University of Lille I
PC first year **list of exercises n° 8**

Structure of vector space

Exercise 1 1. Using the addition + and the multiplication $\cdot$ of two numbers, define, for each set E in the list below :
- an addition $\oplus\ : E \times E \to E$;
- a multiplication by real numbers $\odot\ : \mathbb{R} \times E \to E$.

(a) $E = \mathbb{R}^n$;

(b) $E =$ the set of trajectories of a point particle in $\mathbb{R}^3$;

(c) $E =$ the set of solutions $(x, y, z) \in \mathbb{R}^3$ of the equation $\mathcal{S}_1\ : x - 2y + 3z = 0$;

(d) $E =$ the set of solutions $(x, y, z) \in \mathbb{R}^3$ of the system of equations $\mathcal{S}_2\ : \begin{cases} 2x + 4y - 6z = 0 \\ y + z = 0 \end{cases}$;

(e) $E =$ the set of solutions of the differential equation $y'' + 2y' - 3y = 0$;

(f) $E =$ the set of functions $y(x)$ such that

$$y''(x) \sin x + x^3 y'(x) + y(x) \log x = 0, \ \ \forall x > 0 \ ;$$

(g) $E =$ the set of complex valued functions $\Psi(t, x)$ that are solutions of the Schrödinger equation :

$$i\hbar \frac{\partial}{\partial t} \Psi(t, x) = -\frac{\hbar}{2m} \frac{\partial^2}{\partial x^2} \Psi(x, t) + x^2 \Psi(t, x)$$

where $\hbar$ and m are constants ;

(h) $E =$ the set of sequences $(x_n)_{n \in \mathbb{N}}$ of real numbers ;

(i) $E =$ the set of polynomials $P(x)$ with real coefficients ;

(j) $E =$ the set of polynomials $P(x)$ of degree less than or equal to 3 with real coefficients ;

(k) $E =$ the set of polynomials $P(x)$ with real coefficients divisible by $(x - 1)$;

(l) $E =$ the set of continuous functions on the interval $[0, 1]$ taking real values ;

(m) $E =$ the set of continuous functions on the interval $[0, 1]$ taking real values and whose integral is zero ;

(n) $E =$ the set of differentiable functions on the interval $(0, 1)$ taking real values ;

(o) $E =$ the set of real functions which vanish at $0 \in \mathbb{R}$;

(p) $E =$ the set of real functions having 0 as a limit when x goes to $+\infty$.

2. For the previously defined additions $\oplus$, show that E admits a neutral element (expression to be defined), and that each element in E admits an inverse.

Solution of Exercise 1 :

1. (a) For $E = \mathbb{R}^n$, let us define $\oplus$ and $\odot$ by

$$(x_1, x_2, \dots, x_n) \oplus (y_1, y_2, \dots, y_n) = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n),$$

and

$$\lambda \odot (x_1, x_2, \dots, x_n) = (\lambda \cdot x_1, \lambda \cdot x_2, \dots, \lambda \cdot x_n).$$

The results belong to $\mathbb{R}^n$, hence $\oplus\ : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\odot\ : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ are well-defined.

(b) Let E be the set of trajectories of a point particle in $\mathbb{R}^3$, i.e. the set of functions from $\mathbb{R}^+$ to $\mathbb{R}^3$. Take two such functions f_1 and f_2, and any real number $\lambda \in \mathbb{R}$. Define $f_1 \oplus f_2$ by

$$(f_1 \oplus f_2)(x) = f_1(x) + f_2(x),$$

and $\lambda \odot f_1$ by $(\lambda \odot f_1)(x) = \lambda \cdot f_1(x)$. The results of such operations are again in E, hence $\oplus\ : E \times E \to E$ and $\odot\ : \mathbb{R} \times E \to E$ are well-defined.

(c) – Define the addition $\oplus$ on $E = \{(x, y, z) \in \mathbb{R}^3,\ \ x - 2y + 3z = 0\}$ by the following formula

$$(x_1, y_1, z_1) \oplus (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2),$$

where (x_1, y_1, z_1) and (x_2, y_2, z_2) belong to E. Let us check that this indeed defines a map $\oplus\ : E \times E \to E$. One has

$$(x_1 + x_2) - 2(y_1 + y_2) + 3(z_1 + z_2) = (x_1 - 2y_1 + 3z_1) + (x_2 - 2y_2 + 3z_2) = 0 + 0 = 0,$$

hence $(x_1, y_1, z_1) \oplus (x_2, y_2, z_2)$ belongs to E whenever (x_1, y_1, z_1) and (x_2, y_2, z_2) belong to E.

– Similarly define the product $\odot$ by the formula

$$\lambda \odot (x, y, z) = (\lambda \cdot x, \lambda \cdot y, \lambda \cdot z),$$

where $\lambda \in \mathbb{R}$ and $(x, y, z) \in E$. One has

$$\lambda \cdot x - 2\lambda \cdot y + 3\lambda \cdot z = \lambda(x - 2y + 3z) = \lambda \cdot 0 = 0,$$

hence $\lambda \odot (x, y, z)$ belong to E whenever (x, y, z) belong to E.

