On the classification of infinite-dimensional irreducible Hermitian-symmetric affine coadjoint orbits

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Abstract

In the finite-dimensional setting, every Hermitian-symmetric space of compact type is a coadjoint orbit of a finite-dimensional Lie group. It is natural to ask whether every infinite-dimensional Hermitian-symmetric space of compact type, which is a particular example of an Hilbert manifold, is transitively acted upon by a Hilbert Lie group of isometries. In this paper we give the classification of infinite-dimensional irreducible Hermitian-symmetric affine coadjoint orbits of compact type using the notion of simple roots of non-compact type. The key step is, given an infinite-dimensional symmetric pair \((g, k)\), where \(g\) is a simple \(L^*\)-algebra of compact type and \(k\) a subalgebra of \(g\), to construct an increasing sequence of finite-dimensional subalgebras \(g_n\) of \(g\) together with an increasing sequence of finite-dimensional subalgebras \(k_n\) of \(k\) such that \(g = \bigcup g_n\), \(k = \bigcup k_n\), and such that the pairs \((g_n, k_n)\) are symmetric. Comparing with the classification of Hermitian-symmetric spaces given by W. Kaup, it follows that any Hermitian-symmetric space of compact or non-compact type is an affine-coadjoint orbit of an Hilbert Lie group.

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1 Introduction

The topic of the present paper belongs to the theory of infinite-dimensional Hermitian-symmetric spaces, which are particular examples of symmetric spaces modelled on Banach spaces. The reader will find in [24] the fundamentals of the theory.

The classification of Hermitian-symmetric spaces of arbitrary dimension has been carried out by W. Kaup in [11] using the notion of Jordan triple systems developed in [10], and the equivalence between the category of simply connected, symmetric, complex Banach manifolds with base point and the category of Hermitian Jordan triple systems proved as the Main Theorem in [9]. A Hermitian-symmetric space \(M\) is defined to be a connected complex Banach manifold with a Hermitian structure such that each point in \(M\) is an isolated fixed point of an involutive holomorphic isometry of \(M\). By Theorem (4.2) in [11] and the discussion that follows, every Hermitian-symmetric space is the orthogonal product \(M = M_+ \times M_0 \times M_-\) where \(M_0\) is the quotient of a Hilbert space by a discrete subgroup, and \(M_+\) (resp. \(M_-\)) is a simply-connected Hermitian-symmetric space of compact (resp. non-compact) type. By Theorem (3.9) and the discussion following Theorem (4.2) in [11], every Hermitian-symmetric space of compact (resp. non-compact) type is the orthogonal product of (possibly an infinite number of) irreducible Hermitian-symmetric spaces of compact (resp. non-compact) type. The category of irreducible Hermitian-symmetric spaces of compact type is equivalent to the category of irreducible Hermitian-symmetric spaces of non-compact type ([11]). It is therefore sufficient to classify either the irreducible Hermitian-symmetric spaces of compact type or the irreducible Hermitian-symmetric spaces of non-compact type.

In this paper, we are interested in the classification of irreducible infinite-dimensional Hermitian-symmetric affine coadjoint orbits of compact or non-compact type. In order to state the corresponding results, let us first introduce some notation. For any complex Hilbert space \(F\) endowed with a distinguished basis \(\{f_j\}_{j \in J}\), \(F^\mathbb{R}\) will denote the real Hilbert space with basis \(\{f_j\}_{j \in J}\) and \(F^\mathbb{R}\) the real
Every irreducible infinite-dimensional Hermitian-symmetric affine coadjoint orbit of a
1.1
the Grassmannian
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with respect to the first variable, and
the
\{e_i\}_i \in \mathbb{Z}_-\{0\}. The Hermitian scalar product on \(\mathcal{H}\) will be denoted by \(\langle \cdot , \cdot \rangle_{\mathcal{H}}\) and will be \(\mathbb{C}\)-skew-linear with respect to the first variable, and \(\mathbb{C}\)-linear with respect to the second variable. For a bounded operator \(x\) on \(\mathcal{H}\), denote by \(x^T\) the transpose of \(x\) defined by \(\langle x^T e_i , e_j \rangle_{\mathcal{H}} = \langle e_j , e_i \rangle_{\mathcal{H}}\), and by \(x^*\) the adjoint of \(x\) defined by \(\langle xe_i , e_j \rangle_{\mathcal{H}} = \langle e_j , xe_i \rangle_{\mathcal{H}}\). The closed infinite-dimensional subspace of \(\mathcal{H}\) generated by the \(e_n\)'s for \(n > 0\) will be called \(\mathcal{H}_+\), and its orthogonal complement \(\mathcal{H}_-\). For \(0 < p < +\infty\), the \(p\)-dimensional subspace of \(\mathcal{H}\) generated by the \(e_n\)'s for \(0 < n \leq p\) will be denoted \(\mathcal{H}_p\). Let \(J_0\) be the bounded operator on \(\mathcal{H}\) defined by \(J_0 e_i = -e_{-i}\) if \(i < 0\) and \(J_0 e_i = e_{-i}\) if \(i > 0\). For \(\mathcal{F} = \mathcal{H}, \mathcal{H}_\pm, \mathcal{H}_p,\) or \(\mathcal{H}_p^2\) define the following Hilbert Lie groups and the associated Lie algebras
\[
\begin{align*}
\text{GL}_2(\mathcal{F}) & := \{ g \in \text{GL}(\mathcal{F}) \mid g - \text{id} \in L^2(\mathcal{F}) \}, \\
\text{U}_2(\mathcal{F}) & := \{ g \in \text{U}(\mathcal{F}) \mid g - \text{id} \in L^2(\mathcal{F}) \}, \\
\text{O}_2(\mathcal{F}_\mathbb{R}) & := \{ g \in \text{U}_2(\mathcal{F}) \mid g^T g = \text{id} \}, \\
\mathfrak{o}_2(\mathcal{F}_\mathbb{R}) & := \{ a \in \mathfrak{u}_2(\mathcal{F}) \mid a^T + a = 0 \}.
\end{align*}
\]
At last define
\[
\text{Sp}_2(\mathcal{H}) := \{ g \in \text{U}_2(\mathcal{H}) \mid g^T J_0 g = J_0 \}, \quad \mathfrak{sp}_2(\mathcal{H}) := \{ a \in \mathfrak{u}_2(\mathcal{H}) \mid a^T J_0 + J_0 a = 0 \}.
\]
On the Lie algebras \(\mathfrak{g}\) listed above, the bracket is the commutator of operators and the Hermitian product \(\langle \cdot , \cdot \rangle_{\mathcal{H}}\) is defined using the trace by
\[
\langle A , B \rangle := \text{Tr} A^* B.
\]
These Lie algebras are \(L^*\)-algebras in the sense that the following property is satisfied :
\[
\langle [x , y] , z \rangle = \langle y , [x^* , z] \rangle
\]
for every \(x, y\) and \(z\). In fact, \(\mathfrak{u}_2(\mathcal{H}), \mathfrak{o}_2(\mathcal{H}_\mathbb{R})\) and \(\mathfrak{sp}_2(\mathcal{H})\) are the only separable infinite-dimensional simple \(L^*\)-algebras of compact type modulo isomorphisms (see below for the corresponding definition and [1], [8], or [23] for the proof of this statement). An \(L^*\)-group is a Banach-Lie group whose Lie algebra has a structure of \(L^*\)-algebra (see [7]). The \(L^*\)-groups \(\text{GL}_2(\mathcal{H}), \text{U}_2(\mathcal{H})\) and \(\text{Sp}_2(\mathcal{H})\) are connected, but \(\text{O}_2(\mathcal{H}_\mathbb{R})\) admits two connected components (see Proposition 12.4.2 on page 245 in [15]). The connected component of \(\text{O}_2(\mathcal{H}_\mathbb{R})\) containing the special orthogonal group
\[
\text{SO}_1(\mathcal{H}_\mathbb{R}) := \{ g \in \text{O}_2(\mathcal{H}_\mathbb{R}) \mid g - \text{id} \in L^1(\mathcal{H}), \det(g) = 1 \},
\]
where \(\det\) denotes the Fredholm determinant (see [18]), will be denoted by \(\text{O}_2^+(\mathcal{H})\). The aim of this paper is to prove the following statement.

