Infinite-dimensional hyperkähler manifolds associated with Hermitian-symmetric affine coadjoint orbits

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Abstract

In this paper, we construct a hyperkähler structure on the complexification $\mathcal{O}^{\mathbb{C}}$ of any Hermitian symmetric affine coadjoint orbit $\mathcal{O}$ of a semi-simple L^*-group of compact type, which is compatible with the complex symplectic form of Kirillov-Kostant-Souriau and restricts to the Kähler structure of $\mathcal{O}$. By a relevant identification of the complex orbit $\mathcal{O}^{\mathbb{C}}$ with the cotangent space $T'\mathcal{O}$ of $\mathcal{O}$ induced by Mostow's decomposition theorem, this leads to the existence of a hyperkähler structure on $T'\mathcal{O}$ compatible with Liouville's complex symplectic form and whose restriction to the zero section is the Kähler structure of $\mathcal{O}$. Explicit formulas of the metric in terms of the complex orbit and of the cotangent space are given. As a particular case, we obtain the one-parameter family of hyperkähler structures on a natural complexification of the restricted Grassmannian and on the cotangent space of the restricted Grassmannian constructed by hyperkähler reduction in [29] .

Résumé

Dans cet article, nous construisons une métrique hyperkählerienne sur l'orbite complexifiée $\mathcal{O}^{\mathbb{C}}$ de toute orbite coadjointe affine hermitienne symétrique $\mathcal{O}$ d'un L^*-groupe semi-simple de type compact, qui est compatible avec la forme symplectique complexe de Kirillov-Kostant-Souriau et qui se restreint en la structure kählérienne de $\mathcal{O}$. Grâce à une identification pertinente de l'orbite complexifiée $\mathcal{O}^{\mathbb{C}}$ avec l'espace cotangent $T'\mathcal{O}$ de l'orbite de type compact $\mathcal{O}$ induite par le théorème de décomposition de Mostow, nous en déduisons l'existence d'une structure hyperkählérienne sur $T'\mathcal{O}$ compatible avec la forme symplectique complexe de Liouville et dont la restriction à la section nulle est la structure kählérienne de $\mathcal{O}$. Des formules explicites de la métriques en termes de l'orbite complexifiée et de l'espace cotangent sont données. Comme cas particulier, nous retrouvons la famille à un paramètre de structures hyperkählériennes sur une complexification naturelle de la grassmannienne restreinte et sur l'espace cotangent de la grassmannienne restreinte obtenue par réduction hyperkählérienne en [29].

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1 Introduction

In finite dimension, each (co-)adjoint orbit $\mathcal{O}$ of a compact semi-simple Lie group G is an homogeneous Kähler manifold (hence of dimension $2n$, $n \in \mathbb{N}$). There exists a unique complex semi-simple Lie group $G^{\mathbb{C}}$ such that G embeds into $G^{\mathbb{C}}$ and such that this embedding induces the natural injection of the Lie algebra $\mathfrak{g}$ of G into the complex Lie algebra $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \oplus i\mathfrak{g}$. In this setting, adjoint and coadjoint orbits of G (resp. $G^{\mathbb{C}}$) are identified via the Killing form of $\mathfrak{g}$ (resp. $\mathfrak{g}^{\mathbb{C}}$). The complexification $\mathcal{O}^{\mathbb{C}}$ of $\mathcal{O}$ is defined as the orbit of any element in $\mathcal{O}$ under the coadjoint action of $G^{\mathbb{C}}$. It is natural to ask whether the coadjoint orbit $\mathcal{O}^{\mathbb{C}}$ (which is of dimension $4n$) admits a hyperkähler structure compatible with the complex symplectic form of Kirillov-Kostant-Souriau. In the same circle of idea, one can ask whether the cotangent space of $\mathcal{O}$ (which is again a $4n$-dimensional manifold) admits a hyperkähler structure compatible with Liouville's complex symplectic form. These two questions have been answered by the affirmative by O. Biquard in [6] and independently by A.G. Kovalev in [16]. More precisely, a family of hyperkähler structures on the complex adjoint orbit $\mathcal{O}^{\mathbb{C}}$ of an element $\tau \in \mathfrak{g}$ answering the first question is given by Theorem 1 in [6] and Theorem 1 in [16] applied to the triple $(0, \tau, 0)$. Adding the requirement that the hyperkähler structure should extend the Kähler structure of $G \cdot \tau =: \mathcal{O}$, specifies the hyperkähler structure in the family. A family of hyperkähler structures on the cotangent space of $\mathcal{O}$ answering the second question is given by Theorem 2, 2) in [6] with $\tau^r = i\tau$ and $\tau^c = 0$. The restriction to the zero section of one of these hyperkähler structures is the Kähler structure of $\mathcal{O}$. The aforementioned results are based on the study of different forms of Nahm's equations and extend related results obtained by P. B. Kronheimer ([17], [18]). Unfortunately the hyperkähler metrics involved are not explicit.

In the special case of compact Hermitian-symmetric orbits $\mathcal{O}$, an explicit formula for the unique G-invariant hyperkähler metric on $\mathcal{O}^{\mathbb{C}}$, which restricts to the Kähler metric of $\mathcal{O}$ and is compatible with the complex symplectic form of Kirillov-Kostant-Souriau, is given by O. Biquard and P. Gauduchon in [7] in terms of the curvature of $\mathcal{O}$. Its construction is based on the existence of a fiber bundle structure on $\mathcal{O}^{\mathbb{C}}$ over $\mathcal{O}$. A projection from the complex orbit onto the orbit of compact type exists for general adjoint orbits as a consequence of Mostow's decomposition theorem (see [30]). Nevertheless, only in the Hermitian-symmetric case it has the property of minimizing the distance in $\mathfrak{g}^{\mathbb{C}}$ to the orbit of compact type (with respect to the Hermitian product on $\mathfrak{g}^{\mathbb{C}}$ whose restriction to $\mathfrak{g}$ is the opposite of the Killing form). This metrical characterization is crucial in the proof of the aforementioned result. In [8], the same authors express in terms of the curvature of $\mathcal{O}$ the unique G-invariant hyperkähler metric on the cotangent space $T'\mathcal{O}$ compatible with Liouville's symplectic form, whose restriction to the zero section is the Kähler metric of $\mathcal{O}$. The finishing touches to the picture are given in [9], where the hyperkähler manifolds $\mathcal{O}^{\mathbb{C}}$ and $T'\mathcal{O}$ are identified. In the present work, we extend the aforementioned results of [7], [8] and [9] to the infinite-dimensional setting, considering Hermitian-symmetric affine coadjoint orbits of semi-simple L^*-groups of compact type. As far as we know, the case of a general orbit of an L^*-group is an open problem.

The necessity of considering *affine* coadjoint orbits instead of simply coadjoint orbits is motivated by the example of the connected components of the restricted Grassmannian, which are affine coadjoint orbits of the unitary L^*-group U_2 (see below for the precise definition of this group) but not coadjoint orbits of U_2 in the usual sense. The non-equivalence of these two notions in the infinite-dimensional case is related to the fact that not every derivation of an infinite-dimensional semi-simple L^*-algebra is inner. In other words, every derivation D of a L^*-algebra defines an affine coadjoint orbit $\mathcal{O}_D$ of the corresponding L^*-group, which is a coadjoint orbit if and only if the derivation is inner.

The classification of irreducible infinite-dimensional Hermitian-symmetric affine (co-)adjoint orbits of compact type has been carried out in [31], generalizing the classification given in the finite-dimensional case by J. Wolf in [34]. The classification of Hermitian-symmetric spaces has been obtained by W. Kaup in [15] using the algebraic notion of Hermitian Jordan Tripelsystems (see [14]). It is noteworthy that Hermitian-symmetric affine adjoint orbits of L^*-groups exhaust the set of all Hermitian-symmetric spaces (compare [31] and [15]), so the notion of *affine* coadjoint orbit appear to be the right notion to recover the equivalence between Hermitian-symmetric spaces and coadjoint orbits valid in the finite-dimensional case (see for instance Theorem 8.89 in [5]).

A first step toward the generalization of the results of O. Biquard and P. Gauduchon mentioned above to the infinite-dimensional setting has been carry out by the author in [29]. An infinite-dimensional hyperkähler quotient of a Banach manifold by a Banach Lie group has been used to construct hyperkähler structures on a natural complexification of the restricted Grassmannian and on the cotangent space of the restricted Grassmannian. The approach here is more conceptual and applies to every Hermitian-symmetric affine coadjoint orbit.

A first tool used in this paper is the analogue of Mostow's decomposition theorem for L^*-groups, which has been discussed by the author in [30] and independently by G. Larotonda in [19] (see also [20] for the finite-dimensional proof and [1] for a generalization to some von Neumann algebras). The second tool needed is the theory of strong orthogonal roots, which has to be adapted to the infinite-dimensional setting. With these tools in hand we are able to prove the main Theorems of this work, namely Theorem 3.1, Theorem 4.1 and Theorem 6.1.

The structure of the paper is as follows. The next section contains the notation and definitions used throughout the paper, as well as some known results on which the present work is based. Section 3 is devoted to the proof of the fiber bundle structure of a complexified Hermitian-symmetric affine coadjoint orbit $\mathcal{O}_D^{\mathbb{C}}$ over the corresponding orbit of compact type $\mathcal{O}_D$, precisely described in Theorem 3.1. In section 4, the hyperkähler structure of $\mathcal{O}_D^{\mathbb{C}}$ is constructed (Theorem 4.1). In section 5, a natural isomorphism between the complex orbit $\mathcal{O}_D^{\mathbb{C}}$ and the cotangent space $T'\mathcal{O}_D$ is given (Theorem 5.1). In Theorem 6.1 of section 6, the pull-back of the hyperkähler structure constructed in section 4 by the isomorphism constructed in section 5 is described in terms of the cotangent space $T'\mathcal{O}_D$. The reader will find in the Appendix the general results on strongly orthogonal roots in L^*-algebras that are used in the proves of the Theorems.

2 Preliminaries

In the following, $\mathcal{H}$ will denote a separable infinite-dimensional complex Hilbert space. Let us first recall some basic facts about L^*-algebras and L^*-groups.

An L^*-algebra $\mathfrak{g}$ over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is a Lie-algebra over $\mathbb{K}$ which is also an Hilbert space endowed with an involution $*$ satisfying

$$\langle [x\,,\,y]\,,\,z\rangle = \langle y\,,\,[x^*,\,z]\rangle$$

for every x, y and z in $\mathfrak{g}$. An L^*-algebra $\mathfrak{g}$ is semi-simple if $\mathfrak{g} = [\mathfrak{g}\,,\,\mathfrak{g}]$, and simple if $\mathfrak{g}$ is non-commutative and if every closed ideal in $\mathfrak{g}$ is trivial. Every L^*-algebra decomposes into an Hilbert sum of its center and a sequence of closed simple ideals (this was proved by J.R. Schue in [26]). According to [26], every simple separable infinite-dimensional L^*-algebra over $\mathbb{C}$ is isomorphic to one of the non-isomorphic algebras

$$\mathfrak{gl}_2, \quad \mathfrak{o}_2(\mathbb{C}) \quad \text{or} \quad \mathfrak{sp}_2(\mathbb{C}),$$

where $\mathfrak{gl}_2$ denotes the Lie-algebra of Hilbert-Schmidt operators on $\mathcal{H}$, $\mathfrak{o}_2(\mathbb{C})$ is the subalgebra of $\mathfrak{gl}_2$ consisting of skew-symmetric operators with respect to a given real Hilbert space structure on $\mathcal{H}$, and where $\mathfrak{sp}_2(\mathbb{C})$ is the subalgebra of $\mathfrak{gl}_2$ consisting of operator x whose transpose x^T satisfies

$$x^T = -JxJ^{-1},$$

for the linear operator J defined on a basis $\{e_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ of $\mathcal{H}$ by

$$Je_n = -e_{-n} \quad \text{if} \quad n < 0; \quad Je_n = +e_{-n} \quad \text{if} \quad n > 0.$$

To every L^*-algebra is associated a connected Hilbert-Lie group, called L^*-group (see Theorem 4.2 in [22]). The L^*-group associated to $\mathfrak{gl}_2$, denoted by GL_2, is the group of invertible operators on $\mathcal{H}$ which differ from the identity by Hilbert-Schmidt operators. A non-connected L^*-group with Lie algebra $\mathfrak{o}_2(\mathbb{C})$ is the subgroup $\mathrm{O}_2(\mathbb{C})$ of GL_2 consisting of operators which preserve the $\mathbb{C}$-bilinear symmetric form β defined by

$$\beta(x\,,y) = \text{ Tr } (x^T y),$$

for every x, y in $\mathcal{H}$. The L^*-group $\mathrm{Sp}_2(\mathbb{C})$, whose Lie algebra is $\mathfrak{sp}_2(\mathbb{C})$, is the subgroup of GL_2 preserving the $\mathbb{C}$-bilinear skew-symmetric form γ given by

$$\gamma(x\,,y) = \text{ Tr } (x^T Jy).$$

An L^*-algebra $\mathfrak{g}$ is said to be *of compact type* if $x^* = -x$ for every $x \in \mathfrak{g}$. Every simple separable infinite-dimensional L^*-algebra of compact type is isomorphic to one of the non-isomorphic real L^*-algebras

$$\mathfrak{u}_2 := \{x \in \mathfrak{gl}_2, x^* = -x\}; \qquad \mathfrak{o}_2 := \mathfrak{o}_2(\mathbb{C}) \cap \mathfrak{u}_2; \qquad \mathfrak{sp}_2 := \mathfrak{sp}_2(\mathbb{C}) \cap \mathfrak{u}_2.$$

(This result can be found in [3], [11] and [32].) An Hermitian-symmetric space is a smooth strong Riemannian manifold (M, g) endowed with a g-orthogonal complex structure and which admits, for every x in M, a globally defined isometry s_x (the *symmetry* with respect to x) preserving the complex structure, such that x is a fixed point of s_x, and such that the differential of s_x at x is minus the identity of T_xM. Every infinite-dimensional Hermitian-symmetric space M decomposes into an orthogonal product $M_0 \times M_+ \times M_-$, where M_0 is flat, M_+ is simply-connected with positive sectional curvature and M_- is simply-connected with negative sectional curvature ([15]). An Hermitian-symmetric space with positive (resp. negative) sectional curvature is said to be *of compact type* (resp. *of non-compact type*). An Hermitian-symmetric space is called *irreducible* if it is not flat and not locally isomorphic to a product of Hermitian-symmetric spaces with non-zero dimensions. Every Hermitian-symmetric space of compact or non-compact type can be decomposed into a product of (possibly infinitely many) irreducible pieces. The irreducible infinite-dimensional Hermitian-symmetric spaces have been classified by W. Kaup in [15] using techniques developed in [14]. According to [31], every irreducible infinite-dimensional Hermitian-symmetric space of compact type is an affine coadjoint orbit (see below for the definition) of a simple L^*-group G of compact type.

