Alexander Belyaev (belyaev@mpi-sb.mpg.de, http://www.mpi-sb.mpg.de/∼belyaev/gm04)

Plane and Space Curves. Curvature. Curvature-based Features.

Description of Plane Curves

Plane curves can be described mathematically in
explicit form: $y = f(x)$ (as a function graph);
implicit form: $f(x, y) = 0$;
parametric form: $\mathbf{r}(t) = [x(t), y(t)]$.
Here and everywhere below the vectors are denoted by bold letters.

Arc Length

Let A and B be two points with coordinates $[x_A, y_A]$ and $[x_B, y_B]$, respectively. The distance between A and B is given by

$$\mathrm{dist}(A, B) = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}.$$

Consider a curve $\mathbf{r}(t) = [x(t), y(t)]$. Let us approximate the curve by a polygonal line $P_0, P_1, \ldots, P_n$ such that $P_k = [x(t_k), y(t_k)]$ and $t_{k+1} - t_k = \Delta t$. Now let us approximate the length of the curve

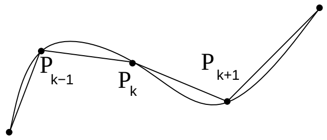


Fig. 1: Approximation of a smooth curve by a polygonal line.

by the length of the polygonal line $P_0, P_1, \ldots, P_n$. The length of the polygonal line is given by the sum

$$\sum_k \sqrt{(x_{k+1} - x_k)^2 + (y_{k+1} - y_k)^2}.$$

Rewriting the sum in a more convenient form and passing to the limit as $t \to 0$ yields

$$\sum_k \sqrt{\left(\frac{x_{k+1} - x_k}{\Delta t}\right)^2 + \left(\frac{y_{k+1} - y_k}{\Delta t}\right)^2} \Delta t \to \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Thus the length of the curve segment between two points $A = [x(t_A), y(t_A)]$ and $B = [x(t_B), y(t_B)]$ is given by

$$\text{dist\_along\_curve}(A, B) = \int_{t_A}^{t_B} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

We will use Newton's dot notations $\dot{x}$ for $\dfrac{dx}{dt}$, $\dot{y}$ for $\dfrac{dy}{dt}$, $\ddot{x}$ for $\dfrac{d^2x}{dt^2}$, etc.

Let $s(t)$ be the length of the curve segment between the point on the curve corresponding to the value $t = t_0$, $[x(t_0), y(t_0)]$, and a general point on the curve $[x(t), y(t)]$. Then

$$s(t) = \int_{t_0}^{t} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt, \quad \frac{ds(t)}{dt} = \dot{s}(t) = \sqrt{\dot{x}^2 + \dot{y}^2} = \left|\frac{d\mathbf{r}}{dt}\right|$$

Note that $s(t)$ measures the distance between points $[x(t_0), y(t_0)]$ and $[x(t), y(t)]$ along the curve and $s(t) > 0$ if $t > t_0$ and $s(t) < 0$ if $t < t_0$. The value $s(t)$ is called arc length of the curve. The value $\dfrac{ds(t)}{dt} = \dot{s}(t) = \sqrt{\dot{x}^2 + \dot{y}^2}$ indicates the rate of change of arc length with respect to the curve parameter t. In other words, $\dot{s}(t)$ gives the speed of the curve parameterization. Note that $dt/ds = 1/\sqrt{\dot{x}^2 + \dot{y}^2}$.

Often it is very convenient to parameterize the curve with respect to its arc length:

$$\mathbf{r}(s) = [x(s), y(s)].$$

Note that $\dfrac{d\mathbf{r}}{ds} = \dfrac{d\mathbf{r}}{dt} \cdot \dfrac{dt}{ds} = \dfrac{d\mathbf{r}}{dt} \Big/ \left|\dfrac{d\mathbf{r}}{dt}\right|$ and, hence, $\left|\dfrac{d\mathbf{r}}{ds}\right| = 1$

Orientation, Curvature, Curvature Vector

Consider a curve $\mathbf{r}(s) = [x(s), y(s)]$ parameterized by arc length s. The parameterization defines a prescribed direction along the curve at which s grows. Let $\mathbf{t}(s)$ be the unit tangent vector associated with the direction. Let $\mathbf{n}(s)$ be the unit normal vector such that $(\mathbf{t}, \mathbf{n})$ forms the counter-clockwise oriented frame.

Denote by $\varphi(s)$ the angle between the tangent at a point $[x(s), y(s)]$ and the positive direction of the x-axis. The **curvature** at a point measures the rate of curving (bending) as the point moves along the curve with unit speed and can be defined as

$$k(s) = \frac{d\varphi(s)}{ds}.$$

Thus the curvature of a straight line is zero. The curvature of a circle oriented by its inner normal is equal to the reciprocal of the radius of the circle.

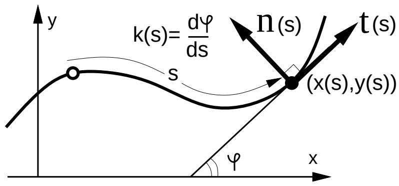


Fig. 2: Definition of curvature.

Points where $k = 0$ are called the **points of inflection**.

Surprisingly, the curvature is all that is needed to define a curve (up to rigid motions).

An oriented curve is a curve such that at every point a unit normal vector $\mathbf{n}$ is defined, provided $\mathbf{n}(s)$ is continuous along the curve. The curvature depends on the orientation. If orientation is changed, the sign of curvature changes. The curvature vector $\mathbf{k} = k(s)\mathbf{n}(s)$ does not depend on the orientation.

The Formulas of Frenet

Let a curve $\mathbf{r}(s) = [x(s), y(s)]$ be parameterized by arc length. The derivative of $\mathbf{r}(s)$ with respect to arc length s, $\mathbf{t}(s) = \dfrac{d\mathbf{r}(s)}{ds}$ is a unit vector tangent to the curve (prove it). Differentiation of the scalar product $\mathbf{t} \cdot \mathbf{t} = 1$ gives $2\dfrac{d\mathbf{t}}{ds} \cdot \mathbf{t} = 0$ and, therefore, the vectors $\mathbf{t}$ and $\dfrac{d\mathbf{t}}{ds}$ are mutually perpendicular: $\mathbf{t} \perp \dfrac{d\mathbf{t}}{ds}$. Since $\mathbf{t} = [\cos\varphi, \sin\varphi]$, see Fig. 2, then

$$\frac{d\mathbf{t}}{d\varphi} = [-\sin\varphi, \cos\varphi] = \mathbf{n}$$

is a unit vector normal to the curve. Differentiating with respect to arc length s gives

$$\frac{d\mathbf{t}}{ds} = \frac{d\mathbf{t}}{d\varphi} \cdot \frac{d\varphi}{ds} = k\mathbf{n} \quad \text{and} \quad \frac{d\mathbf{n}}{ds} = \frac{d\mathbf{n}}{d\varphi} \cdot \frac{d\varphi}{ds} = -k\mathbf{t}$$

We arrive at the famous formulas of Frenet: $\boxed{\dfrac{d\mathbf{t}}{ds} = k\mathbf{n}, \quad \dfrac{d\mathbf{n}}{ds} = -k\mathbf{t}}$

Curvature Computation

Parameterized curves. Consider a parameterized curve $\mathbf{r}(t) = (x(t), y(t))$ and assume that $\langle \mathbf{t}, \mathbf{n} \rangle$ forms a right-hand basis. The curvature $k(t)$ is given by

$$k(t) = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}} \tag{1}$$

Let us derive this formula:

$$\tan\varphi = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\dot{y}(t)}{\dot{x}(t)}, \quad \varphi = \tan^{-1}\left(\frac{\dot{y}(t)}{\dot{x}(t)}\right),$$

$$\frac{d\varphi}{ds} = \frac{d\varphi}{dt} \cdot \frac{dt}{ds} = \frac{1}{1+(\dot y/\dot x)^2} \cdot \frac{d}{dt}\left(\frac{\dot y}{\dot x}\right) \cdot \frac{1}{\sqrt{\dot x^2 + \dot y^2}} =$$
$$= \frac{\dot x^2}{\dot x^2 + \dot y^2} \cdot \frac{\dot x \ddot y - \ddot x \dot y}{\dot x^2} \cdot \frac{1}{\sqrt{\dot x^2 + \dot y^2}} = \frac{\dot x \ddot y - \ddot x \dot y}{(\dot x^2 + \dot y^2)^{3/2}}$$

Curves in explicit form. Consider a plane curve given as the graph of a function: $y = f(x)$. Assume that the curve is oriented by its upward normal $\mathbf{n} = [-f'(x), 1] / \sqrt{1 + f'(x)^2}$ (it is equivalent that we move along the curve from left to right). The curvature of the curve is given by

$$k = \frac{f''(x)}{\left(1 + f'(x)^2\right)^{3/2}} \tag{2}$$

This formula for the curvature is easily derived from the previous one if we represent the curve in the following parametric form

$$x = t, \quad y = f(t)$$

Curves in implicit form. Consider a plane curve given by the equation $F(x, y) = 0$. The curvature vector of the curve is given by

$$\mathbf{k} = k\mathbf{n} = -\frac{F_{xx}F_y^2 - 2F_xF_yF_{xy} + F_{yy}F_x^2}{(F_x^2 + F_y^2)^{3/2}}\,\mathbf{n}, \quad \mathbf{n} = \frac{[F_x, F_y]}{(F_x^2 + F_y^2)^{1/2}} \tag{3}$$

To derive this formula let us consider a point on the curve where $F_y \neq 0$ (we use subindices to denote partial derivatives: F_y means $\frac{\partial F}{\partial y}$, F_{xy} means $\frac{\partial^2 F}{\partial x \partial y}$, etc.). At some vicinity of the point the curve can be represented as the graph of a function $y = f(x)$. Thus

$$F(x, f(x)) = 0.$$

Differentiating this equation with respect to x one and two times gives

$$F_x + F_y f' = 0 \quad \text{and} \quad F_{xx} + F_{xy}f' + F_{xy}f' + F_{yy}f'^2 + F_y f'' = 0$$

respectively. Now expressing f' and f'' through partial derivatives of $F(x, y)$ and substituting them in (2) we arrive at (3).