**Theorem 1.1** Every irreducible infinite-dimensional Hermitian-symmetric affine coadjoint orbit of a connected simple \(L^*\)-group of compact type is isomorphic to one of the following homogeneous space
\begin{enumerate}
\item the Grassmannian \(\text{Gr}_p^{(p)} = \text{U}_2(\mathcal{H})/(\text{U}_2(\mathcal{H}_p) \times \text{U}_2(\mathcal{H}_p^+))\) of \(p\)-dimensional subspaces of \(\mathcal{H}\) with \(\dim(\mathcal{H}_p) < p < +\infty\)
\item the connected component of the restricted Grassmannian \(\text{Gr}^0_{\text{res}} = \text{U}_2(\mathcal{H})/(\text{U}_2(\mathcal{H}_+) \times \text{U}_2(\mathcal{H}_-))\) of the polarized Hilbert space \(\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-\) with \(\dim\mathcal{H}_+ = \dim\mathcal{H}_- = +\infty\)
\item the Grassmannian \(\text{Gr}_1^{(2)} = \text{O}_2^+(\mathcal{H}_\mathbb{R})/(\text{SO}(\mathcal{H}_\mathbb{R}) \times \text{O}_2^+(\mathcal{H}_\mathbb{R}^\perp))\) of oriented 2-planes in \(\mathcal{H}_\mathbb{R}\),
\item the Grassmannian \(\mathcal{Z}(\mathcal{H}) = \text{O}_2^+(\mathcal{H}_\mathbb{R})/\text{U}_2(\mathcal{H})\) of orientation-preserving orthogonal complex structures close to the distinguished complex structure on \(\mathcal{H}\),
\item the Grassmannian \(\mathcal{L}(\mathcal{H}) = \text{Sp}_2(\mathcal{H})/\text{U}_2(\mathcal{H}_+))\) of Lagrangian subspaces close to \(\mathcal{H}_+\).
\end{enumerate}

Since there is a duality between affine coadjoint orbits of compact and non-compact type, Theorem 1.1 gives as a Corollary the classification of every irreducible infinite-dimensional Hermitian-symmetric affine coadjoint orbits of the connected \(L^*\)-groups of non-compact type with simple complexification. Each of these non-compact duals are symmetric Hilbert domains (see Corollary 3.17).
In the finite-dimensional case, every Hermitian-symmetric space of compact type is a coadjoint orbit of its connected group of isometries (see Proposition 8.89 in [3]). In the infinite-dimensional setting, the biggest group of isometries of a given Hermitian-symmetric space is not a Hilbert Lie group in general. For example the restricted unitary group \( U_{\text{res}}(\mathcal{H}) \) (see [13] for its definition) is a Banach Lie group acting by isometries on the restricted Grassmannian. It is a non trivial fact that the unitary Hilbert Lie group \( U_{\text{res}}(\mathcal{H}) \), strictly contained in \( U_{\text{res}}(\mathcal{H}) \), acts transitively on each connected component of the restricted Grassmannian (see Proposition 5.2 in [3]). Theorem 1.1 above compared to the work of W. Kaup ([9], [10], [11]), leads to the following generalization:

**Corollary 1.2** Every Hermitian-symmetric space of compact or non-compact type is an homogeneous space of an Hilbert Lie group. More precisely, every Hermitian-symmetric space of compact or non-compact type is an affine-coadjoint orbit of an \( L^* \)-group.

## 2 Root Theory of complex \( L^* \)-algebra

The root theory of complex \( L^* \)-algebras has been developed by J. R. Schue in [10] and [17]. Let us first recall that an \( L^* \)-algebra \( g \) over \( K \in \{ \mathbb{R}, \mathbb{C} \} \) is a Lie algebra over \( K \), which is also a Hilbert space over \( K \) such that for every element \( x \in g \), there exists \( x^* \in g \) with the following property

\[
\langle [x, y], z \rangle = \langle y, [x^*, z] \rangle, \tag{1}
\]

for every \( y, z \in g \). In the case when \( K = \mathbb{C} \), our convention for the Hermitian product \( \langle \cdot, \cdot \rangle \) is that it is \( \mathbb{C} \)-skew-linear with respect to the first variable, and \( \mathbb{C} \)-linear with respect to the second variable. The first example of \( L^* \)-algebra is a semi-simple finite-dimensional complex Lie algebra \( g_0 \) endowed with an involution \( \sigma \), which defines a compact real form of \( g_0 \). In this example, the involutions \( * \) and \( \sigma \) are related by \( x^* = -\sigma(x) \) and the Hermitian scalar product is given by \( \langle x, y \rangle = B(x^*, y) \), where \( B \) denotes the Killing form of \( g_0 \). An \( L^* \)-algebra is called of compact type if \( x^* = -x \) for every \( x \in g \). It is called of non-compact type otherwise. For a given \( L^* \)-algebra \( g \) the subspace

\[
\mathfrak{k} := \{ x \in g \mid x^* = -x \}
\]

is a real \( L^* \)-algebra of compact type. Thus a complex \( L^* \)-algebra can be thought as an Hilbert Lie algebra together with a distinguished compact real form.

For every subsets \( A \) and \( B \) of an \( L^* \)-algebra \( g \), \( [A, B] \) will denote the closure of the vector space spanned by \( \{ [a, b] \mid a \in A, b \in B \} \). With this notation, an \( L^* \)-algebra is called semi-simple if \( g = [g, g] \), and simple if \( g \) is non-commutative and if every closed ideal of \( g \) is trivial. Every \( L^* \)-algebra can be decomposed into an orthogonal sum of its center and a semi-simple closed ideal (see [10], 2.2.13.). A Cartan subalgebra of a complex semi-simple \( L^* \)-algebra \( g^C \) is defined as a maximal Abelian \(*\)-stable subalgebra of \( g^C \). Note that the condition of being \(*\)-stable is added in comparison to the finite-dimensional setting, hence a Cartan subalgebra may not be maximal in the set of Abelian subalgebras. It is noteworthy that a Cartan subalgebra of an \( L^* \)-algebra is in fact maximal Abelian (see [17], 1.1). Remark that a finite-dimensional Cartan subalgebra of a complex semi-simple Lie algebra \( g^C \) (for the usual definition) is contained in a compact real form of \( g^C \), thus is also a Cartan subalgebra of the corresponding finite-dimensional \( L^* \)-algebra. The existence of Cartan subalgebras of \( L^* \)-algebra is guarantied by Zorn’s Lemma. Every semi-simple \( L^* \)-algebra is an Hilbert sum of closed \(*\)-stable simple ideals (see Theorem 1 in [10] for the complex case and Theorem 1 in [2] for the real case).

In the sequel, \( g^C \) will denote a semi-simple complex \( L^* \)-algebra and \( \mathfrak{h}^C \) a Cartan subalgebra of \( g^C \). A root of \( g^C \) with respect to \( \mathfrak{h}^C \) is defined, as in the finite dimensional case, as an element \( \alpha \) in the dual of \( \mathfrak{h}^C \) such that the corresponding “eigenspace”

\[
V_{\alpha} := \{ v \in g^C \mid \forall h \in \mathfrak{h}^C, [h, v] = \alpha(h)v \}
\]

is non-empty. In the following the set of non-zero roots with respect to a given Cartan subalgebra will be denoted by \( \mathcal{R} \). Let us remark that a root has operator norm less than 1 and that for a non-zero root \( \alpha \), the vector space \( V_{\alpha} \) is one-dimensional (see [10]). The Jacobi identity implies that

\[
[V_{\alpha}, V_{\beta}] \subset V_{\alpha+\beta}. \tag{2}
\]
By relation (1), $V^*_\alpha = V_{-\alpha}$. The main achievement in [17] is to prove that a semi-simple complex $L^*$-algebra $\mathfrak{g}^C$ admits a Cartan decomposition with respect to a given Cartan subalgebra $\mathfrak{h}^C$ in the sense that $\mathfrak{g}^C$ is the Hilbert sum

$$\mathfrak{g}^C = \mathfrak{h}^C \oplus \sum_{\alpha \in \mathcal{R}} V_\alpha. \quad (3)$$

Let us remark that in a separable $L^*$-algebra, the set of root is countable or finite.

By Zorn’s Lemma, one can decompose the set $\mathcal{R}$ of non-zero roots into two disjoint subsets $\mathcal{R}_+$ and $\mathcal{R}_-$ such that $\alpha \in \mathcal{R}_+ \Leftrightarrow -\alpha \in \mathcal{R}_-$. Such a decomposition defines a strict partial ordering on $\mathcal{R}$ by

$$\alpha > \beta \Leftrightarrow \alpha - \beta > 0,$$

where we write $\alpha - \beta > 0$ for $\alpha - \beta \in \mathcal{R}_+$. The elements in $\mathcal{R}_+$ will be called positive roots. In the sequel, a decomposition $\mathcal{R} = \mathcal{R}_+ \cup \mathcal{R}_-$ as before and the induced ordering on the set of non-zero roots will be identified.

For every positive root $\alpha$, one can choose $e_\alpha \in V_\alpha$ such that $\|e_\alpha\| = 1$. Then $e^*_\alpha \in V_{-\alpha}$ and $\|e^*_\alpha\| = 1$. This choice made, we define $e_\alpha := e^*_\alpha$ for $\alpha \in \mathcal{R}_-$, in order to have, for every $\alpha \in \mathcal{R}$, the following relation $e^*_\alpha = -e_\alpha$. By (3), the set $\{e_\alpha \mid \alpha \in \mathcal{R}\}$ is an Hilbert basis of $(\mathfrak{h}^C)^\perp$, and by (2), $[e_\alpha, e^*_\alpha]$ belongs to $\mathfrak{h}^C$. We define the following elements in the Cartan subalgebra $\mathfrak{h}^C$:

$$h_\alpha := [e_\alpha, e^*_\alpha], \quad (4)$$

for $\alpha \in \mathcal{R}_+$. A positive root is called simple if it can not be written as the sum of two positive roots. The set of simple roots will be denoted by $\mathcal{S}$. A subset $\mathcal{N}$ of the set of non-zero roots $\mathcal{R}$ is called a root system, if it satisfies the following conditions:

1. $\alpha \in \mathcal{N} \Rightarrow -\alpha \notin \mathcal{N}$,
2. $(\alpha, \beta \in \mathcal{N} \text{ and } \alpha + \beta \in \mathcal{R}) \Rightarrow \alpha + \beta \in \mathcal{N}$.