(d) Let E be the set of solutions $(x, y, z) \in \mathbb{R}^3$ of the system of equations $\mathcal{S}_2\ :\begin{cases} 2x + 4y - 6z = 0 \\ y + z = 0 \end{cases}$.
As previously define the addition $\oplus$ on E by $(x_1, y_1, z_1) \oplus (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$ for (x_1, y_1, z_1) and (x_2, y_2, z_2) in E, and the product $\odot$ by $\lambda \odot (x, y, z) = (\lambda \cdot x, \lambda \cdot y, \lambda \cdot z)$ for $\lambda \in \mathbb{R}$ and $(x, y, z) \in E$. The same kind of computations as in (a) show that $(x_1, y_1, z_1) \oplus (x_2, y_2, z_2)$ and $\lambda \odot (x_1, y_1, z_1)$ belong to E whenever (x_1, y_1, z_1) and (x_2, y_2, z_2) belong to E and $\lambda \in \mathbb{R}$.

(e) – Let $E = \{y\ : \mathbb{R} \to \mathbb{R},\ y'' + 2y' - 3y = 0\}$. For two functions y_1 and y_2 in E define a new function $y_1 \oplus y_2\ : \mathbb{R} \to \mathbb{R}$ by

$$(y_1 \oplus y_2)(x) = y_1(x) + y_2(x),$$

i.e. $(y_1 \oplus y_2)$ is the sum of the two function y_1 and y_2. Since the derivative of a sum is the sum of the derivatives, $(y_1 \oplus y_2)' = y_1' + y_2'$. Now

$$(y_1 \oplus y_2)'' + 2(y_1 \oplus y_2)' - 3(y_1 \oplus y_2) = (y_1'' + 2y_1' - 3y_1) + (y_1'' + 2y_1' - 3y_1) = 0,$$

hence the sum of two solutions of the given differential equation is again a solution of the same equation.

– For $\lambda \in \mathbb{R}$ and $y \in E$, define $\lambda \odot y$ by

$$(\lambda \odot y)(x) = \lambda \cdot y(x), \forall x \in \mathbb{R},$$

hence $\lambda \odot y$ is the usual product of the function y by the constant λ. Since $(\lambda \odot y)' = \lambda \odot y'$, one has

$$(\lambda \odot y)'' + 2(\lambda \odot y)' - 3(\lambda \odot y)' = \lambda \cdot (y'' + 2y' - 3y) = 0,$$

hence the product of a solution of the given differential equation by a real number is again a solution.

(f) Let E be the set of functions $y(x)$ such that

$$y''(x)\sin x + x^3 y'(x) + y(x)\log x = 0, \ \ \forall x > 0.$$

As in (c), define $y_1 \oplus y_2 \ : \mathbb{R}^+ \to \mathbb{R}$ by $(y_1 \oplus y_2)(x) = y_1(x) + y_2(x)$, and $\lambda \odot y \ : \mathbb{R}^+ \to \mathbb{R}$ by $(\lambda \odot y)(x) = \lambda \cdot y(x), \forall x \in \mathbb{R}^+$. One has

$$\begin{aligned}&(y_1 \oplus y_2)'' \sin x + x^3 (y_1 \oplus y_2)' + (y_1 \oplus y_2)\log x = \\ &(y_1''(x)\sin x + x^3 y_1'(x) + y_1(x)\log x) + (y_2''(x)\sin x + x^3 y_2'(x) + y_2(x)\log x) = 0,\end{aligned}$$

and

$$(\lambda \odot y)'' \sin x + x^3 (\lambda \odot y)' + (\lambda \odot y)\log x = \lambda \cdot (y''(x)\sin x + x^3 y'(x) + y(x)\log x) = 0.$$

Hence the sum of two solutions and the product of a solution by a real number are again solutions of E.

(g) – Let E be the set of complex valued functions $\Psi(t,x)$ that are solutions of the Schrödinger equation :

$$i\hbar \frac{\partial}{\partial t}\Psi(t,x) = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + x^2\Psi(t,x),$$

where $\hbar$ and m are constants. Let us define the addition $\oplus$ on E by the usual sum $+$ of functions. One has to show that the sum of two solutions of the Schrödinger equation also satisfies the same equation. Let Ψ_1 and Ψ_2 be two elements of E. One has

$$\frac{\partial}{\partial t}(\Psi_1 \oplus \Psi_2) = \frac{\partial}{\partial t}\Psi_1 \oplus \frac{\partial}{\partial t}\Psi_2, \tag{1}$$

$$\frac{\partial^2}{\partial x^2}(\Psi_1 \oplus \Psi_2) = \frac{\partial^2}{\partial x^2}\Psi_1 \oplus \frac{\partial^2}{\partial x^2}\Psi_2, \tag{2}$$

and

$$x^2(\Psi_1 \oplus \Psi_2) = x^2\Psi_1 \oplus x^2\Psi_2. \tag{3}$$

Therefore by (1) and the Schrödinger equation applied to Ψ_1 and Ψ_2

$$\begin{aligned}i\hbar\tfrac{\partial}{\partial t}(\Psi_1 \oplus \Psi_2) &= i\hbar\tfrac{\partial}{\partial t}\Psi_1 + i\hbar\tfrac{\partial}{\partial t}\Psi_2 \\ = -\tfrac{\hbar}{2m}\tfrac{\partial^2}{\partial x^2}\Psi_1 &+ x^2\Psi_1 - \tfrac{\hbar}{2m}\tfrac{\partial^2}{\partial x^2}\Psi_2 + x^2\Psi_2.\end{aligned}$$

Now by (2) and (3),

$$i\hbar\frac{\partial}{\partial t}(\Psi_1 \oplus \Psi_2) = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}(\Psi_1 \oplus \Psi_2) + x^2(\Psi_1 \oplus \Psi_2).$$

– Similarly, define the product $\odot$ by the usual product $\cdot$ of a function by real numbers. Using that $\frac{\partial}{\partial t}(\lambda \cdot \Psi) = \lambda \cdot \frac{\partial}{\partial t}\Psi$, $\frac{\partial^2}{\partial x^2}(\lambda \cdot \Psi) = \lambda \cdot \frac{\partial^2}{\partial x^2}\Psi$ and $x^2(\lambda \cdot \Psi) = \lambda \cdot x^2\Psi$, one checks that, for any $\lambda \in \mathbb{R}$ and any $\Psi \in E$, $\lambda \cdot \Psi$ is a solution of the Schrödinger equation.

