Hölderian functional central limit theorem for multi-indexed summation process ⋆

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Abstract

Let $\{X_{\boldsymbol{j}};\ \boldsymbol{j} \in \mathbb{N}^d,\ \boldsymbol{j} \geq \mathbf{1}\}$ be an i.i.d. random field of square integrable centered random elements in the separable Hilbert space $\mathbb{H}$ and $\xi_{\boldsymbol{n}}$, $\boldsymbol{n} \in \mathbb{N}^d$, be the summation processes based on the collection of sets $[0, t_1] \times \cdots \times [0, t_d]$, $0 \leq t_i \leq 1$, $i = 1, \ldots, d$. When $d \geq 2$, we characterize the weak convergence of $(n_1 \ldots n_d)^{-1/2}\xi_{\boldsymbol{n}}$ in the Hölder space $\mathrm{H}^o_\alpha(\mathbb{H})$ by the finiteness of the weak p moment of $\|X_{\mathbf{1}}\|$ for $p = (1/2 - \alpha)^{-1}$. This contrasts with the Hölderian FCLT for $d = 1$ and $\mathbb{H} = \mathbb{R}$ (Račkauskas, Suquet, 2003) where the necessary and sufficient condition is $P(|X_1| > t) = o(t^{-p})$.

Key words: Brownian sheet, Hilbert space valued Brownian sheet, Hilbert space, functional central limit theorem, Hölder space, invariance principle, summation process.
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1 Introduction

Convergence of stochastic processes to some Brownian motion or related process is an important topic in probability theory and mathematical statistics. The first functional central limit theorem by Donsker and Prohorov states

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the $C[0,1]$-weak convergence of $n^{-1/2}\xi_n$ to the standard Brownian motion W. Here ξ_n denotes the random polygonal line process indexed by $[0,1]$ with vertices $(k/n, S_k)$, $k = 0, 1, \dots, n$ and $S_0 := 0$, $S_k := X_1 + \dots + X_k$, $k \geq 1$, are the partial sums of a sequence $(X_i)_{i\geq 1}$ of i.i.d. random variables such that $\mathbf{E}\, X_1 = 0$ and $\mathbf{E}\, X_1^2 = 1$. This theorem implies via continuous maping the convergence in distribution of $f(n^{-1/2}\xi_n)$ to $f(W)$ for any continuous functional $f : \mathrm{C}[0,1] \to \mathbb{R}$. Clearly this provides many statistical applications. On the other hand, considering that the paths of ξ_n are piecewise linear and that W has roughly speaking, an α-Hölder regularity for any exponent $\alpha < 1/2$, it is tempting to look for a stronger topological framework for the weak convergence of $n^{-1/2}\xi_n$ to W. In addition to the satisfaction of mathematical curiosity, the practical interest of such an investigation is to obtain a richer set of continuous functionals of the paths. For instance, Hölder norms of ξ_n are closely related to some test statistics to detect short "epidemic" changes in the distribution of the X_i's, see [21,22].

In 1962, Lamperti [12] obtained the first functional central limit theorem in the separable Banach spaces H_α^o, $0 < \alpha < 1/2$, of functions $x : [0,1] \to \mathbb{R}$ such that

$$\|x\|_\alpha := |x(0)| + \omega_\alpha(x, 1) < \infty,$$

with

$$\omega_\alpha(x,\delta) := \sup_{0<|t-s|\leq\delta} \frac{|x(t) - x(s)|}{|t-s|^\alpha} \xrightarrow[\delta\to 0]{} 0.$$

Assuming that $\mathbf{E}\, |X_1|^q < \infty$ for some $q > 2$, he proved the weak convergence of $n^{-1/2}\xi_n$ to W in the Hölder space H_α^o for any $\alpha < 1/2 - 1/q$. Račkauskas and Suquet in [20] (see also [19]) obtained a necessary and sufficient condition for the Lamperti's functional central limit theorem. Namely for $0 < \alpha < 1/2$, $n^{-1/2}\xi_n$ converges weakly in H_α^o to W if and only if

$$\lim_{t\to\infty} t^{p(\alpha)} P(|X_1| > t) = 0, \tag{1}$$

where

$$p(\alpha) := \frac{1}{\frac{1}{2} - \alpha}. \tag{2}$$

Further extensions of Donsker-Prohorov's functional central limit theorem concern summation processes. Let $|A|$ denote the Lebesgue measure of the Borel subset A of $\mathbb{R}^d$. For a collection $\mathcal{A}$ of Borel subsets of $[0,1]^d$, summation process $\{\xi_{\boldsymbol{n}}(A);\ A \in \mathcal{A}\}$ based on a random field $\{X_{\boldsymbol{j}},\ \boldsymbol{j} \in \mathbb{N}^d\}$, of independent identically distributed real random variables with zero mean is defined by

$$\xi_{\boldsymbol{n}}(A) = \sum_{\mathbf{1}\leq\boldsymbol{j}\leq\boldsymbol{n}} |R_{\boldsymbol{n},\boldsymbol{j}}|^{-1} |R_{\boldsymbol{n},\boldsymbol{j}} \cap A| X_{\boldsymbol{j}}, \tag{3}$$

where $\boldsymbol{j} = (j_1, \dots, j_d)$, $\boldsymbol{n} = (n_1, \dots, n_d)$, $R_{\boldsymbol{n},\boldsymbol{j}}$ is the "rectangle"

$$R_{\boldsymbol{n},\boldsymbol{j}} := \left[\frac{j_1 - 1}{n_1}, \frac{j_1}{n_1}\right) \times \cdots \times \left[\frac{j_d - 1}{n_d}, \frac{j_d}{n_d}\right) \tag{4}$$

and the indexation condition "$\boldsymbol{1} \leq \boldsymbol{j} \leq \boldsymbol{n}$" is understood componentwise : $1 \leq j_1 \leq n_1, \dots, 1 \leq j_d \leq n_d$. Of special interest are the partial sum processes based on the collection of sets $\mathcal{A} = \mathcal{Q}_d$ where

$$\mathcal{Q}_d := \Big\{ [0, t_1] \times \cdots \times [0, t_d];\ \boldsymbol{t} = (t_1, \dots, t_d) \in [0, 1]^d \Big\}, \tag{5}$$

Note that when $d = 1$ the partial sum process ξ_n based on $\mathcal{Q}_d$ is the random polygonal line of Donsker-Prohorov's theorem.

By equipping the collection $\mathcal{A}$ with some pseudo-metric δ, one define the space $C(\mathcal{A})$ of real continuous functions on $\mathcal{A}$, endowed with the norm

$$\|f\|_{\mathcal{A}} := \sup_{A \in \mathcal{A}} |f(A)|.$$

The usual semimetrics are $\delta(A, B) = \sqrt{|A \Delta B|}$, or $\delta(A, B) = \sqrt{m(A \Delta B)}$, for $A, B \in \mathcal{A}$, where m is a probability measure on the σ-algebra of Borel subsets of $[0, 1]^d$. When $\mathcal{A}$ is totally bounded with respect to δ, $C(\mathcal{A})$ is a separable Banach space.

A standard Wiener process indexed by $\mathcal{A}$ is a mean zero Gaussian process W with sample paths in $C(\mathcal{A})$ and

$$\mathbf{E}\, W(A)W(B) = |A \cap B|, \quad A, B \in \mathcal{A}.$$

Existence of such process is proved by placing restrictions on collection $\mathcal{A}$ which are usually expressed by some condition on its metric entropy. For existence of W in Hölder spaces $\mathrm{H}_\rho(\mathcal{A})$ built on some weight function ρ, see Dudley [6] and Erickson [8]. For $\rho(h) = h^\alpha$, Erickson [8] proves that α cannot exceed $1/2$ and it decreases as the entropy of $\mathcal{A}$ increases. The functional central limit theorem (FCLT) in $\mathrm{C}(\mathcal{A})$ or in $\mathrm{H}_\rho(\mathcal{A})$ means the convergence of the summation process $\{\xi_{\boldsymbol{n}}(A); A \in \mathcal{A}\}$, suitably normalized, to a Wiener process indexed by $\mathcal{A}$.

The first FCLT for $\{\xi_n(A); A \in \mathcal{Q}_d\}$ in $\mathrm{C}(\mathcal{Q}_d)$ were established by Kuelbs [10] under some moment restrictions and by Wichura [27] under finite variance condition. In 1983, Pyke [15] derived a FCLT for summation process in $\mathrm{C}(\mathcal{A})$, provided that the collection $\mathcal{A}$ satisfies the bracketing entropy condition. However, his result required moment conditions which depend on the size of the collection $\mathcal{A}$. Bass [2] and simultaneously Alexander and Pyke [1] extended Pyke's result to i.i.d. random fields with finite variance. Further developments were concerned with relaxing entropy conditions on the collection

$\mathcal{A}$, Ziegler [28], and with relaxing i.i.d. condition on the random field $\{X_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^d\}$, Dedecker [4], El Machkouri and Ouchti [7] to name a few.

The FCLT for summation process in $\mathrm{H}_\rho(\mathcal{A})$ is not so extensively studied. Most general results are provided by Erickson [8] who shows that if $\mathbf{E}\,|X_{\boldsymbol{j}}|^q < \infty$ for some $q > 2$ then the FCLT holds in $\mathrm{H}_\rho(\mathcal{A})$ for some ρ which depends on q and properties of $\mathcal{A}$. For $d = 1$ and the class $\mathcal{A}$ of intervals $[0, t]$, $0 \leq t \leq 1$, Erickson's results coincide with Lamperti's ones [12], whereas his case $d > 1$ requires moments of order $q > dp(\alpha)$ with the same $p(\alpha)$ as in (2). In Račkauskas and Zemlys [23], the result by Erickson was improved in the case $d = 2$.

In this paper, we investigate summation processes build from Hilbert space valued random elements. We establish necessary and sufficient conditions for the FCLT to hold in certain Hölder spaces. To illustrate our main result let us state here its particular case which can be considered as Lamperti's functional central limit theorem for summation process $\{\xi_{\boldsymbol{n}}(A) : A \in \mathcal{Q}_d\}$ defined above.

