{s,t} := \sum_{s<u<t} d_u/(u-s)$. Then

$$a_{s,t} = \begin{cases} -2/t + 1/(t-1) - 1/(t-2) \ \le 0, & s < 0, t > 2 \\ 1/(t-1) - 1/(t-2) \ \le 0, & s = 0, t > 2 \\ -1/(t-2) \ \le 0, & s = 1, t > 2 \\ 0, & s < 0, t = 2 \\ -2, & s < 0, t = 1 \end{cases}$$

and $A(D)$ holds. On the other hand, $b_{0,t} = 1/2 > 0$ for $t > 2$ so that $B(D)$ is not satisfied.

Proposition 2.1. *Let $D \notin \mathbb{Z}_- := \{x \in \mathbb{Z} : x \le 0\}$.*
(a) Let a Hurst sequence $(d_t, t \in \mathbb{Z})$ satisfy condition $A(D)$. Then there exists a constant $C < \infty$ such that for all $t \in \mathbb{Z}, j \ge 1$

$$|a_j(t)| \le C|\psi_j(D)|.$$

(b) Let a Hurst sequence $(d_t, t \in \mathbb{Z})$ satisfy condition $B(D)$. Then there exists a constant $C < \infty$ such that for all $t \in \mathbb{Z}, j \ge 1$

$$|b_j(t)| \le C|\psi_j(D)|.$$

Proof. (a) Assume j_0 large enough so that $1/2 < (d_u + k - 1)/(D + k - 1) < 3/2$ holds for all $u \in \mathbb{Z}, k > j_0$. Then

$$\begin{aligned} \Lambda_j(t) := |a_j(t)|/|\psi_j(D)| &= C(t, j_0) \prod_{k=j_0+1}^{j} \frac{d_{t-k} + k - 1}{D + k - 1} \\ &= C(t, j_0) \prod_{k=j_0+1}^{j} \Big(1 + \frac{d_{t-k} - D}{D + k - 1}\Big), \end{aligned}$$

where

$$C(t, j_0) = \left| \prod_{k=1}^{j_0} \frac{d_{t-k} + k - 1}{D + k - 1} \right|$$

is bounded by a constant $C(j_0)$, uniformly in $t \in \mathbb{Z}$, because the Hurst sequence $\mathbf{d}$ is bounded. Using the inequality: $1 + x \le \mathrm{e}^x$ $(x \ge -1)$, this implies

$$\Lambda_j(t) \le C(j_0) \exp\Big\{ \sum_{k=j_0+1}^{j} \frac{d_{t-k} - D}{D + k - 1} \Big\}.$$

Write

$$\sum_{k=j_0+1}^{j} \frac{d_{t-k} - D}{D + k - 1} = \sum_{k=1}^{j} \frac{d_{t-k} - D}{k} + R,$$

where $\sum_{k=1}^{j}(d_{t-k}-D)/k \le 0$ by condition $A(D)$, provided j is large enough. Next,

$$R \ := \ \sum_{k=j_0+1}^{j} (d_{t-k}-D)\Big(\frac{1}{D+k-1}-\frac{1}{k}\Big) \ - \ \sum_{k=1}^{j_0}\frac{d_{t-k}-D}{k} \ =: \ R'-R''$$

where $|R'|$ and $|R''|$ are bounded uniformly in $t\in\mathbb{Z}$ and $j>j_0$. This proves $\Lambda_j(t)\le C$ uniformly in t,j, or part (a).

(b) We omit this proof which is similar to the proof of (a). □

Corollary 2.1. *Let* $\mathbf{d}$ *be a Hurst sequence satisfying condition* $A(D)$ *(respectively, condition* $B(D)$*).*

(i) If $D<0$*, then* $A(\mathbf{d})$ *(respectively,* $B(\mathbf{d})$*) is a bounded operator* $\mathcal{L}^1\to\mathcal{L}^1$*.*

(ii) If $D<1/2$*, then* $A(\mathbf{d})$ *(respectively,* $B(\mathbf{d})$*) is a bounded operator* $\mathcal{L}^2_0\to\mathcal{L}^2$*.*

Corollary 2.1 follows from Proposition 2.1 and well-known properties of FARIMA weights (1.3). Part (i) of this corollary says that there exists a constant $C<\infty$, which depends on $\mathbf{d}$ only, and such that for any random sequence $\{x_t\}$ with $\sup_{t\in\mathbb{Z}}E|x_t|\le 1$, and for any $t\in\mathbb{Z}$

$$E\Big|\sum_{j=0}^{\infty}a_j(t)x_{t-j}\Big| \ \le \ C, \qquad E\Big|\sum_{j=0}^{\infty}b_j(t)x_{t-j}\Big| \ \le \ C,$$

where $a_j(t)$ and $b_j(t)$ are defined in (1.6) and (1.7), respectively. Part (ii) of the above corollary says there exists a constant $C<\infty$, which depends on $\mathbf{d}$ only, such that for any $t\in\mathbb{Z}$

$$E\Big(\sum_{j=0}^{\infty}a_j(t)\varepsilon_{t-j}\Big)^2 \ \le \ C, \qquad E\Big(\sum_{j=0}^{\infty}b_j(t)\varepsilon_{t-j}\Big)^2 \ \le \ C.$$

Proposition 2.2. *(i) Let* $\mathbf{d}$ *be a Hurst sequence and let*

$$\bar d := \limsup_{s,t:t-s\to\infty}\frac{1}{t-s}\sum_{s<u<t}d_u.$$

Then $\mathbf{d}$ *satisfies conditions* $A(D)$ *and* $B(D)$ *with any* $D>\bar d$*.*

(ii) Let $\bar D := \limsup_{|t|\to\infty}d_t$*. Then* $\mathbf{d}$ *satisfies conditions* $A(D)$ *and* $B(D)$ *with any* $D>\bar D$*.*

Proof. (i) Let $\delta>0$ and $D'=\bar d+\delta, D''=\bar d+2\delta$. According to the definition, for any $\delta>0$ there exists $K<\infty$ such that for all $s<t, t-s>K$

$$\alpha_{s,t} := \sum_{s<u<t}(d_u-D')<0. \tag{2.1}$$

Then, for all $s<t, t-s>K$ we have

$$\sum_{s<u<t}\frac{d_u-D''}{t-u} \ = \ -\delta\sum_{s<u<t}\frac{1}{t-u}+\sum_{s<u<t}\frac{d_u-D'}{t-u} \ =: \ -\delta J'_{s,t}+J''_{s,t},$$

where $J'_{s,t}\to\infty$ as $t-s\to\infty$. On the other hand, using (2.1) and summation by parts,

$$J''_{s,t} \ = \ \frac{\alpha_{s,t}}{t-s-1}+\sum_{i=1}^{t-s-2}\alpha_{t-i-1,t}\Big(\frac{1}{i}-\frac{1}{i+1}\Big) \ < \ \sum_{i=1}^{t-s-2}\frac{\alpha_{t-i-1,t}}{i(i+1)} \ =: \ I_{s,t}$$

where

$$I_{s,t} \;=\; \sum_{i=1}^{K} \frac{\alpha_{t-i-1,i}}{i(i+1)} + \sum_{i=K+1}^{t-s-2} \frac{\alpha_{t-i-1,i}}{i(i+1)} \;=:\; I'_{s,t} + I''_{s,t}$$

and where $|I'_{s,t}| \le C$ is uniformly bounded in t for any $K < \infty$ fixed while $I''_{s,t} < 0$ by (2.1). This proves condition $A(D)$ with any $D > \bar{d}$. The proof of condition $B(D)$ follows similarly.
(ii) Follows from (i) and the fact that $\bar{d} \le \bar{D}$. □

3 Invertibility

In this section, the main result is the following theorem.

Theorem 3.1.

(i) Let a Hurst sequence $\mathbf{d} = (d_t, t \in \mathbb{Z})$ satisfy condition $A(D)$ with $D < 1/2$ and the sequence $-\mathbf{d} = (-d_t, t \in \mathbb{Z})$ satisfy condition $B(D)$ with $D < 0$. Then

$$A(\mathbf{d})^{-1} = B(-\mathbf{d}).$$

(ii) Let a Hurst sequence $\mathbf{d} = (d_t, t \in \mathbb{Z})$ satisfy condition $B(D)$ with $D < 1/2$ and the sequence $-\mathbf{d} = (-d_t, t \in \mathbb{Z})$ satisfy condition $A(D)$ with $D < 0$. Then

$$B(\mathbf{d})^{-1} = A(-\mathbf{d}).$$

Proof. (i) The statement is equivalent to $B(-\mathbf{d})A(\mathbf{d}) = I$, where the l.h.s. is well-defined by Corollary 2.1 as an operator from $\mathcal{L}^2_0$ to $\mathcal{L}^1$. To prove it, it is enough to show that for every $n \ge 0$

$$\Delta_n(t) \;=\; \sum_{j=0}^{n} b^*_j(t) a_{n-j}(t-j) = 0 \tag{3.1}$$

where $b^*_j(t)$ are defined by (1.7) with $d_u, u \in \mathbb{Z}$ replaced by $-d_u, u \in \mathbb{Z}$.

For $n = 0$, (3.1) is obvious since $a_0(t) = b^*_0(t) = 1$. Let $n \ge 1$. Note by definition (1.7) that

$$b^*_j(t) = (-1)^j \frac{d_{t-1}}{j!} \prod_{k=1}^{j-1} (d_{t-k-1} - j + k), \quad j \ge 1.$$

We obtain

$$\begin{aligned}
\Delta_n(t) \;&=\; a_n(t) + b^*_1(t) a_{n-1}(t-1) + \sum_{j=2}^{n-1} b^*_j(t) a_{n-j}(t-j) + b^*_n(t) \\
&=\; \frac{1}{n!} \prod_{k=1}^{n} (d_{t-k} + k - 1) - \frac{d_{t-1}}{(n-1)!} \prod_{k=2}^{n} (d_{t-k} + k - 2) \\
&+\; d_{t-1} \sum_{j=2}^{n-1} \frac{(-1)^j}{j!(n-j)!} \prod_{k=1}^{j-1} (d_{t-k-1} + k - j) \prod_{\ell=1}^{n-j} (d_{t-j-\ell} + \ell - 1) \\
&+\; \frac{d_{t-1}(-1)^n}{n!} \prod_{k=1}^{n-1} (d_{t-1-k} + k - n).
\end{aligned}$$

Next,

$$\prod_{k=1}^{j-1}(d_{t-k-1}+k-j)\prod_{\ell=1}^{n-j}(d_{t-j-\ell}+\ell-1) \;=\; \prod_{k=1}^{n-1}(d_{t-k-1}+k-j)$$

leads to

$$\Delta_n(t) \;=\; \frac{d_{t-1}}{n!}\sum_{j=0}^{n}(-1)^j\binom{n}{j}(x_1+n-j)\cdots(x_{n-1}+n-j)$$

with $x_k := d_{t-k-1}-n+k,\ k=1,\dots,n-1$. The result is now deduced from Lemma 3.1.