(h) Let E be the set of sequences $(x_n)_{n\in\mathbb{N}}$ of real numbers. For $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in E and $\lambda \in \mathbb{R}$, set

$$(x_n)_{n\in\mathbb{N}} \oplus (y_n)_{n\in\mathbb{N}} = (x_n + y_n)_{n\in\mathbb{N}}$$

and

$$\lambda \odot (x_n)_{n\in\mathbb{N}} = (\lambda \cdot x_n)_{n\in\mathbb{N}}.$$

The maps $\oplus$ and $\odot$ are well-defined applications from $E \times E$ and $\mathbb{R} \times E$ respectively with values in E.

(i) Let E be the set of polynomials $P(x)$ with real coefficients. One uses that the sum of two polynomials with *real* coefficients is again a polynomial with *real* coefficients, and that multiplying a polynomial with *real* coefficient by a *real* number gives another polynomial with real coefficients.

(j) Let E be the set of polynomials $P(x)$ of degree less than or equal to 3 with real coefficients. For $P, Q \in E$ and $\lambda \in \mathbb{R}$, one has $d^o(P+Q) = d^oP + d^oQ$ and $d^o(\lambda P) = \lambda\, d^oP$, where d^o denotes the degree. Hence E is stable by the sum and the multiplication by real numbers.

(k) Let E be the set of polynomials $P(x)$ with real coefficients divisible by $(x-1)$. Take P and Q in E. By definition, there exists two polynomials $\tilde{P}$ and $\tilde{Q}$ such that $P(x) = (x-1)\tilde{P}(x)$ and $Q(x) = (x-1)\tilde{Q}(x)$. Therefore $P(x)+Q(x) = (x-1)\tilde{P}(x)+(x-1)\tilde{Q}(x) = (x-1)(\tilde{P}+\tilde{Q})(x)$. In other words, $P+Q$ is divisible by $(x-1)$. Moreover, for any $\lambda \in \mathbb{R}$ and any $P \in E$, λP is divisible by $(x-1)$, with $\widetilde{\lambda P} = \lambda\tilde{P}$.

(l) Let E be the set of continuous functions on the interval $[0,1]$ taking real values. Here one uses that the sum of two continuous functions is again a continuous function, and that the product of a real-valued continuous function by a real number is again a real-valued continuous function.

(m) Let E be the set of continuous functions on the interval $[0,1]$ taking real values and whose integral is zero. Here one uses that

$$\int_0^1 (f+g)(x)\,dx = \int_0^1 f(x)\,dx + \int_0^1 g(x)\,dx$$

and

$$\int_0^1 \lambda \cdot f(x)\,dx = \lambda \cdot \int_0^1 f(x)\,dx.$$

(n) Let E be the set of differentiable functions on the interval $(0,1)$ taking real values. Here the point is to remark that the sum of two differentiable real functions is differentiable and real, and that the product of a differentiable real function by a real number is a real-valued differentiable function.

(o) Let E be the set of real functions which vanish at $0 \in \mathbb{R}$. Since for any f and g in E one has $(f+g)(0) = f(0)+g(0) = 0+0 = 0$ and, for any $\lambda \in \mathbb{R}$, $(\lambda \cdot f)(0) = \lambda \cdot f(0) = \lambda \cdot 0 = 0$, the set E is stable by the sum and the product by the reals.

(p) Let E be the set of real functions having 0 as a limit when x goes to $+\infty$. Here the point is that, given two functions f and g having as a limit 0 at $+\infty$

$$\lim_{x\to+\infty}(f+g) = \lim_{x\to+\infty} f + \lim_{x\to+\infty} g = 0+0 = 0,$$

and $\lim_{x\to+\infty} \lambda \cdot f = \lambda \cdot \lim_{x\to+\infty} f = \lambda \cdot 0 = 0$.

2. A neutral element for an addition $\oplus\ : E \times E \to E$ is an element $e \in E$ such that

$$e \oplus x = x \oplus e = e, \forall x \in E.$$

The inverse of an element $x \in E$, is an element $\tilde{x} \in E$ such that $x \oplus \tilde{x} = \tilde{x} \oplus x = e$. The neutral elements e and inverses $\tilde{x}$ of $x \in E$ for the previously defined additions are :

(a) $e = (0,0,0) \in \mathbb{R}^n$, the inverse of $x = (x_1, x_2, \dots, x_n)$ is $\tilde{x} = (-x_1, -x_2, \dots, -x_n)$.