Proposition 1 *Let $0 < \alpha < 1/2$ and $d > 1$. Let $\{X_{\boldsymbol{j}}, \boldsymbol{j} \in \mathbb{N}^d\}$ be a set of i.i.d. random variables with mean zero and variance $\mathbf{E}\, X_{\boldsymbol{j}}^2 = 1$. Let W be a standard Brownian sheet on $[0, 1]^d$. Then normed summation process*

$$\{(n_1 \cdots n_d)^{-1/2} \xi_{\boldsymbol{n}}(A);\ A \in \mathcal{Q}_d\}$$

converge in distribution to W in the space H_α^o if and only if

$$\sup_{t>0} t^{p(\alpha)} P(|X_{\mathbf{1}}| > t) < \infty. \tag{6}$$

As we see, condition (6) does not depend on the dimension d provided $d > 1$ and is weaker than necessary and sufficient condition (1) in the extension by Račkauskas and Suquet of Lamperti's functional central limit theorem. Moreover, we show that summation process considered along the diagonal, namely the sequence $n^{-d/2}\xi_n = n^{-d/2}\xi_{n,\ldots,n}, n \in \mathbb{N}$, converges in H_α^0 if and only if

$$\lim_{t\to\infty} t^{2d/(d-2\alpha)} P(|X_{\mathbf{1}}| > t) = 0. \tag{7}$$

As dimension d increases, this condition weakens. For example, (7) is satisfied for any $d > 1$ provided $\mathbf{E}\, X_{\mathbf{1}}^4 < \infty$. This again shows up a difference between the cases $d = 1$ and $d > 1$ for functional central limit theorems in Hölder spaces.

The rest of the paper is organized in the following way. Section 2 introduces the notations and precise definitions which are needed and states the results. In Section 3 are collected necessary background material on the weak convergence of distributions in Hölder spaces. The proof of the main result is given in

Section 4.

2 Notation and results

In this paper vectors $\boldsymbol{t} = (t_1, \dots, t_d)$ of $\mathbb{R}^d$, $d \geq 2$, are typeset in italic bold. In particular,

$$\mathbf{1} := (1, \dots, 1).$$

For $1 \leq k < l \leq d$, $\boldsymbol{t}_{k:l}$ denotes the "subvector"

$$\boldsymbol{t}_{k:l} := (t_k, t_{k+1}, \dots, t_l).$$

The set $\mathbb{R}^d$ is equipped with the partial order

$$\boldsymbol{s} \leq \boldsymbol{t} \quad \text{if and only if} \quad s_k \leq t_k, \text{ for all } k = 1, \dots, d.$$

As a vector space $\mathbb{R}^d$, is endowed with the norm

$$|\boldsymbol{t}| = \max(|t_1|, \dots, |t_d|), \quad \boldsymbol{t} = (t_1, \dots, t_d) \in \mathbb{R}^d.$$

Together with the usual addition of vectors and multiplication by a scalar, we use also the componentwise multiplication and division of vectors $\boldsymbol{s} = (s_1, \dots, s_d)$, $\boldsymbol{t} = (t_1, \dots, t_d)$ in $\mathbb{R}^d$ defined whenever it makes sense by

$$\boldsymbol{st} := (s_1 t_1, \dots, s_d t_d), \quad \boldsymbol{s}/\boldsymbol{t} := (s_1/t_1, \dots, s_d/t_d).$$

Partial order as well as all these operations are also intended componentwise when one of the two involved vectors is replaced by a scalar. So for $c \in \mathbb{R}$ and $\boldsymbol{t} \in \mathbb{R}^d$, $c \leq \boldsymbol{t}$ means $c \leq t_k$ for $k = 1, \dots, d$, $\boldsymbol{t} + c := (t_1 + c, \dots, t_d + c)$, $c/\boldsymbol{t} := (c/t_1, \dots, c/t_d)$.

For $\boldsymbol{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$, we write

$$\boldsymbol{\pi}(\boldsymbol{n}) := n_1 \dots n_d,$$

and for $\boldsymbol{t} = (t_1, \dots, t_d) \in \mathbb{R}^d$,

$$\mathrm{m}(\boldsymbol{t}) := \min(t_1, \dots, t_d).$$

For any real number x, denote by $[x]$ and $\{x\}$ its integer part and fractional part defined respectively by

$$[x] \leq x < [x] + 1, \quad [x] \in \mathbb{Z} \quad \text{and} \quad \{x\} := x - [x].$$

When applied to vectors $\boldsymbol{t}$ of $\mathbb{R}^d$, these operations are defined componentwise:

$$[\boldsymbol{t}] := ([t_1], \dots, [t_d]), \quad \{\boldsymbol{t}\} := (\{t_1\}, \dots, \{t_d\}).$$

The context should dispel any notational confusion between the fractional part of x (or $\boldsymbol{t}$) and the set having x (or $\boldsymbol{t}$) as unique element.

We denote by $\mathbb{H}$ a separable Hilbert space with norm $\|\cdot\|$ and inner product $\langle\, .\, ,\, .\, \rangle$. For $0 < \alpha < 1$, we define the Hölder space $\mathrm{H}_\alpha^o(\mathbb{H})$ as the vector space of functions $x : [0,1]^d \to \mathbb{H}$ such that

$$\|x\|_\alpha := \|x(0)\| + \omega_\alpha(x, 1) < \infty,$$

with

$$\omega_\alpha(x, \delta) := \sup_{0<|\boldsymbol{t}-\boldsymbol{s}|\leq\delta} \frac{\|x(\boldsymbol{t}) - x(\boldsymbol{s})\|}{|\boldsymbol{t}-\boldsymbol{s}|^\alpha} \xrightarrow[\delta\to 0]{} 0.$$

Endowed with the norm $\|.\|_\alpha$, $\mathrm{H}_\alpha^o(\mathbb{H})$ is a separable Banach space, see [17] or [18].

As we are mainly dealing in this paper with weak convergence in some function spaces, it is convenient to introduce the following notations. Let B be some separable Banach space and $(Y_n)_{n\geq 1}$ and $(Z_{\boldsymbol{n}})_{\boldsymbol{n}\geq \boldsymbol{1}}$ be respectively a sequence and a random field of random elements in B. We write

$$Y_n \xrightarrow[n\to\infty]{B} Y, \qquad Z_{\boldsymbol{n}} \xrightarrow[\mathrm{m}(\boldsymbol{n})\to\infty]{B} Z,$$

for their weak convergence in the space B to the random elements Y or Z, i.e. $\mathbf{E}\, f(Y_n) \to \mathbf{E}\, f(Y)$ for any continuous and bounded $f : B \to \mathbb{R}$ and similarly with $Z_{\boldsymbol{n}}$, the weak convergence of $Z_{\boldsymbol{n}}$ to Z being understood in the net sense.

A $\mathbb{H}$-valued Brownian sheet with covariance operator Γ is a $\mathbb{H}$-valued zero mean Gausian process indexed by $[0,1]^d$ and satisfying

$$\mathbf{E}\, \langle W(\boldsymbol{t}), x\rangle \langle W(\boldsymbol{s}), y\rangle = (t_1 \wedge s_1) \dots (t_d \wedge s_d) \langle \Gamma x, y\rangle$$

for $\boldsymbol{t}, \boldsymbol{s} \in [0,1]^d$ and $x, y \in \mathbb{H}$. As the following estimate

$$\mathbf{E}\, \|W(\boldsymbol{t} + \boldsymbol{h}) + W(\boldsymbol{t} - \boldsymbol{h}) - 2W(\boldsymbol{t})\|^2 \leq c|h| \operatorname{tr} \Gamma,$$

is valid for all $\boldsymbol{t}-\boldsymbol{h}, \boldsymbol{t}, \boldsymbol{t}+\boldsymbol{h} \in [0,1]^d$, it follows from Račkauskas and Suquet [17] that $W(\boldsymbol{t})$ has a version in $\mathrm{H}_\alpha^o(\mathbb{H})$ for any $0 < \alpha < 1/2$.

It is well known that in the Hilbert space $\mathbb{H}$, every random element X such that $\mathbf{E}\, \|X\|^2 < \infty$ is *pregaussian*, i.e. there is a Gaussian random element G in $\mathbb{H}$ with the same covariance operator as X, see [14, Prop. 9.24]. Let the X_i's be i.i.d. copies of X. If moreover $\mathbf{E}\, X = 0$, then $n^{-1/2} \sum_{i=1}^n X_i$ converges weakly to G in $\mathbb{H}$, in other words X satisfies the CLT in $\mathbb{H}$ [14, Th. 10.5].

We establish necessary and sufficient conditions for FCLT in Hölder space $\mathrm{H}^o_\alpha(\mathbb{H})$, where $0 < \alpha < 1/2$ and $d \geq 2$.

When based on the collection $\mathcal{Q}_d$, the summation process $\xi_{\boldsymbol{n}}$ defined by (3) can be canonically identified with a random field with parameter set $[0,1]^d$. Indeed writing

$$[0,\boldsymbol{t}] := [0,t_1] \times \cdots \times [0,t_d] \tag{8}$$

we define

$$\xi_{\boldsymbol{n}}(\boldsymbol{t}) := \xi_{\boldsymbol{n}}([0,\boldsymbol{t}]) = \sum_{\boldsymbol{1}\leq \boldsymbol{j}\leq \boldsymbol{n}} |R_{\boldsymbol{n},\boldsymbol{j}}|^{-1}|R_{\boldsymbol{n},\boldsymbol{j}} \cap [0,\boldsymbol{t}]|X_{\boldsymbol{j}}, \quad \boldsymbol{t} \in [0,1]^d. \tag{9}$$

In subsection 3.3 below we discuss in detail the construction of the random field ξ_n and propose some useful representations. Now we can state our main result which appears as a contrasted extension of the necessary and sufficient condition obtained by Račkauskas and Suquet [19, Th. 1] in the context of Lamperti's Hölderian FCLT.

Theorem 2 *For $0 < \alpha < 1/2$, set $p = p(\alpha) := 1/(1/2 - \alpha)$. For $d \geq 2$, let $\{X_{\boldsymbol{j}};\ \boldsymbol{j} \in \mathbb{N}^d,\ \boldsymbol{j} \geq \boldsymbol{1}\}$ be an i.i.d. random field of square integrable centered random elements in the separable Hilbert space $\mathbb{H}$ and $\xi_{\boldsymbol{n}}$ be the summation process defined by* (9). *Let W be a $\mathbb{H}$-valued Brownian sheet with the same covariance operator as $X_{\boldsymbol{1}}$. Then the convergence*

$$\boldsymbol{\pi}(\boldsymbol{n})^{-1/2}\xi_{\boldsymbol{n}} \xrightarrow[\mathrm{m}(\boldsymbol{n})\to\infty]{\mathrm{H}^o_\alpha(\mathbb{H})} W \tag{10}$$

holds if and only if

$$v_1^p(v_2\cdots v_d)^2 P\Big(\|X_{\boldsymbol{1}}\| > v_1v_2\cdots v_d\Big) \xrightarrow[\mathrm{m}(\boldsymbol{v})\to\infty]{} 0. \tag{11}$$

Moreover (11) is equivalent to the finiteness of the weak p-moment of $X_{\boldsymbol{1}}$, i.e.

$$\sup_{t>0} t^p P(\|X_{\boldsymbol{1}}\| > t) < \infty. \tag{12}$$

At first sight, condition (11) looks asymetric, but it is easy to see that any permutation on the indexes $1, \dots, d$ leads to an equivalent condition.