(ii) The statement is equivalent to $A(-\mathbf{d})B(\mathbf{d})=I$, where the l.h.s. is well-defined by Corollary 2.1 as an operator from $\mathcal{L}^2_0$ to $\mathcal{L}^1$. To prove it, it is enough to show that for every $n\ge 0$

$$\tilde\Delta_n(t) \;:=\; \sum_{j=0}^{n}a^*_j(t)b_{n-j}(t-j) \;=\; 0 \tag{3.2}$$

where $a^*_j(t)$ are defined by (1.6) with $d_u, u\in\mathbb{Z}$ replaced by $-d_u, u\in\mathbb{Z}$.

>From (1.6)-(1.7), for $j=1,\dots,n-2$ we have

$$a^*_j(t)b_{n-j}(t-j) \;=\; \frac{d_{t-1}}{j!(n-j)!}\prod_{k=2}^{j}(k-1-d_{t-k})d_{t-j-1}\prod_{\ell=1}^{n-j-1}(\ell+d_{t-n+\ell-1}).$$

Using the notation $x_k := d_{t-n+k-1}+k,\ k=1,\dots,n-1$, the last product can be rewritten as

$$\prod_{k=2}^{j}(k-1-d_{t-k})d_{t-j-1}\prod_{\ell=1}^{n-j-1}(\ell+d_{t-n+\ell-1}) \;=\; \Big(\prod_{\ell=1}^{n-j-1}x_\ell\Big)(n-j-x_{n-j})\prod_{\ell=n-j+1}^{n-1}(n-x_\ell).$$

Similarly,

$$\begin{aligned}
a^*_0(t)b_n(t) &= \frac{d_{t-1}}{n!}\prod_{\ell=1}^{n-1}x_\ell,\\
a^*_{n-1}(t)b_1(t-n+1) &= \frac{d_{t-1}}{(n-1)!}(1-x_1)\prod_{\ell=2}^{n-1}(n-x_\ell),\\
a^*_n(t)b_0(t-n) &= \frac{d_{t-1}}{n!}(-1)\prod_{\ell=1}^{n-1}(n-x_\ell).
\end{aligned}$$

This yields

$$\begin{aligned}
\tilde\Delta_n(t) &= \frac{d_{t-1}}{n!}\Bigg[\prod_{\ell=1}^{n-1}x_\ell+\sum_{j=1}^{n-2}\binom{n}{j}\Big(\prod_{\ell=1}^{n-j-1}x_\ell\Big)(n-j-x_{n-j})\prod_{\ell=n-j+1}^{n-1}(n-x_\ell)\\
&\qquad+\binom{n}{n-1}(1-x_1)\prod_{\ell=2}^{n-1}(n-x_\ell)-\prod_{\ell=1}^{n-1}(n-x_\ell)\Bigg]\\
&=: \frac{d_{t-1}}{n!}\sum_{j=0}^{n}A_j.
\end{aligned}$$

We claim that for any $m = 1, \dots, n-2$

$$\sum_{j=0}^{m} A_j = \binom{n-1}{m}\Big(\prod_{\ell=1}^{n-m-1} x_\ell\Big) \prod_{k=n-m}^{n-1} (n - x_k). \tag{3.3}$$

Indeed, for $m = 1$,

$$\begin{aligned} A_0 + A_1 &= \prod_{\ell=1}^{n-1} x_\ell + \binom{n}{1}\Big(\prod_{\ell=1}^{n-2} x_\ell\Big)(n-1-x_{n-1}) \\ &= \binom{n-1}{1}\Big(\prod_{\ell=1}^{n-2} x_\ell\Big)(n - x_{n-1}). \end{aligned}$$

The induction step $m-1 \to m$ easily follows:

$$\begin{aligned} \sum_{j=0}^{m} A_j &= \binom{n-1}{m-1}\Big(\prod_{\ell=1}^{n-m} x_\ell\Big) \prod_{k=n-m+1}^{n-1} (n - x_k) \\ &+ \binom{n}{m}\Big(\prod_{\ell=1}^{n-m-1} x_\ell\Big)(n - m - x_{n-m}) \prod_{k=n-m+1}^{n-1} (n - x_k) \\ &= \binom{n-1}{m}\Big(\prod_{\ell=1}^{n-m} x_\ell\Big) \prod_{k=n-m}^{n-1} (n - x_k). \end{aligned}$$

This proves (3.3). Using it, we obtain

$$\sum_{j=1}^{n-1} A_j = \binom{n-1}{n-2} x_1 \prod_{k=2}^{n-1}(n - x_k) + \binom{n}{n-1}(1 - x_1) \prod_{k=2}^{n-1}(n - x_k) = -A_n.$$

This proves (3.2) and Theorem 3.1, too. $\square$

Let $n, k \geq 1$ be integers. Introduce the polynomial $P_{n,k}(x_1, \dots, x_k)$ in $x_1, \dots, x_k$ by

$$P_{n,k}(x_1, \dots, x_k) := \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (j + x_1) \dots (j + x_k). \tag{3.4}$$

Lemma 3.1. *Let $1 \leq k < n$. Then the polynomial function defined in (3.4) satisfy*

$$P_{n,k}(x_1, \dots, x_k) \equiv 0.$$

Proof. For $k < n$, the polynomial satisfies the easily verifiable relations:

$$P_{n+1,k}(x_1, \dots, x_k) = P_{n,k}(x_1 + 1, \dots, x_k + 1) - P_{n,k}(x_1, \dots, x_k), \tag{3.5}$$

$$P_{n+1,k+1}(x_1, \dots, x_{k+1}) = (n+1)P_{n,k}(x_1 + 1, \dots, x_k + 1) + x_{k+1} P_{n+1,k}(x_1, \dots, x_k) \tag{3.6}$$

The lemma is proved by induction on n. Note it is true for $k = 1, n = 2$ as $P_{2,1}(x) = (2+x) - 2(1+x) + x \equiv 0$. Assume that $P_{n,k}(x_1, \dots, x_k) \equiv 0$ for some $n \geq 1$ and all $1 \leq k < n$.

Then from (3.5), we deduce that $P_{n+1,k}(x_1,\ldots,x_k) \equiv 0$ for all $1 \le k < n$. Finally, (3.6) ensures that $P_{n+1,n}(x_1,\ldots,x_n) \equiv 0$, or the validity of the induction step $n \to n+1$. $\square$

Remark 3.1. Lemma 3.1 is *not* true for $k \ge n$. In fact, for $k \ge n \ge 1$,

$$\frac{1}{n!}P_{n,k}(0,\ldots,0) \;=\; \frac{1}{n!}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}j^k \;=\; \left\{ {k \atop n} \right\}$$

is the *Stirling number*, or the number of ways k different objects can be partitioned into n groups.

4 Asymptotic Hurst sequences: convergence to a self-similar process with nonstationary increments

In this section, we consider the case of asymptotic Hurst sequences, that is, sequences $\mathbf{d} = (d_t, t \in \mathbb{Z})$ having limits $d_\pm$ at $\pm\infty$ (see (1.8). Moreover, we assume that $d_\pm \in (0, 1/2)$, that $d_t \notin \mathbb{Z}$ for all t and that there exists a $\delta > 0$ such that

$$d_t \;=\; d_\pm + O(|t|^{-\delta}) \qquad (t \to \pm\infty). \tag{4.1}$$

The main results of this section are Theorems 4.1 and 4.2 which give the limit distributions of partial sums processes for nonstationary processes $X_t = A(\mathbf{d})\varepsilon_t$ and $Y_t = B(\mathbf{d})\varepsilon_t$. Everywhere below $\varepsilon_t, t \in \mathbb{Z}$ denotes a weak white noise, i.e. a sequence of uncorrelated random variables with zero mean and unit variances. By Proposition 2.2 (ii), condition (1.8) implies both conditions $A(D)$ and $B(D)$ with any $D > \max(d_+, d_-)$ so that the processes X_t and Y_t are well-defined provided $\max(d_+, d_-) < 1/2$.

The following Lemmas 4.1 and 4.2 provide the asymptotic behavior of the coefficients $a_{v-u}(v)$ and $b_{v-u}(v)$ as $v-u \to \infty$ and $|v| + |u| \to \infty$ in various ways. These lemmas reflect the connection between the coefficients of filters $A(\mathbf{d})$ and $B(\mathbf{d})$ and FARIMA$(0, d_\pm, 0)$ coefficients for asymptotic Hurst sequences.

Notation. Introduce the following notation. For any integer $v \in \mathbb{Z}$, put

$$\gamma_A(v; d_-) \;:=\; \prod_{u<v}\Big(1 + \frac{d_u - d_-}{d_- + v - u - 1}\Big), \tag{4.2}$$

$$\gamma_B(v; d_+) \;:=\; \prod_{u\ge v}\Big(1 + \frac{d_u - d_+}{d_+ + u - v - 1}\Big). \tag{4.3}$$

Note condition (4.1) implies that the infinite products in (4.2) and (4.3) converge uniformly in $v \le v_0$ and $v \ge v_0$, respectively, for any v_0 fixed. Next, for any integers $u < v$, put

$$\begin{aligned}
\theta^{(1)}_{v,u} &:= a_{v-u}(v)/\psi_{v-u}(d_+),\\
\theta^{(2)}_{v,u} &:= a_{v-u}(v)/\psi_{v-u}(d_-),\\
\theta^{(3)}_{v,u} &:= a_{v-u}(v)\psi_v(d_-)/\psi_{v-u}(d_-)\psi_v(d_+),\\
\theta^{(4)}_{v,u} &:= a_{v-u}(v)/\psi_{v-u}(d_-)\gamma_A(v; d_-)
\end{aligned}$$

and

$$\begin{aligned}
\vartheta^{(1)}_{v,u} &:= b_{v-u}(v)/\psi_{v-u}(d_+),\\
\vartheta^{(2)}_{v,u} &:= b_{v-u}(v)d_-/\psi_{v-u}(d_-)d_{v-1},\\
\vartheta^{(3)}_{v,u} &:= b_{v-u}(v)\psi_{-u}(d_+)d_-/\psi_{v-u}(d_+)\psi_{-u}(d_-)d_{v-1},\\
\vartheta^{(4)}_{v,u} &:= b_{v-u}(v)/\psi_{v-u}(d_+)\gamma_B(u;d_+).
\end{aligned}$$