Reconstruction of Curve from its Curvature

Since $d\mathbf{r}(s)/ds = \mathbf{t}(s) = [\cos\varphi(s), \sin\varphi(s)]$ and $k(s) = d\varphi(s)/ds$, the curvature of a curve determines the curve up to rigid transformations. One can reconstruct the curve from its curvature by integration

$$\varphi(s) = \int k(s)\,ds + \varphi_0, \quad \mathbf{r}(s) = \left[\int \cos\varphi(s)\,ds + a, \int \sin\varphi(s)\,ds + b\right],$$

where φ_0, a, b are constants of integration (a rigid transformation is defined by three parameters).

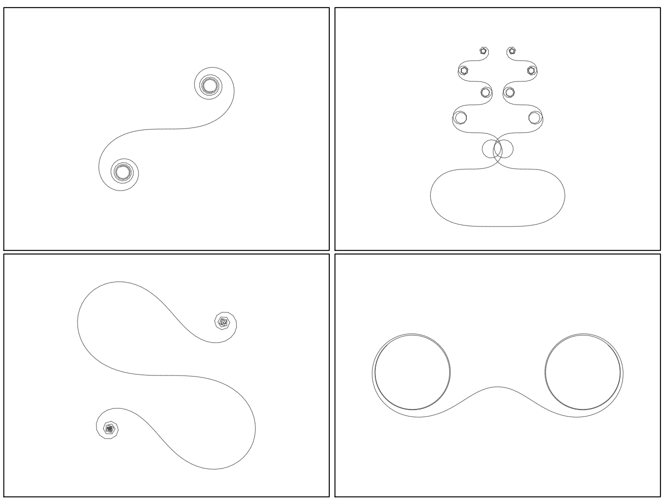


Fig. 3: Curves reconstructed from prescribed curvature functions $k(s)$.

Curve Quality Evaluation via Curvature Profile

The shape of the curvature profile is an important tool for the curve quality evaluation. Fig. 4 displays an original curve $\mathbf{r} = \mathbf{r}(s)$ (the left image) and the curve together with its curvature vectors $k(s)\mathbf{n}(s)$ attached to curve points (the right image).

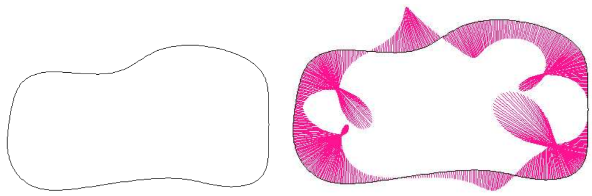


Fig. 4: Curve quality evaluation via curvature profile.

Circle of Curvature

The **circle of curvature** or **osculating circle** at a non-inflection point P (where the curvature is not zero) on a plane curve is the circle that

1. is tangent to the curve at P;
2. has the same curvature as the curve has at P;
3. lies toward the concave or inner side of the curve.

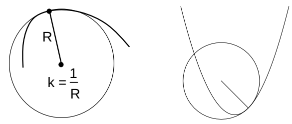


Fig. 5: The circle of curvature.

The center of the circle of curvature is called the **center of curvature**.

Consider a parameterized curve $[x(t), y(t)]$. At each point of the curve $P_0 = [x(t_0), y(t_0)]$ let us measure how closely the curve can be approximated by a circle. To do this let us consider a general circle

$$(x-a)^2 + (y-b)^2 = R^2 \tag{4}$$

through P_0 and the function

$$g(t) = (x(t) - a)^2 + (y(t) - b)^2 - R^2.$$

Since the circle goes through P_0, the equation $g(t) = 0$ has an obvious solution $t = t_0$. In order to find the circle providing with the best approximation of the curve at P_0 we require that $g(t)$ itself and as many as possible derivatives of $g(t)$ vanish at $t = t_0$:

$$\dot g(t_0) = 0, \quad \ddot g(t_0) = 0, \quad \ldots, \quad g^{(k)}(t_0) = 0, \quad \ldots. \tag{5}$$

Actually, since in general a circle depends on three parameters, two coordinates of the center and the radius, at a generic curve point we are able to satisfy only three conditions

$$g(t_0) = 0, \quad \dot g(t_0) = 0, \quad \ddot g(t_0) = 0.$$

Denote $x(t_0), y(t_0), \dot x(t_0), \dot y(t_0), \ddot x(t_0), \ddot y(t_0)$ by $x_0, y_0, \dot x_0, \dot y_0, \ddot x_0, \ddot y_0$ respectively. The equations $\dot g(t_0) = 0$ and $\ddot g(t_0) = 0$ give

$$\begin{cases} \dot x_0\,(x_0 - a) + \dot y_0\,(y_0 - b) = 0 \\ \ddot x_0\,(x_0 - a) + \ddot y_0\,(y_0 - b) = -\dot x_0^2 - \dot y_0^2 \end{cases}$$

The first equation says that (a, b) lies on the normal to the curve at P_0. Solving the system with respect to a and b we arrive at

$$a = x_0 - \frac{1}{k(t_0)} \cdot \frac{\dot y_0}{\sqrt{\dot x_0^2 + \dot y_0^2}}, \quad b = y_0 + \frac{1}{k(t_0)} \cdot \frac{\dot x_0}{\sqrt{\dot x_0^2 + \dot y_0^2}}$$

where $k(t) = \dfrac{\dot x \ddot y - \ddot x \dot y}{(\dot x^2 + \dot y^2)^{3/2}}$ is the curvature of the curve. Note that

$$\left[\frac{-\dot y_0}{\sqrt{\dot x_0^2 + \dot y_0^2}}, \frac{\dot x_0}{\sqrt{\dot x_0^2 + \dot y_0^2}}\right]$$

is a unit vector normal to the curve. Thus we have proved

Theorem 1 *The best approximation of the curve by a circle is given by the circle of curvature (osculating circle).*

A **vertex** of a smooth curve is a point of the curve where the curvature takes an extremum: $k'(s_0) = 0$.

Theorem 2 (Four-Vertex Theorem) *A simple closed curve has at least four vertices.*

A sketch of a simple proof of the theorem will be given in the section devoted to the medial axis.

We will say that the circle (4) has contact of order k or higher with curve $[x(t), y(t)]$ at point $P_0 = [x(t_0), y(t_0)]$ if $g(t_0)$ and the first k derivatives in (5) vanish. Thus the osculating circle has contact of order 2 or higher. It is not difficult to show that if P_0 is a vertex then the osculating circle at P has contact of order 3 or higher.

Evolute

The locus of the centers of curvature of a given curve is called the **evolute** of the curve. The evolute of a smooth curve has singularities (so-called cusps). The cups of the evolute correspond to the points where the curvature takes extremal values.

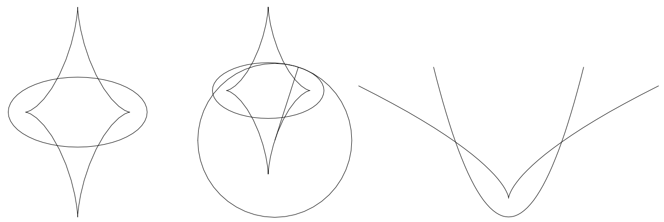


Fig. 6: Left: an ellipse and its evolute. Middle: a circle of curvature is added. Right: a parabola and its evolute.

Given an oriented curve $\mathbf{r} = \mathbf{r}(t)$, its evolute is described by

$$\mathbf{e}(t) = \mathbf{r}(t) + \frac{\mathbf{n}(t)}{k(t)},$$

where $k(t)$ is the curvature of the curve at $\mathbf{r}(t)$.