As condition (12) is weaker than $\mathbf{E}\,\|X_1\|^p < \infty$, then theorem 2 improves when $\mathbb{H} = \mathbb{R}$, Erickson's [8] result for $\mathcal{Q}_d$:

$$(n_1\cdots n_d)^{-1/2}\xi_{\boldsymbol{n}} \xrightarrow[\mathrm{m}(\boldsymbol{n})\to\infty]{\mathrm{H}^o_\alpha(\mathbb{R})} W.$$

if $0 < \alpha < 1/2$ and $\mathbf{E}\,|X_{\mathbf{1}}|^q < \infty$, where $q > dp(\alpha)$.

Considering the convergence of random fields $(\xi_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^d)$ along fixed path $\boldsymbol{n} = (n, \dots, n) \in \mathbb{N}^d$, $n \in \mathbb{N}$ we obtain the following result.

Theorem 3 *The convergence*

$$n^{-d/2}\xi_{(n,\dots,n)} \xrightarrow[n\to\infty]{\mathrm{H}^o_\alpha(\mathbb{H})} W \tag{13}$$

holds if and only if

$$\lim_{t\to\infty} t^{\frac{2d}{d-2\alpha}} P(\|X_{\mathbf{1}}\| > t) = 0, \tag{14}$$

Since $2d/(d-2\alpha) < 2d/(d-1)$ we see that $\mathbf{E}\,\|X_{\mathbf{1}}\|^{2d/(d-1)} < \infty$ yields (14). In particular $\mathbf{E}\,\|X_{\mathbf{1}}\|^4 < \infty$ gives the convergence (13) for any $d \geq 2$ and any $0 < \alpha < 1/2$. This contrasts with the corresponding result for Hölder convergence of the usual Donsker-Prokhorov polygonal line processes where necessarily $\mathbf{E}\,|X_1|^q < \infty$ for any $q < p(\alpha)$ as follows from (1).

Of course, Theorem 3 is only a striking special case and similar results can be obtained adapting the proof of Theorem 2 for summation processes with index going to infinity along some various paths or surfaces.

As passing from n to $n+1$ brings $O(n^{d-1})$ new summands in the summation process of Theorem 3, one may be tempted to look for similar weakening of the assumption in the Hölderian FCLT for $d = 1$, when restricting for subsequences. In fact even so, the situation is quite different: it is easy to see that for any increasing sequence of integers n_k such that $\sup_{k\geq 1} n_{k+1}/n_k < \infty$, the convergence to zero of $n_k^{p(\alpha)} P(|X_1| > n_k)$ when k tends to infinity implies (1). As $n_k^{p(\alpha)} P(|X_1| > n_k) = o(1)$ is a necessary condition for $(\xi_{n_k})_{k\geq 1}$ to satisfy the FCLT in $\mathrm{H}^o_\alpha(\mathbb{R})$ when $d = 1$, there is no hope to obtain this FCLT for $(\xi_{n_k})_{k\geq 1}$ under some condition weaker than (1).

3 Background and tools

3.1 Hölder spaces and Schauder decomposition

We present briefly here some structure property of $\mathrm{H}^o_\alpha(\mathbb{H})$ which is needed to obtain a tightness criterion. For more details, the reader is referred to [17] and [18]. Set

$$W_j = \{k2^{-j};\ 0 \le k \le 2^j\}^d, \quad j = 0, 1, 2, \dots$$

and

$$V_0 := W_0, \qquad V_j := W_j \setminus W_{j-1}, \quad j \ge 1,$$

so V_j is the set of dyadic points $\boldsymbol{v} = (k_1 2^{-j}, \dots, k_d 2^{-j})$ in W_j with at least one k_i odd. Define the *pyramidal functions* $\Lambda_{j,\boldsymbol{v}}$ by

$$\Lambda_{j,\boldsymbol{v}}(\boldsymbol{t}) = \Lambda(2^j(\boldsymbol{t} - \boldsymbol{v})), \quad \boldsymbol{t} \in [0,1]^d,$$

where

$$\Lambda(\boldsymbol{t}) := \max\Big(0,\ 1 - \max_{t_i<0} |t_i| - \max_{t_i>0} t_i\Big), \quad \boldsymbol{t} = (t_1, \dots, t_d) \in [-1,1]^d.$$

The $\mathbb{H}$-valued coefficients $\lambda_{j,\boldsymbol{v}}(x)$ are given by:

$$\begin{aligned} \lambda_{0,\boldsymbol{v}}(x) &= x(\boldsymbol{v}), \quad \boldsymbol{v} \in V_0; \\ \lambda_{j,\boldsymbol{v}}(x) &= x(\boldsymbol{v}) - \frac{1}{2}\Big(x(\boldsymbol{v}^-) + x(\boldsymbol{v}^+)\Big), \quad \boldsymbol{v} \in V_j,\ j \ge 1, \end{aligned}$$

where $\boldsymbol{v}^-$ and $\boldsymbol{v}^+$ are defined as follows. Each $\boldsymbol{v} \in V_j$ admits a unique representation $\boldsymbol{v} = (v_1, \dots, v_d)$ with $v_i = k_i/2^j$, $(1 \le i \le d)$. The points $\boldsymbol{v}^- = (v_1^-, \dots, v_d^-)$ and $\boldsymbol{v}^+ = (v_1^+, \dots, v_d^+)$ are defined by

$$v_i^- = \begin{cases} v_i - 2^{-j}, & \text{if } k_i \text{ is odd;} \\ v_i, & \text{if } k_i \text{ is even} \end{cases} \qquad v_i^+ = \begin{cases} v_i + 2^{-j}, & \text{if } k_i \text{ is odd;} \\ v_i, & \text{if } k_i \text{ is even,} \end{cases}$$

Define the linear operators E_j $(j \ge 0)$

$$E_j x := \sum_{i=0}^{j} \sum_{\boldsymbol{v} \in V_i} \lambda_{i,\boldsymbol{v}}(x) \Lambda_{i,\boldsymbol{v}}, \quad x \in \mathrm{H}_\alpha^o(\mathbb{H}).$$

Introduce the sequential norm

$$\|x\|_\alpha^{\mathrm{seq}} := \sup_{j \ge 0} 2^{\alpha j} \max_{\boldsymbol{v} \in V_j} \|\lambda_{j,\boldsymbol{v}}(x)\|, \quad x \in \mathrm{H}_\alpha^o(\mathbb{H}).$$

From Račkauskas and Suquet [18] this norm is equivalent to norm $\|x\|_\alpha$ on $\mathrm{H}_\alpha^o(\mathbb{H})$. Note also that

$$\|x - E_J x\|_\alpha^{\mathrm{seq}} = \sup_{j > J} \max_{\boldsymbol{v} \in V_j} \|\lambda_{j,\boldsymbol{v}}(x)\|.$$

is non increasing in J.

For proving tightness criteria in $\mathrm{H}^o_\alpha(\mathbb{H})$ we need this result from [18].

Theorem 4 *The space* $\mathrm{H}^o_\alpha(\mathbb{H})$ *has the Schauder decomposition*

$$\mathrm{H}^o_\alpha(\mathbb{H}) = \bigoplus_{i=0}^{\infty} \boldsymbol{W}_i,$$

where $\boldsymbol{W}_i$ *is the closed subspace of* $\mathrm{H}^o_\alpha(\mathbb{H})$ *spanned by the sums* $\sum_{\boldsymbol{v}\in V_i} h_{\boldsymbol{v}}\lambda_{i,\boldsymbol{v}}$, *where the* $h_{\boldsymbol{v}}$ *are arbitrary elements of* $\mathbb{H}$. *This means that the direct sum above is topological, i.e., that the canonical projectors* $\pi_i : \mathrm{H}^o_\alpha(\mathbb{H}) \to \boldsymbol{W}_i$ *are continuous in the strong topology of* $\mathrm{H}^o_\alpha(\mathbb{H})$.

3.2 Tightness criteria

Compacts in separable Banach spaces with Schauder decomposition are characterised by this result from Suquet [24]:

Theorem 5 *Let* $\mathcal{X}$ *be a separable Banach space having a Schauder decomposition* $\bigoplus_{i=0}^{\infty} \boldsymbol{W}_i$. *A subset* K *is relatively compact in* $\mathcal{X}$ *if and only if:*

i) For each $j \in \mathbb{N}$, $E_j K$ *is relatively compact in* $\boldsymbol{V_j} := \bigoplus_{i=0}^{j} \boldsymbol{W}_i$, *where* E_j *is the continuous canonical projector* $\mathcal{X} \to \boldsymbol{V_j}$.
ii) $\sup_{x\in K} \|x - E_j x\| \to 0$ *as* $j \to \infty$.

Since the set $\mathbb{N}^d$ with the binary relation $\boldsymbol{j} \leq \boldsymbol{n}$ is directed, our summation process $\{\xi_{\boldsymbol{n}},\ \boldsymbol{n} \in \mathbb{N}^d\}$ is a net. So to prove convergence we will need the tightness criteria for nets. Due to Prokhorov's theorem for nets, see e.g. [26, th.1.3.9, p.21], we need only asymptotical tightness. For the net of $\mathbb{H}$-valued random elements $\{\zeta_{\boldsymbol{n}},\ \boldsymbol{n} \in \mathbb{N}^d\}$ the asymptotical tightness means that for each $\varepsilon > 0$ there exists a compact set $K_\varepsilon \in \mathrm{H}^o_\alpha(\mathbb{H})$ such that

$$\liminf_{\mathrm{m}(\boldsymbol{n})\to\infty} P(\zeta_{\boldsymbol{n}} \in K_\varepsilon) > 1 - \varepsilon. \tag{15}$$

Now we can prove tightness criterion in $\mathrm{H}^o_\alpha(\mathbb{H})$.

Theorem 6 *Let* $\{\zeta_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^d\}$ *and* ζ *be random elements with values in the space* $\mathrm{H}^o_\alpha(\mathbb{H})$. *Assume that the following conditions are satisfied.*

i) For each dyadic $\boldsymbol{t} \in [0,1]^d$, *the net of* $\mathbb{H}$*-valued random elements* $\{\zeta_{\boldsymbol{n}}(\boldsymbol{t}), \boldsymbol{n} \in \mathbb{N}^d\}$ *is asymptoticaly tight on* $\mathbb{H}$.
ii) For each $\varepsilon > 0$

$$\lim_{J\to\infty} \limsup_{\mathrm{m}(\boldsymbol{n})\to\infty} P\Big(\sup_{j\geq J} 2^{\alpha j} \max_{\boldsymbol{v}\in V_j} |\lambda_{j,\boldsymbol{v}}(\zeta_{\boldsymbol{n}})| > \varepsilon \Big) = 0.$$

Then the net $\{\zeta_{\boldsymbol{n}},\ \boldsymbol{n} \in \mathbb{N}^d\}$ is asymptoticaly tight in the space $\mathrm{H}^o_\alpha(\mathbb{H})$.