A subset $\mathcal{N} \subset \mathcal{R}$ is called indecomposable if it can not be written as the union of two orthogonal non-empty subsets. As in the classical theory, one has the following facts. The set $\mathcal{R}$ of non-zero roots of a simple $L^*$-algebra is indecomposable. If $\mathcal{F}$ is an indecomposable subset of the set of non-zero roots $\mathcal{R}$, then it generates a root system $\mathcal{N}_\mathcal{F}$, which is again indecomposable. The simple $L^*$-algebra generated by $\{e_\alpha \mid \alpha \in \mathcal{N}_\mathcal{F}\}$ will be denoted by $\mathfrak{g}(\mathcal{N}_\mathcal{F})$.

For the classification of Hermitian-symmetric affine coadjoint orbits given in next section, we will need the following results. They were proved by J.R. Schue in [16] in order to classify the complex simple infinite-dimensional $L^*$-algebras.

**Proposition 2.1** ([16]) For every finite subset $\mathcal{F}$ of the set of non-zero roots $\mathcal{R}$ of a simple $L^*$-algebra, there exists a finite indecomposable system of non-zero roots containing $\mathcal{F}$.

**Theorem 2.2** ([16], 3.2) Let $\mathfrak{g}^C$ be a simple complex separable $L^*$-algebra and $\mathcal{R} = \{\alpha_i \mid i \in \mathbb{N} \setminus \{0\}\}$ the set of non-zero roots with respect to a given Cartan subalgebra of $\mathfrak{g}^C$. For every $n \in \mathbb{N} \setminus \{0\}$, set $\mathcal{F}_n := \{\alpha_1, \ldots, \alpha_n\}$. Then there exists a sequence $\{\mathcal{N}_n\}_{n \in \mathbb{N} \setminus \{0\}}$ of finite subsets of $\mathcal{R}$ such that

1. $\mathcal{F}_n \subset \mathcal{N}_n \subset \mathcal{N}_{n+1}$;
2. $\mathcal{N}_n$ is an indecomposable root system;
3. $\mathcal{R} = \bigcup_{n \in \mathbb{N} \setminus \{0\}} \mathcal{N}_n$.
4. the simple subalgebras $\mathfrak{g}(\mathcal{N}_n)$ generated by $\mathcal{N}_n$ form a strictly increasing sequence with

$$\mathfrak{g}^C = \bigcup_{n \in \mathbb{N} \setminus \{0\}} \mathfrak{g}(\mathcal{N}_n);$$

5. The simple complex finite-dimensional algebras $\mathfrak{g}(\mathcal{N}_n)$ are of the same Cartan type A, B, C or D.

**Proposition 2.3** ([16], 3.2) Given a sequence $\{\mathcal{N}_n\}_{n \in \mathbb{N} \setminus \{0\}}$ as in the previous Theorem, there exists a total ordering on the vector space generated by the set of roots such that:

1. $\alpha > 0 \Rightarrow -\alpha < 0$;
2. $\alpha > 0, \beta > 0 \Rightarrow \alpha + \beta > 0$;
3. $\alpha > 0$ and $\alpha \notin \mathcal{N}_n$ then $\alpha > \beta$ for all $\beta \in \mathcal{N}_n$.
4. the induced ordering on $N_n$ is a lexicographical ordering with respect to a basis of roots.

Proposition 2.4 ([16], 3.3) Let $S$ be the set of simple roots of $g^C$ with respect to the ordering defined in the previous Proposition. The following assertions hold:

1. $S \cap N_n$ is a complete system of simple roots of the finite-dimensional algebra $g(N_n)$, i.e. every positive root $\alpha$ of $N_n$ can be written as a linear combination of elements in $S \cap N_n$ with non-negative integral coefficients;

2. If $\alpha$ and $\beta$ belong to $S$, $\alpha - \beta$ is a root if and only if $\alpha = \beta$;

3. the elements in $S$ are linearly independent on the reals and every positive root $\alpha \in R_+$ is a linear combination of elements in $S$ with non-negative integral coefficients which are all zero except for a finite number of them.

3 Classification of irreducible Hermitian-symmetric affine coadjoint orbits

The classification of finite-dimensional Hermitian-symmetric coadjoint orbits using the notion of roots of non-compact type has been carried out by J. A. Wolf in [26]. In this section we use the same technique to classify Hermitian-symmetric affine coadjoint orbits of connected simple $L^*$-groups of compact type, and then deduce a classification result for Hermitian-symmetric affine coadjoint orbits of non-compact type. Affine coadjoint orbits have been introduced in particular by K.-H. Neeb in [12]. Given an $L^*$-group $G$ with Lie algebra $g$, an affine coadjoint action of $G$ is a continuous homomorphism $Ad^*_g$ from $G$ into the affine group of transformations $Aff(g^e) = GL(g^e) \ltimes g^e$ of the continuous dual $g^e$ of $g$ such that $Ad^*_g(g) = (Ad^*(g), \theta(g))$, $g \in G$, where $Ad^*$ is the usual linear coadjoint action. By derivation at the unit element $1 \in G$, it gives an affine coadjoint action of $g$ on $g^e$, i.e. an continuous homomorphism $ad^*_g : g \to aff(g^e) = gl(g^e) \ltimes g^e$ such that $ad^*_g(x) = (ad^*(x), d\theta(x))$, $x \in g$. If $d\theta(x) = \omega(x, \cdot)$ for a continuous 2-cocycle $\omega \in Z^2(gl)$, then the orbits of the affine coadjoint action of $G$ defined by $\theta$ are naturally symplectic (see Theorem 2.4 in [12]) with symplectic form:

$$\Omega_\theta(ad^*_g(x)(\beta), ad^*_g(y)(\beta)) = \beta([x, y]) - \omega(x, y),$$

where $x, y \in g$ and $\beta \in g^e$.

Definition 3.1 An affine coadjoint orbit $O$ of $G$ is called Hermitian-symmetric if it has a $G$-invariant structure of Hermitian-symmetric space.

Remark 3.2 A Hermitian-symmetric affine coadjoint orbit $O$ is in particular (locally-)symmetric, i.e. the Lie algebra $g$ of $G$ splits into $g = \mathfrak{t} \oplus \mathfrak{m}$, where $\mathfrak{t}$ is the Lie algebra of the isotropy group $K$ fixing a given point $o \in O$ and $\mathfrak{m}$ is a $K$-invariant complement of $\mathfrak{t}$ in $g$ such that $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{t}$.

(One also says that $(g, \mathfrak{t})$ is a symmetric pair). Consequently the Levi-Civita connection of an affine coadjoint Hermitian-symmetric orbit $O$ is the homogeneous connection and every $G$-invariant tensor is parallel (see e.g. Proposition 1.9 in [21] and its proof). In particular, $O$ is Kähler since the complex structure is $G$-invariant hence parallel.

Since we are interested in Hermitian-symmetric orbits, which by the previous remark are in particular symplectic, we will consider only affine coadjoint actions such that $d\theta_{\mathfrak{e}}(x) = \omega(x, \cdot)$ for some continuous 2-cocycle $\omega \in Z^2(gl)$. Since the bracket $\langle \cdot, \cdot \rangle$ on the $L^*$-algebra $g$ is non-degenerate, there exists an operator $D$ on $g$ such that:

$$\omega(x, y) = \langle x^*, Dy \rangle,$$

for $x, y \in g$. Since $\omega$ is a cocycle, $D$ is a derivation of the Lie algebra $g$. By Remark 2.5(d) in [12], it is sufficient to consider affine coadjoint orbits of $0 \in g^e$.

In the sequel, $g$ will denote an infinite-dimensional separable simple $L^*$-algebra of compact type and $G_0$ a connected simple $L^*$-group with the Lie algebra $g$. According to [1], [8] or [23], $g$ can be realized as a subalgebra of the $L^*$-algebra $gl_2(H)$ consisting of Hilbert-Schmidt operators on a separable complex
Hilbert space $\mathcal{H}$. We may therefore assume $\mathfrak{g} \subseteq \mathfrak{gl}_2(\mathcal{H})$. By the duality $\mathfrak{g}' = \mathfrak{g}$ given by the trace, we can identify affine adjoint and affine coadjoint orbits of $G_0$.

Suppose that $\mathcal{O}$ is a Hermitian-symmetric affine adjoint orbit of $G_0$ for an action as above. Then it is in particular strongly Kähler and by Theorem 4.4 in [12], there exists $D \in B(\mathcal{H})$ satisfying $D^* = -D$ such that for every $x$ in $\mathfrak{g}$, $\mathbb{D}x = [D, x]$, as well as a Cartan subalgebra $\mathfrak{h}^C$ of $\mathfrak{g}^C$ which is contained in $\ker \mathbb{D}$. To emphasize the relation between the orbit and the bounded operator $D$, we will often write $\mathcal{O} = \mathcal{O}_D$.

Now let $G$ be the group of operators on $\mathcal{H}$ generated by the exponentials of operators in $\mathfrak{g}$, and $G^C$ be the group of operators on $\mathcal{H}$ generated by the exponentials of operators in $\mathfrak{g}^C := \mathfrak{g} \oplus i \mathfrak{g} \subseteq \mathfrak{gl}_2(\mathcal{H})$. That is, $G$ (resp. $G^C$) is the connected 1-component of the classical Hilbert-Lie group whose Lie algebra is $\mathfrak{g}$ (resp. $\mathfrak{g}^C$). Since the center of $G$ reduces to $\{1\}$, it follows that the corresponding adjoint action

$$\text{Ad}_G : G \to \text{Ad}(\mathfrak{g})$$

is an isomorphism of Lie groups, where $\text{Ad}(\mathfrak{g})$ is the adjoint group of the Banach-Lie algebra $\mathfrak{g}$. Recall that the automorphism group $\text{Aut}(\mathfrak{g})$ of $\mathfrak{g}$ has the natural structure of a Banach-Lie group whose Lie algebra consists of all derivations of $\mathfrak{g}$, and $\text{Ad}(\mathfrak{g})$ is the connected (integral) subgroup of $\text{Aut}(\mathfrak{g})$ corresponding to the Lie subalgebra of inner derivations of $\mathfrak{g}$.