Let us consider a curve parameterized by arc length s. To study the shape of the evolute at a small vicinity of a point $\mathbf{r}(s)$ we expand $\mathbf{r}(s+\alpha)$ into Taylor series with respect to α, $\alpha \ll 1$. Let

$$\mathbf{r}' = \mathbf{t} \quad \text{and} \quad \mathbf{n} = \mathbf{t}^{\perp}$$

compose the Frenet frame at $\mathbf{r} = \mathbf{r}(s)$. The evolute is given by

$$\mathbf{e}(s) = \mathbf{r}(s) + \frac{\mathbf{n}(s)}{k(s)},$$

where $k(s)$ is the curvature of the curve at $\mathbf{r}(s)$. According to the Frenet formulas

$$\mathbf{t}' = k\,\mathbf{n}, \quad \mathbf{n}' = -k\,\mathbf{t}$$

we have

$$\mathbf{r}' = \mathbf{t}, \qquad \mathbf{r}'' = \mathbf{t}' = k\,\mathbf{n}, \qquad \mathbf{r}''' = (k\mathbf{n})' = k'\mathbf{n} - k^2\mathbf{t}.$$

It yields

$$\begin{aligned}\mathbf{r}(s+\alpha) &= \mathbf{r} + \alpha\mathbf{r}' + \frac{\alpha^2}{2}\mathbf{r}'' + \frac{\alpha^3}{6}\mathbf{r}''' + O\left(\alpha^4\right) = \\ &= \mathbf{r}(s) + \mathbf{t}\left[\alpha - \frac{\alpha^3}{6}k^2 + O\left(\alpha^4\right)\right] + \\ &\quad + \mathbf{n}\left[\frac{\alpha^2}{2}k + \frac{\alpha^3}{6}k' + O\left(\alpha^4\right)\right].\end{aligned} \tag{6}$$

Similarly,

$$\begin{aligned}\mathbf{n}' &= -k\mathbf{t}, \\ \mathbf{n}'' &= (-k\mathbf{t})' = -k'\mathbf{t} - k^2\mathbf{n}, \\ \mathbf{n}''' &= (-k'' + k^3)\,\mathbf{t} - 3k'k\,\mathbf{n},\end{aligned}$$

$$\begin{aligned}\mathbf{n}(s+\alpha) &= \mathbf{n}(s) + \\ &+ \mathbf{t}\left[-\alpha\,k - \frac{\alpha^2}{2}k' + \frac{\alpha^3}{6}\left(k^3 - k''\right) + O(\alpha^4)\right] + \\ &+ \mathbf{n}\left[-\frac{\alpha^2}{2}k^2 - \frac{\alpha^3}{2}kk' + O(\alpha^4)\right],\end{aligned}$$

$$k(s+\alpha) = k(s) + \alpha\,k' + \frac{\alpha^2}{2}k'' + O(\alpha^3),$$

$$\frac{1}{k(s+\alpha)} = \frac{1}{k(s)} - \alpha\,\frac{k'}{k^2} + \alpha^2\left(\frac{k'^{\,2}}{k^3} - \frac{k''}{2k^2}\right) + O(\alpha^3).$$

For the evolute point $\mathbf{e}(s+\alpha)$ we have

$$\begin{aligned}\mathbf{e}(s+\alpha) &= \mathbf{r}(s+\alpha) + \frac{\mathbf{n}(s+\alpha)}{k(s+\alpha)} = \mathbf{e}(s) + \\ &+ \mathbf{t}\left[\alpha^2\frac{k'}{2k} + \alpha^3\left(\frac{k''}{3k} - \frac{k'^{\,2}}{2k^2}\right) + O(\alpha^4)\right] + \\ &+ \mathbf{n}\left[-\alpha\,\frac{k'}{k^2} + \alpha^2\left(\frac{k'^{\,2}}{k^3} - \frac{k''}{2k^2}\right) + O(\alpha^3)\right]\end{aligned} \tag{7}$$

Thus, if $k'(s) \neq 0$, in a small vicinity of the center of curvature $\mathbf{r}(s) + \mathbf{n}(s)/k(s)$ the evolute is approximated by a parabola tangent to the normal $\mathbf{n}(s)$ (see the left image of Fig. 7). If $k'(s) = 0$, in a small vicinity of the center of curvature the evolute is approximated by a semicubic parabola $y = Ax^{2/3}$ tangent to the normal $\mathbf{n}(s)$ (see the middle and right images of Fig. 7).

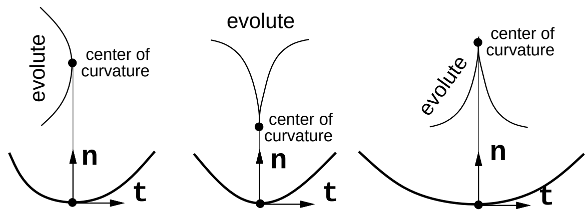


Fig. 7: The evolute is tangent to the curve normals at the centers of curvature. The critical points of the curvature correspond to singularities of the evolute.

We have proven the following theorem.

Theorem 3 *The evolute is tangent to the curve normals at the centers of curvature.*

The middle image of Fig. 6 demonstrates this property of the evolute.

A more detail analysis of expansion (7) leads us to the following result.

Theorem 4 *The evolute of a curve $\mathbf{r}(t)$ has a cusp at t_0 if and only if $\mathbf{r}(t_0)$ is a vertex of the curve. The cusp on the evolute is pointing towards or away from the vertex depending on whether the absolute value of the curvature has a local minimum or maximum at t_0.*

Let $\mathbf{t}$ and $\mathbf{n}$ be the Frenet basic vectors of a given curve $\mathbf{r}(t)$. If $k' > 0$, the vectors $\mathbf{t_e} = \mathbf{n}$ and $\mathbf{n_e} = -\mathbf{t}$ form the Frenet basis of the evolute. Let l deliver arc length parameterization of the evolute. Reparameterization of (7) gives

$$\mathbf{e}(l+\beta) = \mathbf{e}(s+\alpha) = \mathbf{e}(s) + \mathbf{t_e}\left[\beta + O(\beta^2)\right] + \mathbf{n_e}\left[\frac{\beta^2}{2}\frac{k^3}{k'} + O(\beta^3)\right],$$

where $\beta = \alpha k'/k^2$. Comparing the above expansion with (6) we conclude that the curvature of the evolute is is equal to k^3/k'. It follows that the evolute has no inflections. Indeed, if $k^3/k' = 0$ then the curve has an inflection and the corresponding evolute point is located at infinity.

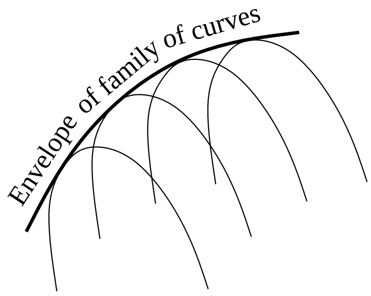


Fig. 8: The envelope of the family of curves is tangent to every curve from the family

Families and Envelopes

Consider a family of plane curves $F(x, y, a) = 0$ parameterized by the parameter a. The **envelope** of the family of curves $F(x, y, a) = 0$ is a curve which is tangent to every curve from the family, as seen in Fig.8

Theorem 5 *The envelope of the family of curves $F(x, y, a) = 0$ is the solution to the system*

$$\left\{ \begin{array}{rcl} F(x,y,a) & = & 0, \\ \dfrac{\partial}{\partial a}F(x,y,a) & = & 0. \end{array} \right.$$

Proof. Let $\mathbf{r} = \mathbf{r}(t)$ be a parameterization of the envelope. Then for each t, $\mathbf{r}(t)$ belongs to some curve $F(x, y, a(t)) = 0$ from the family. Thus a can be considered as a non-constant function of t, $da/dt \neq 0$, since otherwise it would imply the envelope was a member of the family. Differentiating with respect to t yields

$$F_x\,\dot{x} + F_y\,\dot{y} + F_a\,\dot{a} = 0.$$

Since the curve and its envelope have the common tangent at $(x(t), y(t))$, we have

$$F_x\,\dot{x} + F_y\,\dot{y} = 0.$$

Thus $F_a\,\dot{a} = 0$. Since $\dot{a} \neq 0$ we get $F_a = 0$.

Theorem 6 *The envelope of all the normals to a given curve is the locus of centers of curvature – the evolute – of the curve.*

Proof. Let us consider a curve $\mathbf{r}(s) = [x(s), y(s)]$ parameterized by arc length s. The equation of the straight line normal to the curve and passing through a curve point $\mathbf{r}(s) = [x(s), y(s)]$ is given by

$$(\mathbf{p} - \mathbf{r}(s)) \cdot \mathbf{r}'(s) = 0,$$

where $\mathbf{p} = [x, y]$ is a point on the line. Thus the family of all the normals is described by the equation

$$F(\mathbf{p}, s) \equiv (\mathbf{p} - \mathbf{r}(s)) \cdot \mathbf{r}'(s) = 0. \qquad (8)$$

We have:

$$\begin{aligned} \partial F/\partial s \equiv (\mathbf{p} - \mathbf{r}(s)) \cdot \mathbf{r}''(s) - \mathbf{r}'(s) \cdot \mathbf{r}'(s) = \\ = (\mathbf{p} - \mathbf{r}(s)) \cdot \mathbf{n}(s)k(s) - 1 = 0. \end{aligned} \qquad (9)$$

Suppose $\mathbf{p} = \mathbf{r}(s) + \alpha\,\mathbf{t}(s) + \beta\,\mathbf{n}(s)$. From (8) and (9) it follows that $\alpha = 0$ and $\beta = 1/k(s)$. It gives the evolute:

$$\mathbf{p} = \mathbf{r}(s) + \mathbf{n}(s)/k(s).$$

Optically the evolute of a given curve can be described as the locus of points where the light rays emitted by the curve in the normal directions are concentrated. At the singularities of the evolute the concentration is even greater.

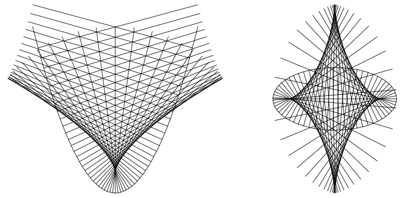


Fig. 9: The envelope of all the normals to a given curve is the evolute of the curve.

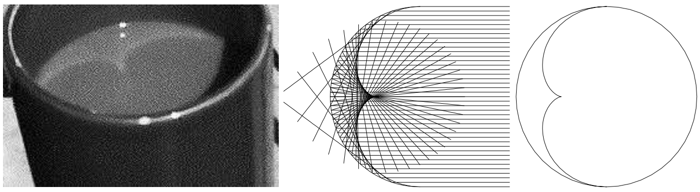


Fig. 10: Coffeecup caustic.