(b) e is the null function $e\ : \mathbb{R}^+ \to \mathbb{R}^3$, $e(t) = (0,0,0)$. The inverse of $x\ : \mathbb{R}^+ \to \mathbb{R}^3$, $x(t) = (x_1(t), x_2(t), x_3(t))$ is $\tilde{x}\ : \mathbb{R}^+ \to \mathbb{R}^3$, $\tilde{x}(t) = (-x_1(t), -x_2(t), -x_3(t))$.

(c) as in (a) for $n = 3$.

(d) as in (a) for $n = 3$.

(e) e is the null function $e\ : \mathbb{R} \to \mathbb{R}$, $e(t) = 0$. The inverse of $x\ : \mathbb{R} \to \mathbb{R}$ is $\tilde{x}\ : \mathbb{R} \to \mathbb{R}$, $\tilde{x}(t) = -x(t)$.

(f) e is the null function $e\ : \mathbb{R}^{+*} \to \mathbb{R}$, $e(t) = 0$. The inverse of $x\ : \mathbb{R}^{+*} \to \mathbb{R}$ is $\tilde{x}\ : \mathbb{R}^{+*} \to \mathbb{R}$, $\tilde{x}(t) = -x(t)$.

(g) e is the null function $e\ : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{C}$, $e(t,x) = 0$. The inverse of $\Psi\ : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{C}$ is $\tilde{\Psi}\ : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{C}$, $\tilde{\Psi}(t,x) = -\Psi(t,x)$.

(h) e is the null sequence $e = (0,0,\dots,0,\dots)$. The inverse of the sequence $(x_n)_{n\in\mathbb{N}}$ is the sequence $(-x_n)_{n\in\mathbb{N}}$.

(i) e is the null polynomial, $e = 0$. The inverse of the polynomial $P(X) = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$ is the polynomial $-P(X) = -a_nX^n - a_{n-1}X^{n-1} - \cdots - a_1X - a_0$.

(j) as in (i).

(k) as in (i).

(l) e is the null function $e \ : [0,1] \to \mathbb{R}$. The inverse of a continuous function $f \ : [0,1] \to \mathbb{R}$ is the function $\tilde{f} : \ [0,1] \to \mathbb{R}$ such that $\tilde{f}(t) = -f(t)$.

(m) as in (l).

(n) e is the null function $e \ : (0,1) \to \mathbb{R}$. The inverse of a continuous function $f \ : (0,1) \to \mathbb{R}$ is the function $\tilde{f} : \ (0,1) \to \mathbb{R}$ such that $\tilde{f}(t) = -f(t)$.

(o) e is the null function $e \ : \mathbb{R} \to \mathbb{R}$. The inverse of the function $f \ : \mathbb{R} \to \mathbb{R}$ is the function $\tilde{f}(t) = -f(t)$.

(p) as in (o).

Exercise 2 What are the obstacles to defining the same operations as before for the following sets E ?

(a) $E =$ the set of solutions $(x,y,z) \in \mathbb{R}^3$ of the equation $\mathcal{S}_3 \ : x - 2y + 3z = 3$;

(b) $E =$ the set of functions $y(x)$ such that $y''(x)\sin x + x^3y^2(x) + y(x)\log x = 0, \forall x > 0$;

(c) $E = \mathbb{N}$;

(d) $E = \mathbb{Z}$;

(e) $E = \mathbb{R}^+$;

(f) $E = \mathbb{Q}^n$;

(g) $E =$ the set of sequences $(x_n)_{n\in\mathbb{N}}$ of non-negative numbers ;

(h) $E =$ the set of real functions taking the value 1 at 0 ;

(i) $E =$ the set of real functions going to $+\infty$ as x goes to $+\infty$.

Solution of Exercise 2 :

(a) $3 + 3 \neq 3$.

(b) $(y_1 + y_2)^2 \neq y_1^2 + y_2^2$.

(c) $n \in \mathbb{N} \Rightarrow (-1) \cdot n \notin \mathbb{N}$.

(d) $n \in \mathbb{Z} \Rightarrow \pi \cdot n \notin \mathbb{Z}$.

(e) $t \in \mathbb{R}^{+*} \Rightarrow (-1)t \notin \mathbb{R}^+$

(f) If $x = (x_1, \dots, x_n) \in \mathbb{Q}^n$, the n-uple $\pi \cdot (x_1, \dots, x_n) \notin \mathbb{Q}^n$. Remark : this could be overcome by restricting the multiplication to $\mathbb{Q} \subsetneq \mathbb{R}$, i.e. by defining the map $\odot$ from $\mathbb{Q} \times E$ into E.

(g) If $(x_n)_{n\in\mathbb{N}} \in E$, the sequence $(-1) \cdot (x_n)_{n\in\mathbb{N}} \notin E$.

(h) $1 + 1 \neq 1$.

(i) $(-1) \cdot +\infty \neq +\infty$.

Exercise 3 In $\mathbb{R}^3$ consider the vectors $\overrightarrow{v_1} = (1,1,0)$, $\overrightarrow{v_2} = (4,1,4)$ and $\overrightarrow{v_3} = (2,-1,4)$.

1. Show that $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ are not collinear. Do the same with $\overrightarrow{v_1}$ and $\overrightarrow{v_3}$, and with $\overrightarrow{v_2}$ and $\overrightarrow{v_3}$.
2. Is the family $(\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3})$ linearly independent ?