Lemma 4.1. *We have*

$$\sup_{u:0<u<v}\left|\theta^{(1)}_{v,u}-1\right| = O\left(v^{-1}+v^{-\delta}\log v\right), \qquad v\to+\infty, \tag{4.4}$$

$$\sup_{u:u<v}\left|\theta^{(2)}_{v,u}-1\right| = O\left(|v|^{-\delta}\log|v|\right), \qquad v\to-\infty, \tag{4.5}$$

$$\sup_{u:u\le 0}\left|\theta^{(3)}_{v,u}-1\right| = O\left(v^{-1}+v^{-\delta}\log v\right), \qquad v\to+\infty, \tag{4.6}$$

$$\sup_{v:u<v}\left|\theta^{(4)}_{v,u}-1\right| = O\left(|u|^{-\delta}\log|u|\right). \qquad u\to-\infty. \tag{4.7}$$

Lemma 4.2. *We have*

$$\sup_{v:u<v}\left|\vartheta^{(1)}_{v,u}-1\right| = O\left(u^{-\delta}\right), \qquad u\to+\infty, \tag{4.8}$$

$$\sup_{v:u<v\le 0}\left|\vartheta^{(2)}_{v,u}-1\right| = O\left(|u|^{-1}+|u|^{-\delta}\log|u|\right), \qquad u\to-\infty, \tag{4.9}$$

$$\sup_{v:v>0}\left|\vartheta^{(3)}_{v,u}-1\right| = O\left(|u|^{-1}+|u|^{-\delta}\log|u|\right), \qquad u\to-\infty, \tag{4.10}$$

$$\sup_{u:0<u<v}\left|\vartheta^{(4)}_{v,u}-1\right| = O\left(v^{-\delta}\log v\right). \qquad v\to+\infty. \tag{4.11}$$

The proofs of both lemmas are relegated to Appendix. These lemmas are used in the proof of Theorems 4.1 and 4.2. They can also be used to obtain various asymptotics of autocovariance functions of nonstationary processes $X_t = A(\mathbf{d})\varepsilon_t$ and $Y_t = B(\mathbf{d})\varepsilon_t$ as in Corollary 4.1 below. In this corollary, we consider the process X_t only and give a partial description of the asymptotics of

$$r(s,t) := EX_sX_t \; = \; \sum_{u<s} a_{s-u}(s)a_{t-u}(t), \qquad s\le t,$$

as $t-s\to\infty$. Let $B(d,1-2d)=\int_0^\infty (u+u^2)^{d-1}\mathrm{d}u$ be the beta-function.

Corollary 4.1.
(i) Let $t,s\to\infty, t-s\to\infty$ and $t/s\to 1$. Then

$$r(s,t) \;\sim\; C_1(t-s)^{2d_+-1}, \qquad C_1 := B(d_+,1-2d_+)/\Gamma(d_+)^2.$$

(ii) Let $s \to -\infty$ and $t \to \infty$. Then

$$r(s,t) \ \sim\ C_2 t^{d_+ - d_-}(t-s)^{2d_- - 1}, \qquad C_2 := B(d_-, 1-2d_-)/\Gamma(d_+)\Gamma(d_-).$$

Proof. (i) Write $r(s,t) = r_1(s,t) + r_2(s,t)$, where

$$r_1(s,t) = \sum_{0 \le u < s} a_{s-u}(s) a_{t-u}(t), \qquad r_2(s,t) = \sum_{u<0} a_{s-u}(s) a_{t-u}(t).$$

Using the notation above, we can rewrite the first sum as $r_1(s,t) = \sum_{0 \le u < s} \psi_{s-u}(d_+)\psi_{t-u}(d_+)$ $\theta^{(1)}_{s,u}\theta^{(2)}_{t,u}$. According to Lemma 4.1 (4.4), the last sum behaves asymptotically as the integral

$$\frac{1}{\Gamma(d_+)^2}(t-s)^{2d_+ - 1}\int_0^{s/(t-s)} v^{d_+ - 1}(1+v)^{d_+ - 1}\mathrm{d}v \ \sim\ C_1 (t-s)^{2d_+ - 1}.$$

Similarly, using Lemma 4.1 (4.6), we see that $r_2(s,t)$ behaves asymptotically as

$$\begin{aligned} &\frac{\psi_s(d_+)\psi_t(d_+)}{\psi_s(d_-)\psi_t(d_-)}\sum_{u<0}\psi_{s-u}(d_-)\psi_{t-u}(d_-) \\ &\sim\ \frac{1}{\Gamma(d_+)^2}(ts)^{d_+ - d_-}(t-s)^{2d_- - 1}\int_{s/(t-s)}^{\infty} v^{d_- - 1}(1+v)^{d_- - 1}\mathrm{d}v \ =\ O(s^{2d_+ - 1}). \end{aligned}$$

As $t - s = o(s)$ and $0 < d_+ < 1/2$, we obtain $r_2(s,t) = o((t-s)^{2d_+ - 1})$ and hence part (i).
(ii) By Lemma 4.1 (4.5)-(4.6), similarly as above, the following equivalence relations hold:

$$\begin{aligned} r(s,t) \ &=\ \frac{\psi_t(d_+)}{\psi_t(d_-)}\sum_{u<s}\psi_{s-u}(d_-)\psi_{t-u}(d_-)\theta^{(2)}_{s,u}\theta^{(3)}_{t,u} \\ &\sim\ \frac{t^{d_+ - d_-}\Gamma(d_-)}{\Gamma(d_-)}\sum_{u<s}\psi_{s-u}(d_-)\psi_{t-u}(d_-) \ \sim\ C_2 t^{d_+ - d_-}(t-s)^{2d_- - 1}. \end{aligned}$$

□

Write $\to_{D[0,1]}$ for the weak convergence of random elements in the Skorohod space $D[0,1]$ endowed with the sup-topology.

Theorem 4.1. *Let $X_t := A(\mathbf{d})\varepsilon_t$, where $\varepsilon_t, t \in \mathbb{Z}$ is a stationary ergodic martingale difference sequence, with zero mean and unit variance. Then*

$$N^{-d_+ - (1/2)}\sum_{t=1}^{[N\tau]} X_t \quad \to_{D[0,1]} \quad \frac{1}{\Gamma(d_+)}\left(C^+_{d_+,0,0}(\tau) + C^-_{d_-,d_+ - d_-,0}(\tau)\right), \tag{4.12}$$

where $C^+_{d_+,0,0}(\tau)$ and $C^-_{d_-,d_+ - d_-,0}(\tau)$ are defined in (1.12) and (1.13), respectively.

Theorem 4.2. *Assume that the assumptions of Theorem 4.1 are satisfied. Let $Y_t := B(\mathbf{d})\varepsilon_t$. Then*

$$N^{-d_+ - (1/2)}\sum_{t=1}^{[N\tau]} Y_t \quad \to_{D[0,1]} \quad \frac{1}{\Gamma(d_+)} C^+_{d_+,0,0}(\tau), \qquad \text{if} \quad d_+ > d_-, \tag{4.13}$$

$$N^{-d_- - (1/2)}\sum_{t=1}^{[N\tau]} Y_t \quad \to_{D[0,1]} \quad \frac{d_+}{d_-\Gamma(d_-)} C^-_{d_+,0,d_- - d_+}(\tau), \ \text{if} \quad d_+ < d_-, \tag{4.14}$$

where $C^{+}_{d_+,0,0}(\tau)$ and $C^{-}_{d_+,0,d_- - d_+}(\tau)$ are defined in (1.12) and (1.13), respectively.

Proof of Theorem 4.1. To prove the convergence of finite dimensional dstributions, we use the so-called scheme of discrete stochastic integrals (Surgailis (1982, 2002)). We shall restrict the proof to the case $\tau = 1$. Let I_N stand for the l.h.s. of (4.12), where $\tau = 1$. We shall write I_N as a "discrete stochastic integral" w.r.t. a stochastic measure with orthogonal increments. Namely,

$$I_N = \int_{\mathbb{R}} f_N(x) Z_N(\mathrm{d}x),$$

where the piecewise constant function $f_N = f_N(x), x \in \mathbb{R}$ is defined by

$$f_N(x) := N^{-d_+} \begin{cases} \sum_{t=1}^{N} a_{t-s}(t), & x \in ((s-1)/N, s/N],\ x \in (-\infty, 1], \\ 0, & \text{otherwise.} \end{cases} \tag{4.15}$$

The stochastic measure Z_N is defined on finite intervals $(x', x'']$, $x' < x''$ by

$$Z_N((x', x'']) := N^{-1/2} \sum_{x' < s/N \le x''} \varepsilon_s.$$

The discrete stochastic integral $\int_{\mathbb{R}} g(x) Z_N(\mathrm{d}x) \equiv \int g \mathrm{d}Z_N$ is defined for each (simple) function $g(x)$ taking a finite number of non-zero constant values g^{Δ} on intervals $\Delta = (s/N, (s+1)/N], s \in \mathbb{Z}$, by $\int g \mathrm{d}Z_N := \sum_{\Delta} g^{\Delta} Z_N(\Delta)$. Because of the orthogonality of the measure Z_N (which is a consequence of the martingale difference property of the ε_s's),

$$E\Big(\int g \mathrm{d}Z_N\Big)^2 = \|g\|_2^2, \tag{4.16}$$

where $\|g\|_2 = (\int_{\mathbb{R}} g^2(x)\mathrm{d}x)^{1/2}$ is the norm in the Hilbert space $L^2(\mathbb{R})$.

According to the classical martingale central limit theorem (Billingsley (1968)), for any $m < \infty$ and any disjoint intervals $(x'_i, x''_i], i = 1, \dots, m$,

$$(Z_N((x'_1, x''_1]), \dots, Z_N((x'_m, x''_m])) \ \to_{\text{law}}\ (Z((x'_1, x'']), \dots, Z((x'_m, x''_m]), \tag{4.17}$$

where $Z(\mathrm{d}x)$ is a standard Gaussian noise with mean zero and variance $\mathrm{d}x$. The r.h.s. of (4.12) (corresponding to $\tau = 1$) is written as $I = \int_{\mathbb{R}} f(x) Z(\mathrm{d}x) \equiv \int f \mathrm{d}Z$, where

$$f(x) \ := \ \frac{1}{\Gamma(d_+)} \begin{cases} \int_0^1 (t-x)_+^{d_+ - 1} \mathrm{d}t, & x \in [0, 1], \\ \int_0^1 (t-x)^{d_- - 1} t^{d_+ - d_-} \mathrm{d}t, & x \in (-\infty, 0), \\ 0, & \text{otherwise.} \end{cases} \tag{4.18}$$

Then convergence $I_N \to_{\text{law}} I$ follows from (4.15)-(4.18) and from

$$\|f_N - f\|_2 \ \to \ 0 \quad (N \to \infty); \tag{4.19}$$

see Surgailis (1982) for details. With Lemma 4.1 in mind, set

$$\tilde{f}_N(x) \ := \ N^{-d_+} \begin{cases} \sum_{t=1}^{N} \psi_{t-s}(d_+), & x \in ((s-1)/N, s/N], x \in [0, 1], \\ \sum_{t=1}^{N} \psi_{t-s}(d_-)(\psi_t(d_+)/\psi_t(d_-)), & x \in ((s-1)/N, s/N], x \in (-\infty, 0), \\ 0, & \text{otherwise.} \end{cases}$$

Clearly the convergence (4.19) follows from

$$\|\tilde{f}_N - f_N\|_2 \to 0 \qquad (N \to \infty) \tag{4.20}$$

and from

$$\tilde{f}_N(x) \ \to \ f(x), \quad |\tilde{f}_N(x)| \ \le \ \bar{f}(x) \qquad (\mathrm{d}x\text{–a.e.}), \tag{4.21}$$

where $\bar{f} \in L^2(\mathbb{R})$.