On the other hand we have the adjoint action

$$\text{Ad}_{G_0} : G_0 \to \text{Ad}(\mathfrak{g}).$$

This Lie group homomorphism is onto and its kernel is equal to the center $Z_{G_0}$ of $G_0$. Since $G_0$ is a simple Lie group, it follows that $Z_{G_0}$ is a discrete subgroup. Thus we get a covering homomorphism

$$\pi = (\text{Ad}_G)^{-1} \circ \text{Ad}_{G_0} : G_0 \to G \hookrightarrow B(\mathcal{H})$$

whose fiber over $1 \in G$ is precisely the center of $G_0$, and for every $D \in B(\mathcal{H})$ the diagram

$$\begin{array}{ccc}
G_0 & \xrightarrow{\pi} & G \\
\text{Ad}_{G_0, \omega_D} \downarrow & & \downarrow \text{Ad}_{G, \omega_D} \\
B(\mathcal{H}) & \xrightarrow{\text{id}} & B(\mathcal{H})
\end{array}$$

is commutative. Here the vertical arrows stand for the corresponding affine coadjoint actions :

$$\text{Ad}_{G_0, \omega_D}(g)X = \text{Ad}_{G_0}(g)X + \pi(g)D\pi(g)^{-1} - D$$

for every $g \in G_0$ and $X \in \mathfrak{g}$, and

$$\text{Ad}_{G, \omega_D}(g)X = \text{Ad}_G(g)X + gDg^{-1} - D$$

for every $g \in G$ and $X \in \mathfrak{g}$.

Since $\pi : G_0 \to G$ is a covering map, it follows by the above commutative diagram that the affine coadjoint orbits of $G_0$ and the ones of $G$ are the same. Thus it suffices to investigate the affine coadjoint orbits of $G$.

Abusing slightly the notation, we will sometimes denote $\mathbb{D}$ by $\text{ad}(D)$. An alternative definition of $\mathcal{O}_D$ is

$$\mathcal{O}_D = \{gDg^{-1} - D \mid g \in G\},$$

and the affine adjoint action of $G$ on $\mathfrak{g}$ is given by

$$g \cdot a = \text{Ad}_G(g)(a) + gDg^{-1} - D$$

where $g \in G$ and $a \in \mathfrak{g}$. The subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ which fixes $0$ is

$$\mathfrak{t} := \{x \in \mathfrak{g} \mid [D, x] = 0\}.$$ 

It is an $L^*$-subalgebra of $\mathfrak{g}$. Let $K$ be the isotropy subgroup of $G$ that fixes $0$. Since $G$ and $\mathcal{O}$ are connected, $K$ is connected. We will denote by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{g}$, which is in particular $K$-invariant.
From the discussion above it follows that it suffices to consider Hermitian-symmetric orbits \( O_D \) of the connected 1-component \( G \) of the classical Hilbert-Lie group whose Lie algebra is the infinite-dimensional separable simple \( L^* \)-algebra of compact type \( g \subseteq \mathfrak{gl}_2(\mathcal{H}) \). Such an orbit is said to be of compact type and admits a dual of non-compact type in the following sense. If \( g = \mathfrak{t} \oplus \mathfrak{m} \) is the decomposition of the Lie algebra of \( G \) as above, then \( g^{\mathbb{C}} = \mathfrak{t} \oplus \mathfrak{m} \) is a real \( L^* \)-subalgebra of the complexification \( g^\mathbb{C} \) of \( g \). Since \( g^\mathbb{C} \) is supposed to be a subalgebra of \( \mathfrak{gl}_2(\mathcal{H}) \), one can define the connected \( L^* \)-group \( G^{\mathbb{C}} \) generated by the exponentials of operators in \( g^{\mathbb{C}} \). Then the dual of \( O_D \) is defined as the affine coadjoint orbit of \( G^{\mathbb{C}} \) for the derivation \( D \). Let \( \mathfrak{t}^\mathbb{C} \) and \( \mathfrak{m}^\mathbb{C} \) denote the complexifications of \( \mathfrak{t} \) and \( \mathfrak{m} \) respectively. Note that \( g^\mathbb{C} \) is the orthogonal sum of \( \mathfrak{t}^\mathbb{C} \) and \( \mathfrak{m}^\mathbb{C} \) with respect to the Hermitian product of the \( L^* \)-algebra \( g^\mathbb{C} \).

**Proposition 3.3** Let \( \mathfrak{h}^\mathbb{C} \) be a Cartan subalgebra of \( g^\mathbb{C} \) that is contained in \( \ker \text{ad} D \), and let

\[
\mathfrak{h}^\mathbb{C} = \mathfrak{h}^\mathbb{C} \oplus \sum_{\alpha \in \mathcal{R}} \mathfrak{V}_\alpha
\]

be the associated Cartan decomposition of \( g^\mathbb{C} \), where \( \mathcal{R} \) denotes the set of non-zero roots with respect to \( \mathfrak{h}^\mathbb{C} \). Suppose that \( O_D \) is Hermitian-symmetric. Then there exists two subsets \( \mathcal{A} \) and \( \mathcal{B} \) of \( \mathcal{R} \) such that \( \mathcal{A} \cup \mathcal{B} = \mathcal{R} \) and

\[
\mathfrak{t}^\mathbb{C} = \mathfrak{h}^\mathbb{C} \oplus \sum_{\alpha \in \mathcal{A}} \mathfrak{V}_\alpha, \quad \mathfrak{m}^\mathbb{C} = \sum_{\alpha \in \mathcal{B}} \mathfrak{V}_\alpha.
\]

**Proof of Proposition 3.3:**

Since \( O_D \) is (locally-)symmetric, one has \( g^\mathbb{C} = \mathfrak{t}^\mathbb{C} \oplus \mathfrak{m}^\mathbb{C} \) with

\[
[\mathfrak{t}^\mathbb{C}, \mathfrak{t}^\mathbb{C}] \subset \mathfrak{t}^\mathbb{C}; \quad [\mathfrak{t}^\mathbb{C}, \mathfrak{m}^\mathbb{C}] \subset \mathfrak{m}^\mathbb{C}; \quad [\mathfrak{m}^\mathbb{C}, \mathfrak{m}^\mathbb{C}] \subset \mathfrak{t}^\mathbb{C}.
\]

Let \( v \) be a non-zero vector in \( \mathfrak{V}_\alpha \), and \( v = v_0 + v_1 \) his decomposition with respect to the direct sum \( g^\mathbb{C} = \mathfrak{t}^\mathbb{C} \oplus \mathfrak{m}^\mathbb{C} \). For every \( h \in \mathfrak{h}^\mathbb{C} \), one has

\[
[h, v] = [h, v_0 + v_1] = \alpha(h)(v_0 + v_1) = \alpha(h)v_0 + \alpha(h)v_1 = [h, v_0] + [h, v_1].
\]

Since \( [\mathfrak{h}^\mathbb{C}, \mathfrak{t}^\mathbb{C}] \subset \mathfrak{t}^\mathbb{C} \) and \( [\mathfrak{h}^\mathbb{C}, \mathfrak{m}^\mathbb{C}] \subset \mathfrak{m}^\mathbb{C} \), it follows that

\[
[h, v_0] = \alpha(h)v_0 \quad \text{et} \quad [h, v_1] = \alpha(h)v_1.
\]

But \( \mathfrak{V}_\alpha \) is one-dimensional, hence either \( v_0 = 0 \), or \( v_1 = 0 \). Consequently \( \mathfrak{V}_\alpha \) is contained either in \( \mathfrak{t}^\mathbb{C} \) or in \( \mathfrak{m}^\mathbb{C} \).

**Proposition 3.4** For every \( \alpha \in \mathcal{R} \), there exists a constant \( c_\alpha \in \mathbb{R} \) such that \( [D, e_\alpha] = ic_\alpha e_\alpha \). Moreover \( c_{-\alpha} = -c_\alpha \).

**Proof of Proposition 3.4:**

For every \( \alpha \in \mathcal{R} \) and every \( h \in \mathfrak{h}^\mathbb{C} \), one has

\[
[h, [D, e_\alpha]] = [[h, D], e_\alpha] + [D, [h, e_\alpha]] = \alpha(h) [D, e_\alpha].
\]

The space \( \mathfrak{V}_\alpha \) being one-dimensional, it follows that \( [D, e_\alpha] \) is proportional to \( e_\alpha \). Since \( D \) satisfies \( D^* = -D \), one has, for every \( \alpha \in \mathcal{R} \), the following relation

\[
\langle [D, e_\alpha], e_\alpha \rangle = -\langle e_\alpha, [D, e_\alpha] \rangle = -\langle [D, e_\alpha], e_\alpha \rangle.
\]

Thus there exists a real constant \( c_\alpha \) such that

\[
[D, e_\alpha] = ic_\alpha e_\alpha.
\]

On the other hand,

\[
[D, e_\alpha]^* = [e_\alpha^*, D]^* = -[e_\alpha^*, D] = [D, e_\alpha^*].
\]

Whence

\[
\langle [D, e_\alpha^*], e_\alpha^* \rangle = \langle e_\alpha, [D, e_\alpha] \rangle = \langle e_\alpha, [D, e_\alpha] \rangle = ic_\alpha.
\]

Consequently \( [D, e_\alpha^*] = -ic_\alpha e_\alpha^* \).
Remark 3.5 Let us denote by \( m_+ \) (resp. \( m_- \)) the closed subspace of \( m^C \) generated by the \( e_\alpha \)'s, where \( \alpha \) runs over the set of roots for which \( c_\alpha > 0 \) (resp. \( c_\alpha < 0 \)). Let \( B_+ \) (resp. \( B_- \)) be the set of roots \( \beta \) in \( B \) such that \( V_\beta \in m_+ \) (resp. \( V_\beta \in m_- \)).