Caustics by Reflection

The envelope of the family of rays is called a **caustic** (i.e. 'burning', since light is concentrated at it). A caustic is clearly visible on the inner surface of a cup when the sun shines on it. See a coffeecup caustic in Fig. 10: drinking from a cylindrical cup in the sunshine one can see a crescent of light as the sunshine reflects from the inside of the cup onto the surface of the drink.

Consider a point O and a curve not passing through O. The locus of the reflections of O with respect to all the tangent lines to the curve is called the **orthotomic** of the curve relative to O, see the left image of Fig. 11.

Let O be a point source of light. The light from O is reflected from the curve according to the usual rule that the reflected ray and the incident ray make equal angles on opposite sides of the normal of the curve.

Theorem 7 *The caustic generated by the reflected rays is the evolute of the orthotomic.*

Proof. Suppose that O is the origin of coordinates. Let the curve be parameterized by arc length s:

$$\mathbf{r} = \mathbf{r}(s) \equiv (x(s), y(s)),$$

$\mathbf{t}(s)$ and $\mathbf{n}(s)$ be the unit tangent and normal vectors respectively, $k(s)$ be the curvature of the curve. Then the orthotomic is given by

$$\rho(s) = 2\,(\mathbf{r}(s) \cdot \mathbf{n}(s))\,\mathbf{n}(s).$$

Differentiating with respect to s yields

$$\dot{\rho}(s) = -2k(s)\,\Big[\big(\mathbf{r}(s) \cdot \mathbf{t}(s)\big)\mathbf{n}(s) + \big(\mathbf{r}(s) \cdot \mathbf{n}(s)\big)\mathbf{t}(s)\Big]$$

and the expression in square brackets is a vector tangent to the orthotomic. Therefore

$$(\mathbf{r}(s) \cdot \mathbf{t}(s))\,\mathbf{t}(s) - (\mathbf{r}(s) \cdot \mathbf{n}(s))\,\mathbf{n}(s)$$

is a vector normal of the orthotomic. Simultaneously this vector coincides with the direction the ray reflected at $\mathbf{r}(s)$. It remains to recall that the evolute can be described as the envelope of the normals. □

Conic Sections and Their Optical Properties

Ellipse. An ellipse given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad a > b > 0,$$

is represented in parametric form by

$$x = a\cos\varphi, \quad y = b\sin\varphi, \qquad 0 \leq \varphi < 2\pi.$$

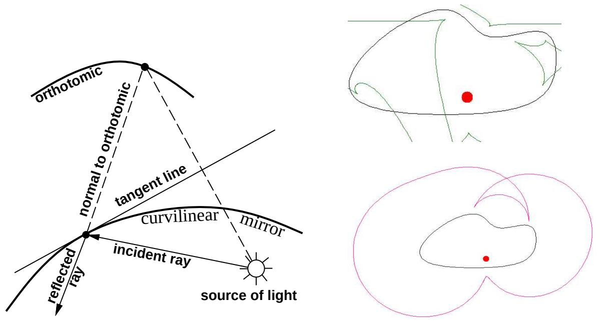


Fig. 11: Left: The orthotomic is formed by the reflections of a given source of light with respect to all the tangent lines. Top-right: A curve (reflector), a source of light, and a caustic generated by the rays reflected by the curve. Bottom-right: The same curve and source of light and the orthotomic curve associated with them.

Let $e = \sqrt{a^2 - b^2}$. The points $\mathbf{f}_\pm = (\pm e, 0)$ are called the **foci** of the ellipse. Let d_+ and d_- be the distances between a point $\mathbf{r}$ on the ellipse and the foci $\mathbf{f}_+$ and $\mathbf{f}_-$, respectively. Then

$$d_+ = a - e\cos\varphi, \quad d_- = a + e\cos\varphi, \quad d_+ + d_- = 2a$$

Therefore an ellipse can be also defined as the locus of all points such that the sum of the distances from each point to two fixed points (the foci) is constant.

The tangent and normal of the ellipse at φ have the equations

$$\frac{\cos\varphi}{a}x + \frac{\sin\varphi}{b}y = 1 \quad \text{and} \quad \frac{a}{\cos\varphi}x + \frac{b}{\sin\varphi}y = e^2,$$

respectively. It is easy to verify that the tangent at $\mathbf{r} = (a\cos\varphi, b\sin\varphi)$ makes equal angles with the line segments connecting $\mathbf{r}$ with the foci $\mathbf{f}_+ = (e, 0)$ and $\mathbf{f}_- = (-e, 0)$.

Thus rays emitted from a focus of an ellipse after being reflected by the ellipse will pass through the other focus of the ellipse.

Hyperbola. A hyperbola given by the equation $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, where $a > b > 0$, is represented in parametric form by

$$x = \frac{a}{\cos\psi}, \quad y = b\tan\psi.$$

Let $e = \sqrt{a^2 + b^2}$. The points $\mathbf{f}_\pm = (\pm e, 0)$ are called the **foci** of the hyperbola. Let d_+ and d_- be the distances between a point r on the hyperbola and the foci $\mathbf{f}_+$ and $\mathbf{f}_-$, respectively. It is easy to demonstrate that $|d_+ - d_-| = 2a$.

Therefore a hyperbola can be defined as a set of points the difference of whose distances from two given points, the foci, is a given positive constant.

The hyperbola has the important property that a ray originating at one focus reflects in such a way that the outgoing path lies along the line from the other focus through the point of intersection.

Parabola. A parabola given by the equation $2ax = y^2$, $a > 0$, can be represented in parametric form by $\{x = 2at^2, y = 2at\}$. The point $\mathbf{f} = (a/2, 0)$ is called the **focus** of the parabola. Calculations show that the distance d between a point $\mathbf{r}(x, y)$ on the parabola and $\mathbf{f}$ is given by $d = x + a/2$. The straight line $x = -a/2$ is called the **focal line** of the parabola.

Thus a parabola can be defined as the locus of all points equidistant from a given line (the focal line) and a given point not on the line (the focus).

It can be shown that rays parallel to the axis of a parabola after being reflected by the parabola will pass through the focus of the parabola.

Ellipse, parabola, and hyperbola for telescope design. Fig. 12 demonstrates how optical properties of the ellipse, hyperbola, and parabola are used in telescope design.

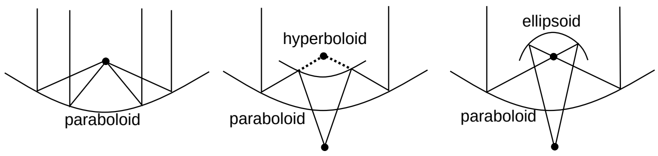


Fig. 12: Telescope configurations: the Newtonian telescope (left) the Cassegrainian telescope (middle), and the Gregorian telescope (right).

Involute

If a line rolls without slipping as a tangent along a curve, then the path of a fixed point on the line forms a new curve called the **involute**. An involute of the curve may be thought as any curve which is always perpendicular to the tangent lines of the original curve.

The involute was introduced by Huygens who used it to improve a pendulum clock mechanism. For a simple pendulum clock, the time of swing of the pendulum is not perfectly constant. It changes slightly if the angle of swing changes. Huygens proposed to change the length of a pendulum in order to make the time of swing of the pendulum the same for all angles. In 1673, Huygens built a clock with a modified pendulum. The top of the pendulum was made of flexible wire which would swing against curved metal cheeks, altering the pendulum's length as it swings. See [1] for Huygens' clock inventions.

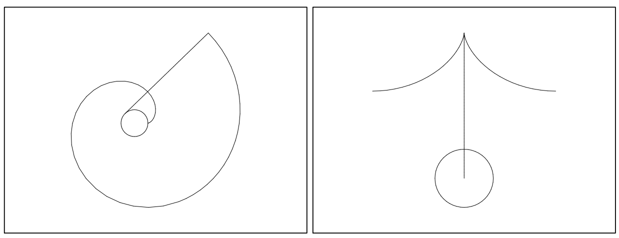


Fig. 13: Left: a circle and its involute. Right: a scheme of Huygens's pendulum with flexible wire and curved metal cheeks.

Dual Curves

I hope to write here about the dual curves in the future.

Offset Curves

There is a set of points, each of which has a constant distance a from a corresponding point of a smooth curve $\mathbf{r} = \mathbf{r}(t) = [x(t), y(t)]$ in the direction of its normal vector at that point. The set forms a curve $\mathbf{o}(t) = \mathbf{r}(t) + \mathbf{n}(t)\,d$, where $\mathbf{n}(t)$ is the unit normal vector at the point $\mathbf{r}(t)$. The set is called an **offset curve** (or d-offset) of the initial curve $\mathbf{r}(t)$.

Offset curves and surfaces arise in many industrial applications, in particular in mathematical modeling of cutting paths for milling machines. For example, let us consider a ball-end milling machine cutting a surface. The trajectory of the center of the machine is R-offset of the surface where R is the radius of the cutting ball.

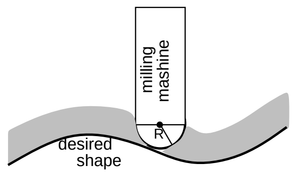


Fig. 14: A ball-end milling machine cutting a desired shape.

Let $k(t)$ be the curvature of the curve $\mathbf{r}(t)$. As long as $d < 1/k(t_0)$, the offset curve $\mathbf{r}(t) + \mathbf{n}(t)\,d$ is smooth at $t = t_0$. If $d = 1/k(t_0)$ then the offset curve is not smooth: a singularity (so called cusp) appears at $t = t_0$. For $d > 1/k(t_0)$ d-offset is not smooth: it has singularities (cusps).