By Lemma 4.1 (4.4)-(4.6) and the dominated convergence theorem, we have $\tilde{f}_N(x) - f_N(x) \to 0$ dx-a.e. Indeed, with $\delta(t) := \sup_{u<0} |\theta^{(3)}_{t,u} - 1| \to 0$ $(t \to \infty)$, for any $x < 0, x \in ((s-1)/N, s/N]$ we obtain

$$\begin{aligned} |\tilde{f}_N(x) - f_N(x)| \ &\le \ CN^{-d_+} \sum_{t=1}^{N} \psi_{t-s}(d_-)(\psi_t(d_+)/\psi_t(d_-))\delta(t) \\ &\le \ CN^{-d_+} \int_0^N (t - Nx)^{d_- -1} t^{d_+ - d_-} \delta([t]) \mathrm{d}t \ = \ o(1) \end{aligned}$$

and the case $0 < x < 1$ follows similarly. From Lemma 4.1 (4.4)-(4.6) it also follow $|\tilde{f}_N(x) - f_N(x)| \le C\tilde{f}_N(x)$, with $C < \infty$ independent of N, x, and therefore (4.20) is a consequence of (4.21) and the dominated convergene theorem. In turn, relations (4.21) can be easily shown using well-known properties of FARIMA$(0, d, 0)$ weights $\psi_t(d)$. (Note $\tilde{f}_N(x) = f_N(x)$ for $\mathbf{d}$ given by (1.14).) This proves finite dimensional convergence in (4.12).

Let us check the tightness in $D[0,1]$. By a well-known criterion, it suffices to show that there exists $\gamma > 0$ such that for $0 \le \tau < \tau + h \le 1$

$$E\left(\sum_{s=[N\tau]}^{[N(\tau+h)]} X_s\right)^2 \ \le \ Ch^{1+\gamma} N^{1+2d_+}, \tag{4.22}$$

Using Lemma 4.1, the proof of (4.22) reduces to

$$\sum_{s=1}^{N} \sum_{t_1,t_2=[N\tau]}^{[N(\tau+h)]} |\psi_{t_1-s}(d_+)\psi_{t_2-s}(d_+)| \ \le \ Ch^{1+\gamma} N^{1+2d_+}, \tag{4.23}$$

$$\sum_{s=-\infty}^{0} \sum_{t_1,t_2=[N\tau]}^{[N(\tau+h)]} \left|\psi_{t_1-s}(d_-)\psi_{t_2-s}(d_-) \frac{\psi_{t_1}(d_+)}{\psi_{t_1}(d_-)} \frac{\psi_{t_2}(d_+)}{\psi_{t_2}(d_-)}\right| \ \le \ Ch^{1+\gamma} N^{1+2d_+}, \tag{4.24}$$

Let us check (4.24); the proof of (4.23) follows similarly and is omitted.

By using integral approximation and the elementary inequality: $\sum_{s\in\mathbb{Z}} |t_1 - s|_+^{-\alpha} |t_2 - s|_+^{-\beta} \le C|t_1 - t_2|_+^{1-\alpha-\beta}$ which is valid for any $0 < \alpha < 1, 0 < \beta < 1, \alpha + \beta > 1$ and any $t_1, t_2 \in \mathbb{Z}$, (4.24) follows from

$$\int_\tau^{\tau+h} \int_\tau^{\tau+h} |u_1 - u_2|^{2d_- -1} u_1^{d_+ - d_-} u_2^{d_+ - d_-} \mathrm{d}u_1 \mathrm{d}u_2 \ \le \ Ch^{1+\gamma}, \tag{4.25}$$

If $d_+ \ge d_-$, (4.25) holds with $\gamma = 2d_- > 0$. This is immediate since $0 \le u_1, u_2 \le 1$. For $d_+ < d_-$, the l.h.s. of (4.25) is monotone in τ and therefore does not exceed

$$\begin{aligned} &2h^{2d_+ -1} \int_0^1 v_1^{d_+ - d_-} \mathrm{d}v_1 \int_0^{v_1} (v_1 - v_2)^{2d_- -1} v_2^{d_+ - d_-} \mathrm{d}v_2 \\ &= 2h^{2d_+ -1} \int_0^1 v_1^{2d_+} \mathrm{d}v_1 \int_0^1 (1-w)^{2d_- -1} w^{d_+ - d_-} \mathrm{d}w \ = \ O(h^{2d_+ -1}), \end{aligned}$$

which yields again (4.24) with $\gamma = 2d_+ > 0$. This concludes the proof of the theorem. $\square$

Proof of Theorem 4.2. It is similar to that of Theorem 4.1, so we just sketch it. Let J_N, J stand for the left and the right hand side of (4.13), respectively, when $\tau = 1$. Then $J_N = \int g_N \mathrm{d}Z_N,\ J = \int g\mathrm{d}Z$, where Z_N, Z are the same as in the proof of Theorem 4.1, and where $g_N = g_N(x), g = g(x),\ x \in \mathbb{R}$ are defined by

$$g_N(x) := N^{-d_+} \begin{cases} \sum_{t=1}^N b_{t-s}(t), & x \in ((s-1)/N, s/N],\ x \in (-\infty, 1], \\ 0, & \text{otherwise.} \end{cases} \tag{4.26}$$

and

$$g(x) := \frac{1}{\Gamma(d_+)} \begin{cases} \int_0^1 (t-x)_+^{d_+-1}\mathrm{d}t, & x \in [0,1], \\ 0, & \text{otherwise.} \end{cases} \tag{4.27}$$

Then, the convergence $J_N \to_{\text{law}} J$ follows from

$$\|g_N - g\|_2 \ \to\ 0. \tag{4.28}$$

Firstly, using Lemma 4.2 (4.8), it is easy to show $\|(g_N - g)\mathbf{1}_{[0,1]}\|_2 \to 0$. Then it remains to prove

$$\|g_N \mathbf{1}_{(-\infty,0)}\|_2 \ \to\ 0. \tag{4.29}$$

Now, by definition of g_N and Lemma 4.2 (4.10),

$$\begin{aligned}
\|g_N \mathbf{1}_{(-\infty,0)}\|_2^2 \ &\le\ CN^{-1-2d_+} \sum_{s<0} \sum_{t_1,t_2=1}^N |b_{t_1-s}(t_1) b_{t_2-s}(t_2)| \\
&\le\ CN^{-1-2d_+} \sum_{s<0} (\psi_{-s}(d_-)/\psi_{-s}(d_+))^2 \sum_{t_1,t_2=1}^N |\psi_{t_1-s}(d_+)(t_1)\psi_{t_2-s}(d_+)| \\
&\le\ CN^{-1-2d_+} \int_0^\infty s^{2(d_- - d_+)}\mathrm{d}s \int_0^N \int_0^N (t_1+s)^{d_+-1}(t_2+s)^{d_+-1}\mathrm{d}t_1\mathrm{d}t_2,
\end{aligned}$$

which converges to zero since the last integral is $O(N^{1+2d_-})$ and $d_- < d_+$. This proves (4.28) and the one-dimensional convergence in (4.13) for $\tau = 1$.

Let us sketch the proof of a similar one-dimensional convergence for $\tau = 1$ in (4.14). Using the same notation, we need again to verify (4.28), where now the functions g_N, g are defined by

$$g_N(x) := N^{-d_-} \begin{cases} \sum_{t=1}^N b_{t-s}(t), & x \in ((s-1)/N, s/N],\ x \in (-\infty, 1], \\ 0, & \text{otherwise,} \end{cases}$$

and

$$g(x) := \frac{d_+}{d_-\Gamma(d_-)} \begin{cases} \int_0^1 (t-x)_+^{d_+-1}\mathrm{d}t, & x \in (-\infty,0), \\ 0, & \text{otherwise.} \end{cases}$$

Using $d_- > d_+$ and Lemma 4.2, (4.8) and (4.10), it follows easily that $\|g_N \mathbf{1}_{[0,1]}\|_2 \to 0$ and $\|(g_N - g)\mathbf{1}_{(-\infty,0)}\|_2 \ \to\ 0$. The remaining details including the proof of tightness in both convergences (4.13) and (4.14) are similar to the proof of Theorem 4.1. Theorem 4.2 is proved. $\square$

5 Almost periodic Hurst sequences: convergence to a fractional Brownian motion

Let us study the limit distribution of partial sums processes of nonstationary processes $X_t = A(\mathbf{d})\varepsilon_t$ and $Y_t = B(\mathbf{d})\varepsilon_t$ corresponding to almost periodic Hurst sequence $\mathbf{d}$.

We assume in this section that $\mathbf{d} = (d_t, t \in \mathbb{Z})$ is almost periodic with the mean value $\bar{d} \in (0, 1/2)$, $d_t \notin \mathbb{Z}_-$ $(t \in \mathbb{Z})$, and that there exists $0 < \delta \le 1$ such that

$$\Big|\frac{1}{t-s}\sum_{u=s}^{t}(d_u - \bar{d})\Big| \;=\; O(|t-s|^{-\delta}) \qquad (t - s \to \infty). \tag{5.1}$$

By Proposition 2.2, these assumptions imply both conditions $A(D)$ and $B(D)$ with any $D > \bar{d}$, and the processes X_t and Y_t are well-defined.