Definition 3.6 The affine coadjoint orbit \( O_D \) is called (isotropy-)irreducible if \( m \) is a non-zero irreducible \( K \)-module.

Proposition 3.7 If the affine adjoint orbit \( O_D \) is irreducible, then \( m_+ \) and \( m_- \) are irreducible \( \text{Ad}(K) \)-modules, and there exists a constant \( c > 0 \) such that \( \text{ad}(D)|_{m_+} = ic \text{id}|_{m_+} \) and \( \text{ad}(D)|_{m_-} = -ic \text{id}|_{m_-} \). In particular, the spectrum of \( \text{ad}(D) \) is \( \{0, ic, -ic\} \), hence \( D \) admits exactly two distinct eigenvalues.

\[ \square \text{ Proof of Proposition 3.7.} \]
For every \( k \in \mathfrak{k} \) and every \( e_\alpha \in m_\pm \), one has
\[
[D, [k, e_\alpha]] = [[D, k], e_\alpha] + [k, [D, e_\alpha]] = ic[e, e_\alpha].
\]
It follows that \( [\mathfrak{t}, m_\pm] \subset m_\pm \) and that \( m_\pm \) is stable under the adjoint action of \( K \). Let us suppose that \( m_+ \) decomposes into a sum of two non-zero \( \text{Ad}(K) \)-modules \( m_1 \) and \( m_2 \). Then
\[
m_- = m_1^+ \oplus m_2^+,
\]
and it follows that \( m \) decomposes also into the sum of two non-zero \( \text{Ad}(K) \)-modules, namely \( \mathfrak{g} \cap (m_1^+ \oplus m_2^+) \) and \( \mathfrak{g} \cap (m_2^+ \oplus m_1^+) \). The orbit \( O_D \) being irreducible, \( m \) is an irreducible \( \text{Ad}(K) \)-module and this leads to a contradiction. So the irreducibility of \( m_\pm \) is proved. Let \( e_\alpha \) be an element in \( m_+ \) and set \( c = e_\alpha : \)
\[
[D, e_\alpha] = ic e_\alpha.
\]
The kernel \( \ker(D - ic) \) being an \( \text{Ad}(K) \)-module of \( m_+ \), one has \( \text{ad}(D)|_{m_+} = ic \text{id}|_{m_+} \). The relation \( c_{-\alpha} = -c_\alpha \) implies that \( \text{ad}(D)|_{m_-} = -ic \text{id}|_{m_-} \). \( \square \)

Definition 3.8 Given an ordering on the set of non-zero roots \( R \) of \( g^C \), a simple root \( \phi \) is called of non-compact type (see [26]) if every root \( \alpha \in R \) is of the form
\[
\alpha = \pm \sum_{\Psi \in S \setminus \{\phi\}} a_\Psi \Psi, \text{ where } a_\Psi \geq 0 \text{ for all } \Psi \in S \setminus \{\phi\},
\]
or of the form
\[
\alpha = \pm \left( \phi + \sum_{\Psi \in S \setminus \{\phi\}} a_\Psi \Psi \right), \text{ where } a_\Psi \geq 0 \text{ for all } \Psi \in S \setminus \{\phi\}.
\]

Lemma 3.9 Let \( O_D \) be a Hermitian-symmetric affine adjoint irreducible orbit of a simple \( L^* \)-algebra \( \mathfrak{g}, \mathfrak{h}^C \) be a Cartan subalgebra of \( \mathfrak{g}^C \) contained in \( \ker \text{ad}D \), and
\[
\mathfrak{g}^C = \mathfrak{h}^C \oplus \sum_{\alpha \in A} V_\alpha \oplus \sum_{\beta \in B} V_\beta
\]
be the associated Cartan decomposition of \( g^C \) with
\[
\mathfrak{t}^C = \mathfrak{h}^C \oplus \sum_{\alpha \in A} V_\alpha, \text{ and } m^C = \sum_{\beta \in B} V_\beta.
\]
For every ordering \( R = R_+ \cup R_- \) on the set of roots, there exists a unique simple root \( \phi \) belonging to \( R \).

\[ \Delta \text{ Proof of Lemma 3.9:} \]
Let \( \{\phi_i, \Psi_j\}_{i \in I, j \in J} \) be the set of simple roots with \( \phi_i \) in \( B \) and \( \Psi_j \) in \( A \). Let us suppose that \( I \) is empty. The relation \( [\mathfrak{t}^C, \mathfrak{t}^C] \subset \mathfrak{t}^C \) implies that every positive root belongs to \( A \) and consequently \( m = \{0\} \), which contradicts the hypothesis that \( m \) is a non-zero irreducible \( \text{Ad}(K) \)-module. Let \( \phi \) be a simple root in \( B \). The closed vector space spanned by the adjoint action of \( \mathfrak{t} \) on \( e_\phi \) is a non-zero irreducible \( \text{Ad}(K) \)-submodule of \( m^C \). It follows that \( \phi \) is necessarily unique. \( \Delta \)
Lemma 3.10 Under the hypothesis of Lemma 3.9, there exists an increasing sequence of finite indecomposable root systems \( N_n \) such that

1. \( R = \bigcup_{n \in \mathbb{N} \setminus \{0\}} N_n \);  
2. all the finite-dimensional subalgebras \( \mathfrak{g}(N_n) \) generated by \( N_n \) belong to the same type \( A, B, C, \) or \( D \) and \( \mathfrak{g}^c \) is the closure of the union of the subalgebras \( \mathfrak{g}(N_n) \);  
3. \( \phi \) is a simple root of non-compact type for each subalgebra \( \mathfrak{g}(N_n) \) with respect to the ordering on the roots of \( \mathfrak{g}(N_n) \) induced by the ordering on \( R \) defined in Proposition 2.3.

\( \Delta \) Proof of Lemma 3.10

Let \( \{\alpha_1, \ldots, \alpha_n, \ldots\} \) be a numbering of the roots in \( A \). Set \( F_n = \{\alpha_1, \ldots, \alpha_n\} \). Let us construct by induction an increasing sequence of finite indecomposable root systems \( N_n \) as follows. By Proposition 2.1 there exists a finite indecomposable root system \( N_1 \) containing \( \{\phi\} \cup F_1 \). Suppose that \( N_{n-1} \) is constructed, then there exists a finite indecomposable root system \( N_n \) containing \( F_n \cup N_{n-1} \). Since every root in \( B \) is the sum of \( \phi \) and roots in \( A \), \( R = \bigcup_{n \in \mathbb{N} \setminus \{0\}} N_n \). The sequence of finite-dimensional simple subalgebras \( \mathfrak{g}(N_n) \) generated by the root systems \( N_n \) is increasing and such that \( \mathfrak{g}^c = \bigcup_{n \in \mathbb{N} \setminus \{0\}} \mathfrak{g}(N_n) \). Since there exists only 9 types of finite-dimensional simple algebras, at least one type occurs an infinite number of times. Since \( \mathfrak{g}^c \) is finite-dimensional and since only the types \( A, B, C, \) or \( D \) corresponds to algebras of arbitrary dimension, at least one of the types \( A, B, C, \) or \( D \) occurs an infinite number of times. It follows that there exists a subsequence \( N_{n_k} \) of \( N_n \) such that all the subalgebras \( \mathfrak{g}(N_{n_k}) \) are of the same type \( A, B, C, \) or \( D \). Let \( S_{n_k} \) be the set of simple roots of \( \mathfrak{g}(N_{n_k}) \) with respect to the ordering induced by the ordering on \( R \) defined in Proposition 2.3. By Proposition 2.4 \( S_{n_k} = S \cap \mathfrak{g}(N_{n_k}) \), where \( S \) is the set of simple roots of \( \mathfrak{g}^c \). For every positive root \( \gamma \) in \( N_{n_k} \), there exists a finite sequence \( \{\gamma_i, i = 1, \ldots, k\} \) of roots in \( S_{n_k} \) such that

\[
\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_k,
\]

and such that the partial sums \( \gamma_1 + \cdots + \gamma_j, 1 \leq j \leq k \) are roots (see [5]). Hence the vector space \( V_{\gamma} \) is generated by

\[
v = [e_{\gamma_1}, [e_{\gamma_1}, e_{\gamma_2}], \ldots, [e_{\gamma_1}, e_{\gamma_k}]]\].