[1] http://www.sciencemuseum.org.uk/collections/exhiblets/huygens/1673.asp

Theorem 8 *For a given curve, the locus of the singularities of the offset curves is the evolute of the curve.*

Proof. Let the curve be parameterized by arc length s: $\mathbf{r} = \mathbf{r}(s)$. Let l be the arc length parameter of the offset curve $\mathbf{o}(s) = \mathbf{r}(s) + \mathbf{n}(s)\, d$. Denote by $\mathbf{t}(s)$ the unit tangent vector of the curve: $\mathbf{t}(s) = \dfrac{d\mathbf{r}}{ds}$. Using the Frenet formulas we get:

$$\frac{d}{ds}\,\mathbf{o}(s) = \frac{d}{ds}\,(\mathbf{r}(s) + \mathbf{n}(s)d) = (1 - k(s)d)\,\mathbf{t}(s).$$

Thus $dl/ds = 1 - k(s)d$,

$$\frac{d\mathbf{t}(s)}{dl} = \frac{ds}{dl}\,\frac{d\mathbf{t}(s)}{ds} = \frac{k(s)\mathbf{n}(s)}{1 - k(s)d}$$

and the curvature of the offset curve equals

$$\frac{k(s)}{(1 - k(s)d}.$$

The denominator vanishes when $d = 1/k(s)$. □

In particular we showed that it is impossible to produce a curve which curvature attains values greater then $1/R$ by a ball-end milling machine with the cutting ball of radius R.

Fig. 15 illustrates properties of the evolute and offset curves.

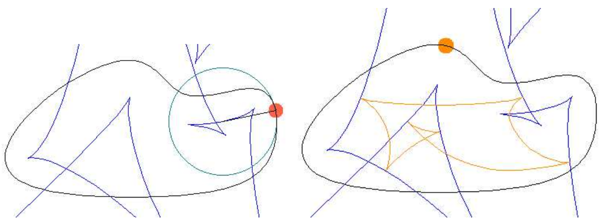


Fig. 15: Left: A curve, its evolute, and an osculating circle; note that the segment connecting the circle center and the tangency point is tangent to the evolute. Right: The same curve, its evolute, and an offset curve; note that the cusps of the offset curve are situated on the evolute.

Medial Axis

The **medial axis** or **skeleton** of a figure F is the set of points inside F that have more than one closest point among the points of ∂F.

We can define the medial axis of F as the locus of centers of circles inside F that are bitangent to ∂F.

The medial axis of F can be also defined as the locus of centers of maximal circles: circles inside F that are not themselves enclosed in any other circle inside F.

Equivalently, the medial axis of F can be defined via the set of points where the offset curves of ∂F meet themselves. Perhaps the most intuitive is the prairie fire analogy. Imagine that the interior of the figure is composed of dry grass and that the exterior, or background, of the figure has no grass. Suppose a fire is set simultaneously at all points along the figure boundary. The fire will propagate, at uniform speed, toward the middle of the figure. At some points, however, the advancing line of the fire from one region of the boundary will intersect the fire front from some other region and the two fronts will extinguish each other. These points are called quench points of the fire; the set of quench points defines the skeleton of the figure.

The above description of the medial axis via first self intersections of the offset curves can be used to demonstrate that the endpoints of the medial axis are located at cusps of the evolute and correspond to curvature extrema.

It is easy to verify the following results:

- The skeleton of a polygon consists only of linear segments and parabolic arcs.
- If the polygon is convex, the skeleton consists only of linear segments.

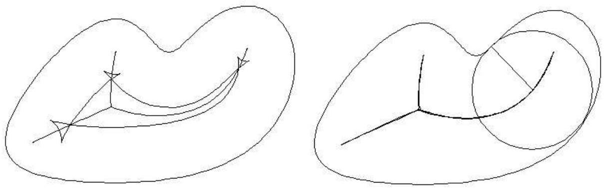


Fig. 16: Left: medial axis is formed by first self-intersections of offset curves. Right: medial axis is formed by centers of inner bitangent circles.

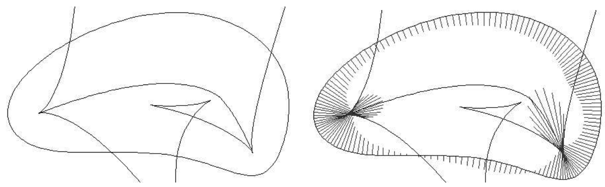


Fig. 17: Left: closed curve, its evolute, and medial axis. Right: curvature profile is added.

- For a figure without holes, the skeleton is a tree (in the graph theoretic sense).

Fig. 18 shows a closed curve and its skeleton.

Medial axis transform (MAT) consists of transforming a given figure to its skeleton. Features of MAT can be used for pattern recognition, e.g. number of nodes, branches etc. Notably, it is applied in the medical field for chromosome identification, see the left image of Fig. 19.

Unfortunately the skeleton of a figure is very sensitive to small perturbations (noise) of the boundary of the figure, as demonstrated in the left image of Fig. 19.

Consider a point on the skeleton and let us move it toward an endpoint of the skeleton. When that moving point approaches the endpoint, the to points of tangency merge and the bitangent circle becomes osculating. That osculating circle has contact of order 3 or higher. Thus the endpoints of the skeleton correspond to the vertices of the curve. Since the skeleton of a figure bounded by a simple closed curve has at least two endpoints, the curve has at least four vertices. This completes the proof of the four-vertex theorem.

Practical Extraction of Medial Axis

Voronoi Diagram. Given a set of points on the plane, the Voronoi diagram of the set is a partition of the plane into regions (Voronoi regions), each of which consists of the points closer to one particular point of the set than to any others. Each Voronoi cell is a convex polygon, and its vertices are called Voronoi vertices. Each Voronoi vertex is equidistant from 3 or more original points.

There are several efficient algorithms to compute the Voronoi diagram of a set of points. [2] [3]

Practical Extraction of Medial Axis via Voronoi Diagram. Since the medial axis of a closed curve and the Voronoi diagram of a polygon have much in common, it is natural to approximate the medial axis by Voronoi vertices of a proper set of points. Given a closed curve,

[2] Joseph O'Rourke. *Computational Geometry in C*. Cambridge Univ. Press.

[3] Mark de Berg, Marc H. Overmars, Otfried Schwarzkopf, Marc Van Kreveld. *Computational Geometry: Algorithms and Applications*. Springer-Verlag.

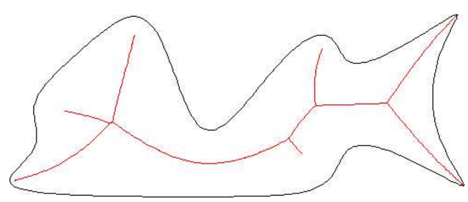


Fig. 18: A closed curve and its skeleton.

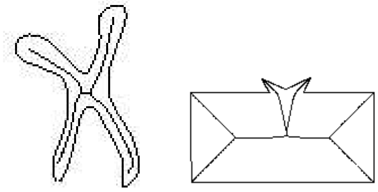


Fig. 19: Left: medial axis transform is used for chromosome identification. Right: the skeleton of a figure is very sensitive to small perturbations of the boundary of the figure.

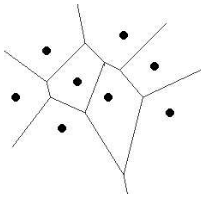


Fig. 20: A set of points and its Voronoi diagram.

let us consider a dense set of points on the curve. The inner Voronoi vertices of the set provides us with a good approximation of the medial axis.

Fig. 21 demonstrates how the approximation of the skeleton via a subset of the Voronoi diagram of the interpolating polygon depends on the sampling rate of the polygonal interpolation.

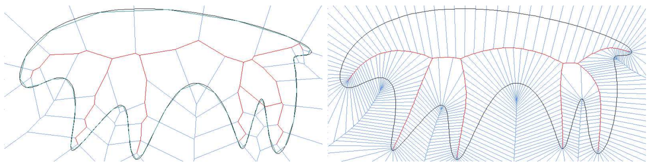


Fig. 21: Left: approximation of the skeleton via a subset of the Voronoi diagram of a sparse set of boundary points. Right: approximation of the skeleton with a dense set of boundary points.

Curve Smoothing

Smoothing reduces small, narrow, and edgy protrusions. Therefore preliminary curve smoothing is very useful for robust extraction of the medial axis and other purposes. The smoothing method presented below is based on curve evolution by curvature. At present it is considered as one of the best smoothing methods.

Consider a smooth curve each point of which moves in the normal direction with speed equal to the curvature at that point. It turns out that such curve evolution has remarkable smoothing properties:

– it reduces quickly small-scale curve oscillations;

– a nonconvex curve becomes convex and then it tends to a circle and simultaneously shrinks into a point;

– no new curvature extrema are generated.

See Fig. 22 where smoothing by curve evolution with speed equal to curvature is combined with rescaling.

Let us describe the curve evolution by curvature mathematically. Consider a family of smooth curves $\mathcal{C}(p,t)$, where p parameterizes the curve and t parameterizes the family, either connecting two given endpoints or closed. We suppose p to be independent of t. The family evolves according to the evolution equation

$$\frac{\partial \mathcal{C}(p,t)}{\partial t} = k\,\mathbf{n}, \qquad \mathcal{C}(p,0) = \mathcal{C}^{(0)}(p), \tag{10}$$

where $\mathbf{n}(p,t)$ is the unit normal vector for $\mathcal{C}(p,t)$, $k(p,t)$ is the curvature of $\mathcal{C}(p,t)$. The family parameter t can be considered as the time duration of the evolution.