Solution of Exercise 3 :

1. Two vectors $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ in $\mathbb{R}^3$ are collinear if and only if their coordinates are proportionnal. One sees that $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ are non-collinear since $\frac{1}{4} \neq \frac{1}{1}$. The vectors $\overrightarrow{v_1}$ and $\overrightarrow{v_3}$ are non-collinear since $\frac{1}{2} \neq \frac{1}{-1}$. The vectors $\overrightarrow{v_2}$ and $\overrightarrow{v_3}$ are non-collinear since $\frac{4}{2} \neq \frac{1}{-1}$.

2. A family of 3 vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}$ in $\mathbb{R}^3$ is linearly independent if and only if $\det(\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}) \neq 0$. One has

$$\det(\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}) = \begin{vmatrix} 1 & 4 & 2 \\ 1 & 1 & -1 \\ 0 & 4 & 4 \end{vmatrix} = -4 \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} + 4 \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} = 12 - 12 = 0.$$

Hence the family $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}\}$ is linearly dependent.

Exercise 4 Are the following families linearly independent ?

1. $\overrightarrow{v_1} = (1, 0, 1)$, $\overrightarrow{v_2} = (0, 2, 2)$ and $\overrightarrow{v_3} = (3, 7, 1)$ in $\mathbb{R}^3$.
2. $\overrightarrow{v_1} = (1, 0, 0)$, $\overrightarrow{v_2} = (0, 1, 1)$ and $\overrightarrow{v_3} = (1, 1, 1)$ in $\mathbb{R}^3$.
3. $\overrightarrow{v_1} = (1, 2, 1, 2, 1)$, $\overrightarrow{v_2} = (2, 1, 2, 1, 2)$, $\overrightarrow{v_3} = (1, 0, 1, 1, 0)$ and $\overrightarrow{v_4} = (0, 1, 0, 0, 1)$ in $\mathbb{R}^5$.
4. $\overrightarrow{v_1} = (2, 4, 3, -1, -2, 1)$, $\overrightarrow{v_2} = (1, 1, 2, 1, 3, 1)$ and $\overrightarrow{v_3} = (0, -1, 0, 3, 6, 2)$ in $\mathbb{R}^6$.
5. $\overrightarrow{v_1} = (2, 1, 3, -1, 4, -1)$, $\overrightarrow{v_2} = (-1, 1, -2, 2, -3, 3)$ and $\overrightarrow{v_3} = (1, 5, 0, 4, -1, 7)$ in $\mathbb{R}^6$.

Solution of Exercise 4 :

1. The vectors $\overrightarrow{v_1} = (1, 0, 1)$, $\overrightarrow{v_2} = (0, 2, 2)$ and $\overrightarrow{v_3} = (3, 7, 1)$ are 3 linearly independent vectors in $\mathbb{R}^3$ if and only if $\det(\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}) \neq 0$. One has

$$\det(\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}) = \begin{vmatrix} 1 & 0 & 3 \\ 0 & 2 & 7 \\ 1 & 2 & 1 \end{vmatrix} = 1 \begin{vmatrix} 2 & 7 \\ 2 & 1 \end{vmatrix} + 1 \begin{vmatrix} 0 & 3 \\ 2 & 7 \end{vmatrix} = -12 - 6 = -18.$$

Since $\det(\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}) \neq 0$, $\overrightarrow{v_1}$, $\overrightarrow{v_2}$, $\overrightarrow{v_3}$ are linearly independent.

2. One has

$$\det(\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}) = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0$$

Hence the vectors $\overrightarrow{v_1} = (1, 0, 0)$, $\overrightarrow{v_2} = (0, 1, 1)$ and $\overrightarrow{v_3} = (1, 1, 1)$ are linearly dependent (one can also argue that $\overrightarrow{v_3} = \overrightarrow{v_1} + \overrightarrow{v_2}$).

3. By definition, p-vectors $\overrightarrow{v_1}$, $\overrightarrow{v_2}$, $\ldots$, $\overrightarrow{v_p}$ in $\mathbb{R}^n$ are linearly independent if and only if the equation

$$\lambda_1 \overrightarrow{v_1} + \lambda_2 \overrightarrow{v_2} + \cdots + \lambda_p \overrightarrow{v_p} = \overrightarrow{0}$$

(whose unknowns are $\lambda_1, \ldots, \lambda_p$) admits a unique solution given by $(\lambda_1, \lambda_2, \ldots, \lambda_p) = (0, 0, \ldots, 0)$. Let us consider the equation

$$\lambda_1 \overrightarrow{v_1} + \lambda_2 \overrightarrow{v_2} + \lambda_3 \overrightarrow{v_3} + \lambda_4 \overrightarrow{v_4} = \overrightarrow{0}.$$

In coordinates, we obtain the following system

$$\left\{ \begin{array}{rcrcrcrl} \lambda_1 & + & 2\lambda_2 & + & \lambda_3 & & & = 0 \\ 2\lambda_1 & + & \lambda_2 & & & + & \lambda_4 & = 0 \\ \lambda_1 & + & 2\lambda_2 & + & \lambda_3 & & & = 0 \\ 2\lambda_1 & + & \lambda_2 & + & \lambda_3 & & & = 0 \\ \lambda_1 & + & 2\lambda_2 & & & + & \lambda_4 & = 0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{rcrcrcrll} \lambda_1 & + & 2\lambda_2 & + & \lambda_3 & & & = 0 & \\ & - & 3\lambda_2 & - & 2\lambda_3 & + & \lambda_4 & = 0 & L_2 \leftarrow L_2 - 2L_1 \\ & & & & & & 0 & = 0 & L_3 \leftarrow L_3 - L_1 \\ & - & 3\lambda_2 & - & \lambda_3 & & & = 0 & L_4 \leftarrow L_4 - 2L_1 \\ & & & - & \lambda_3 & + & \lambda_4 & = 0 & L_5 \leftarrow L_5 - L_1 \end{array} \right. \Leftrightarrow$$