An affine adjoint (resp. coadjoint) action of an L^*-group on its Lie algebra $\mathfrak{g}$ (resp. on the continuous dual $\mathfrak{g}'$ of its Lie algebra) is given by a group homomorphism from G into the affine group of transformations of $\mathfrak{g}$ (resp. $\mathfrak{g}'$), whose linear part is the adjoint action of G on $\mathfrak{g}$ (resp. the coadjoint action of G on $\mathfrak{g}'$). An affine (co-)adjoint orbit of G is the orbit of an element in $\mathfrak{g}$ (resp. $\mathfrak{g}'$) under the affine (co-)adjoint action of G (see section 2 in [22]). For simple L^*-groups, affine coadjoint orbits and affine adjoint orbits are identified by the trace. Every irreducible Hermitian-symmetric affine adjoint orbit of compact type is the orbit of 0 in $\mathfrak{g} \in \{\mathfrak{u}_2, \mathfrak{o}_2, \mathfrak{sp}_2\}$ under the affine adjoint action Ad_D of G given by

$$\begin{array}{rccl} \mathrm{Ad}_D & : & G & \rightarrow & GL(\mathfrak{g}) \rtimes \mathfrak{g} \\ & & g & \mapsto & (\mathrm{Ad}(g), \Theta_D(g)), \end{array} \tag{1}$$

where

$$\begin{array}{rccl} \Theta_D & : & G & \rightarrow & \mathfrak{g} \\ & & g & \mapsto & \Theta_D(g) = gDg^{-1} - D \end{array}$$

for a bounded skew-Hermitian operator D on $\mathcal{H}$ with two different eigenvalues (see Theorem 4.4 in [22] and [31]). For a bounded skew-Hermitian operator D, we will denote by $\mathcal{O}_D$ the orbit of 0 under the affine adjoint action Ad_D of G. The projective space of an infinite-dimensional separable complex Hilbert space, and the connected components of the restricted Grassmannian associated to a polarized Hilbert space are examples of such an orbit.

Throughout in the following $\mathcal{O} = \mathcal{O}_D$ will denote an irreducible Hermitian-symmetric affine adjoint orbit of a (simple) L^*-group of compact type G with Lie algebra $\mathfrak{g}$, and D the corresponding bounded linear operator. In particular

$$\mathcal{O}_D = \{gDg^{-1} - D, g \in G\} = G/K.$$

where K is the isotropy group of $0 \in \mathcal{O}_D$. The Lie algebra of K is

$$\mathfrak{k}_0 := \{x \in \mathfrak{g} \mid [D, x] = 0\}.$$

We will denote by $\mathbb{D}$ the derivation $[D\,, \cdot]$, and use the following notation : $\mathrm{ad}(x)(y) := [x\,, y]$. The tangent space at $0 \in \mathcal{O}_D$ is isomorphic to the orthogonal $\mathfrak{m}_0$ of $\mathfrak{k}_0$ in $\mathfrak{g}$. The complex structure at 0 is given by the operator

$$I := \frac{1}{c}\mathbb{D}_{|\mathfrak{m}_0}$$

on the tangent space $T_0\mathcal{O}_D \simeq \mathfrak{m}_0$, where c is the positive constant defined by

$$[D\,, [D\,, \cdot]]_{|\mathfrak{m}_0} = -c^2 \;\mathrm{id}_{\mathfrak{m}_0}.$$

The orbit $\mathcal{O}_D$ being a homogeneous symmetric space of G, the following commutation relations hold

$$[\mathfrak{k}_0\,, \mathfrak{k}_0] \subset \mathfrak{k}_0; \qquad [\mathfrak{k}_0\,, \mathfrak{m}_0] \subset \mathfrak{m}_0; \qquad [\mathfrak{m}_0\,, \mathfrak{m}_0] \subset \mathfrak{k}_0. \tag{2}$$

For every $x = gDg^{-1} - D$ in $\mathcal{O}_D$, we will denote by $\mathfrak{k}_x$ the Lie subalgebra of $\mathfrak{g}$ which fixes x, and $\mathfrak{m}_x$ its orthogonal in $\mathfrak{g}$. One has $\mathfrak{k}_x = g\mathfrak{k}_0 g^{-1}$ and $\mathfrak{m}_x := g\mathfrak{m}_0 g^{-1}$.

The complexified orbit $\mathcal{O}_D^{\mathbb{C}}$ of $\mathcal{O}_D$ is defined as the complex affine adjoint orbit of 0 under the complex L^*-group $G^{\mathbb{C}}$ with Lie-algebra

$$\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \oplus i\mathfrak{g},$$

for the affine adjoint action which extend naturally Ad_D (and which will be also denoted by Ad_D in the following). Note that the derivation $\mathbb{D} = [D\,,\cdot]$ applies $\mathfrak{g}^{\mathbb{C}}$ onto $\mathfrak{m}_0 \oplus i\mathfrak{m}_0$. Mostow's decomposition theorem (see [20] for the finite-dimensional case, [19] or [30] for infinite-dimensional L^*-groups) states that, for every x in $\mathcal{O}_D$, there exists a homeomorphism

$$G^{\mathbb{C}} \simeq G \exp(i\mathfrak{m}_x) \exp(i\mathfrak{k}_x).$$

The complexified orbit $\mathcal{O}_D^{\mathbb{C}}$ is a strong symplectic manifold for the Kirillov-Kostant-Souriau symplectic form $\omega^{\mathbb{C}}$ defined as the $G^{\mathbb{C}}$-invariant 2-form whose value at the tangent space $T_0\mathcal{O}_D^{\mathbb{C}}$ at 0 is given by

$$\omega^{\mathbb{C}}(X\,,Y) = \langle X^*, [D\,,Y]\rangle \tag{3}$$

for X, Y in $T_0\mathcal{O}_D^{\mathbb{C}}$ (see Theorem 4.4 in [22]). Note that this convention differs from the convention usually used in the finite-dimensional case by the multiplicative constant c^2.

3 The complex orbit $\mathcal{O}^{\mathbb{C}}$ as a fiber bundle over the orbit of compact type $\mathcal{O}$

This section is devoted to the below "fiber bundle Theorem" which specifies the metric properties acquired, in the case of a Hermitian-symmetric orbit, by the projection $\pi\ : \mathcal{O}_D^{\mathbb{C}} \to \mathcal{O}_D$ defined in [30]. It is a generalization of Proposition 1 in [7] to the case of an affine coadjoint action. We give below some details of the proof since traces of operators are involved and the computation as given in [7] does not make sense in our context (recall that $\langle\cdot,\cdot\rangle$ denotes the Hermitian product in the L^*-algebra $\mathfrak{g}$ which is given by the trace). Let us emphasize that the minimizing property described in this theorem and its finite-dimensional counterpart is the key step in the construction of the hyperkähler metrics on Hermitian-symmetric spaces by the method developped in [7], [8] and [9] and which we will follow. For a general complex coadjoint orbit, this key step is missing and the current method can not be applied (for the construction of hyperkähler metrics on complex coadjoint orbits of general type see [17], [18], [6], [16]). At the end of this section, the Proposition 3.2 gives an isomorphism between the tangent space to $\mathcal{O}_D^{\mathbb{C}}$ at any y and the tangent space to $\mathcal{O}_D^{\mathbb{C}}$ at $\pi(y) \in \mathcal{O}_D$. It is the infinite-dimensional version of Lemma 4 in [7], whose proof is too concise to contain all the necessary informations. These identifications of tangent spaces are crucial for a good understanding of the expression of the hyperkähler metrics constructed in sections 4 and 6. For this reason we include a detailed proof.

Theorem 3.1 *Every element y of the complex affine adjoint orbit $\mathcal{O}_D^{\mathbb{C}}$ can be written uniquely as*

$$y = Ad_D\left(e^{i\mathfrak{a}}\right)(x)$$

where x belongs to $\mathcal{O}_D$ and where $\mathfrak{a}$ is in $\mathfrak{m}_x$. The element x is characterized by the property that it minimizes the distance in $\mathfrak{g}^{\mathbb{C}}$ between y and the orbit of compact type $\mathcal{O}_D$. The fibers of the orthogonal projection π which takes y in $\mathcal{O}_D^{\mathbb{C}}$ to the corresponding x in $\mathcal{O}_D$ are the sets of the form $Ad_D\left(G_x^{n.c.}\right)(x)$, where $x \in \mathcal{O}_D$ and where $G_x^{n.c.}$ denotes the connected L^*-group (of non compact type) with Lie algebra $\mathfrak{k}_x \oplus i\mathfrak{m}_x$. Moreover, π is G-equivariant.

■ **Proof of Theorem** 3.1:
Every element $y = \mathrm{Ad}_D(g)(0)$ in the affine adjoint orbit $\mathcal{O}_D^{\mathbb{C}}$ can be written uniquely has

$$y = g\cdot 0 = \mathrm{Ad}_D(e^{iu\mathfrak{d}u^{-1}})(x)$$

where $x := \mathrm{Ad}_D(u)(0) = u\cdot 0$ and where $u\mathfrak{d}u^{-1}$ belongs to $\mathfrak{m}_x = u\mathfrak{m}_0u^{-1}$ (see [30]). Let us show that x minimizes the distance in $\mathfrak{g}^{\mathbb{C}}$ between y and $\mathcal{O}_D$. Every element x' in a neighborhood of x in $\mathcal{O}_D$ can be joint to x by a (minimal) geodesic. Since $\mathcal{O}_D$ is a symmetric space, every geodesic starting from x is of the form $t \mapsto \exp(t\mathfrak{b})\cdot x$, where $\mathfrak{b}$ belongs to $\mathfrak{m}_x$ (see Proposition 8.8 in [2], Corollary 3.33 in [10], or

Proposition 25 p 313 in [24] for a description of the geodesics in finite-dimensional symmetric spaces, or its infinite-dimensional versions as given in Example 3.9 in [23] or in [30]). For

$$x' = \mathrm{Ad}_D(e^{\mathfrak{b}'})(x) = e^{\mathfrak{b}'}uDu^{-1}e^{-\mathfrak{b}'} - D,$$

where $\mathfrak{b}'$ belongs to $\mathfrak{m}_x$, consider the geodesic

$$x_t := \mathrm{Ad}_D(e^{t\mathfrak{b}'})(x) = e^{t\mathfrak{b}'}uDu^{-1}e^{-t\mathfrak{b}'} - D, \quad t \in [0\,,1]$$

from x to x', and the following function

$$f(t) = \frac{1}{2}\|y - x_t\|^2.$$

The explicit expression of f is the following

$$\begin{aligned} f(t) &= \tfrac{1}{2}\|e^{iu\mathfrak{a}u^{-1}}uDu^{-1}e^{-iu\mathfrak{a}u^{-1}} - e^{t\mathfrak{b}'}uDu^{-1}e^{-t\mathfrak{b}'}\|^2 = \tfrac{1}{2}\|e^{i\mathfrak{a}}De^{-i\mathfrak{a}} - e^{t\mathfrak{b}}De^{-t\mathfrak{b}}\|^2 \\ &= \tfrac{1}{2}\left\langle e^{i\mathfrak{a}}De^{-i\mathfrak{a}} - e^{t\mathfrak{b}}De^{-t\mathfrak{b}}, e^{i\mathfrak{a}}De^{-i\mathfrak{a}} - e^{t\mathfrak{b}}De^{-t\mathfrak{b}}\right\rangle, \end{aligned}$$

where we have set $\mathfrak{b} := u^{-1}\mathfrak{b}'u \in \mathfrak{m}_0$. One has

$$f'(t) = \Re\langle e^{i\mathfrak{a}}De^{-i\mathfrak{a}} - D, -[\mathfrak{b}, e^{t\mathfrak{b}}De^{-t\mathfrak{b}}]\rangle + \Re\langle e^{t\mathfrak{b}}De^{-t\mathfrak{b}} - D, [\mathfrak{b}, e^{t\mathfrak{b}}De^{-t\mathfrak{b}}]\rangle. \tag{4}$$

From the commutation relations (2) which characterize a symmetric orbit, one deduce that $e^{i\mathfrak{a}}De^{-i\mathfrak{a}} - D$ belongs to the direct sum $\mathfrak{k}_0 \oplus i\mathfrak{m}_0$, whereas $-[\mathfrak{b}, e^{t\mathfrak{b}}De^{-t\mathfrak{b}}]$ belongs to $\mathfrak{k}_0 \oplus \mathfrak{m}_0$. Hence only the projections on $\mathfrak{k}_0$ are involved in the scalar product. Let us consider each term of the sum (4) separately.