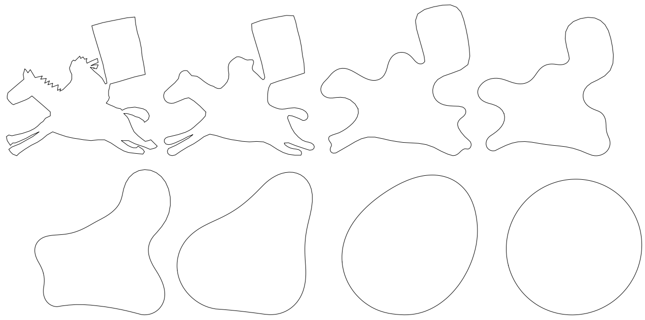


Fig. 22: From left to right, from top to bottom: evolution of a closed curve with speed equal to curvature is combined with rescaling.

To solve (10) numerically we first replace the time derivative term by its forward difference approximation

$$\frac{\partial \mathcal{C}(p,t)}{\partial t} \approx \frac{\mathcal{C}(p,t+\varepsilon) - \mathcal{C}(p,t)}{\varepsilon}, \qquad \varepsilon \ll 1.$$

It gives

$$\mathcal{C}(p,t+\varepsilon) = \mathcal{C}(p,t) + \varepsilon\, k\, \mathbf{n}.$$

We approximate smooth curves by polygonal lines for which we introduce discrete analogs of the curvature, tangent and normal vectors. For a polygonal curve given by its vertices $\{P_i\}_{i=1}^{M}$ we consider the discrete evolution shifting each vertex according to

$$P_i^{(j+1)} = P_i^{(j)} + \varepsilon^{(j)}\, k_i^{(j)}\, \mathbf{n}_i^{(j)}. \tag{11}$$

Here $P_i^{(j)}$ indicates the location of vertex i after k iterations, $\mathbf{n}_i^{(j)}$ is the unit normal vector at $P_i^{(j)}$, $k_i^{(j)}$ is a discrete approximation of the curvature k at $P_i^{(j)}$, the iterative step parameter $\varepsilon^{(j)}$ is sufficiently small. See Fig. 23.

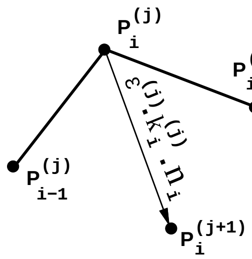


Fig. 23: Discrete curvature flow applied to a polyline.

Practical Curvature Estimation

Circle approximation for curvature estimation. Let us use the circle passing through the points A, O, B as an approximation to the osculating circle to the curve at O, see the left image of Fig. 24. Then

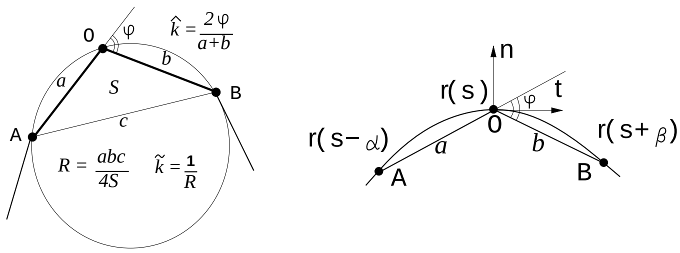


Fig. 24: Circle and angle approximations of curvature.

the inverse value of the radius will serve as an approximation of the curvature at O. Let S denote the area of the triangle AOB. The discrete curvature at O is given by

$$\widetilde{k} = \frac{4S}{abc}. \tag{12}$$

Angle approximation for curvature estimation. Another idea to define the curvature of a polygonal line is based on the definition of the curvature as the rate of change of the angle between the tangent and the positive direction of the x-axis when we proceed along the curve. Let φ denote the turn angle at O (see Fig. 24). Let us define the discrete curvature at O as

$$\widehat{k} = \frac{2\varphi}{a+b}. \tag{13}$$

Curvature vector approximation via derivatives. One can estimate the tangent, normal, and curvature vectors using finite difference approximations of curve derivatives. The first derivative of $\mathbf{r}(s)$ at O can be approximated by

$$\mathbf{r}' = \mathbf{t} \approx \frac{\mathbf{r}(B)-\mathbf{r}(O)}{b} + \frac{\mathbf{r}(O)-\mathbf{r}(A)}{a} - \frac{\mathbf{r}(B)-\mathbf{r}(A)}{a+b}$$

The second derivative of $\mathbf{r}(s)$ at O can be approximated as follows

$$\mathbf{r}'' = \mathbf{t}' = k\,\mathbf{n} \approx \frac{2\mathbf{r}(A)}{a(a+b)} - \frac{2\mathbf{r}(O)}{ab} + \frac{2\mathbf{r}(B)}{b(a+b)} \tag{14}$$

Space Curves

Velocity and acceleration. A space curve can be specified by the parametric equations $x = x(t)$, $y = y(t)$, $z = z(t)$ or, in vector notations, $\mathbf{r} = \mathbf{r}(t)$.

If t is time, then $d\mathbf{r}/dt$ is the velocity vector of the end point of the vector $\mathbf{r}$, and $d^2\mathbf{r}/dt^2$ is the acceleration vector.

Arc length. Consider a space curve described parametrically

$$\mathbf{r}(t) = \big[x(t), y(t), z(t)\big].$$

If $s(t)$ is the arc length from the point on the curve corresponding to the parameter value $t = t_0$ to the general point of the curve then

$$s(t) = \int_{t_0}^{t} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_{t_0}^{t} \left|\frac{d\mathbf{r}}{dt}\right| dt.$$

Osculating plane. For a curve, the tangent can be defined as the limiting position of the line passing through two nearby points of the curve when one of these points approaches the second. For a space curve, the **osculating plane** can be defined as the limiting position of the plane passing through three nearby points of the curve when two of these points approaches the third.

Let $P_1 = \mathbf{r}(t_1)$, $P_2 = \mathbf{r}(t_2)$, $P_3 = \mathbf{r}(t_3)$ be three points on a curve given parametrically

$$\mathbf{r}(t) = \big[x(t), y(t), z(t)\big]$$

Consider the plane $\mathbf{p}\cdot\mathbf{a} = b$ passing through the points P_1, P_2, P_3. Here the vector $\mathbf{p}$ defines a generic point $P = (x, y, z)$ on the plane, the vector $\mathbf{a}$ is orthogonal to the plane, and b is constant. Then the function

$$f(t) = \mathbf{r}(t)\cdot\mathbf{a} - b$$

has zeros at $t = t_1$, $t = t_2$, and $t = t_3$:

$$f(t_1) = 0, \quad f(t_2) = 0, \quad f(t_3) = 0.$$

Hence, according to Rolle's theorem,

$$f'(t_4) = 0 = f'(t_5), \quad f''(t_6) = 0$$

for some $t_4 \in (t_1, t_2)$, $t_5 \in (t_2, t_3)$, and $t_6 \in (t_4, t_5)$. When P_1, P_2, P_3 approaches some P_0 as $t_1, t_2, t_3 \to t_0$, we obtain, for the limiting values of $\mathbf{a}$ and b, the conditions:

$$\begin{aligned} f(t_0) &= \mathbf{r}(t_0)\cdot\mathbf{a} - b = 0 \\ f'(t_0) &= \dot{\mathbf{r}}(t_0)\cdot\mathbf{a} = 0 \\ f''(t_0) &= \ddot{\mathbf{r}}(t_0)\cdot\mathbf{a} = 0 \end{aligned}$$

Thus the vectors $\mathbf{p} - \mathbf{r}(t_0)$, $\dot{\mathbf{r}}(\mathbf{t_0})$, $\ddot{\mathbf{r}}(\mathbf{t_0})$ belong to the same plane and, therefore, are linearly dependent. It gives the following equation

$$\begin{vmatrix} x - x(t_0) & y - y(t_0) & z - z(t_0) \\ \dot{x}(t_0) & \dot{y}(t_0) & \dot{z}(t_0) \\ \ddot{x}(t_0) & \ddot{y}(t_0) & \ddot{z}(t_0) \end{vmatrix} = 0$$

describing the osculating plane.

Principal normal. Unlike the planar case, there is an infinity normals to the curve at each point. The line in the osculating plane at P perpendicular to the tangent line is called the **principal normal**. In its direction let us place a unit vector $\mathbf{n}$ such that **n**(P) changes continuously when P moves along the curve.

Curvature vector. If the curve $\mathbf{r}(s) = \big[x(s), y(s), z(s)\big]$ is parameterized by arc length s, then $\mathbf{t}(s) = d\mathbf{r}(s)/ds$ is a unit tangent vector:

$$\left|\frac{d\mathbf{r}}{dt}\right| = \frac{ds}{dt} \implies \left|\frac{d\mathbf{r}}{ds}\right| = 1.$$

The vector $\mathbf{k} = d\mathbf{t}(s)/ds$ belongs to the osculating plane and perpendicular to $\mathbf{t}$. We can therefore introduce a proportionality factor

$$\mathbf{k} = d\mathbf{t}(s)/ds = k\,\mathbf{n}.$$

The vector $\mathbf{k}$, which express the rate of change of the tangent when we proceed along the curve, is called the **curvature vector**. The factor k is called the **curvature**.