Consider the infinite products (c.f. (4.2), (4.3)):

$$\gamma_A(v) := \prod_{u<v}\Big(1 + \frac{d_u - \bar{d}}{\bar{d} + v - u - 1}\Big), \qquad \gamma_B(v) := \prod_{u \ge v}\Big(1 + \frac{d_u - \bar{d}}{\bar{d} + u - v - 1}\Big). \tag{5.2}$$

Lemma 5.1. *The products (5.2) converge for any $v \in \mathbb{Z}$ and define almost periodic functions.*

Proof. Set, for $n \ge 1$,

$$\begin{aligned}
\gamma_A^n(v) \;&:=\; \prod_{p=1}^{n}\Big(1 + \frac{d_{v-p} - \bar{d}}{\bar{d} + p - 1}\Big) \;=:\; \prod_{p=1}^{n}(1 + \beta_p(v)),\\
\gamma_B^n(u) \;&:=\; \prod_{p=1}^{n}\Big(1 + \frac{d_{u+p-1} - \bar{d}}{\bar{d} + p}\Big) \;=:\; \prod_{p=1}^{n}(1 + \lambda_p(u)).
\end{aligned}$$

The functions β_p and λ_p being almost periodic for any $p \ge 1$ implies that γ_A^n and γ_B^n are almost periodic for any $n \ge 1$. It is well-known that uniform limits of almost periodic functions are almost periodic. The statement of the lemma thus follows from

$$\sup_{v \in \mathbb{Z}}|\gamma_A^n(v) - \gamma_A^m(v)| \;\to\; 0, \qquad \sup_{v \in \mathbb{Z}}|\gamma_B^n(v) - \gamma_B^m(v)| \;\to\; 0, \tag{5.3}$$

as $m, n \to \infty$.

Since $(d_t, t \in \mathbb{Z})$ is bounded, there exists n_0 such that $|\beta_p(v)| < 1/2$ and $|\lambda_p(v)| < 1/2$ holds for any $p \ge n_0, v \in \mathbb{Z}$. Now, uniform convergence of γ_n (respectively, λ_n) is equivalent to the uniform convergence of γ_n/γ_{n_0} (respectively, λ_n/λ_{n_0}), and we can suppose in the sequel that $n_0 = 1$, or that $|\beta_p(v)| < 1/2$ and $|\lambda_p(v)| < 1/2$ hold for all $p \ge 1$. Then $\gamma_n(v) > 0$ and $\lambda_n(v) > 0$, $\forall n \ge 1$, and

$$|\gamma_A^n(v) - \gamma_A^m(v)| \;\le\; \gamma_A^n(v)\left|\prod_{p=n+1}^{m}(1 + \beta_p(v)) - 1\right| \qquad (n < m).$$

Using the inequalities $e^{x-x^2} \le 1 + x \le e^x$ for $|x| < 1/2$ we obtain

$$\gamma_A^n(v) \;\le\; \exp\Big\{\sum_{p=1}^{n}\beta_p(v)\Big\},$$

$$\exp\Big\{\sum_{p=n+1}^{m}\beta_p(v) - \sum_{p=n+1}^{m}\beta_p^2(v)\Big\} \;\le\; \prod_{p=n+1}^{m}(1 + \beta_p(v)) \;\le\; \exp\Big\{\sum_{p=n+1}^{m}\beta_p(v)\Big\}.$$

Here, $\sum_{p=n+1}^{m} \beta_p^2(v) \le Cn^{-1}$, due to $\beta_p^2(v) \le Cp^{-2}$. Let $D_{n,m}(v) := \sum_{p=n+1}^{m}(d_{v-p} - \bar{d})$. Using summation by parts,

$$\sum_{p=n+1}^{m} \beta_p(v) \;=\; \frac{D_{n,m}(v)}{\bar{d}+m-1} \;+\; \sum_{p=n+1}^{m-1} D_{n,p}(v)\Big(\frac{1}{\bar{d}+p-1} - \frac{1}{\bar{d}+p}\Big). \tag{5.4}$$

By (5.1), $|D_{n,m}(v)| \le C|n-m|^{1-\delta}$ where the constant does not depend on v. The first term on the r.h.s. of (5.4) does not exceed $C|n-m|^{1-\delta}m^{-1} \le Cn^{-\delta}$, while

$$\sum_{p=n+1}^{m-1} |D_{n,p}(v)|\Big|\frac{1}{\bar{d}+p-1} - \frac{1}{\bar{d}+p}\Big| \le C\sum_{p=n+1}^{\infty} |p-n|^{1-\delta}p^{-2} \le Cn^{-\delta}.$$

Therefore, for n sufficiently large,

$$\mathrm{e}^{-Cn^{-\delta}} \le \prod_{p=n+1}^{m} (1+\beta_p(v)) \le \mathrm{e}^{Cn^{-\delta}}, \tag{5.5}$$

for some constant $C > 0$ independent of n and $m > n$. It also follows that $\sup_{n\ge 1, v\in\mathbb{Z}} \gamma_A^n(v) < \infty$. This proves the first relation of (5.2) and the second follows similarly. Lemma 5.1 is proved. $\square$

Lemma 5.2. *We have*

$$\begin{aligned} a_{v-u}(v) &= \gamma_A(v)\psi_{v-u}(\bar{d})\theta_{u,v}, \\ b_{v-u}(v) &= \gamma_B(u)(d_{v-1}/\bar{d})\psi_{v-u}(\bar{d})\vartheta_{u,v}, \end{aligned}$$

where $|\theta_{u,v} - 1| + |\vartheta_{u,v} - 1| = O(|v-u|^{-\delta})$ *as* $v - u \to +\infty$.

Proof. By definition,

$$\theta_{u,v} = a_{v-u}/\gamma_A(v)\psi_{v-u}(\bar{d}) = \prod_{p=v-u}^{\infty} (1+\beta_p(v))^{-1},$$

Then from (5.5), $\mathrm{e}^{-C(v-u)^{-\delta}} \le \theta_{u,v} \le \mathrm{e}^{C(v-u)^{-\delta}}$ for all $v-u$ sufficiently large, yielding $|\theta_{u,v} - 1| = O(|v-u|^{-\delta})$. The proof for $b_{v-u}(v)$ and $\vartheta_{u,v}$ is similar. $\square$

Let $\overline{\gamma_A}$, $\overline{\gamma_B^2}$ denote the mean values of the almost periodic functions $\gamma_A(v)$, $\gamma_B^2(v)$, respectively. Let

$$\kappa_A := \overline{\gamma_A}, \qquad \kappa_B := \Big\{\overline{\gamma_B^2}\Big\}^{1/2}. \tag{5.6}$$

Proposition 5.1. *Let* $r_A(s,t)$ *and* $r_B(s,t)$ *denote the autocovariance functions of* $X_t = A(\mathbf{d})\varepsilon_t$ *and* $Y_t = B(\mathbf{d})\varepsilon_t$*, respectively, where* $\varepsilon_t, t \in \mathbb{Z}$ *is a weak white noise. Then, uniformly in* $s < t$*, as* $t - s \to \infty$*,*

$$r_A(s,t) \;\sim\; \frac{B(\bar{d}, 1-2\bar{d})}{\Gamma(\bar{d})^2}\gamma_A(s)\gamma_A(t)(t-s)^{2\bar{d}-1}, \tag{5.7}$$

$$r_B(s,t) \;\sim\; \frac{\kappa_B^2 B(\bar{d}, 1-2\bar{d})}{\bar{d}^2\Gamma(\bar{d})^2} d_{s-1}d_{t-1}(t-s)^{2\bar{d}-1}. \tag{5.8}$$

Proof. Let $s < t$. Then

$$r_A(s,t) = \sum_{u \le s} a_{s-u} a_{t-u} = \gamma_A(s)\gamma_B(t) \sum_{u \le s} \psi_{s-u}(\bar d)\psi_{t-u}(\bar d) + R_A(s,t),$$

where $\sum_{u\le s} \psi_{s-u}(\bar d)\psi_{t-u}(\bar d) \sim (B(\bar d, 1-2\bar d)/\Gamma(\bar d)^2)(t-s)^{2\bar d-1}$ $(t-s\to\infty)$ (Hosking (1981)). Using Lemma 5.2,

$$\begin{aligned} |R_A(s,t)| &= \Big|\gamma_A(s)\gamma_B(t)\sum_{u\le s}\psi_{s-u}(\bar d)\psi_{t-u}(\bar d)(\theta_{s,u}\theta_{t,u}-1)\Big| \\ &\le C\sum_{u\le s}\psi_{s-u}(\bar d)\psi_{t-u}(\bar d)(1+s-u)^{-\delta} \ = \ o((t-s)^{2\bar d-1}) \end{aligned}$$

uniformly in $s < t$. This proves (5.7).

Let us prove (5.8). We have

$$r_B(s,t) = \sum_{u\le s} b_{s-u}b_{t-u} = \frac{d_{s-1}d_{t-1}}{\bar d^2}\sum_{u\le s}\psi_{s-u}(\bar d)\psi_{t-u}(\bar d)\gamma_B(u)^2 + R_B(s,t),$$

where $R_B(s,t) = o((t-s)^{2\bar d-1})$ uniformly in $s<t$, similarly as in the proof of (5.7) above. Then (5.8) follows from

$$I(s,t) := \sum_{u\le s}\psi_{s-u}(\bar d)\psi_{t-u}(\bar d)\rho(u) = o((t-s)^{2\bar d-1}). \tag{5.9}$$

where $\rho(u) := \gamma_B(u)^2 - \kappa_B^2, u\in\mathbb{Z}$ is almost periodic with zero mean value $\overline{\rho} = 0$.

By the well-known properties of almost periodic functions, for any $\epsilon > 0$ one can find a periodic function $\rho_\epsilon(u)$ such that $\sup_{u\in\mathbb{Z}}|\rho(u)-\rho_\epsilon(u)| < \epsilon$ and $|\overline{\rho_\epsilon}| < \epsilon$. Then $I(s,t) = \sum_{k=1}^3 I_k(s,t)$, where $I_1(s,t) := \sum_{u\le s}\psi_{s-u}(\bar d)\psi_{t-u}(\bar d)(\rho_\epsilon(u) - \overline{\rho_\epsilon})$ and

$$\begin{aligned} |I_2(s,t)| &= \Big|\sum_{u\le s}\psi_{s-u}(\bar d)\psi_{t-u}(\bar d)(\rho(u)-\rho_\epsilon(u))\Big| \le C\epsilon|t-s|^{2\bar d-1}, \\ |I_3(s,t)| &= \Big|\overline{\rho_\epsilon}\sum_{u\le s}\psi_{s-u}(\bar d)\psi_{t-u}(\bar d)\Big| \le C\epsilon|t-s|^{2\bar d-1}, \end{aligned}$$

where $\epsilon > 0$ can be taken arbitrarily small. Therefore it suffices to prove (5.8) for a periodic $\rho(u)$ with zero mean value. The last fact clearly follows from the following observation: for any fixed integers p and $T\ge 1$, uniformly in $s<t$

$$\sum_{u\le s: u=p+kT}\psi_{s-u}(\bar d)\psi_{t-u}(\bar d) \ \sim \ \frac{B(\bar d, 1-2\bar d)}{T\Gamma(\bar d)^2}(t-s)^{2\bar d-1} \qquad (t-s\to\infty), \tag{5.10}$$

where the sum is taken over all integers $u\le s$ of the form $u = p+kT, k\in\mathbb{Z}$. The proof of (5.10) follows by the asymptotics of $\psi_t(\bar d)$ and a standard integral approximation. This proves (5.8) and the proposition. □

We recall that a strictly stationary sequence $\xi_t, t \in \mathbb{Z}$ is said *weakly mixing* (see e.g. Halmos (1960, p. 38)) if for any random variable Z with $EZ^2 < \infty$ and any measurable function f with $Ef(\xi_1)^2 < \infty$,

$$N^{-1} \sum_{t=1}^{N} |E(f(\xi_t)Z) - Ef(\xi_1)EZ| \ \to \ 0.$$

In Theorem 5.1, similarly as in Theorems 4.1 and 4.2, we assume that $\varepsilon_t, t \in \mathbb{Z}$ is a strictly stationary martingale difference sequence with zero mean and unit variance.