First,

$$\begin{aligned} \Re\left\langle e^{i\mathfrak{a}}De^{-i\mathfrak{a}} - D, -\left[\mathfrak{b}, e^{t\mathfrak{b}}De^{-t\mathfrak{b}}\right]\right\rangle &= \Re\left\langle \tfrac{\cosh \mathrm{ad}(i\mathfrak{a}) - 1}{\mathrm{ad}(i\mathfrak{a})^2}\left[\mathfrak{a}, [D, \mathfrak{a}]\right], \tfrac{\sin \mathrm{ad}(it\mathfrak{b})}{\mathrm{ad}(it\mathfrak{b})}\left[t\mathfrak{b}, [D, \mathfrak{b}]\right]\right\rangle \\ &= \tfrac{c^2}{t}\Re\left\langle \tfrac{\cosh \mathrm{ad}(i\mathfrak{a}) - 1}{\mathrm{ad}(i\mathfrak{a})^2}\left[\mathfrak{a}, I\mathfrak{a}\right], \tfrac{\sin \mathrm{ad}(it\mathfrak{b})}{\mathrm{ad}(it\mathfrak{b})}\left[t\mathfrak{b}, It\mathfrak{b}\right]\right\rangle, \end{aligned}$$

where, for any analytic function f, the notation $f\left(\mathrm{ad}(i\mathfrak{a})\right)$ denotes the operator obtained by applying the expansion of f to the Hermitian operator $\mathrm{ad}(i\mathfrak{a})$. From Lemma A.9 in the Appendix of the present paper, it follows that

$$\begin{aligned} \Re\left\langle e^{i\mathfrak{a}}De^{-i\mathfrak{a}} - D, -\left[\mathfrak{b}, e^{t\mathfrak{b}}De^{-t\mathfrak{b}}\right]\right\rangle &= \tfrac{c^2}{t}\Re\left\langle [\mathfrak{a}'', I\mathfrak{a}''], [\mathfrak{b}'', I\mathfrak{b}'']\right\rangle, \\ &= \tfrac{c^2}{t}\left(\|\,[\mathfrak{a}'', \mathfrak{b}'']\,\|^2 + \|\,[\mathfrak{a}'', I\mathfrak{b}'']\,\|^2\right) \end{aligned}$$

where $\mathfrak{a}'' := \sqrt{\frac{\cosh \mathrm{ad}(iI\mathfrak{a}) - 1}{\mathrm{ad}(iI\mathfrak{a})^2}}(\mathfrak{a})$ and $\mathfrak{b}'' := \sqrt{\frac{\sin \mathrm{ad}(iIt\mathfrak{b})}{\mathrm{ad}(iIt\mathfrak{b})}}(t\mathfrak{b})$, the latter expression being valid only for $t \leq \frac{\pi}{2\|b\|}$.

Secondly, let us remark that $\langle e^{t\mathfrak{b}}De^{-t\mathfrak{b}} - D, [\mathfrak{b}, e^{t\mathfrak{b}}De^{-t\mathfrak{b}} - D]\rangle$ is purely imaginary. It follows that

$$\Re\left\langle e^{t\mathfrak{b}}De^{-t\mathfrak{b}} - D, [\mathfrak{b}, e^{t\mathfrak{b}}De^{-t\mathfrak{b}}]\right\rangle = \Re\left\langle e^{t\mathfrak{b}}De^{-t\mathfrak{b}} - D, [\mathfrak{b}, D]\right\rangle.$$

Using the commutation relations (2), note that $e^{t\mathfrak{b}}De^{-t\mathfrak{b}} - D$ belongs to $\mathfrak{k}_0 \oplus \mathfrak{m}_0$, and $[\mathfrak{b}, D]$ is in $\mathfrak{m}_0$. One has

$$\begin{aligned} \Re\left\langle e^{t\mathfrak{b}}De^{-t\mathfrak{b}} - D, [\mathfrak{b}, D]\right\rangle &= \Re\left\langle \tfrac{\sin \mathrm{ad}(it\mathfrak{b})}{\mathrm{ad}(it\mathfrak{b})}[t\mathfrak{b}, D], [\mathfrak{b}, D]\right\rangle \\ &= tc^2\Re\langle \tfrac{\sin \mathrm{ad}(it\mathfrak{b})}{\mathrm{ad}(it\mathfrak{b})}I\mathfrak{b}, I\mathfrak{b}\rangle, \end{aligned}$$

which is positive for t in $(0, \frac{\pi}{2\|b\|})$ since $\frac{\sin \mathrm{ad}(it\mathfrak{b})}{\mathrm{ad}(it\mathfrak{b})}$ is an Hermitian operator.

We conclude that both terms in the sum (4) are positive for t in $(0, \frac{\pi}{2\|\mathfrak{b}\|})$, whence $f'(t) > 0$ for t in this interval. The second derivative of f at 0 is given by

$$\begin{aligned}
f''(0) &= \Re \left\langle e^{i\mathfrak{a}} D e^{-i\mathfrak{a}} - D, -[\mathfrak{b},[\mathfrak{b},D]] \right\rangle + \Re \left\langle [\mathfrak{b},D],[\mathfrak{b},D] \right\rangle \\
&= \Re \left\langle \left[\mathfrak{b}, e^{i\mathfrak{a}} D e^{-i\mathfrak{a}} - D\right], [\mathfrak{b},D] \right\rangle + \Re \left\langle [\mathfrak{b},D],[\mathfrak{b},D] \right\rangle \\
&= \Re \left\langle \left[\mathfrak{b}, \tfrac{\cosh(\mathrm{ad}(i\mathfrak{a}))-1}{\mathrm{ad}(i\mathfrak{a})^2}([i\mathfrak{a},[i\mathfrak{a},D]])\right], [\mathfrak{b},D] \right\rangle + \Re \left\langle [\mathfrak{b},D],[\mathfrak{b},D] \right\rangle \\
&= \Re \left\langle \tfrac{\cosh(\mathrm{ad}(i\mathfrak{a}))-1}{\mathrm{ad}(i\mathfrak{a})}([\mathfrak{a},[D,\mathfrak{a}]]), [\mathfrak{b},[D,\mathfrak{b}]] \right\rangle + c^2\|\mathfrak{b}\|^2 \\
&= c^2 \Re \left\langle \tfrac{\cosh(\mathrm{ad}(i\mathfrak{a}))-1}{\mathrm{ad}(i\mathfrak{a})^2}[\mathfrak{a}, I\mathfrak{a}], [\mathfrak{b}, I\mathfrak{b}] \right\rangle + c^2\|\mathfrak{b}\|^2.
\end{aligned} \tag{5}$$

Using again Lemma A.9, one has

$$\begin{aligned}
f''(0) &= c^2 \Re \left\langle [\mathfrak{a}'', I\mathfrak{a}''], [\mathfrak{b}, I\mathfrak{b}] \right\rangle + c^2\|\mathfrak{b}\|^2 \\
&= c^2 \left(\|\mathfrak{a}'', \mathfrak{b}]\|^2 + \|[\mathfrak{a}'', I\mathfrak{b}]\|^2 + \|\mathfrak{b}\|^2 \right),
\end{aligned}$$

where $\mathfrak{a}'' = \sqrt{\frac{\cosh \mathrm{ad}(iI\mathfrak{a})-1}{\mathrm{ad}(iI\mathfrak{a})^2}}(\mathfrak{a})$. Hence the second derivative of f at 0 is positive. Let us define the function

$$\begin{array}{rccl} f_y : & \mathcal{O}_D & \to & \mathbb{R} \\ & x' & \mapsto & \frac{1}{2}\|y - x'\|^2. \end{array}$$

From the second line of computation (5), the Hessian of f_y at 0 is positive-definite and has the following expression

$$\mathrm{Hess}(X^{\mathfrak{c}}, X^{\mathfrak{d}}) = \Re \langle [\mathfrak{c}, e^{i\mathfrak{a}} D e^{-i\mathfrak{a}}], [\mathfrak{d}, D] \rangle, \tag{6}$$

where $X^{\mathfrak{c}}$ and $X^{\mathfrak{d}}$ are the vectors induced at 0 by the infinitesimal action of $\mathfrak{c}, \mathfrak{d} \in \mathfrak{m}_0$ respectively. It follows that x minimizes the distance between y and $\mathcal{O}_D$. In the finite dimensional case, the discussion above would be sufficient to conclude that x is the *unique* minimum of the distance between y and $\mathcal{O}_D$ because Hopf-Rinow Theorem guaranties that every element x' in $\mathcal{O}_D$ can be reached by a geodesic of $\mathcal{O}_D$ starting at x, and because f is strictly increasing along a minimizing geodesic. In the infinite-dimensional setting, Hopf-Rinow Theorem does not hold anymore, thus an argument implying the uniqueness of the minimum has to be added. We give this argument below, but let us first remark that the fiber of the projection π over x is the set of y' such that $y' = \mathrm{Ad}_D\left(e^{i\mathfrak{a}}\right)(x)$ for some $\mathfrak{a}$ in $\mathfrak{m}_x$. Therefore it is the orbit of x under the group $G_x^{n.c.}$. The G-equivariance of π is a direct consequence of the definition and implies that it remains only to prove that 0 is the unique minimum of the distance between a given element y in the fiber $\pi^{-1}(0)$ and $\mathcal{O}_D$. Let $\mathfrak{a}$ be the element in $\mathfrak{m}_0$ such that $y = e^{i\mathfrak{a}} D e^{-i\mathfrak{a}} - D$. By density of the maximal Abelian subalgebras of $\mathfrak{m}_0$ spanned by maximal sets of strong orthogonal roots, for every $\varepsilon > 0$, there exists

$$\mathfrak{a}_\varepsilon = \sum_{\alpha \in \Psi_\varepsilon} a_\alpha \, x_\alpha$$

in a Abelian subalgebra

$$\mathfrak{A}_\varepsilon := \oplus_{\alpha \in \Psi_\varepsilon} \mathbb{R} x_\alpha$$

spanned by a maximal set of strong orthogonal roots Ψ_ε, such that $\|\Re y - \Re\left(e^{i\mathfrak{a}_\varepsilon} \cdot 0\right)\| < \varepsilon$. For every $\alpha \in \Psi_\varepsilon$, set $y_\alpha = I x_\alpha$ and $h_\alpha = \frac{1}{2i}[x_\alpha, y_\alpha]$. For every $\alpha, \beta \in \Psi_\varepsilon$, the following commutation relations hold :

$$[x_\alpha, y_\beta] = 2ih_\alpha \delta_{\alpha\beta}; \quad [h_\alpha, x_\beta] = -2iy_\alpha \delta_{\alpha\beta}; \quad [h_\alpha, y_\beta] = 2ix_\alpha \delta_{\alpha\beta}.$$

It follows that

$$\Re\left(e^{i\mathfrak{a}_\varepsilon} \cdot 0\right) = \sum_{\alpha \in \Phi_\varepsilon} 2ih_\alpha \left(\cosh(2a_\alpha) - 1\right).$$

This implies that $\Re y$ belongs to the following convex set

$$\mathcal{C}_\varepsilon := \{v \in \mathfrak{k}_0 \mid \langle h_\alpha, v \rangle > -\varepsilon\} \cap \{v \in \mathfrak{k}_0, \mathrm{dist}\left(v, \mathrm{span}\{h_\alpha, \alpha \in \Psi_\varepsilon\}\right) < \varepsilon\}.$$

It is easy to check that for ε small enough, 0 is the unique element in $\mathcal{C}_\varepsilon \cap \mathcal{O}_D$, and this complete the proof. $\blacksquare$

Proposition 3.2 *For $y = Ad_D(e^{i\mathfrak{a}})(x) \in \mathcal{O}_D^{\mathbb{C}}$, where $x \in \mathcal{O}_D$, and $\mathfrak{a} \in \mathfrak{m}_x$, the map*

$$\begin{array}{rccc} \rho: & \mathfrak{m}_x \oplus i\mathfrak{m}_x & \to & T_y\mathcal{O}_D^{\mathbb{C}} \\ & \mathfrak{c} & \mapsto & X^{\mathfrak{c}} \end{array}$$

is an isomorphism. The kernel of $\pi_* : T_y\mathcal{O}_D^{\mathbb{C}} \to T_x\mathcal{O}_D$ is $V_y := \{X^{i\mathfrak{c}}, \mathfrak{c} \in \mathfrak{m}_x\}$, and π_* induces an isomorphism from $H_y := \{X^{\mathfrak{c}}, \mathfrak{c} \in \mathfrak{m}_x\}$ onto $\mathfrak{m}_x$.