Proof. For fixed positive η, put $\eta_l = 2^{-l}$, $l = 1, 2, \dots$ and choose a sequence (ε_l) decreasing to zero. By (ii) there is and integer J_l and index $\boldsymbol{n}_0 \in \mathbb{N}^d$ such that for the set

$$A_l := \{x : \sup_{j \geq J_l} 2^{\alpha j} \max_{\boldsymbol{v} \in V_j} ||\lambda_{j,\boldsymbol{v}}(x)|| < \varepsilon_l\},$$

$P(\zeta_{\boldsymbol{n}} \in A_l) > 1 - \eta_l$, for all $\boldsymbol{n} \geq \boldsymbol{n}_0$. It is easily seen that $\boldsymbol{V}_j = \oplus_{i=0}^{j} \boldsymbol{W}_i$ is isomorphic to the Cartesian product of a finite number of copies of $\mathbb{H}$. Thus from (i) there exists a compact $K_l \in \mathrm{H}^o_\alpha(\mathbb{H})$ such that for all $\boldsymbol{n} \geq \boldsymbol{n}_0$, $P(\zeta_n \in B_l) > 1 - \eta_l$, where

$$B_l := \{x \in \mathrm{H}^o_\alpha(\mathbb{H}) : E_{J_l} x \in K_l\}.$$

Take K the closure of $\cap_{l=1}^{\infty}(A_l \cap B_l)$. Then $P(K) > 1 - 2\eta$, and K is compact due to theorem 5. $\square$

3.3 Summation processes

We discuss now the construction of the summation process random field ξ_n. Let us start with the case $d = 1$ where ξ_n is the Donsker-Prohorov polygonal line wich interpolates linearly between the vertices $(k/n, S_k)$. Expressing t as a barycenter of $[nt]/n$ and $([nt] + 1)/n$ we have

$$t = (1 - \{nt\})\frac{[nt]}{n} + \{nt\}\frac{[nt] + 1}{n}. \tag{16}$$

As $\xi_n([nt]) = S_{[nt]}$, the linear interpolation between the vertices $([nt]/n, S_{[nt]})$ and $(([nt] + 1)/n, S_{[nt]+1})$ leads to

$$\xi_n(t) = (1 - \{nt\})S_{[nt]} + \{nt\}S_{[nt]+1}. \tag{17}$$

This expression can be rewritten under the forms

$$\begin{aligned} \xi_n(t) &= S_{[nt]} + \{nt\}(S_{[nt]+1} - S_{[nt]}) && (18)\\ &= S_{[nt]} + \{nt\}X_{[nt]+1} && (19)\\ &= \sum_{1 \leq i \leq n} n\Big|[(i-1)/n, i/n] \cap [0, t]\Big| X_i. && (20) \end{aligned}$$

Formula (17) comes directly from barycentric representation of t and linear interpolation. Formula (18) is useful to control the increments of ξ_n, (19) is the classical expression of ξ_n and (20) gives the interpretation of ξ_n in terms of

$\mathcal{Q}_1$ indexed summation process. Our aim is to generalize these representations when $d > 1$. Our first step will be to generalize (16) expressing $\boldsymbol{t} \in [0,1]^d$ as a barycenter of the vertices of some "rectangle" $R_{\boldsymbol{n},\boldsymbol{i}}$ containing $\boldsymbol{t}$. This leads to the extension of (17) and we shall check that it also coincides with the initial definition (9), so extending (20). Finally we shall extend (18). There is no extension of (19), at least with a single $X_{\boldsymbol{i}}$ outside $S_{[nt]}$, as it is already clear from the case $d = 2$.

For every $\boldsymbol{n} \geq \mathbf{1}$ in $\mathbb{N}^d$, put

$$S_{\boldsymbol{n}} := \sum_{\mathbf{1} \leq \boldsymbol{i} \leq \boldsymbol{n}} X_{\boldsymbol{i}}. \tag{21}$$

Proposition 7 *Let us write any $\boldsymbol{t} \in [0,1]^d$ as the barycenter of the 2^d vertices*

$$V(\boldsymbol{u}) := \frac{[\boldsymbol{nt}]}{\boldsymbol{n}} + \frac{\boldsymbol{u}}{\boldsymbol{n}}, \quad \boldsymbol{u} \in \{0,1\}^d, \tag{22}$$

of the rectangle $R_{\boldsymbol{n},[\boldsymbol{nt}]+\mathbf{1}}$ with some weights $w(\boldsymbol{u}) \geq 0$ depending on $\boldsymbol{t}$, i.e.,

$$\boldsymbol{t} = \sum_{\boldsymbol{u} \in \{0,1\}^d} w(\boldsymbol{u}) V(\boldsymbol{u}), \qquad \textit{where} \qquad \sum_{\boldsymbol{u} \in \{0,1\}^d} w(\boldsymbol{u}) = 1. \tag{23}$$

Using this representation, define the random field ξ_n^* by

$$\xi_{\boldsymbol{n}}^*(\boldsymbol{t}) = \sum_{\boldsymbol{u} \in \{0,1\}^d} w(\boldsymbol{u}) S_{[\boldsymbol{nt}]+\boldsymbol{u}}, \quad \boldsymbol{t} \in [0,1]^d.$$

Then $\xi_{\boldsymbol{n}}^*$ coincides with the summation process defined by (9).

Proof. For fixed $\boldsymbol{n} \geq \mathbf{1} \in \mathbb{N}^d$, any $\boldsymbol{t} \neq \mathbf{1} \in [0,1]^d$ belongs to a unique rectangle $R_{\boldsymbol{n},\boldsymbol{j}}$, defined by (4), namely $R_{\boldsymbol{n},[\boldsymbol{nt}]+\mathbf{1}}$. Then the 2^d vertices of this rectangle are clearly the points $V(\boldsymbol{u})$ given by (22), recalling that in this formula the division of vector is componentwise. To simplify notations, put

$$\boldsymbol{s} = \{\boldsymbol{nt}\}, \quad \text{whence} \quad \boldsymbol{t} = \frac{[\boldsymbol{nt}]}{\boldsymbol{n}} + \frac{\boldsymbol{s}}{\boldsymbol{n}}. \tag{24}$$

For any non empty subset L of $\{1,\dots,d\}$, we denote by $\{0,1\}^L$ the set of binary vectors indexed by L. In particular $\{0,1\}^d$ is an abriged notation for $\{0,1\}^{\{1,\dots,d\}}$. Now define the non negative weights

$$w_L(\boldsymbol{u}) := \prod_{l \in L} s_l^{u_l}(1-s_l)^{1-u_l}, \quad \boldsymbol{u} \in \{0,1\}^L$$

and when $L = \{1,\dots,d\}$, simplify this notation in $w(\boldsymbol{u})$. For fixed L, the sum of all these weigths is one since

$$\sum_{\boldsymbol{u}\in\{0,1\}^L} w_L(\boldsymbol{u}) = \prod_{l\in L}\Big(s_l + (1-s_l)\Big) = 1. \tag{25}$$

The special case $L = \{1,\dots,d\}$ gives the second equality in (23). From (25) we immediatly deduce that for any K non empty and strictly included in $\{1,\dots,d\}$, with $L := \{1,\dots,d\}\setminus K$,

$$\sum_{\substack{\boldsymbol{u}\in\{0,1\}^d,\\ \forall k\in K,\, u_k=1}} w(\boldsymbol{u}) = \prod_{k\in K} s_k \sum_{\boldsymbol{u}\in\{0,1\}^L} s_l^{u_l}(1-s_l)^{1-u_l} = \prod_{k\in K} s_k. \tag{26}$$

Formula (26) remains obviously valid in the case where $K = \{1,\dots,d\}$.

Now let us observe that

$$\sum_{\boldsymbol{u}\in\{0,1\}^d} w(\boldsymbol{u})V(\boldsymbol{u}) = \sum_{\boldsymbol{u}\in\{0,1\}^d} w(\boldsymbol{u})\Big(\frac{[\boldsymbol{nt}]}{\boldsymbol{n}} + \frac{\boldsymbol{u}}{\boldsymbol{n}}\Big) = \frac{[\boldsymbol{nt}]}{\boldsymbol{n}} + \sum_{\boldsymbol{u}\in\{0,1\}^d} w(\boldsymbol{u})\frac{\boldsymbol{u}}{\boldsymbol{n}}.$$

Comparing with the expression of $\boldsymbol{t}$ given by (24), we see that the first equality in (23) will be established if we check that

$$\boldsymbol{s}' := \sum_{\boldsymbol{u}\in\{0,1\}^d} w(\boldsymbol{u})\boldsymbol{u} = \boldsymbol{s}. \tag{27}$$

This is easily seen componentwise using (26) because for any fixed $l \in \{1,\dots,d\}$,

$$s'_l = \sum_{\substack{\boldsymbol{u}\in\{0,1\}^d,\\ u_l=1}} w(\boldsymbol{u}) = \prod_{k\in\{l\}} s_k = s_l.$$

Next we check that $\xi_{\boldsymbol{n}}(\boldsymbol{t}) = \xi^*_{\boldsymbol{n}}(\boldsymbol{t})$ for every $\boldsymbol{t}\in[0,1]^d$. Recalling (8), introduce the notation

$$D_{\boldsymbol{t},\boldsymbol{u}} := \mathbb{N}^d \cap \Big(\big[0,[\boldsymbol{nt}]+\boldsymbol{u}\big]\setminus\big[0,[\boldsymbol{nt}]\big]\Big).$$

Then we have

$$\xi^*_{\boldsymbol{n}}(\boldsymbol{t}) = \sum_{\boldsymbol{u}\in\{0,1\}^d} w(\boldsymbol{u})\Big(S_{[\boldsymbol{nt}]} + (S_{[\boldsymbol{nt}]+\boldsymbol{u}} - S_{[\boldsymbol{nt}]})\Big) = S_{[\boldsymbol{nt}]} + \sum_{\boldsymbol{u}\in\{0,1\}^d} w(\boldsymbol{u}) \sum_{\boldsymbol{i}\in D_{\boldsymbol{t},\boldsymbol{u}}} X_{\boldsymbol{i}}.$$

Now in view of (9), the proof of $\xi_{\boldsymbol{n}}(\boldsymbol{t}) = \xi^*_{\boldsymbol{n}}(\boldsymbol{t})$ reduces clearly to that of

$$\sum_{\boldsymbol{u}\in\{0,1\}^d} w(\boldsymbol{u}) \sum_{\boldsymbol{i}\in D_{\boldsymbol{t},\boldsymbol{u}}} X_{\boldsymbol{i}} = \sum_{\boldsymbol{i}\in I} |R_{\boldsymbol{n},\boldsymbol{i}}|^{-1}|R_{\boldsymbol{n},\boldsymbol{i}}\cap[0,\boldsymbol{t}]|X_{\boldsymbol{i}}, \tag{28}$$

where

$$I := \{\boldsymbol{i} \le \boldsymbol{n};\ \forall k \in \{1,\dots,d\},\ i_k \le [n_k t_k] + 1 \text{ and } \exists l \in \{1,\dots,d\},\ i_l = [n_l t_l] + 1\}. \tag{29}$$