Osculating circle. For a space curve, the **osculating circle** or **circle of curvature** can be defined as the limiting position of the circle passing through three nearby points of the curve when two of these points approaches the third. For a point P_0 on the curve, the center of the osculating circle lies on the principal normal at distance $R = |k|^{-1}$ from P_0, where k is the curvature at P_0. The center of the osculating circle is called the **center of curvature** and $R = |k|^{-1}$ is called the **curvature radius**.

It can be shown that the best approximation of the curve by a circle is given by the osculating circle.

Torsion. The curvature measures the rate of change of the tangent when moving along the curve. Let us now introduce, the **torsion**, a quantity measuring the rate of change of the osculating plane.

A curve normal orthogonal to the osculating plane is called the **binormal**. It is given by $\mathbf{b} = \mathbf{t}\times\mathbf{n}$.

It is easy to show that $d\mathbf{b}/ds$ is parallel to $\mathbf{n}$:

$$\mathbf{b} \perp \frac{d\mathbf{b}}{ds} = \frac{d\,(\mathbf{t}\times\mathbf{n})}{ds} = \frac{d\mathbf{t}}{ds}\times\mathbf{n} + \mathbf{t}\times\frac{d\mathbf{n}}{ds} = \mathbf{t}\times\frac{d\mathbf{n}}{ds} \perp \mathbf{t}$$

The torsion τ is defined by the formula

$$\frac{d\mathbf{b}}{ds} = -\tau\mathbf{n}.$$

If $\tau = 0$ for every point on a given space curve, then the curve is planar.

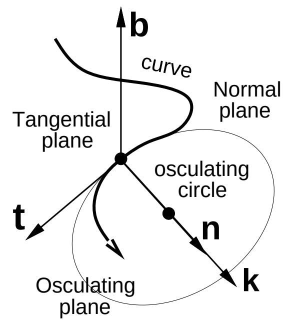


Fig. 25: Geometric attributes of space curve.

Frenet formulas. The orthogonal frame

$$\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$$

is known as the **Frenet frame** at the point on the curve. The Frenet frame rotates as the point moves along the curve.

The orthonormal expansion of $d\mathbf{n}/ds$ relative to the Frenet frame yields

$$\frac{d\mathbf{n}}{ds} = \left(\frac{d\mathbf{n}}{ds} \cdot \mathbf{t}\right) \mathbf{t} + \left(\frac{d\mathbf{n}}{ds} \cdot \mathbf{n}\right) \mathbf{n} + \left(\frac{d\mathbf{n}}{ds} \cdot \mathbf{b}\right) \mathbf{b}.$$

Note that

$$\frac{d\mathbf{n}}{ds} \cdot \mathbf{t} = -\mathbf{n}\frac{d\mathbf{t}}{ds} = -k, \quad \frac{d\mathbf{n}}{ds} \cdot \mathbf{n} = 0, \quad \frac{d\mathbf{n}}{ds} \cdot \mathbf{b} = -\mathbf{n}\frac{d\mathbf{b}}{ds} = \tau.$$

Thus

$$\frac{d\mathbf{n}}{ds} = -k\mathbf{t} + \tau\mathbf{b}.$$

Grouping the above formulas together we arrive at the Frenet formulas for the space curves:

$$\begin{cases} d\mathbf{t}/ds &= k\mathbf{n}, \\ d\mathbf{n}/ds &= -k\mathbf{t} + \tau\mathbf{b}, \\ d\mathbf{b}/ds &= -\tau\mathbf{n}. \end{cases}$$

Curvature and torsion for arbitrary parameterized curve. Consider a space curve $\mathbf{r}(t) = \left[x(t), y(t), z(t)\right]$ parameterized by a parameter t. Since $(ds/dt = |\dot{\mathbf{r}}|$, where s is arc length, we have

$$\mathbf{t} = \frac{\dot{\mathbf{r}}(t)}{|\dot{\mathbf{r}}(t)|}, \quad \mathbf{b} = \frac{\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t)}{|\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t)|}, \quad \mathbf{n} = \mathbf{b} \times \mathbf{t}.$$

Notice that

$$\ddot{\mathbf{r}}(t) = \frac{d^2\mathbf{r}}{ds^2}\left(\frac{ds}{dt}\right)^2 + \frac{d\mathbf{r}}{ds}\frac{d^2s}{dt^2} = k\mathbf{n}|\mathbf{r}|^2 + \mathbf{t}\frac{d^2s}{dt^2}. \tag{15}$$

Thus $\dfrac{d^2s}{dt^2} = \ddot{\mathbf{r}} \cdot \mathbf{t}$ and therefore the curvature vector k is given by

$$\mathbf{k} = k\mathbf{n} = \frac{\ddot{\mathbf{r}} - \mathbf{t}\left(\ddot{\mathbf{r}} \cdot \mathbf{t}\right)}{\dot{\mathbf{r}}^2}$$

The curvature $k(t)$ can be also computed if we multiply (15) by $\mathbf{n}$

$$k = \frac{\ddot{\mathbf{r}} \cdot \mathbf{n}}{\dot{\mathbf{r}}^2}$$

Notice also that $\mathbf{t} \times \ddot{\mathbf{r}} = k\mathbf{t} \times \mathbf{n}\,(ds/dt)^2 = k\mathbf{b}|\dot{\mathbf{r}}|^2$ and thus

$$k\mathbf{b} = \frac{\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t)}{|\dot{\mathbf{r}}(t)|^3} \tag{16}$$

The torsion $\tau(t)$ is computed in a similar way. Differentiating (15) by t yields

$$\dddot{\mathbf{r}} = \frac{d^3\mathbf{r}}{dt^3} = \mathbf{n}(\ldots) + k\frac{d\mathbf{n}}{ds}\left(\frac{ds}{dt}\right)^3 + \frac{d\mathbf{t}}{ds}\frac{ds}{dt}\frac{d^2s}{dt^2} + \mathbf{t}\frac{d^3s}{dt^2}.$$

In view of the Frenet formulas the scalar product between the above equation and $\mathbf{b}$ yields

$$\mathbf{b} \cdot \dddot{\mathbf{r}} = k\tau|\dot{\mathbf{r}}|^3.$$

Taking into account (16) we have

$$k\tau|\dot{\mathbf{r}}|^3 = \frac{(\dot{\mathbf{r}} \times \ddot{\mathbf{r}}) \cdot \dddot{\mathbf{r}}}{|\dot{\mathbf{r}}|^3}, \quad k = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3}$$

and finally arrive at

$$\tau = \frac{(\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t)) \cdot \dddot{\mathbf{r}}(t)}{|\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t)|^2}.$$

Notice that $k(t)$ is a second-derivative quantity while $\tau(t)$ is a third-derivative quantity.

Local shape of space curve. Let us approximate shape of a space curve $\mathbf{r}(s)$ parameterized by arc length s. By expanding $\mathbf{r}(s)$ into the Taylor series, with $\mathbf{r}(0)$ as the origin, we get:

$$\mathbf{r}(s) = \mathbf{r}'s + \mathbf{r}''\frac{s^2}{2} + \mathbf{r}'''\frac{s^3}{6} + \ldots = \left(s - k^2\frac{s^3}{6} + \ldots\right)\mathbf{t} +$$

$$+ \left(k\frac{s^2}{2} + k'\frac{s^3}{6} + \ldots\right)\mathbf{n} + \left(k\tau\frac{s^3}{6} + \ldots\right)\mathbf{b},$$

where a prime mark indicates differentiation with respect to arc length s. To express the shape of the curve near the origin with $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ as the coordinate axes, let us take the first term of each coefficient of $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ in the previous equation and replace $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ by x, y and z respectively. It gives

$$x = s, \quad y = k\frac{s^2}{2}, \quad z = k\tau\frac{s^3}{6}.$$

Eliminating s we obtain

$$y = \frac{1}{2}kx^2, \quad y^3 = \frac{9}{2}\frac{k}{\tau^2}z^2, \quad z = \frac{1}{6}k\tau x^3.$$

From these equations we can analyze curve local shape. The projection of the curve on the xy plane (the osculating plane) is a symmetrical second-degree curve tangent to the x axis at the origin. The projection on the yz plane (the normal plane) is symmetrical with respect to the z axis and has a cusp at the origin. The projection on the zx plane (the tangential plane) is anti-symmetrical with respect to the origin and tangent to the x axis. See Fig. 26 below.

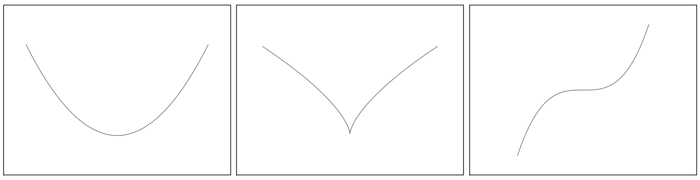


Fig. 26: Projection of a space curve onto the osculating plane (left), normal plane (middle), and tangential plane (right).

Applications to Image Processing

A **grey-scale** image consists of an 2D array of pixels where every pixel is assigned a positive number characterizing the pixel intensity. For example, the following image

consists of 24×9 pixels and is represented by the array shown below.