Theorem 5.1.
(i) Let $X_t = A(\mathbf{d})\varepsilon_t$. Then

$$N^{-\bar{d}-(1/2)} \sum_{t=1}^{[N\tau]} X_t \quad \to_{D[0,1]} \quad \frac{\kappa_A}{\Gamma(\bar{d})} B_{\bar{d}}(\tau). \tag{5.11}$$

(ii) Let $X_t = B(\mathbf{d})\varepsilon_t$. Assume in addition that $\varepsilon_t, t \in \mathbb{Z}$ is weakly mixing. Then

$$N^{-\bar{d}-(1/2)} \sum_{t=1}^{[N\tau]} X_t \quad \to_{D[0,1]} \quad \frac{\kappa_B}{\Gamma(\bar{d})} B_{\bar{d}}(\tau). \tag{5.12}$$

In (5.11) and (5.12), $B_{\bar{d}}(\tau)$ is a fractional Brownian motion as in (6.2).

Proof. (i) The proof is similar to that of Theorem 4.1. We shall prove one-dimensional convergence at $\tau = 1$ only. Write Let I_N, I stand for the left and the right hand side of (5.11), respectively, where $\tau = 1$. Then $I_N = \int f_N \mathrm{d}Z_N$, $I = \int f \mathrm{d}Z$, where Z_N, Z are the same as in the proof of Theorem .1, and where

$$\begin{aligned} f(x) \ &:= \ \frac{\kappa_A}{\Gamma(\bar{d})} \begin{cases} \int_0^1 (t-x)_+^{\bar{d}-1} \mathrm{d}x, & x \le 1, \\ 0, & \text{otherwise}, \end{cases} \\ f_N(x) \ &:= \ N^{-\bar{d}} \begin{cases} \sum_{t=1}^N a_{t-s}(t), & x \in ((s-1)/N, s/N], \ x \in (-\infty, 1], \\ 0, & \text{otherwise}. \end{cases} \end{aligned}$$

Then $I_N \to_{\mathrm{law}} I$ follows from $\|f_N - f\|_2 \to 0$. By using Lemma 5.2, it suffices to show $\|\tilde{f}_N - f\|_2 \to 0$, where

$$\tilde{f}_N(x) := N^{-\bar{d}} \begin{cases} \sum_{t=1}^N \gamma_A(t)\psi_{t-s}(\bar{d}), & x \in ((s-1)/N, s/N], x \in (-\infty, 1], \\ 0, & \text{otherwise}. \end{cases}$$

Next, split $\tilde{f}_N = f'_N + f''_N$, where

$$f'_N(x) := N^{-\bar{d}} \kappa_A \sum_{t=1}^{N} \psi_{t-s}(\bar{d}), \qquad f''_N(x) := N^{-\bar{d}} \sum_{t=1}^{N} (\gamma_A(t) - \kappa_A)\psi_{t-s}(\bar{d}),$$

for $x \in ((s-1)/N, s/N], x \in (-\infty, 1$ and $f'_N(x) = f''_N(x) := 0$ otherwise. It suffices to show

$$\|f'_N - f\|_2 \to 0, \qquad \|f''_N\|_2 \to 0. \tag{5.13}$$

The first relation of (5.13) is well-known. We need to check the second relation of (5.13) only, which is equivalent to

$$R_N \ := \ \sum_{s=-\infty}^{N} \Big(\sum_{t=1}^{N} (\gamma_A(t) - \kappa_A)\psi_{t-s}(\bar d) \Big)^2 \ = \ o(N^{1+2\bar d}). \tag{5.14}$$

We split the sum $R_N = \sum_{s=-\infty}^{0} \ldots + \sum_{s=1}^{N} \ldots =: R'_N + R''_N$. Let $G(t) := \sum_{u=1}^{t} (\gamma_A(u) - \kappa_A)$, then $\epsilon(t) := |G(t)/t| \to 0$ $(t \to \infty)$ since $\gamma_A - \kappa_A$ is almost periodic with zero mean value.

Let $s \le 0$. Using summation by parts we obtain

$$\sum_{t=1}^{N} (\gamma_A(t) - \kappa_A)\psi_{t-s}(\bar d) \ = \ G(N)\psi_{N-s}(\bar d) + \sum_{t=1}^{N-1} G(t)(\psi_{t+1-s}(\bar d) - \psi_{t-s}(\bar d)).$$

Using $|\psi_{N-s}(\bar d)| \le C(N-s)^{\bar d - 1}$, $|\psi_{t+1-s}(\bar d) - \psi_{t-s}(\bar d))| \le C|t-s|^{\bar d - 2}$, we obtain

$$R'_N \ \le \ C\epsilon^2(N) N^2 \sum_{s=0}^{\infty} (N+s)^{2\bar d - 2} + C \sum_{s=0}^{\infty} \Big(\sum_{t=1}^{N} \epsilon(t) t (t+s)^{\bar d - 2} \Big)^2.$$

Here, $N^2 \sum_{s=0}^{\infty} (N+s)^{2\bar d - 2} = O(N^{1+2\bar d})$ while the second sum on the r.h.s. above is dominated by

$$CN^{2\bar d + 1} \int_0^\infty \Big(\sum_0^1 \epsilon(t/N) t (t+s)^{\bar d - 2} \mathrm{d}t \Big)^2 \mathrm{d}s.$$

Since $\int_0^\infty \Big(\sum_0^1 t(t+s)^{\bar d - 2} \mathrm{d}t \Big)^2 \mathrm{d}s < \infty$, the above inequalities and the dominated convergence theorem imply $R'_N = o(N^{1+2\bar d})$. Relation $R''_N = o(N^{1+2\bar d})$ follows similarly, with $G(t)$ replaced by $G(t) - G(s)$. This proves (5.13) and the finite dimensional convergence in (5.11). The proof of tightness in $D[0,1]$ is standard, using Lemma 5.2 and boundedness of γ_A.

(ii) The proof is similar to (i) with the difference that Z_N must be replaced by

$$\tilde Z_N((x', x'']) := \kappa_B^{-1} N^{-1/2} \sum_{x' < s/N \le x''} \gamma_B(s)\varepsilon_s,$$

$x' < x''$, $x', x'' \in \mathbb{R}$. The random measure $\tilde Z_N$ satisfies similar orthogonality property as Z_N and the central limit theorem, namely, for any $m < \infty$ and any disjoint intervals $(x'_i, x''_i], i = 1, \ldots, m$,

$$(\tilde Z_N((x'_1, x''_1]), \ldots, \tilde Z_N((x'_m, x''_m])) \ \to_{\mathrm{law}} \ (Z((x'_1, x'']), \ldots, Z((x'_m, x''_m]), \tag{5.15}$$

c.f. (4.17), where $Z(\mathrm{d}x)$ is the same as in (4.17). The proof of (5.15) follows the argument in Billlingsley (1968, proof of Theorem 23.1) together with the convergence in mean:

$$E\Big| N^{-1} \sum_{t=1}^{N} \gamma_B^2(t)\varepsilon_t^2 - \overline{\gamma_B^2} \Big| \ \to \ 0.$$

The last fact, following from the weak mixing property of ε_t^2 and almost periodicity of γ_B^2, is proved in Zitikis and Davydov (2003).

The remaining details are similar as in the proof of (i). Theorem 5.1 is proved. □

6 A class of self-similar processes with asymptotically stationary or asymptotically vanishing increments

Definition 6.1. *We say that a stochastic process*

(i) $\{W_\tau\} = \{W_\tau, \tau \ge 0\}$ *is self-similar with index* $H > 0$ *($H-$self-similar for short) if* $\{W_{a\tau}, \tau \ge 0\} \ =_{\text{f.d.d.}} \ \{a^H W_\tau, \tau \ge 0\}$ *for any* $a > 0$*;*

(ii) $\{W_\tau\}$ *has stationary increments* $\{W_{T+\tau} - W_T, \tau \ge 0\} \ =_{\text{f.d.d.}} \ \{W_\tau - W_0, \tau \ge 0\}$ *for any* $T > 0$*;*

(iii) $\{W_\tau\}$ *has asymptotically stationary increments if*

$$\{W_{T+\tau} - W_T, \tau \ge 0\} \ \to_{\text{f.d.d.}} \ \{\tilde{W}_\tau, \tau \ge 0\} \qquad (T \to \infty), \tag{6.1}$$

where $\{\tilde{W}_\tau, \tau \ge 0\}$ *is a nontrivial stochastic process;*

(iv) $\{W_\tau\}$ *has asymptotically vanishing increments if (6.1) holds with* $\tilde{W}_\tau = 0, \tau \ge 0$*.*

It is easy to show that if a process $\{W_\tau\}$ has asymptotically stationary increments (respectively, is $H-$self-similar), then the limit process $\{\tilde{W}_\tau\}$ in (6.1) has stationary increments (respectively, is $H-$self-similar). An interesting problem seems the characterization of all Gaussian self-similar processes with asymptotically stationary (asymptotically vanishing) increments. One may expect also that if a process $\{W_\tau\}$ has asymptotically stationary increments and the process $\{K_a^{-1} W_{a\tau}, \tau \ge 0\}$ tends to some limit process $\{Z_\tau\}$ as $a \to \infty$, in the sense of weak convergence of finite dimensional distributions, under some (nonrandom) normalization $K_a \to \infty$, then $\{Z_\tau\}$ is $H-$self-similar with some $H > 0$ and also has stationary increments. The last fact is true and is well-known (due to Lamperti (1962)) if $\{W_\tau\}$ is assumed to have stationary increments.