Clearly I is the union of all $D_{\boldsymbol{t},\boldsymbol{u}}$, $\boldsymbol{u} \in \{0,1\}^d$, so we can rewrite the left hand side of (28) under the form $\sum_{\boldsymbol{i}\in I} a_{\boldsymbol{i}} X_{\boldsymbol{i}}$. For $\boldsymbol{i} \in I$, put

$$K(\boldsymbol{i}) := \Big\{k \in \{1,\dots,d\};\ \boldsymbol{i}_k = [n_k t_k] + 1\Big\}. \tag{30}$$

Then observe that for $\boldsymbol{i} \in I$, the $\boldsymbol{u}$'s such that $\boldsymbol{i} \in D_{\boldsymbol{t},\boldsymbol{u}}$ are exactly those which satisfy $u_k = 1$ for every $k \in K(\boldsymbol{i})$. Using (26), this gives

$$\forall \boldsymbol{i} \in I, \qquad a_{\boldsymbol{i}} = \sum_{\substack{\boldsymbol{u}\in\{0,1\}^d,\\ \forall k \in K(\boldsymbol{i}),\, u_k = 1}} w(\boldsymbol{u}) = \prod_{k\in K(\boldsymbol{i})} s_k. \tag{31}$$

On the other hand we have for every $\boldsymbol{i} \in I$,

$$|R_{\boldsymbol{n},\boldsymbol{i}} \cap [0,\boldsymbol{t}]| = \prod_{k\in K(\boldsymbol{i})} \left(t_k - \frac{[n_k t_k]}{n_k}\right) \prod_{k\notin K(\boldsymbol{i})} \frac{1}{n_k} = \frac{1}{\boldsymbol{\pi}(\boldsymbol{n})} \prod_{k\in K(\boldsymbol{i})} s_k = \frac{a_{\boldsymbol{i}}}{\boldsymbol{\pi}(\boldsymbol{n})}. \tag{32}$$

As $|R_{\boldsymbol{n},\boldsymbol{i}}|^{-1} = \boldsymbol{\pi}(\boldsymbol{n})$, (28) follows and the proof is complete. □

Extending formula (18) to the case $d > 1$ requires the introduction of some more notations. For any finite subset A of $\mathbb{N}^d$, we put

$$S(A) := \sum_{\boldsymbol{i}\in A} X_{\boldsymbol{i}}.$$

Note that when $A = ([0, n_1] \times \cdots \times [0, n_d]) \cap \mathbb{N}^d$ with $\boldsymbol{n} = (n_1,\dots,n_d) \in \mathbb{N}^d$, $S(A)$ is the sum $S_{\boldsymbol{n}}$ defined by (21). For any Cartesian product $C = C_1 \times \cdots \times C_d$ of finite subsets C_i of $\mathbb{N}$, $i = 1,\dots,d$, let us define

$$\Delta_k^{(j)} C := C_1 \times \cdots \times C_{j-1} \times \{k\} \times C_{j+1} \times \cdots \times C_d \tag{33}$$

and

$$\Big(\Delta_k^{(j)} S\Big)(C) := S\Big(\Delta_k^j C\Big). \tag{34}$$

Clearly the operators $\Delta_k^{(j)}$'s commute for different j's. It is worth noticing that

$$\Delta_k^{(j)} S_{\boldsymbol{n}} = S_{(n_1,\dots,n_{j-1},k,n_{j+1}\dots,n_d)} - S_{(n_1,\dots,n_{j-1},k-1,n_{j+1}\dots,n_d)} \tag{35}$$

and that for $\mathbf{1} \le \boldsymbol{i} \le \boldsymbol{n}$,

$$X_{\boldsymbol{i}} = \Delta_{i_1}^{(1)} \dots \Delta_{i_d}^{(d)} S_{\boldsymbol{n}}. \tag{36}$$

Note that when applied to $S_{\boldsymbol{n}}$, $\Delta_k^{(j)}$ is really a difference operator acting on the j-th argument of a function with d arguments. Also since k defines the differencing, $\Delta_k^{(j)} S_{\boldsymbol{n}}$ does not depend on n_j.

Recalling the notations (29), (30) and formula (32), we have

$$\xi_{\boldsymbol{n}}(\boldsymbol{t}) = S_{[\boldsymbol{nt}]} + \sum_{\boldsymbol{i}\in I} |R_{\boldsymbol{n},\boldsymbol{i}}|^{-1} |R_{\boldsymbol{n},\boldsymbol{i}} \cap [0, \boldsymbol{t}]| X_{\boldsymbol{i}} = S_{[\boldsymbol{nt}]} + \sum_{\boldsymbol{i}\in I} \left(\prod_{k\in K(\boldsymbol{i})} s_k \right) X_{\boldsymbol{i}}.$$

This can be recast as

$$\xi_{\boldsymbol{n}}(\boldsymbol{t}) = S_{[\boldsymbol{nt}]} + \sum_{l=1}^{d} T_l(\boldsymbol{t}) \tag{37}$$

with

$$T_l(\boldsymbol{t}) := \sum_{\substack{\boldsymbol{i}\in I \\ \sharp K(\boldsymbol{i})=l}} \left(\prod_{k\in K(\boldsymbol{i})} s_k \right) X_{\boldsymbol{i}}. \tag{38}$$

Now we observe that

$$T_l(\boldsymbol{t}) = \sum_{\substack{K\subset\{1,\dots,d\} \\ \sharp K=l}} \sum_{\substack{\boldsymbol{i}\in I \\ K(\boldsymbol{i})=K}} \left(\prod_{k\in K} s_k \right) X_{\boldsymbol{i}} = \sum_{\substack{K\subset\{1,\dots,d\} \\ \sharp K=l}} \left(\prod_{k\in K} s_k \right) \sum_{\substack{\boldsymbol{i}\in I \\ K(\boldsymbol{i})=K}} X_{\boldsymbol{i}}.$$

From (33) and (34), it should be clear that

$$\sum_{\substack{\boldsymbol{i}\in I \\ K(\boldsymbol{i})=K}} X_{\boldsymbol{i}} = \left(\prod_{k\in K} \Delta^{(k)}_{[n_k t_k]+1} \right) S_{[\boldsymbol{nt}]},$$

where the symbol Π is intended as the composition product of differences operators. Recalling that $s_k = \{n_k t_k\}$, this leads to

$$T_l(\boldsymbol{t}) = \sum_{\substack{K\subset\{1,\dots,d\} \\ \sharp K=l}} \left(\prod_{k\in K} \{n_k t_k\} \right) \left(\prod_{k\in K} \Delta^{(k)}_{[n_k t_k]+1} \right) S_{[\boldsymbol{nt}]}. \tag{39}$$

Finally we obtain the representation

$$\xi_{\boldsymbol{n}}(\boldsymbol{t}) = S_{[\boldsymbol{nt}]} + \sum_{l=1}^{d} \sum_{1\le i_1<i_2<\dots<i_l\le d} \left(\prod_{k=1}^{l} \{n_{i_k} t_{i_k}\} \right) \left(\prod_{k=1}^{l} \Delta^{(i_k)}_{[n_{i_k} t_{i_k}]+1} \right) S_{[\boldsymbol{nt}]}. \tag{40}$$

3.4 Finite dimensional distributions

As (11) implies (14) and $d/(d/2-\alpha) > 2$, for $0 < \alpha < 1/2$ we have that $\mathbf{E}\,\|X_{\mathbf{1}}\|^2 < \infty$. In what follows we assume $\mathbf{E}\,\|X_{\mathbf{1}}\|^2 = 1$.

Define $A_{\boldsymbol{t}} = [0, t_1] \times \cdots \times [0, t_d]$ and the jump summation process by

$$\zeta_{\boldsymbol{n}}(\boldsymbol{t}) = \sum_{\boldsymbol{j} \le \boldsymbol{n}} \chi\{\boldsymbol{j}/\boldsymbol{n} \in A_{\boldsymbol{t}}\} X_{\boldsymbol{j}}.$$

For any Borel set $A \subset [0,1]^d$ define for $\varepsilon > 0$

$$A^{\varepsilon} := \Big\{ y \in \mathbb{R}^d,\ \exists x \in A;\ |x - y| < \varepsilon \Big\}, \quad A^{-\varepsilon} := \mathbb{R}^d \setminus (\mathbb{R}^d \setminus A)^{\varepsilon}.$$

Lemma 8 *Put* $\varepsilon_{\boldsymbol{n}} := \mathrm{m}(\boldsymbol{n})^{-1}$ *and* $\beta_{\boldsymbol{n}}(\boldsymbol{t}) := |A_{\boldsymbol{t}}^{\varepsilon_{\boldsymbol{n}}} \setminus A_{\boldsymbol{t}}^{-\varepsilon_{\boldsymbol{n}}}|$ *for each* $\boldsymbol{t} \in [0,1]^d$. *Then*

$$\mathbf{E}\, \|\boldsymbol{\pi}(\boldsymbol{n})^{-1/2}(\xi_{\boldsymbol{n}}(\boldsymbol{t}) - \zeta_{\boldsymbol{n}}(\boldsymbol{t}))\|^2 \le K \beta_{\boldsymbol{n}}(\boldsymbol{t}) \xrightarrow[\mathrm{m}(\boldsymbol{n}) \to \infty]{} 0.$$

Proof. For each $\boldsymbol{t}$ we can write $\boldsymbol{\pi}(\boldsymbol{n})^{-1/2}(\xi_{\boldsymbol{n}}(\boldsymbol{t}) - \zeta_{\boldsymbol{n}}(\boldsymbol{t})) = \sum_{\boldsymbol{j} \le \boldsymbol{n}} \alpha_{\boldsymbol{j}} X_{\boldsymbol{j}}$, where

$$\alpha_{\boldsymbol{j}} := \boldsymbol{\pi}(\boldsymbol{n})^{1/2}(|R_{\boldsymbol{n},\boldsymbol{j}} \cap A_{\boldsymbol{t}}| - \boldsymbol{\pi}(\boldsymbol{n})^{-1} \chi\{\boldsymbol{j}/\boldsymbol{n} \in A_{\boldsymbol{t}}\}).$$