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	20	20	20	0	12	0	0	0	12	0	0	18	18	0	0	24	24	24	0	31	31	31	0
0	0	20	0	0	12	12	0	12	12	0	18	0	18	0	24	0	0	0	0	31	0	0	0
0	0	20	0	0	12	0	12	0	12	0	18	0	18	0	24	0	0	0	0	31	0	0	0
0	0	20	0	0	12	0	0	0	12	0	18	18	18	0	24	0	24	24	0	31	31	0	0
0	0	20	0	0	12	0	0	0	12	0	18	0	18	0	24	0	0	24	0	31	0	0	0
0	0	20	0	0	12	0	0	0	12	0	18	0	18	0	24	0	0	24	0	31	0	0	0
0	20	20	20	0	12	0	0	0	12	0	18	0	18	0	24	24	24	24	0	31	31	31	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

For the above image, the intensity values vary from 0 (black) to 31 (white). Usually the range of intensity values extends from 0 (black) to 255 (white).

Often it is convenient to forget that a grey-scale image has a discrete nature and think that the image is given by a positive continuous function $I(x, y)$ defined on a rectangular domain $[0, a] \times [0, b]$ where the intensity (brightness) at (x, y) is given $I(x, y)$.

A **binary image** is a special case where at each pixel the intensity function takes on one of two values, black and white. Plane figures are often represented by binary images.

Active contour models (snakes) in image processing. Curvature-driven curve evolutions are used in image processing for detection of object boundaries.

Let us introduce a positive function $g(x,y)$ that achieves its minimal values where $\nabla I(x,y)$ is large. For example, such a function can be defined by

$$g(x,y) = \frac{1}{1 + |\nabla I(x,y)|^2}.$$

Consider a contour $\mathbf{c}^{(0)}(p)$ oriented by its inner normal and enclosing an object whose boundary we want to detect. Let us move all the points of the contour along the contour normals with speed equal to $g(x,y)(k+c)$ where k is the curvature of the contour and c is a positive constant, see Fig. 27.

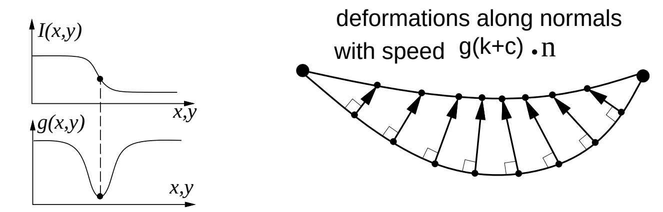


Fig. 27: Geometric idea of using active contours for detection of object boundaries.

We can represent the contour evolution as a family of contours $\mathbf{r}(p,t)$ where t parameterizes the family and p parameterizes the curve $\mathbf{r}(p,t)$. The family of contours $\mathbf{r}(p,t)$ satisfy the following initial value problem

$$\left.\begin{array}{rcl} \partial\mathbf{r}/\partial t &=& g(\mathbf{r})(k(\mathbf{r})+c)\mathbf{n}(\mathbf{r},t), \\ \mathbf{r}(p,0) &=& \mathbf{r}^{(0)}(p), \end{array}\right\} \tag{17}$$

where c is constant.

Let the evolving contour $\mathbf{r}(p,t)$ converges to a limit contour $\mathbf{r}^{(\infty)}(p)$. Thus $g(x,y)(k+c) = 0$ along $\mathbf{r}^{(\infty)}$. So $\mathbf{r}^{(\infty)}$ is close to object boundary.

A similar but more complex contour evolution provides a user with an efficient tool for detecting objects in grey-scale images. Fig. 28 demonstrates tumor area detection in a shaded MRI image of human brain.

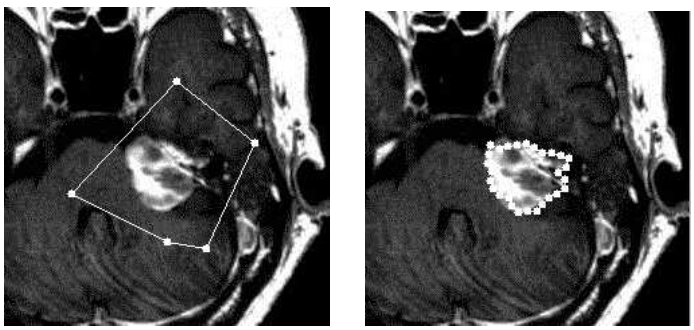


Fig. 28: Tumor area detection with active contours.

Practical skeleton extraction. Let a figure F is represented by a binary image $I(x,y)$ such that $I(x,y) = 0$ (black) if the pixel (x,y) is inside F and $I(x,y) = 1$ (white) otherwise.

The **distance transform** is an operator that maps a binary image into a grey-level image. The pixel values of the grey level image represent a distance from the border of the binary image. The three most common distances for this transform are: Euclidean, city-block (4-connected), and chess-board (8-connected) distances.

Let $I_0(x,y)$ be a binary image. It is used to compute grey-scale images $I_1(x,y), I_2(x,y), \ldots, I_k(x,y)$ according to the following iterative algorithm:

1. $I_k(x,y) = I_0(x,y) + \min I_{k-1}(p,q)$, where minimum is taken for all (p,q) such that $|(p,q)-(x,y)| \le 1$

2. Terminate if $I_k(x,y) = I_{k-1}(x,y)$.

The skeleton of $I_0(x,y)$ is given by all points (x,y) such that $I_k(x,y) \ge I_k(p,q)$ for all (p,q) such that $|(p,q)-(x,y)| \le 1$.

$$\begin{array}{cccccccc} 0&0&0&0&0&0&0&0\\ 0&1&1&1&1&1&1&0\\ 0&1&1&1&1&1&1&0\\ 0&1&1&1&1&1&1&0\\ 0&1&1&1&1&1&1&0\\ 0&1&1&1&1&1&1&0\\ 0&0&0&0&0&0&0&0 \end{array} \longrightarrow \begin{array}{cccccccc} \mathbf{0}&0&0&0&0&0&0&\mathbf{0}\\ 0&\mathbf{1}&1&1&1&1&\mathbf{1}&0\\ 0&1&\mathbf{2}&2&2&\mathbf{2}&1&0\\ 0&1&2&\mathbf{3}&\mathbf{3}&2&1&0\\ 0&1&\mathbf{2}&2&2&\mathbf{2}&1&0\\ 0&\mathbf{1}&1&1&1&1&\mathbf{1}&0\\ \mathbf{0}&0&0&0&0&0&0&\mathbf{0} \end{array}$$

Additional Sources of Information

1. Internet

2. M. Hosaka. *Modeling of Curves and Surfaces in CAD/CAM*. Springer-Verlag, 1992. Chapter 5.

3. D. J. Struik. *Lectures on Classical Differential Geometry*. Dover, 1988. Chapter 1.

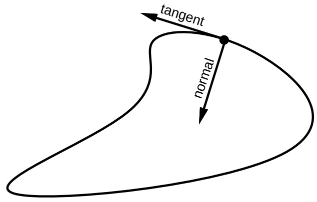
4. J. Gallier. *Geometric Methods and Applications for Computer Science and Engineering*. Springer-Verlag, 2000.

5. M. Berger and B. Gostiaux. *Differential Geometry: Manifolds, Curves, and Surfaces*. Graduate Texts in Mathematics, Vol 115. Springer-Verlag, 1988.

6. J. W. Bruce and P. J. Giblin. *Curves and Singularities: A Geometrical Introduction to Singularity Theory*. Cambridge University Press, 1992.

Problems

1. Consider a closed curve $\mathbf{r}(t)$ from the figure below. The curve is oriented with respect to its inward normal. Let $k(t)$ be the curvature of the curve. Sketch the graph of $k(t)$. Indicate the curvature maxima, minima, and zeros in figure below.



2. Find the curvature and an equation for the circle of curvature of the curve $y = \sin x$ at the point $(\pi/2, 1)$.

3. Fig. 3 shows curves reconstructed from the following curvature functions

$$k_1(s) = \frac{s^2 - 1}{s^2 + 1},\ k_2(s) = s,\ k_3(s) = s^3 - 4s,\ k_4(s) = \sin(s)s.$$

   Determine which curve corresponds to which curvature. Briefly explain your reasoning.

4. Find the curvature of the curve $\mathbf{r}(t) = (\cos t, t, \sin t)$.

5. Consider a family of segments of length $2a$ such that for any segment one of its end-point lies on the x-axis and another end-point lies on the y-axis. Find the curvature of the curve consisting of the centers of the segments.

6. A circular disk of radius 1 in the plane rolls without slipping along the x-axis. The curve described by a point on the circumference is called a **cycloid**.

   (a) Obtain a parameterization of the cycloid.

   (b) Find the curvature at the highest point of the cycloid.

   (c) Find the length of a part of the cycloid corresponding to a complete rotation of a disk.

7. Consider a Bézier curve of degree n and its endpoint. Show that the curvature of the curve at the endpoint is determined by the triangle formed by the endpoint and the next two control points. Find a geometric formula for the curvature.

8. Find the the envelope of the family of straight lines $2ax - y = a^2$, where a parameterizes the family.

9. Find the envelope of the family of curves $t(y + t^2) - x^3 = 0$, where t parameterizes the family.

10. Prove that the evolute of an involute of a given curve is the curve itself.

11. Show that the sum of the distances from each point of an ellipse to the foci is constant.

12. Prove that the rays emitted from a focus of an ellipse after being reflected by the ellipse will pass through the other focus of the ellipse.

13. Suppose the incident rays are parallel to a given vector $\mathbf{v}$. For a given curve find the caustic generated by the reflected rays.

14. Let the reflector be a circle and a source of light be a point inside the circle. Plot the caustic generated by the reflected rays.

15. Propose a method to estimate the torsion of a space curve approximated by a polyline.

16. A space curve can be reconstructed (up to rigid transformations) from its curvature and torsion functions via the Frenet equations. Describe a curve with constant curvature and torsion.