Let $C^+_{h_1,h_2,h_3}(\tau),\ C^-_{h_1,h_2,h_3}(\tau),\ \tau \ge 0$ be the stochastic integrals as defined in (1.12) and (1.13). Note that the processes $\{C^+_{h_1,h_2,h_3}(\tau)\}$ and $\{C^-_{h_1,h_2,h_3}(\tau)\}$ are Gaussian and mutually independent. The process

$$B_d(\tau) \ := \ C^+_{d,0,0}(\tau) + C^-_{d,0,0}(\tau), \quad \tau \ge 0 \tag{6.2}$$

is a usual fractional Brownian motion with stationary increments, which is well-defined for $d \in (0, 1/2)$ and is self-similar with index $H = d + (1/2)$.

Proposition 6.1.

(i) The process $\{C^+_{h_1,h_2,h_3}(\tau)\}$ *is well-defined if and only if* $h_1 > 0$ *and either* $h_1 + h_2 \ge 0,\ h_3 > -1/2$*, or* $h_1 + h_2 < 0,\ h_1 + h_2 + h_3 > -1/2$*;*

(ii) The process $\{C^-_{h_1,h_2,h_3}(\tau)\}$ *is well-defined if and only if* $h_1 > -1/2,\ h_1 + h_3 < 1/2$ *and either* $h_1 + h_2 \ge 0,\ h_3 > -1/2$*, or* $h_1 + h_2 < 0,\ h_1 + h_2 + h_3 > -1/2$*;*

(iii) Both processes $\{C^+_{h_1,h_2,h_3}(\tau)\}$ *and* $\{C^-_{h_1,h_2,h_3}(\tau)\}$ *are $H-$self-similar with* $H = h_1 + h_2 + h_3 + (1/2)$*.*

Proof. (i) It suffices to consider the case $\tau = 1$. The integral $C^+_{h_1,h_2,h_3}(1)$ is well-defined if and only if

$$\int_0^1 s^{2h_3} \mathrm{d}s \Big(\int_s^1 t^{h_2} (t-s)^{h_1-1} \mathrm{d}t \Big)^2 < \infty.$$

Clearly, $h_1 > 0$ is a necessary condition for this inequality. Then it is equivalent to

$$\int_0^1 s^{2(h_1+h_2+h_3)} \mathrm{d}s \Big(\int_0^{1/s} (1+u)^{h_2} u^{h_1-1} \mathrm{d}u \Big)^2 < \infty. \tag{6.3}$$

Let $h_1 + h_2 < 0$, then the inner integral converges as $s \to 0$ and (6.3) is equivalent to $h_1 + h_2 + h_3 > -1/2$. Next, let $h_1 + h_2 > 0$, then the inner integral is $O((1/s)^{h_1+h_2})$ and (6.3) is equivalent to $h_1 > -1/2$. The remaining details are left to the reader.

(ii) Again, it suffices to consider case $\tau = 1$ only. The integral $C^-_{h_1,h_2,h_3}(1)$ is well-defined if and only if

$$\int_0^\infty s^{2h_3} \mathrm{d}s \Big(\int_0^1 t^{h_2} (t+s)^{h_1-1} \mathrm{d}t \Big)^2 < \infty.$$

Write the last integral as $I_1 + I_2$, where $I_1 = \int_1^\infty s^{2(h_1+h_2+h_3)} \mathrm{d}s \Big(\int_0^{1/s} u^{h_2} (1+u)^{h_1-1} \mathrm{d}u \Big)^2$, $I_2 = \int_0^1 s^{2(h_1+h_2+h_3)} \mathrm{d}s \Big(\int_0^{1/s} u^{h_2} (1+u)^{h_1-1} \mathrm{d}u \Big)^2$. Clearly, $I_1 < \infty$ if and only if $h_2 > -1/2, h_1 + h_3 < 1/2$ hold. Next, as $s \to 0$, so $1/s \to \infty$ and $\int_0^{1/s} u^{h_2}(1+u)^{h_1-1} \mathrm{d}u = O(1)$ if $h_1 + h_2 < 0$; $= O((1/s)^{h_1+h_2})$ if $h_1 + h_2 > 0$. Hence the statement (ii) follows.

(iii) Follows by scaling property $Z(\mathrm{d}(au)) =_{\mathrm{law}} a^{1/2} Z(\mathrm{d}u)$ of white noise and scaling properties of the kernels of integrals in (1.12) and (1.13). □

Proposition 6.2. *Let $d, d_+, d_- \in (0, 1/2)$. Then:*

(i) The processes $\{C^-_{d_+,0,d_- - d_-}(\tau)\}$ and $\{C^-_{d_-,d_+ - d_-,0}(\tau)\}$ have asymptotically vanishing increments;

(ii) The process $\{C^+_{d,0,0}(\tau)\}$ has asymptotically stationary increments tending to those of a fractional Brownian motion $\{B_d(\tau)\}$ of (6.2).

Proof. (i) Note $\{C^-_{d_+,0,d_- - d_+}(\tau)\}$ and $\{C^-_{d_-,d_+ - d_-,0}(\tau)\}$ are well-defined for any $d_+, d_- \in (0, 1/2)$ by Proposition 6.1 (ii). It suffices to show

$$\int_0^\infty s^{2(d_- - d_+)} \mathrm{d}s \Big(\int_T^{T+\tau} (t+s)^{d_+ - 1} \mathrm{d}t \Big)^2 \quad \to \quad 0, \tag{6.4}$$

$$\int_0^\infty \mathrm{d}s \Big(\int_T^{T+\tau} t^{d_+ - d_-} (t+s)^{d_- - 1} \mathrm{d}t \Big)^2 \quad \to \quad 0, \tag{6.5}$$

for each $\tau > 0$, as $T \to \infty$. It suffices to consider $\tau = 1$ only. Then the inner integral in (6.4) does not exceed $(T+s)^{d_+ - 1}$ and therefore the l.h.s. of (6.4) does not exceed $T^{2d_- - 1} \int_0^\infty s^{2(d_- - d_+)} (1+s)^{2d_+ - 2} \mathrm{d}s \le C T^{2d_- - 1} = o(1)$, proving (6.4). The proof of (6.5) follows similarly.

(ii) Write $C^+_{d,0,0}(\tau) = B_d(\tau) - C^-_{d,0,0}(\tau)$, then by (i) the process $\{C^-_{d,0,0}(\tau)\}$ has asymptotically vanishing increments, while $\{B_d(\tau)\}$ has stationary increments. Therefore $\{C^+_{d,0,0}(\tau)\}$ has

asymptotically stationary increments $\{C^+_{d,0,0}(\tau+T)-C^+_{d,0,0}(T), \tau\ge 0\} \to_{\text{f.d.d.}} \{B_d(\tau), \tau \ge 0\}$.
$\square$

Proposition 6.3. *The trajectories of processes* $\{C^-_{d_+,0,d_--d_-}(\tau)\}$ *and* $\{C^-_{d_-,d_+-d_-,0}(\tau)\}$ *are a.s. infinitely differentiable on* $(0,\infty)$*, for any* $d_+, d_- \in (0,1/2)$.

Proof. For a Gaussian process of the form $W(\tau) = \int_0^\infty g(\tau,s)Z(\mathrm{d}s)$, the a.s. differentiability on $(0,\infty)$ follows if we show that for any $k\ge 1$, the partial derivatives $g^{(k)}(\tau,s) = \partial^k g(\tau,s)/\partial\tau^k$ exists for each $\tau, s>0$ and are square integrable in $s\in(0,\infty)$ and the integrals $\int_0^\infty (g^{(k)}(\tau,s))^2\mathrm{d}s$ are bounded on compact intervals of $(0,\infty)$ (see e.g. Kutoyants (1985)). The process $C^-_{d_-,d_+-d_-,0}(\tau)$ admits this form of representation with

$$g(\tau,s) := \int_0^\tau t^{d_+-d_-}(t+s)^{d_--1}\mathrm{d}t, \quad \tau, s>0.$$

We have

$$g^{(k)}_\tau(\tau,s) = \sum_{\ell=0}^{k-1}\binom{k-1}{\ell} g_1^{(\ell)}(\tau) g_2^{(k-1-\ell)}(\tau+s),$$

where $g_1(\tau) := \tau^{d_+-d_-}$, $g_2(\tau) := \tau^{d_--1}$. Then $|g_1^{(\ell)}(\tau)| \le C\tau^{d_+-d_--\ell}$, $|g^{(k-1-\ell)}(\tau+s)| \le C(\tau+s)^{d_--k+\ell}$, implying

$$\begin{aligned}
\int_0^\infty (g^{(k)}(\tau,s))^2\mathrm{d}s &\le C\sum_{\ell=0}^{k-1}\tau^{2(d_+-d_--\ell)}\int_0^\infty(\tau+s)^{2(d_--k+\ell)}\mathrm{d}s \\
&\le C\sum_{\ell=0}^{k-1}\tau^{2(d_+-d_--\ell)}\int_0^\infty(\tau+s)^{2(d_--1)}\mathrm{d}s \\
&= C\sum_{\ell=0}^{k-1}\tau^{2(d_+-\ell)-1}\int_0^\infty(1+s)^{2(d_--1)}\mathrm{d}s,
\end{aligned}$$

where the last expression is bounded on $\tau\in(\epsilon,\infty)$ for any $\epsilon>0$ fixed. This proves the a.s. differentiability of $\{C^-_{d_-,d_+-d_-,0}(\tau)\}$. For the process $\{C^-_{d_+,0,d_--d_-}(\tau)\}$ the proof is similar by taking the function $g(\tau,s) = s^{d_--d_+}\int_0^\tau(t+s)^{d_+-1}\mathrm{d}t$. $\square$

7 Appendix: proofs of Lemmas 4.1 and 4.2

In the sequel, C denotes a constant which may be different at each occurence.

Proof of Lemma 4.1. Let us prove (4.4). By definition,

$$\theta^{(1)}_{v,u} = \frac{a_{v-u}(v)}{\psi_{v-u}(d_+)} = \prod_{u\le k<v}\Big(1+\frac{d_k-d_+}{d_++v-k-1}\Big).$$

Note that for v is large enough, $\beta^+_{v,k} := (d_k-d_+)/(d_++v-k-1)$ is arbitrary small uniformly in $k<v$. Indeed, by boundedness of $(d_t, t\in\mathbb{Z})$, for any $\epsilon>0$ one can find $K<\infty$ such that $|\beta^+_{v,k}|<\epsilon$ for any $v\in\mathbb{Z}, k<v-K$. On the other hand, $\max_{v-K\le k<v}|\beta^+_{v,k}| \to 0$ $(v\to+\infty)$ as

$\max_{v-K\le k\le v}|d_k-d_+|\to 0$ $(v\to+\infty)$. Consequently using the inequalities $\mathrm{e}^{-2|x|}\le 1+x\le \mathrm{e}^{2|x|}$ for $|x|$ small enough, for v sufficiently large we obtain

$$\exp\Big\{-2\sum_{u\le k<v}|\beta^+_{v,k}|\Big\}\ \le\ \theta^{(1)}_{v,u}\ \le\ \exp\Big\{2\sum_{u\le k<v}|\beta^+_{v,k}|\Big\},$$

where, by (4.1),

$$\sum_{u\le k<v}|\beta^+_{v,k}|\le C\sum_{1\le k<v}k^{-\delta}(v-k)^{-1}.$$

As v tends to infinity, the r.h.s. is dominated by

$$C\int_1^{v-1}x^{-\delta}(v-x)^{-1}\mathrm{d}x\ =\ Cv^{-\delta}\int_{1/v}^{1-1/v}x^{-\delta}(1-x)^{-1}\mathrm{d}x\ \le\ C(v^{-\delta}\log v+v^{-1}).$$

This proves (4.4).