Then

$$\mathbf{E}\, \|\boldsymbol{\pi}(\boldsymbol{n})^{-1/2}(\xi_{\boldsymbol{n}} - \zeta_{\boldsymbol{n}})\|^2 = \sum_{\boldsymbol{i} \le \boldsymbol{n}} \sum_{\boldsymbol{j} \le \boldsymbol{n}} \alpha_{\boldsymbol{i}} \alpha_{\boldsymbol{j}} \mathbf{E}\, \langle X_{\boldsymbol{i}}, X_{\boldsymbol{j}} \rangle = \mathbf{E}\, \|X_{\boldsymbol{1}}\|^2 \sum_{\boldsymbol{j} \le \boldsymbol{n}} \alpha_{\boldsymbol{j}}^2,$$

since the $X_{\boldsymbol{j}}$'s are i.i.d. with zero mean. Now from Erickson [8, th. 7.3.] we have

$$\sum_{\boldsymbol{j} \le \boldsymbol{n}} \alpha_{\boldsymbol{j}}^2 \le \beta_{\boldsymbol{n}}(\boldsymbol{t}).$$

And this upper bound tends to zero since the Lebesgue measure of $A_{\boldsymbol{t}}^{\varepsilon_{\boldsymbol{n}}} \setminus A_{\boldsymbol{t}}^{-\varepsilon_{\boldsymbol{n}}}$ is clearly $O(\varepsilon_{\boldsymbol{n}}) = O(\mathrm{m}(\boldsymbol{n})^{-1})$. □

Combined with the estimate $P(\|X_1\| > r) \le r^{-2} \mathbf{E}\, \|X_1\|^2$, lemma 8 gives

$$\|\boldsymbol{\pi}(\boldsymbol{n})^{-1/2}(\xi_{\boldsymbol{n}}(\boldsymbol{t}) - \zeta_{\boldsymbol{n}}(\boldsymbol{t}))\| \xrightarrow[\mathrm{m}(\boldsymbol{n}) \to \infty]{\mathrm{Pr}} 0.$$

By Slutsky's lemma, this implies the asymptotical equality of finite dimensional distributions of both processes $\boldsymbol{\pi}(\boldsymbol{n})^{-1/2} \xi_{\boldsymbol{n}}$ and $\boldsymbol{\pi}(\boldsymbol{n})^{-1/2} \zeta_{\boldsymbol{n}}$.

Lemma 9 *Let* $\widetilde{\zeta}_{\boldsymbol{n}} := \boldsymbol{\pi}(\boldsymbol{n})^{-1/2} \zeta_{\boldsymbol{n}}$. *The convergence*

$$\widetilde{\zeta}_{\boldsymbol{n}}(\boldsymbol{t}) \xrightarrow[\mathrm{m}(\boldsymbol{n}) \to \infty]{\mathbb{H}} W(\boldsymbol{t}) \tag{41}$$

holds for each $\boldsymbol{t} \in [0,1]^d$.

Proof. Let

$$J(\boldsymbol{n}) := \{\boldsymbol{j} \in \mathbb{N}^d : \boldsymbol{j}/\boldsymbol{n} \in A_{\boldsymbol{t}}\}$$

and let $l(\boldsymbol{n})$ denote the number of elements in $J(\boldsymbol{n})$. Then

$$\widetilde{\zeta}_{\boldsymbol{n}}(\boldsymbol{t}) = \boldsymbol{\pi}(\boldsymbol{n})^{-1/2} \sum_{\boldsymbol{j} \in J(\boldsymbol{n})} X_{\boldsymbol{j}}.$$

Since $l(\boldsymbol{n}) \to \infty$, as $\mathrm{m}(\boldsymbol{n}) \to \infty$, the central limit theorem in Hilbert space gives

$$l(\boldsymbol{n})^{-1/2} \sum_{\boldsymbol{j} \in J(\boldsymbol{n})} X_{\boldsymbol{j}} \xrightarrow[\mathrm{m}(\boldsymbol{n})\to\infty]{\mathbb{H}} G, \tag{42}$$

where G is a zero mean Gaussian random element in $\mathbb{H}$ with the same covariance operator as $X_{\boldsymbol{1}}$. If $U_{\boldsymbol{n}}$ is random variable uniformly distributed on the points $\boldsymbol{j}/\boldsymbol{n}$, $\boldsymbol{1} \leq \boldsymbol{j} \leq \boldsymbol{n}$, then

$$\frac{l(\boldsymbol{n})}{\boldsymbol{\pi}(\boldsymbol{n})} = \frac{1}{\boldsymbol{\pi}(\boldsymbol{n})} \sum_{\boldsymbol{j} \leq \boldsymbol{n}} \chi\{\boldsymbol{j}/\boldsymbol{n} \in A_{\boldsymbol{t}}\} = P(U_{\boldsymbol{n}} \in A_{\boldsymbol{t}}) \xrightarrow[\mathrm{m}(\boldsymbol{n})\to\infty]{} |A_{\boldsymbol{t}}| = t_1 \dots t_d.$$

This together with (42) gives the convergence (41) for every $\boldsymbol{t} \in [0,1]^d$ since $W(\boldsymbol{t})$ has the same distribution as $|A_{\boldsymbol{t}}|^{1/2} G$. $\square$

Lemma 10 *The convergence*

$$\left(\widetilde{\zeta}_{\boldsymbol{n}}(\boldsymbol{t}_1), \dots, \widetilde{\zeta}_{\boldsymbol{n}}(\boldsymbol{t}_q)\right) \xrightarrow[\mathrm{m}(\boldsymbol{n})\to\infty]{\mathbb{H}^q} \left(W(\boldsymbol{t}_1), \dots, W(\boldsymbol{t}_q)\right)$$

holds for each $q \geq 1$ *and each* $\boldsymbol{t}_1, \dots, \boldsymbol{t}_q \in [0,1]^d$.

Proof. Because $\mathbb{H}^q$ is equipped with product topology, the tightness of the net $(\widetilde{\zeta}_{\boldsymbol{n}}(\boldsymbol{t}_1), \dots, \widetilde{\zeta}_{\boldsymbol{n}}(\boldsymbol{t}_q))$ in $\mathbb{H}^q$ follows from the tightness in $\mathbb{H}$ of the q nets $(\widetilde{\zeta}_{\boldsymbol{n}}(\boldsymbol{t}_i))$.

Denote by $\langle \cdot, \cdot \rangle_q$ the scalar product in $\mathbb{H}^q$ which is defined by

$$\langle h, g \rangle_q := \sum_{i=1}^{q} \langle h_i, g_i \rangle, \quad h = (h_1, \dots, h_q),\ g = (g_1, \dots, g_q) \in \mathbb{H}^q.$$

Accounting the above mentionned tightness, it remains only to check for each $h \in \mathbb{H}^q$, the weak convergence

$$V_{\boldsymbol{n}} := \left\langle \left(\widetilde{\zeta}_{\boldsymbol{n}}(\boldsymbol{t}_1), \dots, \widetilde{\zeta}_{\boldsymbol{n}}(\boldsymbol{t}_q)\right), h \right\rangle_q \xrightarrow[\mathrm{m}(\boldsymbol{n})\to\infty]{\mathbb{R}} \left\langle \left(W(\boldsymbol{t}_1), \dots, W(\boldsymbol{t}_q)\right), h \right\rangle_q. \tag{43}$$

This will be done through Lindeberg theorem. The first step is to establish the convergence of the variance $b_{\boldsymbol{n}} := \mathbf{E}\, V_{\boldsymbol{n}}^2$ using the decomposition

$$V_{\boldsymbol{n}} = \sum_{k=1}^{q} \langle \widetilde{\zeta}_{\boldsymbol{n}}(\boldsymbol{t}_k), h_k \rangle = \boldsymbol{\pi}(\boldsymbol{n})^{-1/2} \sum_{\boldsymbol{i}\le\boldsymbol{n}} \sum_{k=1}^{q} \chi\{\boldsymbol{i}/\boldsymbol{n} \in A_{\boldsymbol{t}_k}\} \langle X_{\boldsymbol{i}}, h_k \rangle.$$

Denoting by Γ the covariance operator of $X_{\mathbf{1}}$, we get

$$\begin{aligned} b_{\boldsymbol{n}} &= \frac{1}{\boldsymbol{\pi}(\boldsymbol{n})} \sum_{\boldsymbol{i}\le\boldsymbol{n}} \sum_{\boldsymbol{j}\le\boldsymbol{n}} \sum_{k=1}^{q} \sum_{l=1}^{q} \chi\{\boldsymbol{i}/\boldsymbol{n} \in A_{\boldsymbol{t}_k}\} \chi\{\boldsymbol{j}/\boldsymbol{n} \in A_{\boldsymbol{t}_l}\} \mathbf{E} \left(\langle X_{\boldsymbol{i}}, h_k \rangle \langle X_{\boldsymbol{j}}, h_l \rangle \right) \\ &= \sum_{k=1}^{q} \sum_{l=1}^{q} \langle \Gamma h_k, h_l \rangle \frac{1}{\boldsymbol{\pi}(\boldsymbol{n})} \sum_{\boldsymbol{i}\le\boldsymbol{n}} \chi\{\boldsymbol{i}/\boldsymbol{n} \in A_{\boldsymbol{t}_k} \cap A_{\boldsymbol{t}_l}\} \\ &= \sum_{k=1}^{q} \sum_{l=1}^{q} \langle \Gamma h_k, h_l \rangle P(U_{\boldsymbol{n}} \in A_{\boldsymbol{t}_k} \cap A_{\boldsymbol{t}_l}), \end{aligned}$$

where the discrete random variable $U_{\boldsymbol{n}}$ is uniformly distributed on the grid $\boldsymbol{i}/\boldsymbol{n}$, $\mathbf{1} \le \boldsymbol{i} \le \boldsymbol{n}$. Under this form it is clear that when $\mathrm{m}(\boldsymbol{n})$ goes to infinity, $b_{\boldsymbol{n}}$ converges to b given by

$$b := \sum_{k=1}^{q} \sum_{l=1}^{q} \langle \Gamma h_k, h_l \rangle |A_{\boldsymbol{t}_k} \cap A_{\boldsymbol{t}_l}| = \mathbf{E} \left(\sum_{k=1}^{q} \langle W(\boldsymbol{t}_k), h_k \rangle \right)^2.$$