The orbit \( O_D \) being irreducible, \( [D, e_{\phi}] = e_{\phi} i e_{\phi} \) with \( e_{\phi} = \pm 1 \) if \( \phi \subset \mathfrak{m}_+ \) (resp. \( \mathfrak{m}_- \)). Whence

\[
[D, v] = \text{card} \{(i, \gamma_i = \phi)\} e_{\phi} i e_{\phi} v.
\]

Since \( \text{ad}(D) \) preserves \( \mathfrak{m}_+ \) and \( \mathfrak{m}_- \), it follows that for \( \gamma \in A \cap \mathfrak{r}_+ \), \( \text{card} \{(i, \gamma_i = \phi)\} = 0 \) and for \( \gamma \in B \cap \mathfrak{r}_+ , \text{card} \{(i, \gamma_i = \phi)\} = 1 \). Consequently \( \phi \) is of non-compact type. \( \Delta \)

Proposition 3.11 Let \( O = G/K \) be a Hermitian-symmetric irreducible affine coadjoint orbit of an \( L^+ \)-group \( G \) of compact type, and \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \) the associated decomposition of the Lie algebra \( \mathfrak{g} \) of \( G \), where \( \mathfrak{k} \) is the Lie algebra of the isotropy group \( K \). Then there exists an increasing sequence of finite-dimensional subalgebras \( \mathfrak{g}_n \) of \( \mathfrak{g} \) of the same type \( A, B, C, \) or \( D \), and an increasing sequence of subalgebras \( \mathfrak{t}_n \) of \( \mathfrak{k} \) such that

1. \( \mathfrak{g} = \bigcup_{n \in \mathbb{N} \setminus \{0\}} \mathfrak{g}_n \)  
2. \( \mathfrak{t} = \bigcup_{n \in \mathbb{N} \setminus \{0\}} \mathfrak{t}_n \)  
3. for every \( n \in \mathbb{N} \setminus \{0\} \), the orthogonal complement \( \mathfrak{m}_n \) of \( \mathfrak{t}_n \) in \( \mathfrak{g}_n \) satisfies

\[
[\mathfrak{t}_n, \mathfrak{m}_n] \subset \mathfrak{m}_n \quad \text{and} \quad [\mathfrak{m}_n, \mathfrak{m}_n] \subset \mathfrak{t}_n,
\]

hence \( (\mathfrak{g}_n, \mathfrak{t}_n) \) is a symmetric pair.

\( \square \) Proof of Proposition 3.11:

This is a direct consequence of Lemma 3.10, with \( \mathfrak{g}_n = \mathfrak{g} \cap \mathfrak{g}(N_n) \) and \( \mathfrak{t}_n = \mathfrak{k} \cap \mathfrak{g}(N_n) \). \( \square \)

From the discussion above it follows that the classification of Hermitian-symmetric irreducible affine coadjoint orbits of \( L^+ \)-groups of compact or non-compact type can be deduced from the knowledge of the simple roots of non-compact type of finite-dimensional simple complex algebras (see the proof of Theorem 1.1 below). A simple root of a simple finite-dimensional complex algebra is of non-compact
Table 1: Simple roots of non-compact type in the simple finite-dimensional Lie algebras of type A, B, C and D.

Type A: $\alpha_2 \rightarrow \cdots \rightarrow \alpha_{n-3} \rightarrow \alpha_{n-1} \rightarrow \alpha_n$

Every root $\alpha_i$ is of non-compact type.

Type B: $\alpha_2 \rightarrow \cdots \rightarrow \alpha_{n-3} \rightarrow \alpha_{n-1} \rightarrow \alpha_n$

Only the root $\alpha_n$ is of non-compact type.

Type C: $\alpha_2 \rightarrow \cdots \rightarrow \alpha_{n-3} \rightarrow \alpha_{n-1} \rightarrow \alpha_n$

Only the root $\alpha_1$ is of non-compact type.

Type D: $\alpha_2 \rightarrow \cdots \rightarrow \alpha_{n-3} \rightarrow \alpha_{n-1}$

Only the roots $\alpha_1$, $\alpha_2$ and $\alpha_n$ are of non-compact type.

type if and only if it appears with the coefficient +1 in the expression of the greatest root. We recall the list of simple roots of non-compact type in the finite-dimensional Lie algebras of type A, B, C, or D in tabular 1 (see 6 or 26).

■ Proof of Theorem 3.11:

By Lemma 3.9, there exists a unique simple root $\phi$ in $B$ regardless to the ordering chosen on the set of non-zero roots $R$. By Lemma 3.10 part 3, $\phi$ is a simple root of non-compact type for each finite-dimensional subalgebras $g(N_\alpha)$ constructed in Lemma 3.10 part 2, when $R$ is endowed with the particular ordering constructed in Proposition 2.3. For this ordering, simple roots of $g(N_\alpha)$ are simple roots of $g^C$. It follows that the set of possible roots $\phi$ can be deduced from tabular 1. Such a root $\phi$ defines a unique symmetric pair of compact type $(g, \mathfrak{t})$ with $g = \{a \in g^C \mid a + a^* = 0\}$, and $\mathfrak{t} = \{a \in g^C \mid a + a^* = 0\}$, where $\mathfrak{t}^C$ is the $L^\ast$-algebra whose Dynkin diagram is obtained by removing $\phi$ from the Dynkin diagram of $g^C$ ($\mathfrak{t}^C$ is the orthogonal complement of the vector space generated by the $e_{\phi + a} \ast$’s). One sees immediately that such a root $\phi$ defines also a unique symmetric pair of non-compact type, the dual of $(g, \mathfrak{t})$, namely $(g^\ast, \mathfrak{t})$, where $g^\ast \mathfrak{t} := \mathfrak{t} \oplus \mathfrak{m}$ and $\mathfrak{m}$ denotes the orthogonal complement to $\mathfrak{t}$ in $g$.

Example 3.12 The Grassmannian $Gr^{(p)} = U_2(\mathcal{H})/(U_2(\mathcal{H}_p) \times U_2(\mathcal{H}_p^\perp))$ of $p$-dimensional subspaces of $\mathcal{H}$ with $\dim(\mathcal{H}_p) = p < +\infty$, is the affine adjoint orbit of $U_2(\mathcal{H})$ for the derivations defined by the bounded operators $D_{k,l}^{(p)} = ikp_{\mathcal{H}_p} - ilp_{\mathcal{H}_p^\perp}$, where $k, l \in \mathbb{R}$, $k \neq -l$, and $p_{\mathcal{H}_p}$ (resp. $p_{\mathcal{H}_p^\perp}$) is the orthogonal projection onto $\mathcal{H}_p$ (resp. $\mathcal{H}_p^\perp$). The homogeneous space $Gr^{(p)}$ is therefore endowed with a one-parameter family of Kähler structures (encoded by $(k + l)$). The derivation $D_{k,l}^{(p)}$ is inner if and only if $l = 0$. For $p = 1$, $Gr^{(p)}$ is the projective space of $\mathcal{H}$.

The dual symmetric space of $Gr^{(p)}$ is the homogeneous space $U_2(\mathcal{H}_p, \mathcal{H}_p^\perp)/(U_2(\mathcal{H}_p) \times U_2(\mathcal{H}_p^\perp))$ where $U_2(\mathcal{H}_p, \mathcal{H}_p^\perp)$ is the subgroup of $GL_2(\mathcal{H})$ which preserves the indefinite Hermitian form $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$ defined by:

$$\langle u, v \rangle = -\langle u_1, v_1 \rangle_{\mathcal{H}_p^\perp} + \langle u_2, v_2 \rangle_{\mathcal{H}_p},$$

where $u = u_1 + u_2$, $v = v_1 + v_2$ with $u_1, v_1 \in \mathcal{H}_p^\perp$ and $u_2, v_2 \in \mathcal{H}_p$. It is the affine adjoint orbit of $U_2(\mathcal{H}_p, \mathcal{H}_p^\perp)$ for the derivations $D_{k,l}^{(p)}$. It can be identified with the symmetric Hilbert domain:

$$\mathcal{A}^{(p)} = \{ Z \in L^2(\mathcal{H}_p, \mathcal{H}_p^\perp), -Z^*Z + \text{id} > 0 \},$$
where the notation $-Z^*Z + \text{id}$ means that the operator $-Z^*Z + \text{id}$ is positive definite. In particular, for $p = 1$, $A^{(1)}$ is the open unit ball in $\mathcal{H}_{+}^\perp$. Let us remark that $A^{(p)}$ is star-shaped hence connected and simply-connected. To see that $A^{(p)}$ is diffeomorphic to the homogeneous space $U_2(\mathcal{H}_p, \mathcal{H}_p^\perp)/(U_2(\mathcal{H}_p) \times U_2(\mathcal{H}_p^\perp))$, note that

$$U_2(\mathcal{H}_p, \mathcal{H}_p^\perp) = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{GL}_2(\mathcal{H}) \mid A^*A - C^*C = \text{id}_{\mathcal{H}_p^\perp}, D^*D - B^*B = \text{id}_{\mathcal{H}_p}, A^*B = C^*D \right\},$$

where the block decomposition of $g$ is relative to the Hilbert sum $\mathcal{H} = \mathcal{H}_p^\perp \oplus \mathcal{H}_p$. In particular, for $Z \in A^{(p)}$, one has

$$-(AZ + B)^*(AZ + B) + (CZ + D)^*(CZ + D) = -Z^*Z + 1 > 0,$$

which implies that $(CZ + D)^*(CZ + D)$ is positive definite hence $(CZ + D) \in \text{GL}(\mathcal{H}_p)$. It follows that one can define an action of $U_2(\mathcal{H}_p, \mathcal{H}_p^\perp)$ on $A^{(p)}$ by:

$$g \cdot Z = (AZ + B)(CZ + D)^{-1},$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_p^\perp \oplus \mathcal{H}_p$. This action is transitive since every $Z \in A^{(p)}$ can be written as

$$Z = \exp \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} \cdot 0,$$

where

$$B = Z^{\frac{\text{argth}(Z^*Z)}{2}} \in L^2(\mathcal{H}_p, \mathcal{H}_p^\perp),$$

(this expression follows from Remark 6.5 in [13]). Another proof for the transitivity of the action (5) of $U_2(\mathcal{H}_p, \mathcal{H}_p^\perp)$ on $A^{(p)}$ can be found in [13], Theorem III.9. Since the isotropy of $0 \in A^{(p)}$ is $U_2(\mathcal{H}_p) \times U_2(\mathcal{H}_p^\perp)$, one has:

$$A^{(p)} = U_2(\mathcal{H}_p, \mathcal{H}_p^\perp)/(U_2(\mathcal{H}_p) \times U_2(\mathcal{H}_p^\perp)).$$

The Hermitian-symmetric space of non-compact type $A^{(p)}$ is a particular example of Finsler-Cartan-Hadamard manifold (see the Definition on p 124, Proposition 3.16, Proposition 3.15, and Theorem 3.6(iii) in [13]). It follows either from the general theory (Theorem 3.14 or Theorem 1.10 in [13]) or from equation (6) that

$$\exp : \left\{ \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} \mid B \in L^2(\mathcal{H}_p, \mathcal{H}_p^\perp) \right\} \longrightarrow A^{(p)} \xrightarrow{\quad X \quad} \exp X \cdot 0$$

is a diffeomorphism.