□ **Proof of Proposition** 3.2:
By G-equivariance, it is sufficient to consider the case where x is equal to 0. Let us consider an element $y = e^{i\mathfrak{a}}De^{-i\mathfrak{a}} - D$ in $\pi^{-1}(0)$ where $\mathfrak{a}$ belongs to $\mathfrak{m}_0$. A tangent vector to $\mathcal{O}_D^{\mathbb{C}}$ at y is given by the action of an element $\mathfrak{c}$ in the complex Lie algebra $\mathfrak{g} \oplus i\mathfrak{g}$, i.e. is the derivative at 0 of the function

$$\Phi^{\mathfrak{c}}(t) = e^{t\mathfrak{c}}e^{i\mathfrak{a}}De^{-i\mathfrak{a}}e^{-t\mathfrak{c}} - D.$$

It is therefore of the form

$$X^{\mathfrak{c}} = [\mathfrak{c}, e^{i\mathfrak{a}}De^{-i\mathfrak{a}}] = e^{i\mathfrak{a}}[e^{-i\mathfrak{a}}\mathfrak{c}e^{i\mathfrak{a}}, D]e^{-i\mathfrak{a}}.$$

For $\mathfrak{c} \in \mathfrak{m}_0 \oplus i\mathfrak{m}_0$,

$$\begin{aligned} [e^{-i\mathfrak{a}}\mathfrak{c}e^{i\mathfrak{a}}, D] &= [\mathrm{Ad}(e^{-i\mathfrak{a}})(\mathfrak{c}), D] = [\exp\left(\mathrm{ad}(-i\mathfrak{a})\right)(\mathfrak{c}), D] \\ &= -cI\cosh\left(\mathrm{ad}(-i\mathfrak{a})\right)(\mathfrak{c}). \end{aligned}$$

Note that the operator $\cosh\left(\mathrm{ad}(-i\mathfrak{a})\right)$ from $\mathfrak{g}^{\mathbb{C}}$ to $\mathfrak{g}^{\mathbb{C}}$ is Hermitian and one-to-one, thus an isomorphism, and preserves the subspace $\mathfrak{m}_0 \oplus i\mathfrak{m}_0$. Since the tangent space to $\mathcal{O}_D^{\mathbb{C}}$ at y is $e^{i\mathfrak{a}}(\mathfrak{m}_0 \oplus i\mathfrak{m}_0)e^{-i\mathfrak{a}}$, it follows that ρ is an isomorphism.

Let us show that for $\mathfrak{c} \in \mathfrak{m}_0$, one has $\pi_*(X^{i\mathfrak{c}}) = 0$. Consider the curve

$$\Phi^{i\mathfrak{c}}(t) = e^{it\mathfrak{c}}\,e^{i\mathfrak{a}} \cdot 0.$$

By Mostow's decomposition theorem (see [20], [19], [30]), for every $t \in \mathbb{R}$, there exists u_t in G, $\mathfrak{b}_t$ in $\mathfrak{m}_0$ and $\mathfrak{d}_t$ in $\mathfrak{k}_0$ such that

$$e^{it\mathfrak{c}}\,e^{i\mathfrak{a}} = u_t\,e^{i\mathfrak{b}_t}\,e^{i\mathfrak{d}_t}.$$

It follows that

$$\begin{aligned} \pi\left(\Phi^{i\mathfrak{c}}(t)\right) &= \pi\left(e^{it\mathfrak{c}}\,e^{i\mathfrak{a}} \cdot 0\right) = \pi\left(u_te^{i\mathfrak{b}_t} \cdot 0\right) \\ &= \pi\left(e^{iu_t\mathfrak{b}_tu_t^{-1}} \cdot (u_t \cdot 0)\right) = u_t \cdot 0, \end{aligned}$$

since $u_t\mathfrak{b}_tu_t^{-1}$ belongs to the subspace $\mathfrak{m}_{u_t\cdot 0}$. Hence

$$\pi_*\left(X^{i\mathfrak{c}}(y)\right) := \frac{d}{dt}_{|t=0}\pi\left(\Phi^{i\mathfrak{c}}(t)\right) = \frac{d}{dt}_{|t=0}(u_t) \cdot 0.$$

But the curve $\Phi^{i\mathfrak{c}}(t)$ belongs to $\mathfrak{k}_0 \oplus i\mathfrak{m}_0$ for all $t \in \mathbb{R}$, thus its derivative at $t = 0$ also. One has

$$\frac{d}{dt}_{|t=0}\Phi^{i\mathfrak{c}}(t) = i\mathfrak{c} \cdot (e^{i\mathfrak{a}} \cdot 0) = \frac{d}{dt}_{|t=0}(u_t) \cdot (e^{i\mathfrak{a}} \cdot 0) + \frac{d}{dt}_{|t=0}(e^{i\mathfrak{b}_t}) \cdot 0.$$

Note that for $t = 0$, u_0 is the unit element in G and that $\mathfrak{b}_0 = \mathfrak{a}$. Since $\mathfrak{b}_t$ belongs to $\mathfrak{m}_0$ for all t, the curve $e^{i\mathfrak{b}_t} \cdot 0$ belongs to $\mathfrak{k}_0 \oplus i\mathfrak{m}_0$ for all $t \in \mathbb{R}$, hence its derivative at $t = 0$ also. It follows that

$$\frac{d}{dt}_{|t=0}(u_t) \cdot (e^{i\mathfrak{a}} \cdot 0) := \left[\frac{d}{dt}_{|t=0}(u_t)\,,\, e^{i\mathfrak{a}} \cdot 0\right]$$

belongs to $\mathfrak{k}_0 \oplus i\mathfrak{m}_0$. From this, one deduces that the component of $\frac{d}{dt}_{|t=0}(u_t)$ along $\mathfrak{m}_0$ vanishes because it has to stabilize $e^{i\mathfrak{a}} \cdot 0$ and because $\mathfrak{m}_0 \cap e^{i\mathfrak{a}}\,\mathfrak{k}_0\,e^{-i\mathfrak{a}} = \{0\}$. Whence $\frac{d}{dt}_{|t=0}(u_t)$ belongs to $\mathfrak{k}_0$ thus $\pi_*(X^{i\mathfrak{c}}) = 0$.

Let us now show that for $\mathfrak{c} \in \mathfrak{m}_0$, one has $\pi_*\left(X^{\mathfrak{c}}(y)\right) = \mathfrak{c} \cdot 0$. One has

$$\pi\left(\Phi^{\mathfrak{c}}(t)\right) = \pi\left(e^{t\mathfrak{c}}\,e^{i\mathfrak{a}} \cdot 0\right) = \pi\left(e^{\mathrm{Ad}(e^{t\mathfrak{c}})(i\mathfrak{a})} \cdot (e^{t\mathfrak{c}} \cdot 0)\right) = e^{t\mathfrak{c}} \cdot 0.$$

It follows that $\pi_*\left(X^{\mathfrak{c}}(y)\right) = \mathfrak{c} \cdot 0$ and the proof is complete. □

4 Hyperkähler structure on the complex orbit $\mathcal{O}^{\mathbb{C}}$

In this section, we will use the particular property of the projection π of minimizing the distance in $\mathfrak{g}^{\mathbb{C}}$ to the orbit of compact type in order to construct a hyperkähler structure on $\mathcal{O}_D^{\mathbb{C}}$ and thereby generalize Theorem 3 in [7] to the case of complexifications of Hermitian-symmetric affine adjoint orbits of L^*-groups of compact type. Note that it is sufficient to consider the case of an irreducible orbit $\mathcal{O}_D^{\mathbb{C}}$. The notation we introduce in Theorem 4.1 below is in correspondence with the one of Theorem 3 in [7], and, using this correspondence, the proof of Theorem 3 in [7] can be formally followed without substantial changes. For this reason we omit the details in the proof. Let us however emphasize that the objects handled in our setting are conceptually different to the ones appearing in the finite-dimensional theory : a based point in the infinite-dimensional orbit is de facto distinguished (the element $0 \in \mathcal{O}_D$), and an element y in $\mathcal{O}_D^{\mathbb{C}}$ is of the form $gDg^{-1} - D$, where $g \in G$ and where D does not necessarily belong to $\mathfrak{g}$. For further comments, see remark 4.2.

Theorem 4.1 *The complex affine adjoint orbit $\mathcal{O}_D^{\mathbb{C}}$ admits a G-invariant hyperkähler structure compatible with the complex symplectic form $\omega^{\mathbb{C}}$ of Kirillov-Kostant-Souriau and extending the natural Kähler structure of the Hermitian-symmetric affine adjoint orbit of compact type $\mathcal{O}_D$. The Kähler form ω_1 associated with the complex structure i of $\mathcal{O}_D^{\mathbb{C}}$ is given by $\omega_1 = dd^cK$, where the potential K has the following expression*

$$K(y) = c\Re\langle y, \pi(y)\rangle, \tag{7}$$

for every y in $\mathcal{O}_D^{\mathbb{C}}$. The explicit expressions of the symplectic form ω_1 and the Riemannian metric g *are the following*

$$\omega_1(X^{\mathfrak{c}+i\mathfrak{c}'}, X^{\mathfrak{d}+i\mathfrak{d}'}) = c\Im\left(\langle X^{i\mathfrak{c}'}, \pi_*(X^{\mathfrak{d}})\rangle - \langle X^{i\mathfrak{d}'}, \pi_*(X^{\mathfrak{c}})\rangle\right)$$

$$\mathrm{g}(X^{\mathfrak{c}}, X^{\mathfrak{d}}) = \mathrm{g}(X^{i\mathfrak{c}}, X^{i\mathfrak{d}}) = c\Re\langle X^{\mathfrak{c}}(y), X^{\mathfrak{d}}(\pi(y))\rangle, \qquad \mathrm{g}(X^{\mathfrak{c}}, X^{i\mathfrak{d}}) = 0,$$

where $\mathfrak{c}$, $\mathfrak{c}'$, $\mathfrak{d}$ and $\mathfrak{d}'$ belong to $\mathfrak{m}_{\pi(y)}$. The complex structure I_2 is given at $y \in \pi^{-1}(0)$ by

$$I_2X^{\mathfrak{d}} = X^{[\frac{D}{c}, \mathfrak{d}]}, \qquad I_2X^{i\mathfrak{d}} = -X^{i[\frac{D}{c}, \mathfrak{d}]},$$

where $\mathfrak{c}$ and $\mathfrak{d}$ belong to $\mathfrak{m}_0$.

■ **Proof of Theorem** 4.1:
The formulas for ω_1 and g appearing in the Theorem can easily be computed following [7]. The G-equivariance of π implies the G-invariance of g. To check that g is positive-definite, it is therefore sufficient to consider g at an element $y = e^{i\mathfrak{a}}De^{-i\mathfrak{a}} - D$ in the fiber $\pi^{-1}(0)$. In this case, one has

$$\mathrm{g}(X^{\mathfrak{c}}, X^{\mathfrak{d}}) = \mathrm{g}(X^{i\mathfrak{c}}, X^{i\mathfrak{d}}) = c\Re\langle[\mathfrak{c}, e^{i\mathfrak{a}}De^{-i\mathfrak{a}}], [\mathfrak{d}, D]\rangle, \tag{8}$$

which, according to equation (6) in the proof of Theorem 3.1, is equal to the Hessian at 0 of the function f_y modulo the positive multiplicative constant c. It follows that g is positive-definite. It remains to show that g is hyperkähler and compatible with $\omega^{\mathbb{C}}$. For this, we will use (as it has been done in [7]) lemma 6.8 of Hitchin's paper [13], which implies that it is sufficient to show that the endomorphism I_2 defined by

$$\mathrm{g}(X, Y) = \Re\omega^{\mathbb{C}}(X, I_2Y)$$

satisfies $(I_2)^2 = -1$. Recall that the natural complex symplectic form $\omega^{\mathbb{C}}$ on $\mathcal{O}_D^{\mathbb{C}}$ is the G-invariant 2-form whose value at $0 \in \mathcal{O}_D^{\mathbb{C}}$ is given by

$$\omega^{\mathbb{C}}(X, Y) = \langle X^*, [D, Y]\rangle, \tag{9}$$

where X, Y belong to $T_0\mathcal{O}_D^{\mathbb{C}}$. By the G-invariance of g and $\omega^{\mathbb{C}}$, the problem reduces to the study of I_2 at an element of the fiber over 0. An easy computation then leads to

$$\mathrm{g}(X^{\mathfrak{c}}, X^{\mathfrak{d}}) \quad = \Re\omega^{\mathbb{C}}\left(X^{\mathfrak{c}}, X^{[\frac{D}{c}, \mathfrak{d}]}\right)$$

for $\mathfrak{c}$ and $\mathfrak{d}$ in $\mathfrak{m}_0$. Hence, for $\mathfrak{d} \in \mathfrak{m}_0$, the expression of I_2 is $I_2X^{\mathfrak{d}} = X^{[\frac{D}{c}, \mathfrak{d}]}$. A similar computation gives $I_2X^{i\mathfrak{d}} = -X^{i[\frac{D}{c}, \mathfrak{d}]}$, where $\mathfrak{d} \in \mathfrak{m}_0$. Since the operator $I := [\frac{D}{c}, .]$ is the complex structure of the orbit of compact type, thus of square -1, it follows that $(I_2)^2 = -1$. ■

Remark 4.2 Let us make a few comments on formula (7) in comparison to the formula given in the finite-dimensional case in Theorem 3 of [7]. First, as mentioned above, the convention for the definition of the complex symplectic form $\omega_{\mathbb{C}}$ in the infinite-dimensional case given by (9) differs from the usual convention for the finite-dimensional case by the multiplicative constant c^2. This explain the different multiplicative constants in the expressions of the potentials ($1/\kappa$ in the finite-dimensional formula, and c in the infinite-dimensional formula, with $\kappa = c$). Secondly, despite the fact that formula (7) looks very similarly to its finite-dimensional version, it differs by a non-trivial element in the kernel of the operator dd^c which encodes the affine structure of the orbit. Indeed, the elements y and $\pi(y)$ in $\mathcal{O}_D$ represent the *differences* between a conjugate of D and D itself. Note in particular that the values of the potential (7) and its derivative vanish along the fiber $\pi^{-1}(0)$.