Let us prove (4.5). We have

$$\theta^{(2)}_{v,u}\ =\ \frac{a_{v-u}(v)}{\psi_{v-u}(d_-)}\ =\ \prod_{u\le k<v}\Big(1+\frac{d_k-d_-}{d_-+v-k-1}\Big).$$

Put $\beta^-_{v,k}:=(d_k-d_-)/(d_-+v-k-1)$. Then similarly as above, one can show that $\beta^-_{v,k}$ is arbitrarily small uniformly in $k<v$, as $v\to-\infty$. Therefore for all $u<v<v_0<0$ and some $-v_0$ large enough, we have

$$\exp\Big\{-2\sum_{u\le k<v}|\beta^-_{v,k}|\Big\}\ \le\ \theta^{(2)}_{v,u}\ \le\ \exp\Big\{2\sum_{u\le k<v}|\beta^-_{v,k}|\Big\},$$

where, by (4.1),

$$\begin{aligned}\sum_{u\le k<v}|\beta^-_{v,k}|\ &\le\ C\sum_{k<v}|k|^{-\delta}(v-k)^{-1}\le C\sum_{p>0}(p+|v|)^{-\delta}p^{-1}\\ &\le\ C\int_1^\infty(x+|v|)^{-\delta}x^{-1}\mathrm{d}x\ \le\ C|v|^{-\delta}\log|v|.\end{aligned}$$

Hence, (4.5) is proved.

Next, let us prove (4.6). We have for $u<0<v$

$$\theta^{(3)}_{v,u}\ =\ \prod_{u\le p<0}\Big(1+\frac{d_p-d_-}{d_-+v-p-1}\Big)\prod_{0\le k<v}\Big(1+\frac{d_k-d_+}{d_++v-k-1}\Big).$$

As above, for any $u<0$ and some $v>v_0>0$ large enough,

$$\exp\Big\{-2\sum_{u\le p<0}|\beta^-_{v,p}|-2\sum_{0\le k<v}|\beta^+_{v,k}|\Big\}\ \le\ \theta^{(3)}_{v,u}\ \le\ \exp\Big\{2\sum_{u\le p<0}|\beta^-_{v,p}|+2\sum_{0\le k<v}|\beta^+_{v,k}|\Big\},$$

Here, by (4.1),

$$\begin{aligned}\sum_{u\le p<0}|\beta^-_{v,p}|\ &\le\ C\sum_{u\le p<0}|p|^{-\delta}(v-p)^{-1}\le C\sum_{k>0}k^{-\delta}(v+k)^{-1}\\ &\le\ C\int_1^\infty x^{-\delta}(v+x)^{-1}\mathrm{d}x\ \le\ C(v^{-\delta}\log v+v^{-1})\end{aligned}$$

and

$$\sum_{0\le k<v} |\beta^+_{v,k}| \le |\beta^+_{v,0}| + C\sum_{k=1}^{v-1} k^{-\delta}(v-k)^{-1} \le C(v^{-1}+v^{-\delta}\log v),$$

where the last sum was estimated in the proof of (4.2). This proves (4.6).

Finally, let us prove (4.7). We have

$$\theta^{(4)}_{v,u} \;=\; \frac{\prod_{u\le k<v}\Big(1+\frac{d_k-d_-}{d_-+v-k-1}\Big)}{\prod_{k<v}\Big(1+\frac{d_k-d_-}{d_-+v-k-1}\Big)} \;=\; \prod_{k<u}\Big(1+\frac{d_k-d_-}{d_-+v-k-1}\Big)^{-1}.$$

Then noting that $\beta^-_{v,k}$ is arbitrarily small uniformly in $k<u<v$, as $u\to-\infty$, for any $u<v$ and $|u|$ large enough $(u>0)$ we have

$$\exp\Big\{-2\sum_{k<u}|\beta^-_{v,k}|\Big\} \;\le\; \theta^{(4)}_{v,u} \;\le\; \exp\Big\{2\sum_{k<u}|\beta^-_{v,k}|\Big\},$$

where by (4.1),

$$\sum_{k<u}|\beta^-_{v,k}| \;\le\; C\sum_{k<u}|k|^{-\delta}(u-k)^{-1} \;\le\; C\sum_{k=1}^{\infty}k^{-1}(|u|+k)^{-\delta} \;\le\; C|u|^{-\delta}\log|u|.$$

Thus, we obtain (4.7) and the lemma. □

Proof of Lemma 4.2. Let us prove (4.8). By definition,

$$\vartheta^{(1)}_{v,u} \;=\; \frac{d_{v-1}}{d_+}\prod_{u\le k\le v-2}\Big(1+\frac{d_k-d_+}{d_++k-u+1}\Big).$$

Note $d_{v-1}/d_+ \to 1$ as $u\to+\infty$ uniformly in $v>u$, and that for u large enough, $\lambda^+_{u,k} := (d_k-d_+)/(d_++k-u+1)$ is arbitrary small uniformly in $k\ge u$. Consequently for any $\epsilon>0$ and all u sufficiently large we obtain

$$(1-\epsilon)\exp\Big\{-2\sum_{u\le k\le v-2}|\lambda^+_{u,k}|\Big\} \;\le\; \vartheta^{(1)}_{v,u} \;\le\; (1+\epsilon)\exp\Big\{2\sum_{u\le k\le v-2}|\lambda^+_{u,k}|\Big\},$$

where, by (4.1),

$$\sum_{u\le k\le v-2}|\lambda^+_{u,k}| \;\le\; C\sum_{u\le k}k^{-\delta}(k-u+1)^{-1} \;\le\; Cu^{-\delta},$$

proving (4.8).

Let us prove (4.9). We have

$$\vartheta^{(2)}_{v,u} \;=\; \prod_{u\le k\le v-2}\Big(1+\frac{d_k-d_-}{d_-+k-u+1}\Big).$$

Put $\lambda^-_{u,k} := (d_k-d_-)/(d_-+k-u+1)$. Then similarly as in the proof of (4.4), one can show that $\lambda^-_{u,k}$ is arbitrarily small uniformly in $k>u$, as $u\to-\infty$, and therefore for $u<0, |u|$ large enough and $u<v$

$$\exp\Big\{-2\sum_{u\le k\le v-2}|\lambda^-_{u,k}|\Big\} \;\le\; \vartheta^{(2)}_{v,u} \;\le\; \exp\Big\{2\sum_{u\le k\le v-2}|\lambda^-_{u,k}|\Big\},$$

where, by (4.1),

$$\sum_{u\le k\le v-2} |\lambda^-_{u,k}| \;\le\; C \sum_{u\le k\le -1} |k|^{-\delta}(k-u+1)^{-1} \;\le\; C\left(|u|^{-1}+|u|^{-\delta}\log|u|\right),$$

proving (4.9).

Next, let us prove (4.10). We have for $u<0<v$

$$\begin{aligned}\vartheta^{(3)}_{v,u} &= \frac{d_-\psi_{-u}(d_+)}{d_+\psi_{-u}(d_-)}\prod_{u\le k\le v-2}\Big(1+\frac{d_k-d_+}{d_++k-u+1}\Big)\\ &= \prod_{u\le p<-1}\Big(1+\frac{d_p-d_-}{d_-+p-u+1}\Big)\prod_{-1\le k<v-2}\Big(1+\frac{d_k-d_+}{d_++k-u+1}\Big).\end{aligned}$$

As above, for any $|u|$ large enough, $u<0$, and all $v>0$ we have

$$\exp\Big\{-2\sum_{p=u}^{-2}|\lambda^-_{u,p}|-2\sum_{k=-1}^{v-2}|\lambda^+_{u,k}|\Big\} \;\le\; \vartheta^{(3)}_{v,u} \;\le\; \exp\Big\{2\sum_{p=u}^{-2}|\lambda^-_{u,p}|+2\sum_{k=-1}^{v-2}|\lambda^+_{u,k}|\Big\}.$$

Here, by (4.1),

$$\sum_{p=u}^{-2}|\lambda^-_{u,p}| \;\le\; C\sum_{p=u}^{-2}|p|^{-\delta}(p-u+1)^{-1} \;\le\; C\left(|u|^{-1}+|u|^{-\delta}\log|u|\right)$$

and

$$\sum_{k=-1}^{v-2}|\lambda^+_{u,k}| \;\le\; |\lambda^+_{u,-1}|+|\lambda^+_{u,0}|+C\sum_{k=1}^{\infty}k^{-\delta}(k+|u|)^{-1} \;\le\; C\left(|u|^{-1}+|u|^{-\delta}\log|u|\right).$$

This proves (4.10)

Finally, let us prove (4.11). We have

$$\vartheta^{(4)}_{v,u} \;=\; \frac{d_{v-1}}{d_+}\frac{\prod_{u\le k\le v-2}\left(1+\frac{d_k-d_+}{d_++k-u+1}\right)}{\prod_{u\le k}\left(1+\frac{d_k-d_+}{d_++k-u+1}\right)} \;=\; \frac{d_{v-1}}{d_+}\prod_{k\ge v-1}\Big(1+\frac{d_k-d_+}{d_++k-u+1}\Big)^{-1}.$$

Then noting that $d_{v-1}/d_+\to 1$ and that $\lambda^+_{u,k}$ is arbitrarily small uniformly in $k\ge v-1>u$, as $v\to\infty$, for any $\delta>0, 0<u<v$ and all v large enough we have

$$(1-\epsilon)\exp\Big\{-2\sum_{k\ge v-1}|\lambda^+_{u,k}|\Big\} \;\le\; \vartheta^{(4)}_{v,u} \;\le\; (1+\epsilon)\exp\Big\{2\sum_{k\ge v-1}|\lambda^+_{u,k}|\Big\},$$

where by (4.1),

$$\sum_{k\ge v-1}|\lambda^+_{u,k}| \;\le\; C\sum_{k\ge v-1}k^{-\delta}(k-v+d_+)^{-1} \;\le\; Cv^{-\delta}\log v.$$

This proves (4.11) and the lemma. $\square$

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