**Example 3.13** The restricted Grassmannian $\text{Gr}_{\text{res}}$ has been studied in [15] and [27]. The connected component $\text{Gr}_{\text{res}}^0$ of $\text{Gr}_{\text{res}}$ containing $\mathcal{H}_+$ is the affine adjoint orbit of $U_2(\mathcal{H})$ for the derivations defined by the bounded operators $D_{k,l}^{\infty} = ik p_+ - il p_-$, where $k, l \in \mathbb{R}$, $k \neq -l$, and $p_\pm$ is the orthogonal projection onto $\mathcal{H}_\pm$. None of these derivations is inner.

As in the previous case, the dual Hermitian-symmetric space of the connected component $\text{Gr}_{\text{res}}^0$ of the restricted Grassmannian is the homogeneous space $U_2(\mathcal{H}_+, \mathcal{H}_-)/(U_2(\mathcal{H}_+) \times U_2(\mathcal{H}_-))$ where $U_2(\mathcal{H}_+, \mathcal{H}_-)$ is the subgroup of $\text{GL}_2(\mathcal{H})$ which preserves the indefinite Hermitian form $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$ defined by:

$$\langle u, v \rangle = -\langle u_1, v_1 \rangle_{\mathcal{H}_-} + \langle u_2, v_2 \rangle_{\mathcal{H}_+},$$

where $u = u_1 + u_2$, $v = v_1 + v_2$ with $u_1, v_1 \in \mathcal{H}_-$ and $u_2, v_2 \in \mathcal{H}_+$. It is the affine adjoint orbit of $U_2(\mathcal{H}_+, \mathcal{H}_-)$ for the derivations $D_{k,l}^{\infty}$. It can be identified with the symmetric Hilbert domain:

$$A^{(\infty)} = \{ Z \in L^2(\mathcal{H}_+, \mathcal{H}_-) \mid -Z^*Z + \text{id} > 0 \}.$$
Example 3.14 Denote by \( g \) the real part of the Hermitian scalar product on \( \mathcal{H} \). The Grassmannian \( \mathcal{Z}(\mathcal{H}) = O_2^+(\mathbb{H})/U_2(\mathcal{H}) \) is the space of complex structures \( I \) on \( \mathbb{H} \) such that

\[
g(I X, I Y) = g(X, Y),
\]

defining the same orientation as the distinguished complex structure \( I_0 \) on \( \mathcal{H} \) and being closed to it. For every \( k \neq 0 \), the space \( \mathcal{Z}(\mathcal{H}) \) can be identified with the \( O_2^+(\mathbb{H}) \)-spine extension of \( \{kI \}_0 \) for the bounded operator \( D_k^{(0)} = kI_0 \). Denote by \( \mathcal{H}^C \) the \( \mathbb{C} \)-extension of \( \mathbb{H} \) and by \( Z_{\pm} \) the eigenspace of the \( \mathbb{C} \)-linear extension of \( I_0 \) with eigenvalue \( \pm i \). One has \( \mathcal{H}^C = Z_+ \oplus Z_- \) as orthogonal sum with respect to the Hermitian scalar product on \( \mathcal{H}^C \) which restricts to \( g \) on \( \mathbb{H} \). The homogeneous space \( \mathcal{Z}(\mathcal{H}) \) injects into the restricted Grassmannian of the polarized Hilbert space \( \mathcal{H}^C = Z_+ \oplus Z_- \) via the application which maps a complex structure \( I \) to the subspace of \( \mathcal{H}^C \) consisting of \((1,0)\)-type vectors \( X \) with respect to \( I \), i.e. satisfying \( IX = iX \). This realizes \( \mathcal{Z}(\mathcal{H}) \) as the totally geodesic submanifold of \( \mathbb{G}_0^0 \) consisting of maximal isotropic subspaces for the \( \mathbb{C} \)-linear extension \( g^C \) of \( g \). Starting with a basis \( \{e_n\}_{n \in \mathbb{Z} \setminus \{0\}} \) of \( \mathcal{H} \), endow \( \mathcal{H}^C \) with the basis \( \{e_n\}_{n \in \mathbb{Z} \setminus \{0\}} \cup \{I_0e_n\}_{n \in \mathbb{Z} \setminus \{0\}} \). Then \( Z_{\pm} \) is the \( \mathbb{C} \)-linear subspace of \( \mathcal{H}^C \) generated by \( \{1/\sqrt{2}(e_n \mp iI_0e_n)\}_{n \in \mathbb{Z} \setminus \{0\}} \). With respect to these basis, the symmetric \( \mathbb{C} \)-bilinear form \( g^C \) and the \( \mathbb{C} \)-linear extension of the operator \( D_k^{(0)} \) have the following decompositions as endomorphisms of \( \mathcal{H}^C = Z_+ \oplus Z_- \):

\[
g^C = \begin{pmatrix} 0 & \text{id} \\ \text{id} & 0 \end{pmatrix},
\]

\[
P_k^{(0)} = \begin{pmatrix} ik & 0 \\ 0 & -ik \end{pmatrix}.
\]

It is easy to see that \( O_2^+(\mathbb{H}) \) (as defined in the Introduction) is conjugate to the connected component of \( O_2(\mathcal{H}^C) \cap U_2(\mathcal{H}^C) \) where \( O_2(\mathcal{H}^C) \) denotes the complex \( L^* \)-group preserving \( g^C \).

The dual symmetric space of \( \mathcal{H} \) is the homogeneous space \( (O_2(\mathcal{H}^C) \cap U_2(Z_+, Z_-))/U_2(Z_+) \) where \( U_2(Z_+, Z_-) \) is the subgroup of \( \text{GL}_2(\mathcal{H}^C) \) which preserves the indefinite Hermitian form \( \langle \cdot, \cdot \rangle \) on \( \mathcal{H}^C \) defined by:

\[
\langle u, v \rangle = -\langle u_1, v_1 \rangle_{Z_-} + \langle u_2, v_2 \rangle_{Z_+},
\]

where \( u = u_1 + u_2 \) and \( v = v_1 + v_2 \) with \( u_1, v_1 \in Z_- \) and \( u_2, v_2 \in Z_+ \). It is the affine adjoint orbit of \( O_2(\mathcal{H}^C) \cap U_2(Z_+, Z_-) \) for the derivations \( D_k^{(0)} \), \( k \neq 0 \). It can be identified with the symmetric Hilbert domain:

\[
\mathcal{B}^{(\infty)} = \{ Z \in \mathbb{L}^2(Z_+, Z_-) \mid Z^T + Z = 0, -Z^T Z + \text{id} > 0 \}.
\]

Example 3.15 The Grassmannian \( \mathcal{L}(\mathcal{H}) = \text{Sp}_2(\mathcal{H})/U_2(\mathcal{H}_+^\perp) \) of Lagrangian subspaces close to \( \mathcal{H}_+^\perp \) is identified with \( \mathcal{L}(\mathcal{H}) \)-affine adjoint orbit of \( 0 \) for the derivations given by the bounded operators \( D_l^{(\infty)} = il_0 P_{-l} - il_0 P_0 \), \( l \neq 0 \). It is a totally geodesic submanifold of the restricted Grassmannian \( \mathbb{G}_0^0 \).

The dual symmetric space of \( \mathcal{L}(\mathcal{H}) \) is the homogeneous space \( \text{Sp}_2(\mathcal{H}, \mathcal{C}) \cap U_2(\mathcal{H}_+^\perp, \mathcal{H}_-^\perp)/U_2(\mathcal{H}_+^\perp) \), where \( \text{Sp}_2(\mathcal{H}, \mathcal{C}) \) is the complex \( L^* \)-group preserving the \( \mathbb{C} \)-antisymmetric form \( \omega(X, Y) = X^T J_0 Y \). It is the affine adjoint orbit of \( \text{Sp}_2(\mathcal{H}, \mathcal{C}) \cap U_2(\mathcal{H}_+^\perp, \mathcal{H}_-^\perp) \) for the derivations \( D_l^{(\infty)} \), \( l \neq 0 \). It can be identified with the symmetric Hilbert domain:

\[
\mathcal{C}^{(\infty)} = \{ Z \in \mathbb{L}^2(Z_+, Z_-) \mid Z^T = Z, -Z^T Z + \text{id} > 0 \}.
\]

Note that \( \text{Sp}_2(\mathcal{H}, \mathcal{C}) \cap U_2(\mathcal{H}_+^\perp, \mathcal{H}_-^\perp) \) is conjugate to

\[
\text{Sp}_2(\mathbb{H}_2) := \{ g \in \text{GL}_2(\mathbb{H}_2) \mid g^T J_0 g = J_0 \},
\]

hence \( \mathcal{C}^{(\infty)} = \text{Sp}_2(\mathbb{H}_2)/U_2(\mathbb{H}_2, J_0) \) where \( U_2(\mathbb{H}_2, J_0) \) denotes the unitary group of the Hilbert space \( \mathbb{H}_2 \) endowed with the complex structure \( J_0 \).