5 From the complex affine coadjoint orbit $\mathcal{O}^{\mathbb{C}}$ to the cotangent space $T'\mathcal{O}$

Let us denote by $\Re y$ (resp. $\Im y$) the projection on the first (resp. second) factor $\mathfrak{g}$ in the direct sum $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ of an element $y \in \mathfrak{g}^{\mathbb{C}}$. The following Theorem is the infinite-dimensional analogue of Theorem 3 (iv) in [9]. It gives a relevant identification of $\mathcal{O}^{\mathbb{C}}$ and $T\mathcal{O}$, which will be used in next section to transport the hyperkähler structure of $\mathcal{O}_D^{\mathbb{C}}$ constructed in Theorem 4.1 to the (co-)tangent bundle of $\mathcal{O}_D$. We give a self-contained proof of this Theorem since the proof in [9] uses a compactness argument which fails in our setting (lemma 5 appearing in the proof of Theorem 3 (iv) in [9] is based on the completeness of a vector field, derived from the compactness of the orbit $\mathcal{O}$ (Proposition 5 in [9]), which can not be showed easily in our context).

Theorem 5.1 *The map*

$$\begin{array}{rccc} \Upsilon : & \mathcal{O}_D^{\mathbb{C}} & \rightarrow & T\mathcal{O}_D \\ & y & \mapsto & -\frac{1}{c} I_{\pi(y)} \Im y \end{array}$$

is an isomorphism which commutes with the natural projections $\pi : \mathcal{O}_D^{\mathbb{C}} \rightarrow \mathcal{O}_D$ and $p : T\mathcal{O}_D \rightarrow \mathcal{O}_D$.

■ **Proof of Theorem** 5.1:
Let us remark that for every $y \in \mathcal{O}_D^{\mathbb{C}}$ in the fiber $\pi^{-1}(x)$ over $x \in \mathcal{O}_D$, $\Im y$ belongs to $\mathfrak{m}_x$, thus can be viewed as an element of $T_x\mathcal{O}_D$. The G-equivariance of the projection π and of the complex structure I of $\mathcal{O}_D$ imply that Υ is G-equivariant and commutes with the projections π and p. To show that Υ is bijective, it is therefore sufficient to show that Υ identifies the fiber $\pi^{-1}(0)$ with $\mathfrak{m}_0$.

Let us define the function $f_1 : \mathfrak{m}_0 \rightarrow \mathfrak{m}_0$ by $f_1(\mathfrak{a}) = \Upsilon(y)$ where $y = e^{i\mathfrak{a}}De^{-i\mathfrak{a}} - D$. One has

$$f_1(\mathfrak{a}) := -\tfrac{1}{c} I \Im y \quad = \tfrac{i}{c} I \sinh\left(\operatorname{ad}(i\mathfrak{a})\right)(D) = \tfrac{i}{c} I \tfrac{\sinh \operatorname{ad}(i\mathfrak{a})}{\operatorname{ad}(i\mathfrak{a})}([i\mathfrak{a}, D]) = I \tfrac{\sinh \operatorname{ad}(i\mathfrak{a})}{\operatorname{ad}(i\mathfrak{a})} I\mathfrak{a}.$$

The eigenvalues of the operator $\frac{\sinh \operatorname{ad}(i\mathfrak{a})}{\operatorname{ad}(i\mathfrak{a})}$ from $\mathfrak{g}$ to $\mathfrak{g}$ being greater or equal to 1, the condition $\Im y = 0$ implies $\mathfrak{a} = 0$, hence $y = 0$.

Let $V \in \mathfrak{m}_0 \simeq T_0\mathcal{O}_D$. Let us show that there exists $y \in \mathcal{O}_D^{\mathbb{C}}$ such that $\Im y = cIV$. To do this, let us first suppose that V belongs to a maximal Abelian subalgebra $\mathfrak{A}$ of $\mathfrak{m}_0$ generated by a maximal subset Ψ of strongly orthogonal roots x_α :

$$\mathfrak{A} := \oplus_{\alpha \in \Psi} \mathbb{R} x_\alpha$$

For every $\alpha \in \Psi$, set $y_\alpha = Ix_\alpha$ and $h_\alpha = \frac{1}{2i}[x_\alpha, y_\alpha]$. For every $\alpha, \beta \in \Psi$, the following commutation relations hold :

$$[x_\alpha, y_\beta] = 2ih_\alpha \delta_{\alpha\beta}; \quad [h_\alpha, x_\beta] = -2iy_\alpha \delta_{\alpha\beta}; \quad [h_\alpha, y_\beta] = 2ix_\alpha \delta_{\alpha\beta}.$$

Now, for $\mathfrak{a} \in \mathfrak{A}$ with decomposition

$$\mathfrak{a} = \sum_{\alpha \in \Psi} a_\alpha x_\alpha$$

with respect to the basis x_α, one has

$$\operatorname{ad}(i\mathfrak{a})^{2n} I\mathfrak{a} = \sum_{\alpha \in \Psi} (2a_\alpha)^{2n} a_\alpha y_\alpha,$$

and consequently

$$\tfrac{i}{c}I\sinh \mathrm{ad}(i\mathfrak{a})(D) \quad = I\tfrac{\sinh \mathrm{ad}(i\mathfrak{a})}{\mathrm{ad}(i\mathfrak{a})}I\mathfrak{a} = \tfrac{1}{2}I\textstyle\sum_{\alpha\in\Psi}\sinh(2a_\alpha)y_\alpha = \tfrac{1}{2}\sum_{\alpha\in\Psi}\sinh(2a_\alpha)x_\alpha.$$

Thus, for any V in $\mathfrak{A}$ with decomposition

$$V = \sum_{\alpha\in\Psi} v_\alpha x_\alpha$$

with respect to the basis x_α, the element y in $\mathcal{O}_D^{\mathbb{C}}$ defined by $y = e^{i\mathfrak{a}}De^{-i\mathfrak{a}} - D$ where

$$\mathfrak{a} := \frac{1}{2}\sum_{\alpha\in\Psi}\operatorname{argsinh}(2v_\alpha)x_\alpha$$

satisfies $-\frac{1}{c}I\Im y = V$. It follows from the computation above that

$$\mathfrak{a} = I\frac{\operatorname{argsinh}\left(\mathrm{ad}(iV)\right)}{\mathrm{ad}(iV)}(IV).$$

Let us define the function $f_2 \ : \mathfrak{m}_0 \to \mathfrak{m}_0$ by

$$f_2(V) := I\tfrac{\operatorname{argsinh}(\mathrm{ad}(iV))}{\mathrm{ad}(iV)}(IV).$$

One has $f_1\circ f_2 = f_2\circ f_1 = \mathrm{Id}$ on $\mathfrak{A}$. To conclude the proof of the Theorem, let us remark that the union of maximal Abelian subalgebras of $\mathfrak{m}_0$ generated by a system of strongly orthogonal roots are dense in $\mathfrak{m}_0$ (indeed $\mathfrak{m}_0 = \overline{\cup_{g\in K}\mathrm{Ad}(g)(\mathfrak{A})}$, see the Appendix). It follows that the range of the restriction of Υ to the fiber $\pi^{-1}(0)$ is dense in $T_0\mathcal{O}_D$. From the arguments above, it also follows that $f_2\circ f_1 = \mathrm{Id}$ and $f_1\circ f_2 = \mathrm{Id}$ on $\mathrm{Ad}(K)\mathfrak{A}$. From the continuity of f_1 and f_2, this implies that $f_2\circ f_1 = \mathrm{Id}$ and $f_1\circ f_2 = \mathrm{Id}$ on $\mathfrak{m}_0$. Hence Υ identifies the fiber $\pi^{-1}(0)$ of $\mathcal{O}_D^{\mathbb{C}}$ with $T_0\mathcal{O}_D$. ■

6 The hyperkähler metric on the cotangent space $T'\mathcal{O}$

In Theorem 6.1 below, we give explicitly the hyperkähler structure of $T'\mathcal{O}_D$ (identified with the tangent space $T\mathcal{O}_D$ by the trace) obtained from the hyperkähler structure of $\mathcal{O}_D^{\mathbb{C}}$ via the map Υ defined in Theorem 5.1. By a standard argument as in Lemma 2.1 in [8], the metric $\tilde{\mathrm{g}}$ obtained is in fact the unique metric on $T'\mathcal{O}_D \simeq T\mathcal{O}_D$ which restricts to the Kähler metric on $\mathcal{O}_D$, is compatible with the Liouville complex symplectic form of $T'\mathcal{O}_D$ and for which the natural horizontal and vertical distributions Hor_V and Ver_V (see below) are $\tilde{\mathrm{g}}$-orthogonal. Let us mention that the last condition on $\tilde{\mathrm{g}}$ has to be a priori added in comparison to the finite-dimensional case to ensure uniqueness (in the proof of Lemma 2.1 in [8], α can be chosen H-invariant because H is compact, but this averaging procedure can not be applied in our case). We recall this uniqueness property in Proposition 6.2. The formulas for the metric given in Theorem 6.1 are identical to the ones appearing in Theorem 1.1 in [8]. The proof is however completely different and has no finite-dimensional analogue in the work of O. Biquard and P. Gauduchon. Moreover it provides a shortcut which avoids the computations of section 4 in [8]. Let us first state the Theorem. We will denote by $\mathrm{g}_{\mathcal{O}}$ the Kähler metric of the affine adjoint orbit of compact type $\mathcal{O}_D$ whose expression at 0 is the following

$$\mathrm{g}_{\mathcal{O}}(X^{\mathfrak{c}}, X^{\mathfrak{d}}) = c\Re\left\langle[\mathfrak{c}, D], [\mathfrak{d}, D]\right\rangle = c^3\Re\left\langle\mathfrak{c}, \mathfrak{d}\right\rangle,$$

where $\mathfrak{c}$ and $\mathfrak{d}$ are in $\mathfrak{m}_0$. This metric is strongly Kähler. This implies in particular that the Levi-Civita connection ∇ is well-defined. At an element V of the tangent space $T\mathcal{O}_D$, the space $T_V(T\mathcal{O}_D)$ splits into the Hilbert direct sum $\mathrm{Hor}_V \oplus \mathrm{Ver}_V$, where Ver_V is the tangent space to the fiber of the natural projection $p : T(T\mathcal{O}_D) \to T\mathcal{O}_D$, and where Hor_V is the horizontal space at V associated with the connection ∇. For any V in the fiber $p^{-1}(x)$, $x \in \mathcal{O}_D$, the space Ver_V will be naturally identified with $i\mathfrak{m}_x$, the vertical element $\mathfrak{c}^V$ corresponding to $\mathfrak{c} \in i\mathfrak{m}_x$ being $\mathfrak{c}^V = i\mathfrak{c}$. The horizontal space Hor_V will be identified with $\mathfrak{m}_x$ via the differential of p. For $\mathfrak{c} \in \mathfrak{m}_x$, the horizontal lift of $\mathfrak{c}\cdot x$ will be denoted by $\mathfrak{c}^H \in \mathrm{Hor}_V$. Let us denote by g_0 the metric on $T(T\mathcal{O}_D)$ obtained from the metric $\mathrm{g}_{\mathcal{O}}$ on $\mathcal{O}_D$ by these identifications together with the requirement that Hor_V and Ver_V are g_0-orthogonal. The pull-back by Υ^{-1} of the hyperkähler metric g will be denoted by $\tilde{\mathrm{g}}$.

Theorem 6.1 *The hyperkähler metric* $\tilde{\mathrm{g}}$ *on the tangent space* $T\mathcal{O}_D$ *is obtained from* g_0 *by the endomorphism whose decomposition with respect to the direct sum* $T_V(T\mathcal{O}_D) = Hor_V \oplus Ver_V$ *is the following*

$$\begin{pmatrix} A_V & 0 \\ 0 & A_V^{-1} \end{pmatrix}$$

with

$$A_V = Id + IR_{I\varphi(IR_{IV,V})(V),\varphi(IR_{IV,V})(V)},$$

where

$$\varphi(\mathrm{x}) = \left(\frac{\sqrt{1+\mathrm{x}}-1}{\mathrm{x}}\right)^{\frac{1}{2}}.$$

Proposition 6.2 (Lemma 2.1 in [8]) *The metric* $\tilde{\mathrm{g}}$ *is the unique hyperkähler metric on* $T\mathcal{O}_D$ *which restricts to the Kähler metric of* $\mathcal{O}_D$*, is compatible with the pull-back of Liouville's complex symplectic form by the identification* $T^*\mathcal{O}_D \simeq T\mathcal{O}_D$*, and for which the horizontal and vertical distributions* Hor_V *and* Ver_V *are* $\tilde{\mathrm{g}}$*-orthogonal.* □

Let us proceed to the proof of Theorem 6.1. We will need the following Lemmas.