$$\left\{ \begin{array}{rcrcrcrll} \lambda_1 & + & 2\lambda_2 & + & \lambda_3 & & & = 0 & \\ & - & 3\lambda_2 & - & 2\lambda_3 & + & \lambda_4 & = 0 & \\ & & & & \lambda_3 & - & \lambda_4 & = 0 & L_4 \leftarrow L_4 - L_2 \\ & & & - & \lambda_3 & + & \lambda_4 & = 0 & \end{array} \right. \Leftrightarrow \left\{ \begin{array}{rcrcrcrll} \lambda_1 & + & 2\lambda_2 & + & \lambda_3 & & & = 0 & \\ & - & 3\lambda_2 & - & 2\lambda_3 & + & \lambda_4 & = 0 & \\ & & & & \lambda_3 & - & \lambda_4 & = 0 & \\ & & & & & & 0 & = 0 & L_4 + L_3 \end{array} \right.$$

Let us parametrize the set of solutions of the system by $t = \lambda_4 \in \mathbb{R}$. One has

$$\left\{\begin{array}{rcrcrcl} \lambda_1 & + & 2\lambda_2 & + & \lambda_3 & = & 0 \\ & - & 3\lambda_2 & - & 2\lambda_3 & = & -t \\ & & & & \lambda_3 & = & t \\ & & & & \lambda_4 & = & t \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcl} \lambda_1 & = & -2\lambda_2 - t \\ -3\lambda_2 & = & -t + 2t = t \\ \lambda_3 & = & t \\ \lambda_4 & = & t \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcl} \lambda_1 & = & \frac{2}{3}t - t = -\frac{1}{3}t \\ \lambda_2 & = & -\frac{1}{3}t \\ \lambda_3 & = & t \\ \lambda_4 & = & t \end{array}\right.$$

Since the solution of the system is the line d generated by the vector $\begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \\ 1 \end{pmatrix}$, or in other words

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} \in \mathbb{R} \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \\ 1 \end{pmatrix},$$

the vectors $\overrightarrow{v_1} = (1,2,1,2,1)$, $\overrightarrow{v_2} = (2,1,2,1,2)$, $\overrightarrow{v_3} = (1,0,1,1,0)$ and $\overrightarrow{v_4} = (0,1,0,0,1)$ are not linearly independent.
(A quicker argument is to point out that $\overrightarrow{v_1} + \overrightarrow{v_2} = 3 \cdot (\overrightarrow{v_3} + \overrightarrow{v_4})$.)

4. Let us consider the equation

$$\lambda_1\overrightarrow{v_1} + \lambda_2\overrightarrow{v_2} + \lambda_3\overrightarrow{v_3} = \overrightarrow{0}.$$

In coordinates, this gives rise to the following system :

$$\left\{\begin{array}{rcrcrl} 2\lambda_1 & + & \lambda_2 & & & = 0 \\ 4\lambda_1 & + & \lambda_2 & - & \lambda_3 & = 0 \\ 3\lambda_1 & + & 2\lambda_2 & & & = 0 \\ -\lambda_1 & + & \lambda_2 & + & 3\lambda_3 & = 0 \\ -2\lambda_1 & + & 3\lambda_2 & + & 6\lambda_3 & = 0 \\ \lambda_1 & + & \lambda_2 & + & 2\lambda_3 & = 0 \end{array}\right.$$

By the Gauss algorithm, the system is equivalent to

$$\left\{\begin{array}{rcrcrl} \lambda_1 & + & \lambda_2 & + & 2\lambda_3 & = 0 \\ 2\lambda_1 & + & \lambda_2 & & & = 0 \\ 4\lambda_1 & + & \lambda_2 & - & \lambda_3 & = 0 \\ 3\lambda_1 & + & 2\lambda_2 & & & = 0 \\ -\lambda_1 & + & \lambda_2 & + & 3\lambda_3 & = 0 \\ -2\lambda_1 & + & 3\lambda_2 & + & 6\lambda_3 & = 0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcrcrll} \lambda_1 & + & \lambda_2 & + & 2\lambda_3 & = 0 & \\ & - & \lambda_2 & - & 4\lambda_3 & = 0 & L_2 \leftarrow L_2 - 2L_1 \\ & - & 3\lambda_2 & - & 9\lambda_3 & = 0 & L_3 \leftarrow L_3 - 4L_1 \\ & - & \lambda_2 & - & 6\lambda_3 & = 0 & L_4 \leftarrow L_4 - 3L_1 \\ & + & 2\lambda_2 & + & 5\lambda_3 & = 0 & L_5 \leftarrow L_5 + L_1 \\ & + & 5\lambda_2 & + & 10\lambda_3 & = 0 & L_6 \leftarrow L_6 + 2L_1 \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{rcrcrll} \lambda_1 & + & \lambda_2 & + & 2\lambda_3 & = 0 & \\ & - & \lambda_2 & - & 4\lambda_3 & = 0 & \\ & & & + & 3\lambda_3 & = 0 & L_3 \leftarrow L_3 - 3L_2 \\ & & & - & 2\lambda_3 & = 0 & L_4 \leftarrow L_4 - L_2 \\ & & & - & 3\lambda_3 & = 0 & L_5 \leftarrow L_5 + 2L_2 \\ & & & - & 10\lambda_3 & = 0 & L_6 \leftarrow L_6 + 5L_1 \end{array}\right.$$

It follows that the unique solution of the system is $(\lambda_1, \lambda_2, \lambda_3) = (0,0,0)$, consequently the vectors $\overrightarrow{v_1} = (2,4,3,-1,-2,1)$, $\overrightarrow{v_2} = (1,1,2,1,3,1)$ and $\overrightarrow{v_3} = (0,-1,0,3,6,2)$ are linearly independent.