When $b = 0$, the convergence (43) is obvious. When $b > 0$, let us introduce the real random variables

$$Y_{\boldsymbol{n},\boldsymbol{i}} := \sum_{k=1}^{q} \boldsymbol{\pi}(\boldsymbol{n})^{-1/2} \chi\{\boldsymbol{i}/\boldsymbol{n} \in A_{\boldsymbol{t}_k}\} \langle X_{\boldsymbol{i}}, h_k \rangle,$$

which have both zero mean and finite variance and note that $V_{\boldsymbol{n}} = \sum_{\boldsymbol{i}\le\boldsymbol{n}} Y_{\boldsymbol{n},\boldsymbol{i}}$. To obtain (43) we have to check, by Lindeberg theorem, that for each $\varepsilon > 0$,

$$L(\boldsymbol{n}) := \frac{1}{b_{\boldsymbol{n}}} \sum_{\boldsymbol{i}\le\boldsymbol{n}} \mathbf{E} \left(Y_{\boldsymbol{n},\boldsymbol{i}}^2 \chi\{|Y_{\boldsymbol{n},\boldsymbol{i}}| > \varepsilon b_{\boldsymbol{n}}^{1/2}\} \right) \xrightarrow[\mathrm{m}(\boldsymbol{n})\to\infty]{} 0. \tag{44}$$

Now we have

$$\begin{aligned} Y_{\boldsymbol{n},\boldsymbol{i}}^2 &= \frac{1}{\boldsymbol{\pi}(\boldsymbol{n})} \sum_{k=1}^{q} \sum_{l=1}^{q} \chi\{\boldsymbol{i}/\boldsymbol{n} \in A_{\boldsymbol{t}_k}\} \chi\{\boldsymbol{i}/\boldsymbol{n} \in A_{\boldsymbol{t}_l}\} \langle X_{\boldsymbol{i}}, h_k \rangle \langle X_{\boldsymbol{i}}, h_l \rangle \\ &\le \frac{1}{\boldsymbol{\pi}(\boldsymbol{n})} \sum_{k=1}^{q} \sum_{l=1}^{q} \|X_{\boldsymbol{i}}\|^2 \|h_k\| \|h_l\| \\ &= \frac{1}{\boldsymbol{\pi}(\boldsymbol{n})} \left(\sum_{k=1}^{q} \|h_k\| \right)^2 \|X_{\boldsymbol{i}}\|^2 = \frac{c_h}{\boldsymbol{\pi}(\boldsymbol{n})} \|X_{\boldsymbol{i}}\|^2. \end{aligned}$$

Recalling that the number of terms in $\sum_{\boldsymbol{i}\le\boldsymbol{n}}$ is exactly $\boldsymbol{\pi}(\boldsymbol{n})$ and choosing

$\mathrm{m}(\boldsymbol{n})$ large enough to have $b_{\boldsymbol{n}} > b/2$, we obtain :

$$L(\boldsymbol{n}) \leq \frac{2}{b}\mathbf{E}\left(\|X_{\mathbf{1}}\|^2\chi\Big\{\|X_{\mathbf{1}}\|^2 > \frac{b\varepsilon^2}{2c_h}\boldsymbol{\pi}(\boldsymbol{n})\Big\}\right),$$

which gives (44) by square integrability of $X_{\mathbf{1}}$. $\square$

To conclude this section, let us retain that from lemmas 8 and 10, the finite dimensional distributions of $\boldsymbol{\pi}(\boldsymbol{n})^{-1/2}\xi_{\boldsymbol{n}}$ converge to finite dimensional distributions of the Wiener sheet W.

3.5 Rosenthal inequality in Hilbert space

Since the Hilbert space $\mathbb{H}$ has cotype 2, it satisfies the following vector valued version of Rosenthal's inequality for every $q \geq 2$, see [13, Th. 2.6]. For any finite set $(Y_i)_{i\in I}$ of independent random elements in $\mathbb{H}$ with zero mean and such that $\mathbf{E}\,\|Y_i\|^q < \infty$ for every $i \in I$,

$$\mathbf{E}\left\|\sum_{i\in I} Y_i\right\|^q \leq C'_q\left(\mathbf{E}\left\|\sum_{i\in I} G(Y_i)\right\|^q + \sum_{i\in I}\mathbf{E}\,\|Y_i\|^q\right), \qquad (45)$$

where the constant C'_q depends only on q and the $G(Y_i)$ are centered Gaussian independant random elements in $\mathbb{H}$ such that for every $i \in I$, $G(Y_i)$ has the same covariance structure as Y_i. In the i.i.d. case with $N = \sharp I$, we note that $\sum_{i\in I} G(Y_i)$ is Gaussian with the same distribution as $N^{1/2}G(Y_1)$ and using the equivalence of moments for Gaussian random elements, see [14, Cor. 3.2], we obtain

$$\mathbf{E}\left\|\sum_{i\in I} Y_i\right\|^q \leq C''_q\left(N^{q/2}\Big(\mathbf{E}\,\|G(Y_1)\|^2\Big)^{q/2} + N\mathbf{E}\,\|Y_1\|^q\right),$$

where C''_q depends on q and does not depend on the distribution of Y_1. Since $\mathbb{H}$ has also the type 2, there is a constant a depending only on $\mathbb{H}$ such that $\mathbf{E}\,\|G(Y_1)\|^2 \leq a\mathbf{E}\,\|Y_1\|^2$, see [14, Prop. 9.24]. Finally there is a constant C_q depending on $\mathbb{H}$, q, but not on the distribution of the Y_i's, such that

$$\mathbf{E}\left\|\sum_{i\in I} Y_i\right\|^q \leq C_q\left(N^{q/2}\Big(\mathbf{E}\,\|Y_1\|^2\Big)^{q/2} + N\mathbf{E}\,\|Y_1\|^q\right), \quad (N = \sharp I). \qquad (46)$$

3.6 An extension of Doob inequality

For i.i.d. Hilbert space valued random field $\{X_{\boldsymbol{j}}, \boldsymbol{j} \in \mathbb{N}^d\}$ introduce d one parameter filtrations, $\mathcal{F}^i = (\mathcal{F}^i_k, k = 0, 1, \dots)$, $i = 1, \dots, d$, where $\mathcal{F}^i_k = \sigma(X_{\boldsymbol{j}}, \boldsymbol{j} \in \mathbb{N}^d, j_i \leq k)$.

Assume that $\mathbf{E}\,\|X_1\| < \infty$, then the X_j's are Bochner integrable and according to [25] we can introduce conditional expectations with respect to $\mathcal{F}^i$, $i = 1, \dots, d$. Let $\mathbf{E}\,X_{\boldsymbol{j}} = 0$. Denote $M_{\boldsymbol{n}} = \|S_{\boldsymbol{n}}\|$. Since the norm is a continuous convex functional we have for $i = 1, \dots, d$, $\boldsymbol{n} \in \mathbb{N}^d$ and $k = 0, 1, \dots$

$$\mathbf{E}\,(\|S_{\boldsymbol{n}}\| | \mathcal{F}^i_k) \geq \|\mathbf{E}\,(S_{\boldsymbol{n}} | \mathcal{F}^i_k)\| = \left\| \sum_{\boldsymbol{j} \leq \boldsymbol{n}} \mathbf{E}\,(X_{\boldsymbol{j}} | \mathcal{F}^i_k) \right\| = \|S_{(n_1, \dots, n_{i-1}, k, n_{i+1}, \dots, n_d)}\|.$$

Hence for each $i = 1, \dots, d$, $n_i \to M_{\boldsymbol{n}}$ is a one parameter submartingale with respect to the filtration $\mathcal{F}^i$. Thus $M_{\boldsymbol{n}}$ is a orthosubmartingale according to [9]. Since $M_{\boldsymbol{n}}$ is nonnegative, we can apply Cairoli's strong (p,p) inequality [9, th. 2.3.1] for nonnegative orthosubmartingales. Thus for all $p > 1$ and $\boldsymbol{n} \in \mathbb{N}^d$

$$\mathbf{E} \max_{0 \leq \boldsymbol{j} \leq \boldsymbol{n}} \|S_{\boldsymbol{j}}\|^p \leq \left(\frac{p}{p-1} \right)^{dp} \mathbf{E}\,\|S_{\boldsymbol{n}}\|^p. \tag{47}$$

4 Proofs of Theorems 2 and 3

This section is mainly devoted to the proof of Theorem 2 which is detailed in subsections 4.1 to 4.3. In subsection 4.4, Theorem 3 is established by a simple adaptation of the previous proof.

4.1 Equivalence of conditions (11) and (12)

First we note that (11) is equivalent to the convergence

$$F(m) \xrightarrow[m \to \infty]{} 0, \tag{48}$$

where

$$F(m) := \sup_{\mathrm{m}(\boldsymbol{v}) \geq m} v_1^p (v_2 \cdots v_d)^2 P\Big(\|X_{\boldsymbol{1}}\| > v_1 v_2 \cdots v_d \Big).$$

Now introducing the function $g(t) := P(\|X_{\boldsymbol{1}}\| > t)$ and the sets

$$H_{t,m} := \{\boldsymbol{v} \in \mathbb{R}^d;\ \boldsymbol{v} \geq m,\ v_1 v_2 \cdots v_d = t\},$$

we have

$$F(m) = \sup_{t \geq m^d} \sup_{\boldsymbol{v} \in H_{t,m}} v_1^{p-2} t^2 g(t) = \sup_{t \geq m^d} t^2 g(t) \sup_{\boldsymbol{v} \in H_{t,m}} v_1^{p-2}.$$

When $t \geq m^d$, $H_{t,m}$ is non empty and on this set, $v_1 = t(v_2 \cdots v_d)^{-1}$ is maximal for $v_2 = \cdots = v_d = m$, so

$$t^2 g(t) \sup_{\boldsymbol{v} \in H_{t,m}} v_1^{p-2} = t^p g(t) m^{-(d-1)(p-2)}.$$

Finally

$$F(m) = m^{-(d-1)(p-2)} \sup_{t \geq m^d} t^p g(t).$$

Recalling that $d > 1$ and $p > 2$, this reduces the convergence (48) to the finiteness of $\sup_{t \geq m_0^d} t^p g(t)$ for some $m_0 > 0$. As $t^p g(t)$ is bounded on any interval $[0, a]$ for $a < \infty$, this finiteness is equivalent to (12).