Lemma 6.3 *For any* $\mathfrak{a}$ *in* $\mathfrak{m}_0$*, one has*

$$\frac{\cosh(\mathrm{ad}(i\mathfrak{a}))-1}{\mathrm{ad}(i\mathfrak{a})^2}([I\mathfrak{a},\mathfrak{a}]) = \frac{\sqrt{1+\mathrm{ad}(iV)^2}-1}{\mathrm{ad}(iV)^2}[IV,V], \tag{10}$$

where $\mathfrak{a}$ *and* V *are related by* $\Upsilon\left(\mathrm{Ad}_D(e^{i\mathfrak{a}})(0)\right) = V$ *or equivalently* $V = f_1(\mathfrak{a}) = I\frac{\sinh \mathrm{ad}(i\mathfrak{a})}{\mathrm{ad}(i\mathfrak{a})}I\mathfrak{a}$.

△ **Proof of Lemma** 6.3:
By continuity of the operators involved and density of maximal Abelian subalgebras of $\mathfrak{m}_0$ generated by maximal subsets of strongly orthogonal roots, it is sufficient to verify equation (10) for an element $\mathfrak{a}$ in a maximal Abelian subalgebra $\mathfrak{A}$ generated by a basis x_α, $\alpha \in \Psi$, where Ψ is a system of maximal strongly orthogonal roots. Using the notation introduced in the proof of Theorem 5.1, one has

$$V = \sum_{\alpha\in\Psi} v_\alpha x_\alpha,$$

and

$$\mathfrak{a} = \sum_{\alpha\in\Psi} a_\alpha x_\alpha = \frac{1}{2}\sum_{\alpha\in\Psi} \mathrm{argsinh}(2v_\alpha) x_\alpha.$$

For $\varphi(\mathrm{x}) = \frac{\cosh(\mathrm{x})-1}{\mathrm{x}^2}$, the following is true

$$\begin{aligned}
\varphi\left(\mathrm{ad}(i\mathfrak{a})\right)([I\mathfrak{a},\mathfrak{a}]) &= \textstyle\sum_{\alpha\in\Psi}\varphi(2a_\alpha)[a_\alpha y_\alpha, a_\alpha x_\alpha] = \sum_{\alpha\in\Psi}\frac{\cosh(2a_\alpha)-1}{(2a_\alpha)^2}[a_\alpha y_\alpha, a_\alpha x_\alpha] \\
&= \textstyle\sum_{\alpha\in\Psi}\frac{1}{4}\left(\cosh\left(\mathrm{argsinh}(2v_\alpha)-1\right)\right)[y_\alpha, x_\alpha] = \sum_{\alpha\in\Psi}\frac{\sqrt{1+(2v_\alpha)^2}-1}{(2v_\alpha)^2}[v_\alpha y_\alpha, v_\alpha x_\alpha] \\
&= \frac{\sqrt{1+\mathrm{ad}(iV)^2}-1}{\mathrm{ad}(iV)^2}[IV,V].
\end{aligned}$$

△

Lemma 6.4 *For any* $V \in \mathfrak{m}_0$ *and any positive analytic function* φ*, one has*

$$\varphi\left(\mathrm{ad}(iIV)^2\right)(V) = \varphi\left(IR_{IV,V}\right)(V).$$

△ **Proof of Lemma** 6.4:
With the notations introduced above,

$$IR_{IV,V} = I[IV,V] = I\sum_{\alpha\in\Psi}[v_\alpha y_\alpha, v_\alpha x_\alpha] = I\sum_{\alpha\in\Psi} v_\alpha^2(-2i)h_\alpha,$$

and

$$\begin{aligned}(IR_{IV,V})V &= I\sum_{\alpha\in\Psi} v_\alpha^2(-2i)[h_\alpha, v_\alpha x_\alpha] = I\sum_{\alpha\in\Psi} v_\alpha^2(-2i)(-2i)v_\alpha y_\alpha\\ &= -I\sum_{\alpha\in\Psi}(2v_\alpha)^2 v_\alpha y_\alpha = \sum_{\alpha\in\Psi}(2v_\alpha)^2 v_\alpha x_\alpha.\end{aligned}$$

On the other hand,

$$\begin{aligned}(\mathrm{ad}(iIV))^2(V) &= \mathrm{ad}(iIV)\left(\sum_{\alpha\in\Psi} i[v_\alpha y_\alpha, v_\alpha x_\alpha]\right) = \mathrm{ad}(iIV)\left(\sum_{\alpha\in\Psi} 2v_\alpha^2 h_\alpha\right)\\ &= \sum_{\alpha\in\Psi} 2iv_\alpha^3[y_\alpha, h_\alpha] = \sum_{\alpha\in\Psi}(2v_\alpha)^2 v_\alpha x_\alpha\\ &= (IR_{IV,V})V.\end{aligned}$$

Hence, it follows that

$$(IR_{IV,V})^n(V) = \sum_{\alpha\in\Psi}(2v_\alpha)^{2n} v_\alpha x_\alpha = (\mathrm{ad}(iIV))^{2n}(V)$$

Consequently, for any positive function φ, one has

$$\varphi\left(\mathrm{ad}(iV)^2\right)[IV,V] = \varphi\left(IR_{IV,V}\right)(V).$$

$\triangle$

■ **Proof of Theorem** 6.1:
Let us recall that the tangent space to $\mathcal{O}_D^{\mathbb{C}}$ at $y = \mathrm{Ad}_D(e^{i\mathfrak{a}})(x)$ ($x \in \mathcal{O}_D$, $\mathfrak{a} \in \mathfrak{m}_x$) is the subspace $e^{i\mathfrak{a}}(\mathfrak{m}_x \oplus i\mathfrak{m}_x)e^{-i\mathfrak{a}}$ of $\mathfrak{g}^{\mathbb{C}}$. It is identified with $\mathfrak{m}_x \oplus i\mathfrak{m}_x$ by the application ρ defined in Proposition 3.2,

$$\begin{array}{rccc}\rho: & \mathfrak{m}_x \oplus i\mathfrak{m}_x & \to & T_y\mathcal{O}_D^{\mathbb{C}}\\ & \mathfrak{c} & \mapsto & X^{\mathfrak{c}}.\end{array}$$

The vertical space $V_y := \rho(i\mathfrak{m}_x)$ is the kernel of π, and ρ restricts to an isomorphism from $\mathfrak{m}_x$ onto the horizontal space $H_y := \rho(\mathfrak{m}_x)$. The metric g is G-invariant and its expression at a point $y = e^{i\mathfrak{a}}De^{-i\mathfrak{a}} - D$ in the fiber $\pi^{-1}(0)$ over 0 is

$$\mathrm{g}\left(\rho(\mathfrak{c}), \rho(\mathfrak{d})\right) = \mathrm{g}\left(\rho(i\mathfrak{c}), \rho(i\mathfrak{d})\right) = c\Re\langle[\mathfrak{c}, e^{i\mathfrak{a}}De^{-i\mathfrak{a}}], [\mathfrak{d}, D]\rangle, \tag{11}$$

where $\mathfrak{c}$, $\mathfrak{d} \in \mathfrak{m}_0$. It follows that for any $\mathfrak{c}$ and $\mathfrak{d}$ in $\mathfrak{m}_0$, one has

$$\begin{aligned}\mathrm{g}\left(\rho(\mathfrak{c}), \rho(\mathfrak{d})\right) &= c\Re\left\langle[\mathfrak{c}, \cosh\left(\mathrm{ad}\left(i\mathfrak{a}\right)\right)(D)], [\mathfrak{d}, D]\right\rangle\\ &= c\Re\left\langle[\mathfrak{c}, D], [\mathfrak{d}, D]\right\rangle + c\Re\left\langle\left[\mathfrak{c}, \tfrac{\cosh(\mathrm{ad}(i\mathfrak{a}))-1}{\mathrm{ad}(i\mathfrak{a})^2}\left([i\mathfrak{a}, [i\mathfrak{a}, D]]\right)\right], [\mathfrak{d}, D]\right\rangle\\ &= c^3\Re\left\langle\mathfrak{c}, \mathfrak{d}\right\rangle + c^2\Re\left\langle\left[\mathfrak{c}, \tfrac{\cosh(\mathrm{ad}(i\mathfrak{a}))-1}{\mathrm{ad}(i\mathfrak{a})^2}\left([\mathfrak{a}, I\mathfrak{a}]\right)\right], [\mathfrak{d}, D]\right\rangle\\ &= c^3\Re\left\langle\mathfrak{c}, \mathfrak{d}\right\rangle + c^3\Re\left\langle I\left[\tfrac{\cosh(\mathrm{ad}(i\mathfrak{a}))-1}{\mathrm{ad}(i\mathfrak{a})^2}\left([I\mathfrak{a}, \mathfrak{a}]\right), \mathfrak{c}\right], \mathfrak{d}\right\rangle.\end{aligned} \tag{12}$$

The identification Υ of $\mathcal{O}_D^{\mathbb{C}}$ and $T\mathcal{O}_D$ commutes with the projections $\pi : \mathcal{O}_D^{\mathbb{C}} \to \mathcal{O}_D$ and $p : T\mathcal{O}_D \to \mathcal{O}_D$. It follows that the differential of Υ maps the vertical space V_y onto the vertical space Ver_V, where y and V are related by $V = \Upsilon(y)$. The horizontal space H_y is identified with $\mathfrak{m}_x$ by ρ^{-1} and Hor_V is identified with $\mathfrak{m}_x$ by dp. The G-invariance of the metrics g and g_0 allows us to suppose that y belongs to the fiber $\pi^{-1}(0)$. By Lemma 6.3, one has

$$\mathrm{g}\left(\rho(\mathfrak{c}), \rho(\mathfrak{d})\right) = c^3\Re\left\langle\mathfrak{c}, \mathfrak{d}\right\rangle + c^3\Re\left\langle I\left[\frac{\sqrt{1+\mathrm{ad}(iV)^2}-1}{\mathrm{ad}(iV)^2}[IV, V], \mathfrak{c}\right], \mathfrak{d}\right\rangle.$$

From Lemma A.9 in Appendix A, it follows that

$$\mathrm{g}\left(\rho(\mathfrak{c}), \rho(\mathfrak{d})\right) = c^3\Re\langle\mathfrak{c}, \mathfrak{d}\rangle + c^3\Re\langle I[[IV', V'], \mathfrak{c}], \mathfrak{d}\rangle,$$

with

$$V' = \left(\frac{\sqrt{1+\mathrm{ad}(iIV)^2}-1}{\mathrm{ad}(iIV)^2}\right)^{\frac{1}{2}}(V).$$

Hence

$$\mathrm{g}\left(\rho(\mathfrak{c}), \rho(\mathfrak{d})\right) = c^3\Re\langle\mathfrak{c}, \mathfrak{d}\rangle + c^3\Re\langle IR_{IV',V'}\mathfrak{c}, \mathfrak{d}\rangle.$$

By Lemma 6.4, it follows that

$$\mathrm{g}\left(\rho(\mathfrak{c}), \rho(\mathfrak{d})\right) = c^3\Re\langle\left(\mathrm{Id} + IR_{I\varphi(IR_{IV,V})(V),\varphi(IR_{IV,V})(V)}\right)\mathfrak{c}, \mathfrak{d}\rangle,$$

where $\varphi(\mathrm{x}) = \left(\frac{\sqrt{1+\mathrm{x}}-1}{\mathrm{x}}\right)^{\frac{1}{2}}$. Since Υ is G-equivariant, for any $\mathfrak{c} \in \mathfrak{m}_x$, $\Upsilon_*\rho(\mathfrak{c})$ is horizontal. Since both $\Upsilon_*\rho(\mathfrak{c})$ and $\mathfrak{c}^H$ projects on $\mathfrak{c}\cdot x$ by p_*, one has $\Upsilon_*\rho(\mathfrak{c}) = \mathfrak{c}^H$. Consequently for any $\mathfrak{c}$ and $\mathfrak{d}$ in $\mathfrak{m}_0$, the metric $\tilde{\mathrm{g}}$ applied to the horizontal lifts $\mathfrak{c}^H$ and $\mathfrak{d}^H$ is equal to

$$\tilde{\mathrm{g}}(\mathfrak{c}^H, \mathfrak{d}^H) = \mathrm{g}\left(\rho(\mathfrak{c}), \rho(\mathfrak{d})\right) = \mathrm{g}_0(A_V\mathfrak{c}, \mathfrak{d})$$

with

$$A_V = \mathrm{Id} + IR_{I\varphi(IR_{IV,V})(V),\varphi(IR_{IV,V})(V)},$$

where

$$\varphi(\mathrm{x}) = \left(\frac{\sqrt{1+\mathrm{x}}-1}{\mathrm{x}}\right)^{\frac{1}{2}}.$$

Hence the Theorem is proved in the horizontal directions. Further the orthogonality of the subspaces H_y and V_y implies the orthogonality of Hor_V and Ver_V. It follows that the hyperkähler metric $\tilde{\mathrm{g}}$ can be deduced from the metric g_0 via an operator of the form

$$\left(\begin{array}{cc} A & 0 \\ 0 & B \end{array}\right),$$

where B defines the metric in the directions tangent to the fibers of the projection p. Let us remark that for any $\mathfrak{c}$ and $\mathfrak{d}$ in $i\mathfrak{m}_x$, one has

$$\mathrm{g}(\rho(\mathfrak{c}), \rho(\mathfrak{d})) = \mathrm{g}(\rho(-i\mathfrak{c}), \rho(-i\mathfrak{d})).$$