5. The equation

$$\lambda_1\overrightarrow{v_1} + \lambda_2\overrightarrow{v_2} + \lambda_3\overrightarrow{v_3} = \overrightarrow{0}$$

written in coordinates gives rise to the following system :

$$\left\{\begin{array}{rcrcrl} 2\lambda_1 & - & \lambda_2 & + & \lambda_3 & = 0 \\ \lambda_1 & + & \lambda_2 & + & 5\lambda_3 & = 0 \\ 3\lambda_1 & - & 2\lambda_2 & & & = 0 \\ -\lambda_1 & + & 2\lambda_2 & + & 4\lambda_3 & = 0 \\ 4\lambda_1 & - & 3\lambda_2 & - & \lambda_3 & = 0 \\ -\lambda_1 & + & 3\lambda_2 & + & 7\lambda_3 & = 0 \end{array}\right.$$

By the Gauss algorithm, the system is equivalent to

$$\left\{\begin{array}{rcrcrl} \lambda_1 & + & \lambda_2 & + & 5\lambda_3 & = 0 \\ 2\lambda_1 & - & \lambda_2 & + & \lambda_3 & = 0 \quad L_2 \leftrightarrow L_1 \\ 3\lambda_1 & - & 2\lambda_2 & & & = 0 \\ -\lambda_1 & + & 2\lambda_2 & + & 4\lambda_3 & = 0 \\ 4\lambda_1 & - & 3\lambda_2 & - & \lambda_3 & = 0 \\ -\lambda_1 & + & 3\lambda_2 & + & 7\lambda_3 & = 0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcrcrll} \lambda_1 & + & \lambda_2 & + & 5\lambda_3 & = 0 & \\ & - & 3\lambda_2 & - & 9\lambda_3 & = 0 & L_2 \leftarrow L_2 - 2L_1 \\ & - & 5\lambda_2 & - & 15\lambda_3 & = 0 & L_3 \leftarrow L_3 - 3L_1 \\ & + & 3\lambda_2 & + & 9\lambda_3 & = 0 & L_4 \leftarrow L_4 + L_1 \\ & - & 7\lambda_2 & - & 21\lambda_3 & = 0 & L_5 \leftarrow L_5 - 4L_1 \\ & + & 4\lambda_2 & + & 12\lambda_3 & = 0 & L_6 \leftarrow L_6 + L_1 \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{rcrcrll} \lambda_1 & + & \lambda_2 & + & 5\lambda_3 & = 0 & \\ & & \lambda_2 & + & 3\lambda_3 & = 0 & -\frac{1}{3}L_2 \\ & & \lambda_2 & + & 3\lambda_3 & = 0 & -\frac{1}{5}L_3 \\ & & \lambda_2 & + & 3\lambda_3 & = 0 & \frac{1}{3}L_4 \\ & & \lambda_2 & + & 3\lambda_3 & = 0 & -\frac{1}{7}L_5 \\ & & \lambda_2 & + & 3\lambda_3 & = 0 & \frac{1}{4}L_6 \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcrcrl} \lambda_1 & + & \lambda_2 & + & 5\lambda_3 & = 0 \\ & & \lambda_2 & + & 3\lambda_3 & = 0 \end{array}\right.$$

A nontrivial solution is therefore given by $\lambda_2 = -3\lambda_3$ and $\lambda_1 = -\lambda_2 - 5\lambda_3 = -2\lambda_3$, i.e. $(\lambda_1, \lambda_2, \lambda_3)$ collinear with $(-2, -3, 1)$. The vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}$ are not linearly independent (and one can check that $2\overrightarrow{v_1} + 3\overrightarrow{v_2} = \overrightarrow{v_3}$).

Exercise 5 One supposes that $v_1, v_2, v_3, \ldots, v_n$ are linearly independent vectors in $\mathbb{R}^n$.

1. Are the vectors $v_1 - v_2, v_2 - v_3, v_3 - v_4, \ldots, v_n - v_1$ linearly independent ?
2. Are the vectors $v_1 + v_2, v_2 + v_3, v_3 + v_4, \ldots, v_n + v_1$ linearly independent ?
3. Are the vectors $v_1, v_1 + v_2, v_1 + v_2 + v_3, v_1 + v_2 + v_3 + v_4, \ldots, v_1 + v_2 + \cdots + v_n$ linearly independent ?