Example 3.16 Recall that \( \mathbb{H}_2 \) is a real Hilbert space with basis \( \{e_n\}_{n \in \mathbb{Z} \setminus \{0\}} \) and that \( \mathbb{H}_2 \) denotes the real subspace generated by \( e_1 \) and \( e_2 \). The space \( \text{Gr}_2^{(2)}(\mathbb{H}_2) = O_2^+(\mathbb{H}_2)/((\text{SO}(\mathbb{H}_2)) \times O_2^+(\mathbb{H}_2)) \) is the Grassmannian of oriented 2-planes in \( \mathbb{H}_2 \) and the \( O_2^+(\mathbb{H}_2) \)-adjoint orbit of \( kJ \) where \( k \neq 0 \) and \( J \) is the natural complex structure on \( \mathbb{H}_2 \). Via the map which assigns to an oriented 2-plane of \( \mathbb{H}_2 \) with
orthonormal basis \( \{u, v\} \) the complex line \( \mathbb{C}(u + iv) \in \mathbb{P}(\mathcal{H}) \), the Grassmannian \( \text{Gr}^{(2)}_{\text{or}} \) can be identified as complex manifold with the quadric \( \mathcal{C} \) in the complex projective space \( \mathbb{P}(\mathcal{H}) \) defined by

\[
\mathcal{C} := \left\{ [z] = \left[ \sum_{i \in \mathbb{Z} \setminus \{0\}} z_i e_i \right] \in \mathbb{P}(\mathcal{H}) \mid \sum_{i \in \mathbb{Z} \setminus \{0\}} z_i^2 = 0 \right\}.
\]

The dual Hermitian-symmetric space of \( \text{Gr}^{(2)}_{\text{or}} \) is the homogeneous space \( O^+_2((\mathcal{H}_2)_{\mathbb{R}}) / (SO((\mathcal{H}_2)_{\mathbb{R}}) \times O^+_2((\mathcal{H}_2)_{\mathbb{R}})) \)

where \( O^+_2((\mathcal{H}_2)_{\mathbb{R}}) \) is the subgroup of \( \text{GL}_2(\mathbb{H}_{\mathbb{R}}) \) which preserves the indefinite symmetric form \( \langle , \rangle \) on \( \mathbb{R}_2 \) defined by:

\[
\langle u, v \rangle = u_1v_1 + u_2v_2 - \sum_{i \in \mathbb{Z} \setminus \{0, 1, 2\}} u_i v_i,
\]

where \( u = \sum_{i \in \mathbb{Z} \setminus \{0\}} u_i e_i \) and \( v = \sum_{i \in \mathbb{Z} \setminus \{0\}} v_i e_i \). It is the \( O^+_2((\mathcal{H}_2)_{\mathbb{R}}) \)-adjoint orbit for the derivations given by the bounded operators \( kJ \) for \( k \neq 0 \). It can be identified with the symmetric Hilbert domain:

\[
\mathcal{D} = \{ Z \in \mathcal{H}_2^+ \mid 1 + |Z^T Z|^2 - 2Z^* Z > 0, -Z^* Z + 1 > 0 \},
\]


**Corollary 3.17** Every infinite-dimensional irreducible Hermitian-symmetric affine (co-)adjoint orbit of a connected \( L^* \)-group of non-compact type with simple complexification is isomorphic to one of the following symmetric Hilbert domains:

1. \( \mathcal{A}^{(p)} = \{ Z \in L^2(\mathcal{H}_p, \mathcal{H}_p^+ \mid -Z^* Z + \text{id} > 0 \} \);
2. \( \mathcal{A}^{(\infty)} = \{ Z \in L^2(\mathcal{H}_+, \mathcal{H}_-) \mid -Z^* Z + \text{id} > 0 \} \);
3. \( \mathcal{B}^{(\infty)} = \{ Z \in L^2(Z_+, Z_-) \mid -Z^* Z + \text{id} > 0, Z^T + Z = 0 \} \);
4. \( \mathcal{C}^{(\infty)} = \{ Z \in L^2(Z_+, Z_-) \mid -Z^* Z + \text{id} > 0, Z^T = Z \} \);
5. \( \mathcal{D}^{2} = \{ Z \in \mathcal{H}_2^+ \mid -Z^* Z + 1 > 0, 1 + |Z^T Z|^2 - 2Z^* Z > 0 \} \).

\( \square \)** Proof of Corollary 3.17:**

Let \( O^{n.c.} \) be an infinite-dimensional irreducible Hermitian-symmetric affine (co-)adjoint orbit of a non-compact \( L^* \)-group \( G^{n.c.} \) with simple complexification. Let \( g^{n.c.} \) be the Lie algebra of \( O^{n.c.} \). Since \( (g^{n.c.})^C \) is simple, \( g^{n.c.} \) is also simple. Moreover we can suppose w.l.o.g. (16) that \( (g^{n.c.})^C \) is either \( \mathfrak{gl}_2(\mathcal{H}) \), \( \mathfrak{o}_2(\mathcal{H}, \mathcal{C}) \) or \( \mathfrak{sp}_2(\mathcal{H}, \mathcal{C}) \), where \( \mathfrak{o}_2(\mathcal{H}, \mathcal{C}) \) (resp. \( \mathfrak{sp}_2(\mathcal{H}, \mathcal{C}) \)) is the Lie algebra of \( O_2(\mathcal{H}^C) \) (resp. \( \mathfrak{Sp}_2(\mathcal{H}, \mathcal{C}) \)) introduced in Example 3.14 (resp. in Example 3.15).

Since \( O^{n.c.} \) is in particular strongly symplectic, by Theorem 4.4 in [12], the derivation defining \( O^{n.c.} \) can be written as \( D_x = [D, x] \) where \( D \) is a skew-Hermitian operator with finite spectrum. It follows that \( t := \text{Ker} D \) is \( + \)-invariant. Since \( g^{n.c.} \) is in particular semi-simple, one has \( \langle x, y \rangle = \langle y^*, x^* \rangle \) for every \( x, y \) in \( g^{n.c.} \). Hence the orthogonal complement \( n \) of \( t \) in \( g^{n.c.} \) is also \( + \)-invariant. Denote by \( K \) the isotropy subgroup of \( G^{n.c.} \) fixing \( 0 \). Since \( n \) is an irreducible \( K \)-module, the bilinear form \( b \) on \( n \) defining the Riemannian metric of \( O^{n.c.} \) is proportional to the trace, that is: \( b(x, y) = \lambda \text{Tr} xy \), for \( x, y \) in \( n \) for some non-zero \( \lambda \in \mathbb{R} \). The condition \( b(x, x) > 0 \) for \( x \neq 0 \) together with the \( + \)-invariance of \( n \) then implies that either \( n \subseteq \{ x \in g^{n.c.} \mid x^* = x \} \) or \( n \subseteq \{ x \in g^{n.c.} \mid x^* = -x \} \). Since \( g^{n.c.} \) is non-compact, the second possibility is fulfilled. Hence \( g := t \oplus n \) is a \( L^* \)-algebra of compact type, which is simple.

Let \( G \) be the connected \( L^* \)-group generated by the exponentials of operator in \( g \). The affine coadjoint orbit of \( G \) defined by the derivation \( [D, .] \) is infinite-dimensional irreducible and Hermitian-symmetric, hence is isomorphic to one of the affine adjoint orbits listed in Theorem 1.11. The corollary then follows by duality (see Examples 3.12, 3.13, 16).

\( \square \)** Proof of Corollary 1.2:**

By Theorem (3.9) and the discussion after Theorem (4.2) in [11], every Hermitian-symmetric space of compact (resp. non-compact) type is isomorphic to the orthogonal product of irreducible Hermitian-symmetric spaces of compact (resp. non-compact) type. The irreducible pieces are of type I-VI and described in paragraph 3 in [10]. The types V and VI correspond to the exceptional Lie groups E6.
and E7, which are of finite dimension. By the finite-dimensional theory, finite-dimensional Hermitian-symmetric spaces are coadjoint orbits of their groups of isometries (see e.g. Theorem 8.89 in \[4\]). An infinite-dimensional irreducible Hermitian-symmetric space is of type I, II, III or IV, and is isomorphic (see paragraph 3 in \[10\]) to one of the affine coadjoint orbits listed in Theorem \[1.1\] or Corollary \[3.17\]. Both the restricted Grassmannian and the Grassmannian of \(p\)-dimensional subspaces of a separable Hilbert space, with \(p < +\infty\), are Hermitian-symmetric spaces of type I. Now the theorem follows by taking the product of the \(L^\ast\)-groups acting on each irreducible pieces.

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References