The multiplication by i exchanges V_y and H_y and induces a complex structure on the tangent space $T\mathcal{O}_D$ at V whose expression with respect to g_0 is given by an endomorphism J_3 exchanging Ver_V and Hor_V, i.e whose expression with respect to the direct sum $T_V(T\mathcal{O}_D) = \mathrm{Hor}_V \oplus \mathrm{Ver}_V$ has the following form

$$J_3 = \left(\begin{array}{cc} 0 & C \\ D & 0 \end{array}\right).$$

Let us recall that the real symplectic form $\omega_1 = \mathrm{g}(i\cdot\,,\cdot)$ associated to the complex structure i on $\mathcal{O}_D^{\mathbb{C}}$ has the following expression

$$\omega_1\left(\rho(\mathfrak{c}+\mathfrak{c}'), \rho(\mathfrak{d}+\mathfrak{d}')\right) = c\Im\left(\langle\rho(\mathfrak{c}'), \pi_*\rho(\mathfrak{d})\rangle - \langle\rho(\mathfrak{d}'), \pi_*\rho(\mathfrak{c})\rangle\right),$$

where $\mathfrak{c}, \mathfrak{d}$ belong to $\mathfrak{m}_x$, and $\mathfrak{c}', \mathfrak{d}'$ belong to $i\mathfrak{m}_x$. Note that only the projections of $\rho(\mathfrak{c}')$ and $\rho(\mathfrak{d}')$ on $i\mathfrak{m}_x$ contribute in the above formula. Denoting by $p_+ : \mathfrak{g}^{\mathbb{C}} \to i\mathfrak{m}_x$ the orthogonal projection onto $i\mathfrak{m}_x$, one has for $\mathfrak{c}' \in i\mathfrak{m}_x$, $\Upsilon_*\rho(\mathfrak{c}') = \frac{i}{c}I_{\pi(y)}p_+\left(\rho(\mathfrak{c}')\right)$, hence $p_+ = i\,c\,I_{\pi(y)}\Upsilon_*$ on V_y. It follows that $p_+\left(\Upsilon_*^{-1}\left((\mathfrak{c}')^V\right)\right) = i\,c\,I_{\pi(y)}(\mathfrak{c}')^V = i\,c\,I_{\pi(y)}(i\mathfrak{c}') = \mathfrak{c}'\cdot x$. Since moreover $\pi_*\Upsilon_*^{-1}\mathfrak{d}^H = p_*\mathfrak{d}^H$, it follows that the symplectic form $\Omega_3 = \Upsilon_*\omega_1$ on $T\mathcal{O}_D$ associated with the complex structure J_3 is Liouville 2-form

$$\Omega_3\left(\mathfrak{c}^H + (\mathfrak{c}')^V, \mathfrak{d}^H + (\mathfrak{d}')^V\right) = c^3\Re\left(\langle i\mathfrak{c}', \mathfrak{d}\rangle - \langle i\mathfrak{d}', \mathfrak{c}\rangle\right),$$

where $\mathfrak{c}, \mathfrak{d}$ belong to $\mathfrak{m}_x$, and $\mathfrak{c}', \mathfrak{d}'$ belong to $i\mathfrak{m}_x$. The symplectic form Ω_3 can be deduce from g_0 via an endomorphism whose block decomposition with respect to the direct sum $T_V(T\mathcal{O}_D) = \mathrm{Hor}_V \oplus \mathrm{Ver}_V$ is

$$\left(\begin{array}{cc} 0 & i \\ i & 0 \end{array}\right).$$

The equation $\tilde{\mathrm{g}}(J_3\cdot\,,\cdot) = \Omega_3(\cdot\,,\cdot)$ implies the followings conditions on the operators A, B, C and D:

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

i.e $AC = i$ and $BD = i$. On the other hand, the condition $(I_3)^2 = -1$ implies $CD = -1$. It follows that $B = A^{-1}$, and J_3 is represented by the following operator

$$J_3 = \begin{pmatrix} 0 & iA^{-1} \\ iA & 0 \end{pmatrix}.$$

■

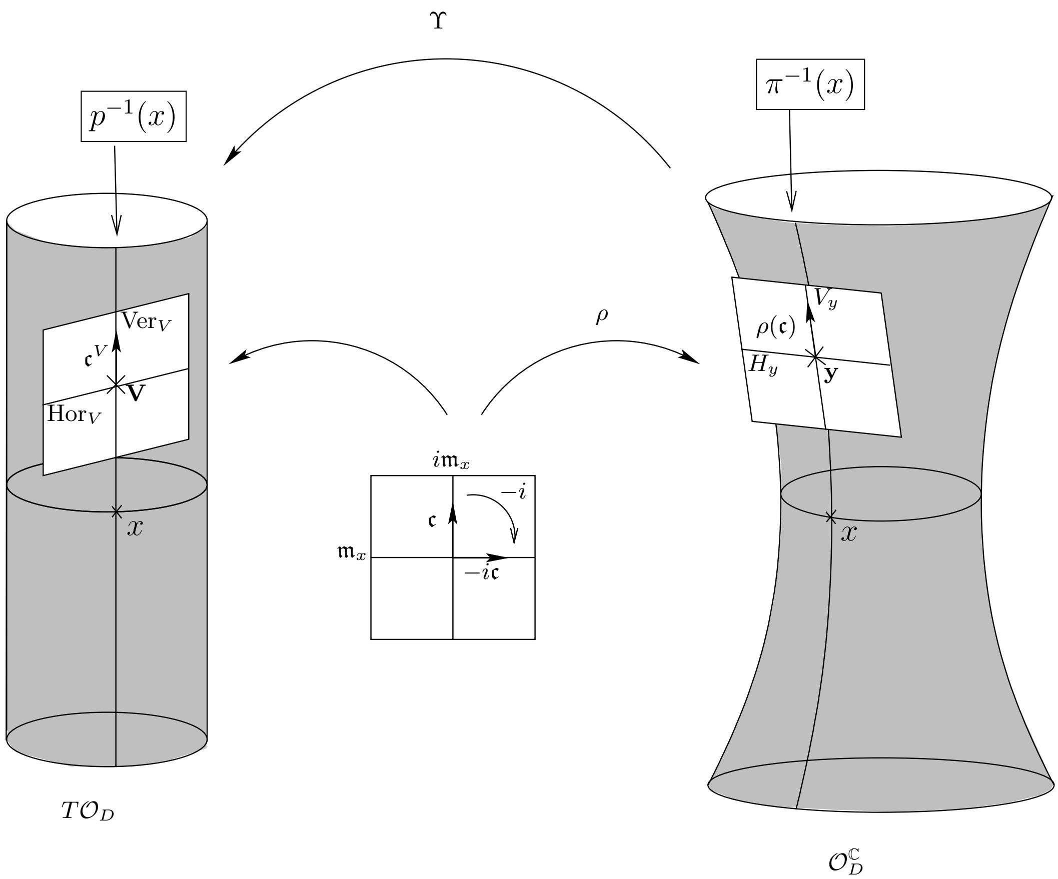

Figure 1: *The expression of the hyperkähler metric on $T\mathcal{O}_D$ can be easily deduced from the expression of the hyperkähler metric on $\mathcal{O}_D^{\mathbb{C}}$*

Remark 6.5 The restricted Grassmannian $\mathrm{Gr}_{\mathrm{res}}(\mathcal{H}_+, \mathcal{H}_-)$ of a polarized Hilbert space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ (where $\mathcal{H}_+$ and $\mathcal{H}_-$ are infinite-dimensional closed orthogonal subspaces of $\mathcal{H}$) is defined as the set of closed subspaces P of $\mathcal{H}$ such that the orthogonal projection from P on $\mathcal{H}_+$ is Fredholm and the orthogonal projection from P on $\mathcal{H}_-$ is a Hilbert-Schmidt operator (for further information on this manifold see [25] and [36]). The connected component $\mathrm{Gr}^0_{\mathrm{res}}(\mathcal{H}_+, \mathcal{H}_-)$ of $\mathrm{Gr}_{\mathrm{res}}(\mathcal{H}_+, \mathcal{H}_-)$ containing the subspace $\mathcal{H}_+$ is an homogeneous space of the unitary group

$$\mathrm{U}_2 = \left\{ u \in \mathrm{U}(H) \quad | \quad u - \mathrm{id} \in L^2(H) \right\}$$

which is a simple L^*-group of compact type (a geometrical proof of this fact is given in [4]). The manifold $\mathrm{Gr}^0_{\mathrm{res}}(\mathcal{H}_+, \mathcal{H}_-)$ can be identified with a family of affine adjoint orbits of the Lie algebra $\mathfrak{u}_2$ of U_2. The corresponding derivations $\mathbb{D}_k = [D_k, \cdot]$ are the following

$$D_k := ik\,(p_+ - p_-)\,,$$

where $p_\pm$ is the orthogonal projection onto $\mathcal{H}_\pm$. The Kähler structures on $\mathrm{Gr}^0_{\mathrm{res}}(\mathcal{H}_+,\mathcal{H}_-)$ obtained by these identifications are proportional to the standard one as defined in [25] or [36]. The complexified orbit $\mathcal{O}^{\mathbb{C}}_{D_k}$ is the set of skew-Hermitian bounded operator on $\mathcal{H}$ with two eigenvalues ik and $-ik$ such that the corresponding eigenspaces P_{ik} and P_{-ik} belong respectively to $\mathrm{Gr}^0_{\mathrm{res}}(\mathcal{H}_+,\mathcal{H}_-)$ and $\mathrm{Gr}^0_{\mathrm{res}}(\mathcal{H}_-,\mathcal{H}_+)$. It can be identified with a natural complexification $\left(\mathrm{Gr}^0_{\mathrm{res}}(\mathcal{H}_+,\mathcal{H}_-)\right)^{\mathbb{C}}$ of $\mathrm{Gr}^0_{\mathrm{res}}(\mathcal{H}_+,\mathcal{H}_-)$ consisting of pairs of subspaces (P,Q) such that $P\in\mathrm{Gr}^0_{\mathrm{res}}(\mathcal{H}_+,\mathcal{H}_-)$, $Q\in\mathrm{Gr}^0_{\mathrm{res}}(\mathcal{H}_-,\mathcal{H}_+)$ and $P\cap Q=\{0\}$. The family of hyperkähler structures on $\left(\mathrm{Gr}^0_{\mathrm{res}}(\mathcal{H}_+,\mathcal{H}_-)\right)^{\mathbb{C}}$ and $T'\mathrm{Gr}^0_{\mathrm{res}}(\mathcal{H}_+,\mathcal{H}_-)$ obtained by applying Theorem 4.1 and Theorem 6.1 to $\mathcal{O}_{D_k}$, $k\neq 0$, was obtained by hyperkähler reduction in [29].

A Strongly orthogonal roots in L^*-algebras

We refer to [33] for more information on the fine structure of finite-dimensional Hermitian-symmetric orbits. Let $\mathcal{O}_D = G/K$ be a Hermitian-symmetric affine coadjoint orbit of an L^*-group G. Denote be $\mathfrak{g}$ the Lie algebra of G, $\mathfrak{k}$ the Lie algebra of K, and $\mathfrak{m}$ the orthogonal of $\mathfrak{k}$ in $\mathfrak{g}$. The following commutation relations hold :

$$[\mathfrak{k},\mathfrak{k}]\subset\mathfrak{k},\qquad [\mathfrak{k},\mathfrak{m}]\subset\mathfrak{m}\qquad [\mathfrak{m},\mathfrak{m}]\subset\mathfrak{k}. \tag{13}$$

If $\mathfrak{A}$ is a subalgebra of $\mathfrak{g}$ contained in $\mathfrak{m}$, then the third commutation relation in (13) implies that $\mathfrak{A}$ is commutative. Abusing slightly the terminology, one says that $\mathfrak{A}$ in an Abelian subalgebra of $\mathfrak{m}$. The next Lemma generalizes Theorem 8.6.1 (iii) in [35] or Lemma 6.3 (iii) in [12] to the case of a Hermitian-symmetric affine coadjoint orbit of an L^*-group.