Solution of Exercise 5 :

1. Consider the following system

$$\alpha_1(\overrightarrow{v_1} - \overrightarrow{v_2}) + \alpha_2(\overrightarrow{v_2} - \overrightarrow{v_3}) + \cdots + \alpha_n(\overrightarrow{v_n} - \overrightarrow{v_1}) = \overrightarrow{0} \tag{4}$$

$$\Leftrightarrow (\alpha_1 - \alpha_n)\overrightarrow{v_1} + (\alpha_2 - \alpha_1)\overrightarrow{v_2} + (\alpha_3 - \alpha_2)\overrightarrow{v_3} \ldots (\alpha_n - \alpha_{n-1})\overrightarrow{v_n} = \overrightarrow{0}.$$

Since $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}, \ldots, \overrightarrow{v_n}$ are linearly independent vectors in $\mathbb{R}^n$, the previous system is equivalent to

$$\left\{\begin{array}{c} \alpha_1 - \alpha_n = 0 \\ \alpha_2 - \alpha_1 = 0 \\ \vdots \\ \alpha_n - \alpha_{n-1} = 0 \end{array}\right. \Leftrightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n.$$

Hence the equation (4) admits a line of solutions given by $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$. Consequently the vectors $v_1 - v_2, v_2 - v_3, v_3 - v_4, \ldots, v_n - v_1$ are not linearly independent.

2. Consider the following system

$$\alpha_1(\overrightarrow{v_1} + \overrightarrow{v_2}) + \alpha_2(\overrightarrow{v_2} + \overrightarrow{v_3}) + \cdots + \alpha_n(\overrightarrow{v_n} + \overrightarrow{v_1}) = \overrightarrow{0} \tag{5}$$

$$\Leftrightarrow (\alpha_1 + \alpha_n)\overrightarrow{v_1} + (\alpha_2 + \alpha_1)\overrightarrow{v_2} + (\alpha_3 + \alpha_2)\overrightarrow{v_3} \ldots (\alpha_n + \alpha_{n-1})\overrightarrow{v_n} = \overrightarrow{0}.$$

Since $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}, \dots, \overrightarrow{v_n}$ are linearly independent vectors in $\mathbb{R}^n$, the previous system is equivalent to

$$\begin{cases} \alpha_1 + \alpha_n = 0 \\ \alpha_2 + \alpha_1 = 0 \\ \alpha_3 + \alpha_2 = 0 \\ \vdots \\ \alpha_n + \alpha_{n-1} = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha_1 = -\alpha_n \\ \alpha_2 = -\alpha_1 \\ \alpha_3 = -\alpha_2 \\ \vdots \\ \alpha_n = -\alpha_{n-1} \end{cases}$$

There are two cases :

(a) The previous system implies

$$\Rightarrow \begin{cases} \alpha_1 = (-1)^n \alpha_1 \\ \alpha_2 = (-1)^n \alpha_2 \\ \alpha_3 = (-1)^n \alpha_3 \\ \vdots \\ \alpha_n = (-1)^n \alpha_n \end{cases}.$$

It follows that if n is odd, $\alpha_1 = 0$, $\alpha_2 = 0$, $\dots \alpha_n = 0$, hence the vectors $v_1 + v_2, v_2 + v_3, v_3 + v_4, \dots, v_n + v_1$ are linearly independent.

(b) if n is even, the system is equivalent to

$$\begin{cases} \alpha_1 = -\alpha_n \\ \alpha_2 = (-1)^2 \alpha_n \\ \alpha_3 = (-1)^3 \alpha_n \\ \vdots \\ \alpha_{n-1} = (-1)^{n-1} \alpha_n \end{cases},$$

hence the set of solutions of the system is the line given by

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_j \\ \vdots \\ \alpha_n \end{pmatrix} = \lambda \begin{pmatrix} -1 \\ 1 \\ -1 \\ \vdots \\ (-1)^j \\ \vdots \\ 1 \end{pmatrix},$$

consequently the vectors $v_1 + v_2, v_2 + v_3, v_3 + v_4, \dots, v_n + v_1$ are not linearly independent.

3. Consider the following vectors

$$\begin{cases} v_1' = v_1 \\ v_2' = v_1 + v_2 \\ v_3' = v_1 + v_2 + v_3 \\ v_4' = v_1 + v_2 + v_3 + v_4 \\ \vdots \\ v_n' = v_1 + v_2 + \dots + v_n \end{cases}$$

One has

$$\begin{pmatrix} v_1' \\ v_2' \\ v_3' \\ \vdots \\ v_{n-1}' \\ v_n' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n-1} \\ v_n \end{pmatrix}.$$

Consequently, the equation $\lambda_1 v_1' + \lambda_2 v_2' + \ldots + \lambda_n v_n' = 0$ can be written

$$\begin{pmatrix} v_1' & v_2' & \ldots & v_n' \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} = 0 \Leftrightarrow \begin{pmatrix} v_1 & v_2 & \ldots & v_n \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 1 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \ldots & 1 & 0 \\ 1 & 1 & 1 & 1 & \ldots & 1 & 1 \end{pmatrix}^T \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_{n-1} \\ \lambda_n \end{pmatrix} = 0.$$

Since the vectors $v_1, v_2, \ldots, v_n$ are linearly independent, it follows that

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 1 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \ldots & 1 & 0 \\ 1 & 1 & 1 & 1 & \ldots & 1 & 1 \end{pmatrix}^T \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_{n-1} \\ \lambda_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

which implies

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_{n-1} \\ \lambda_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & \ldots & 1 & 1 \\ 0 & 1 & 1 & 1 & \ldots & 1 & 1 \\ 0 & 0 & 1 & 1 & \ldots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

In conclusion, the vectors $v_1, v_1 + v_2, v_1 + v_2 + v_3, v_1 + v_2 + v_3 + v_4, \ldots, v_1 + v_2 + \cdots + v_n$ are linearly independent.