Lemma A.1 *Let $\mathfrak{A}$ be a maximal Abelian subalgebra of $\mathfrak{m}$. Then*

$$\mathfrak{m}=\overline{\cup_{g\in K}\mathrm{Ad}(g)\mathfrak{A}}.$$

$\triangle$ **Proof of Lemma** A.1:
Since $\mathcal{O}_D$ can be decomposed in a product of irreducible pieces, it is sufficient to consider the case where $\mathcal{O}_D$ is an irreducible Hermitian-symmetric coadjoint orbit of a simple L^*-group G. There exists an increasing sequence $\{\mathfrak{g}_n\}_{n\in\mathbb{N}}$ of finite-dimensional subalgebras of $\mathfrak{g}$ and an increasing sequence $\{\mathfrak{k}_n\}_{n\in\mathbb{N}}$ of finite-dimensional subalgebras of $\mathfrak{k}$ such that ([29] and [31])

$$\mathfrak{g}=\overline{\cup_{n\in\mathbb{N}}\mathfrak{g}_n}$$

$$\mathfrak{k}=\overline{\cup_{n\in\mathbb{N}}\mathfrak{k}_n}$$

$$[\mathfrak{k}_n,\mathfrak{m}_n]\subset\mathfrak{m}_n\qquad[\mathfrak{m}_n,\mathfrak{m}_n]\subset\mathfrak{k}_n,$$

where $\mathfrak{m}_n$ denotes the orthogonal of $\mathfrak{k}_n$ in $\mathfrak{g}_n$. Let K_n be the subgroup of G with Lie algebra $\mathfrak{k}_n$. For all $n\in\mathbb{N}$, $\mathfrak{A}_n:=\mathfrak{A}\cap\mathfrak{g}_n$ is a maximal Abelian subalgebra of $\mathfrak{g}_n$. From the finite-dimensional theory (see Theorem 8.6.1 (iii) in [35], or Lemma 6.3 in [12]), one has

$$\mathfrak{m}_n=\mathrm{Ad}(K_n)(\mathfrak{A}_n).$$

Since $\mathfrak{m}=\overline{\cup_{n\in\mathbb{N}}\mathfrak{m}_n}$, and $\cup_{n\in\mathbb{N}}\mathrm{Ad}(K_n)(\mathfrak{A}_n)\subset\mathrm{Ad}(K)(\mathfrak{A})$ and since $\mathfrak{m}\supset\mathrm{Ad}(K)(\mathfrak{A})$, one has

$$\mathfrak{m}=\overline{\cup_{g\in K}\mathrm{Ad}(g)\mathfrak{A}}.$$

$\triangle$

Remark A.2 In the finite-dimensional case, every maximal Abelian subalgebra of $\mathfrak{m}$ is the centralizer of one of its elements and every maximal Abelian subalgebras of $\mathfrak{m}$ are conjugate. In particular, the Cartan subalgebras of a compact semi-simple Lie group are conjugate. This is no longer true in the infinite-dimensional case (see [4]).

In this subsection, $\mathcal{O}_D$ will denote a Hermitian-symmetric affine coadjoint orbit *of compact type* associated with the derivation $\mathbb{D}:=[D,\cdot]$. Let $\mathfrak{g}^{\mathbb{C}}$ be the L^*-algebra $\mathfrak{g}\oplus i\mathfrak{g}$, $\mathfrak{k}^{\mathbb{C}}$ the L^*-algebra $\mathfrak{k}\oplus i\mathfrak{k}$, and $\mathfrak{m}^{\mathbb{C}}$ the complex closed vector subspace $\mathfrak{m}\oplus i\mathfrak{m}$. The subspace $\mathfrak{m}^{\mathbb{C}}$ decomposes into $\mathfrak{m}^{\mathbb{C}}=\mathfrak{m}^+\oplus\mathfrak{m}^-$, where $\mathfrak{m}^{\pm}$ is the direct sum of eigenspaces $V_{\pm c_\alpha}$ of $\mathbb{D}$ with eigenvalues $\pm i c_\alpha$, $c_\alpha > 0$. The natural complex structure of $\mathcal{O}_D$ is given by

$$I := \sum_\alpha \frac{1}{c_\alpha} \mathbb{D}_{|V_{c_\alpha} \oplus V_{-c_\alpha}}$$

Let $\mathfrak{h}$ be a Cartan subalgebra contained in $\mathfrak{k}$ (see Theorem 4.4 in [22] for the existence of such a Cartan subalgebra), $\mathcal{R}$ the set of roots and

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \mathcal{A}} V^\alpha \oplus \bigoplus_{\beta \in \mathcal{B}_+} (V^\beta + V^{-\beta})$$

the decomposition of $\mathfrak{g}^{\mathbb{C}}$ into eigenspaces of $\operatorname{ad}(\mathfrak{h})$, where the notation V^α stand for the eigenspace corresponding to α, and where $\mathcal{A}$ and $\mathcal{B}$ are subsets of $\mathcal{R}$ such that (see [29] and [31])

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \bigoplus\nolimits_{\alpha \in \mathcal{A}} V^\alpha; \quad \mathfrak{m}_\pm = \oplus_{\beta \in \mathcal{B}_\pm} V^\beta.$$

Definition A.3 Two roots α and β are called strongly orthogonal if neither $\alpha + \beta$ nor $\alpha - \beta$ is a root.

Remark A.4 Two strongly orthogonal roots are orthogonal for the scalar product of $\mathfrak{h}'$.

Remark A.5 By Zorn's Lemma, there exists maximal sets of (mutually) strongly orthogonal roots.

Remark A.6 If $\mathcal{O}_D$ is irreducible, then, for every order on the set of roots, there exists a unique simple root in $\mathcal{B}$ (see [29] and [31]). Let $\mathcal{R}_+$ (resp. $\mathcal{R}_-$) be the set of positive (resp. negative) roots. Exchanging $\mathcal{R}_+$ and $\mathcal{R}_-$ if necessary, one can suppose that $\mathcal{B}_+ \subset \mathcal{R}_+$. Then, for every root α, there exists $(h_\alpha, e_\alpha, e_{-\alpha}) \in i\mathfrak{h} \times V^\alpha \times V^{-\alpha}$ such that $[h_\alpha, e_{\pm\alpha}] = \pm 2 e_\alpha$, $[e_\alpha, e_{-\alpha}] = h_\alpha$ and $x_\alpha := e_\alpha - e_{-\alpha} \in \mathfrak{g}$. Set $y_\alpha := I x_\alpha$. One has

$$[x_\alpha, y_\alpha] = 2 i h_\alpha; \quad [h_\alpha, x_\alpha] = -2 i y_\alpha; \quad [h_\alpha, y_\alpha] = 2 i x_\alpha.$$

Proposition A.7 *If Ψ is a maximal set of strongly orthogonal roots, then the Hilbert sum*

$$\mathfrak{A} := \oplus_{\alpha \in \Psi} \mathbb{R} x_\alpha$$

defines a maximal Abelian subalgebra $\mathfrak{m}$ such that

$$[\mathfrak{A}, I\mathfrak{A}] = \oplus_{\alpha \in \Psi} \mathbb{R} i h_\alpha.$$

In particular, $\mathfrak{m} = \overline{Ad(K)(\mathfrak{A})}$.

□ **Proof of Proposition** A.7:
This follows directly from the commutation relation $[V^\alpha, V^\beta] \subset V^{\alpha+\beta}$ and from the hypothesis that Ψ is a maximal set of strongly orthogonal roots. □

Proposition A.8 *With the notation above, the curvature R of the symmetric orbit $\mathcal{O}_D$ satisfies*

$$\begin{aligned} &R_{x_\alpha, I x_\alpha} x_\alpha = 4 I x_\alpha \\ &R_{x_\alpha, I x_\alpha} x_\beta = 0 \\ &R_{x_\alpha, I x_\beta} = R_{x_\alpha, x_\beta} = 0, \end{aligned}$$

for every α and β, $\alpha \neq \beta$, in a maximal set Ψ of strongly orthogonal roots.

□ **Proof of Proposition** A.8:
This is an easy consequence of the expression of the curvature of a symmetric homogeneous space (see [5]). In particular,

$$R_{x_\alpha, I x_\alpha} x_\alpha = [[x_\alpha, I x_\alpha], x_\alpha].$$

□

The following Lemma is the infinite-dimensional analogue of Lemma 2 in [8].

Lemma A.9 *For every* $\mathfrak{a}$, $\mathfrak{b}$ *in* $\mathfrak{m}$, *one has*

$$\langle[\mathfrak{a}, I\mathfrak{a}], [\mathfrak{b}, I\mathfrak{b}]\rangle = \|[\mathfrak{a}, \mathfrak{b}]\|^2 + \|[\mathfrak{a}, I\mathfrak{b}]\|^2.$$

Moreover if ϕ *is an analytic positive function such that* $\phi(x) = \phi(-x)$, *then*

$$\phi(\mathrm{ad}(i\mathfrak{a}))[\mathfrak{a}, I\mathfrak{a}] = [\mathfrak{a}', I\mathfrak{a}'],$$

where $\mathfrak{a}' = \sqrt{\phi}(\mathrm{ad}(iI\mathfrak{a}))(\mathfrak{a})$.

$\triangle$ **Proof of Lemma** A.9:
By product, it is enough to consider the case where $\mathfrak{g}$ is simple and $\mathcal{O}_D$ irreducible. In this case, the complex structure is $I = \frac{1}{c}D$ for some positive constant c, and

$$[\mathfrak{a}, I\mathfrak{b}] = \frac{1}{c}[\mathfrak{a}, [D, \mathfrak{b}]] = \frac{1}{c}[[\mathfrak{a}, D], \mathfrak{b}] + \frac{1}{c}[D, [\mathfrak{a}, \mathfrak{b}]].$$

Since $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$, for $\mathfrak{a}$, $\mathfrak{b} \in \mathfrak{m}$, one has

$$[\mathfrak{a}, I\mathfrak{b}] = -[I\mathfrak{a}, \mathfrak{b}].$$

In the same way, for $\mathfrak{a}$, $\mathfrak{b} \in \mathfrak{m}$, one has

$$[I\mathfrak{a}, I\mathfrak{b}] = \frac{1}{c^2}[[D, \mathfrak{a}], [D, \mathfrak{b}]] = \frac{1}{c^2}[D, [\mathfrak{a}, [D, \mathfrak{b}]]] - \frac{1}{c^2}[\mathfrak{a}, [D, [D, \mathfrak{b}]]] = [\mathfrak{a}, \mathfrak{b}].$$

Since every element of $\mathfrak{g}$ is skew-symmetric, it follows that

$$\begin{aligned}\langle[\mathfrak{a}, I\mathfrak{a}], [\mathfrak{b}, I\mathfrak{b}]\rangle &= -\langle I\mathfrak{a}, [\mathfrak{a}, [\mathfrak{b}, I\mathfrak{b}]]\rangle \\ &= -\langle I\mathfrak{a}, [[\mathfrak{a}, \mathfrak{b}], I\mathfrak{b}]\rangle - \langle I\mathfrak{a}, [\mathfrak{b}, [\mathfrak{a}, I\mathfrak{b}]]\rangle \\ &= \langle[I\mathfrak{a}, I\mathfrak{b}], [\mathfrak{a}, \mathfrak{b}]\rangle + \langle[\mathfrak{b}, I\mathfrak{a}], [\mathfrak{a}, I\mathfrak{b}]\rangle \\ &= \|[\mathfrak{a}, \mathfrak{b}]\|^2 + \|[\mathfrak{a}, I\mathfrak{b}]\|^2.\end{aligned}$$

To prove the second assertion of the Lemma, let us first consider the case when $\mathfrak{a}$ belongs to a maximal Abelian subalgebra in $\mathfrak{m}$ of the form

$$\mathfrak{A} := \oplus_{\alpha\in\Psi}\mathbb{R}x_\alpha$$

where Ψ is a maximal set of strongly orthogonal roots. With the notation introduced above, $\mathfrak{a} = \sum_\alpha a_\alpha x_\alpha$, $I\mathfrak{a} = \sum_\alpha a_\alpha y_\alpha$ and $[\mathfrak{a}, I\mathfrak{a}] = \sum_\alpha a_\alpha^2 2ih_\alpha$. Using the commutation relations

$$[x_\alpha, y_\beta] = 2ih_\alpha\delta_{\alpha\beta}; \quad [h_\alpha, x_\beta] = -2iy_\alpha\delta_{\alpha\beta}; \quad [h_\alpha, y_\beta] = 2ix_\alpha\delta_{\alpha\beta},$$

one has

$$\mathrm{ad}(i\mathfrak{a})^{2n}[\mathfrak{a}, I\mathfrak{a}] = \sum_\alpha (2a_\alpha)^{2n}(a_\alpha^2 2ih_\alpha).$$

Thus for every positive analytic function ϕ such that $\phi(\mathrm{x}) = \phi(-\mathrm{x})$, one has

$$\begin{aligned}\phi(\mathrm{ad}(i\mathfrak{a}))[\mathfrak{a}, I\mathfrak{a}] &= \textstyle\sum_\alpha \phi(2a_\alpha)a_\alpha^2 2ih_\alpha \\ &= \textstyle\sum_\alpha \phi(2a_\alpha)a_\alpha^2[x_\alpha, y_\alpha] \\ &= \textstyle\sum_\alpha [\sqrt{\phi}(2a_\alpha)a_\alpha x_\alpha, \sqrt{\phi}(2a_\alpha)a_\alpha y_\alpha].\end{aligned}$$

Moreover, the adjoint action of the element $iI\mathfrak{a}$ is given by

$$\mathrm{ad}(iI\mathfrak{a})^{2n}(\mathfrak{a}) = \sum_\alpha (2a_\alpha)^{2n} a_\alpha x_\alpha.$$

Thus $\sum_\alpha \sqrt{\phi}(2a_\alpha)a_\alpha x_\alpha = \sqrt{\phi}(\mathrm{ad}(iI\mathfrak{a})(\mathfrak{a})$, which conclude the proof of the second assertion of the Lemma for $\mathfrak{a}$ in $\mathfrak{A}$. By adjoint action of K, this assertion is still true for $\mathfrak{a}$ belonging in $\cup_{g\in K}\mathrm{Ad}(g)(\mathfrak{A})$. The continuity of ϕ and of the bracket then imply that it is true for every $\mathfrak{a}$ in $\mathfrak{m} = \overline{\mathrm{Ad}(K)(\mathfrak{A})}$. $\triangle$

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