4.2 Necessity of condition (11)

It is easily checked that condition (11) is equivalent to

$$n_1 \cdots n_d P\Big(\|X_{\boldsymbol{1}}\| > n_1^{1/p} n_2^{1/2} \cdots n_d^{1/2}\Big) \xrightarrow[\mathrm{m}(\boldsymbol{n}) \to \infty]{} 0. \tag{49}$$

Recall that $p = (1/2 - \alpha)^{-1}$. Since $\{X_{\boldsymbol{k}},\ \boldsymbol{k} \leq \boldsymbol{n}\}$ are independent and identically distributed, we have for each $t > 0$

$$\begin{aligned} P\Big(n_1^{-1/p} n_2^{-1/2} \dots n_d^{-1/2} \max_{\boldsymbol{1} \leq \boldsymbol{k} \leq \boldsymbol{n}} \|X_{\boldsymbol{k}}\| > t\Big) &= \\ = 1 - \Big(1 - P\Big(\|X_{\boldsymbol{1}}\| > t n_1^{1/p} n_2^{1/2} \dots n_d^{1/2}\Big)\Big)^{n_1 n_2 \dots n_d}&. \end{aligned} \tag{50}$$

Hence (49) is equivalent to

$$n_1^{-1/p} n_2^{-1/2} \dots n_d^{-1/2} \max_{\boldsymbol{1} \leq \boldsymbol{k} \leq \boldsymbol{n}} \|X_{\boldsymbol{k}}\| \xrightarrow[\mathrm{m}(\boldsymbol{n}) \to \infty]{\mathrm{Pr}} 0. \tag{51}$$

For every $\boldsymbol{1} \leq \boldsymbol{k} = (k_1, \dots, k_d) \leq \boldsymbol{n} = (n_1, \dots, n_d)$ we have

$$X_{\boldsymbol{k}} = \Delta_{k_1}^{(1)} \dots \Delta_{k_d}^{(d)} S_{\boldsymbol{k}}$$

Let $\delta > 0$ be an arbitrary positive number. Applying this representation with any $\boldsymbol{n}$ such that $|\boldsymbol{1}/\boldsymbol{n}| = \mathrm{m}(\boldsymbol{n})^{-1} < \delta$, we deduce for each $t > 0$

$$\begin{aligned} P(n_1^{-1/p} n_2^{-1/2} \dots n_d^{-1/2} \max_{\boldsymbol{1} \leq \boldsymbol{k} \leq \boldsymbol{n}} \|X_{\boldsymbol{k}}\| > t) &= P((n_1 \dots n_d)^{-1/2} \max_{\boldsymbol{1} \leq \boldsymbol{k} \leq \boldsymbol{n}} \frac{\|X_{\boldsymbol{k}}\|}{n_1^{-\alpha}} > t) \\ &\leq P\Big(2^{d-1} (n_1 \dots n_d)^{-1/2} \max_{|\frac{\boldsymbol{k}-\boldsymbol{l}}{\boldsymbol{n}}| = |\frac{\boldsymbol{1}}{\boldsymbol{n}}|} \frac{\|S_{\boldsymbol{k}} - S_{\boldsymbol{l}}\|}{|(\boldsymbol{k} - \boldsymbol{l})/\boldsymbol{n}|^\alpha} > t\Big) \\ &\leq P\Big(w_\alpha((n_1 \dots n_d)^{-1/2} \xi_{\boldsymbol{n}}, \delta) > 2^{1-d} t\Big). \end{aligned} \tag{52}$$

Since the function $w_\alpha(\,\cdot\,,\delta)$ is continuous on $H^o_\alpha(\mathbb{H})$, by continuous mapping theorem it follows that

$$\lim_{\boldsymbol{n}\to\infty} P(w_\alpha((n_1\dots n_d)^{1/2}\xi_{\boldsymbol{n}},\delta) > a) = P(w_\alpha(W_d,\delta) > a) \tag{53}$$

for each continuity point a of distribution function of the random variable $w_\alpha(W_d,\delta)$. Since paths of W_d lie in $\mathrm{H}^o_\alpha(\mathbb{H})$,

$$P(w_\alpha(W_d,\delta) > t) \to 0 \quad \text{as} \quad \delta \to 0. \tag{54}$$

Combining (52) – (54) we easily deduce (51).

4.3 Sufficiency of condition (11)

In view of the convergence of finite dimensional distributions established in subsection 3.4, we only have to check the tightness of the net $(\boldsymbol{\pi}(\boldsymbol{n})^{-1/2}\xi_{\boldsymbol{n}})_{\boldsymbol{n}\geq\mathbf{1}}$ using theorem 6. By lemma 9 and the separability of $\mathbb{H}$, the net $(\boldsymbol{\pi}(\boldsymbol{n})^{-1/2}\xi_{\boldsymbol{n}}(\boldsymbol{t}))_{\boldsymbol{n}\geq\mathbf{1}}$ is asymptoticaly tight for each $\boldsymbol{t}\in[0,1]^d$. Thus condition (i) of theorem 6 is satisfied.

To check condition (ii), consider with $\boldsymbol{s}=(s_2,\dots,s_d)$,

$$\Delta_{\boldsymbol{n}}(t,t';\boldsymbol{s}) := \|\xi_{\boldsymbol{n}}(t',s_2,\dots,s_d) - \xi_{\boldsymbol{n}}(t,s_2,\dots,s_d)\|.$$

Lemma 11 *For any $t',t\in[0,1], t'>t$, we have*

$$\sup_{s\in[0,1]} \Delta_{\boldsymbol{n}}(t,t';s) \leq 3^d\chi\Big\{t'-t\geq\frac{1}{n_1}\Big\}\psi_{\boldsymbol{n}}(t',t) + 3^d\min\Big(1,n_1(t'-t)\Big)Z_{\boldsymbol{n}},$$

where

$$\psi_{\boldsymbol{n}}(t',t) := \max_{\mathbf{1}_{2:d}\leq\boldsymbol{k}_{2:d}\leq\boldsymbol{n}_{2:d}} \Big\|\sum_{i=[n_1t]+1}^{[n_1t']} \Delta^{(1)}_i S_{(i,\boldsymbol{k}_{2:d})}\Big\|, \tag{55}$$

$$Z_{\boldsymbol{n}} := \max_{1\leq\boldsymbol{k}\leq\boldsymbol{n}} \|\Delta^{(1)}_{k_1} S_{\boldsymbol{k}}\|. \tag{56}$$

Proof. Put $\boldsymbol{u}:=(t,\boldsymbol{s})$, $\boldsymbol{u'}:=(t',\boldsymbol{s})$, so $u_1=t$, $u'_1=t'$ and $\boldsymbol{u}_{2:d}=\boldsymbol{u'}_{2:d}=\boldsymbol{s}$. Recalling (37), we have

$$\xi_{\boldsymbol{n}}(\boldsymbol{u'}) - \xi_{\boldsymbol{n}}(\boldsymbol{u}) = S_{[\boldsymbol{nu'}]} - S_{[\boldsymbol{nu}]} + \sum_{l=1}^d \Big(T_l(\boldsymbol{u'}) - T_l(\boldsymbol{u})\Big). \tag{57}$$

To estimate this $\xi_{\boldsymbol{n}}$'s increment we discuss according to the different possible configurations.

Case 1. $0 < t' - t < 1/n_1$.

Case 1.a. $[n_1 t'] = [n_1 t]$, whence $[\boldsymbol{nu'}] = [\boldsymbol{nu}]$. Consider first the increment $T_1(\boldsymbol{u'}) - T_1(\boldsymbol{u})$ and note that by (39) with $l = 1$,

$$T_1(\boldsymbol{u}) = \sum_{1\le k\le d} \{n_k u_k\} \Delta^{(k)}_{[n_k u_k]+1} S_{[\boldsymbol{nu}]}.$$

Because $\boldsymbol{u}_{2:d} = \boldsymbol{u'}_{2:d}$ and $[\boldsymbol{nu'}] = [\boldsymbol{nu}]$, all the terms indexed by $k \ge 2$ disappear in the difference $T_1(\boldsymbol{u'}) - T_1(\boldsymbol{u})$. Note also that $\{n_1 t'\} - \{n_1 t\} = n_1(t' - t)$. This leads to the factorization

$$T_1(\boldsymbol{u'}) - T_1(\boldsymbol{u}) = n_1(t' - t)\Delta^{(1)}_{[n_1 t]+1} S_{[\boldsymbol{nu}]}.$$

For $l \ge 2$, $T_l(\boldsymbol{u})$ is expressed by (39) as

$$T_l(\boldsymbol{u}) = \sum_{1\le i_1<\dots<i_l\le d} \{n_{i_1} u_{i_1}\}\dots\{n_{i_l} u_{i_l}\}\Delta^{(i_1)}_{[n_{i_1} u_{i_1}]+1}\dots\Delta^{(i_l)}_{[n_{i_l} u_{i_l}]+1} S_{[\boldsymbol{nu}]}.$$

As above, all the terms for which $i_1 \ge 2$ disappear in the difference $T_l(\boldsymbol{u'}) - T_l(\boldsymbol{u})$ and we obtain

$$T_l(\boldsymbol{u'}) - T_l(\boldsymbol{u}) = n_1(t' - t) \sum_{1<i_2<\dots<i_l\le d} \{n_{i_2} s_{i_2}\}\dots\{n_{i_l} s_{i_l}\} \\ \Delta^{(1)}_{[n_1 t]+1}\Delta^{(i_2)}_{[n_{i_2} s_{i_2}]+1}\dots\Delta^{(i_l)}_{[n_{i_l} s_{i_l}]+1} S_{[\boldsymbol{nu}]}.$$

Since $\{n_{i_2} s_{i_2}\}\dots\{n_{i_l} s_{i_l}\} < 1$ and

$$\left\|\Delta^{(1)}_{[n_1 t]+1}\Delta^{(i_2)}_{[n_{i_2} s_{i_2}]+1}\dots\Delta^{(i_l)}_{[n_{i_l} s_{i_l}]+1} S_{[\boldsymbol{nu}]}\right\| = \left\|\Delta^{(1)}_{[n_1 t]+1}\sum_{\boldsymbol{i}\in I}\varepsilon_{\boldsymbol{i}} S_{\boldsymbol{i}}\right\| \\ \le \sum_{\boldsymbol{i}\in I}\left\|\Delta^{(1)}_{[n_1 t]+1} S_{\boldsymbol{i}}\right\|,$$

where $\varepsilon_{\boldsymbol{i}} = \pm 1$ and I is some appropriate subset of $[0, \boldsymbol{n}] \cap \mathbb{N}^d$ with 2^{l-1} elements. Hence with $Z_{\boldsymbol{n}}$ defined by (56), we obtain for $l \ge 2$

$$\|T_l(\boldsymbol{u'}) - T_l(\boldsymbol{u})\| \le n_1(t' - t)\binom{d-1}{l-1} 2^{l-1} Z_{\boldsymbol{n}}.$$

Clearly this estimate holds true also for $l = 1$, so going back to (57) and recalling that in the case under consideration $[\boldsymbol{nu'}] = [\boldsymbol{nu}]$, we obtain

$$\|\xi_{\boldsymbol{n}}(\boldsymbol{u'}) - \xi_{\boldsymbol{n}}(\boldsymbol{u})\| \le \sum_{l=1}^{d} n_1(t' - t)\binom{d-1}{l-1} 2^{l-1} Z_{\boldsymbol{n}} = 3^{d-1} n_1(t' - t) Z_{\boldsymbol{n}}. \quad (58)$$

Case 1.b. $n_1 t < [n_1 t'] \le n_1 t'$. Using chaining to exploit the result of case 1.a, we obtain

$$\begin{aligned}\|\xi_{\boldsymbol{n}}(\boldsymbol{u'}) - \xi_{\boldsymbol{n}}(\boldsymbol{u})\| &\le \left\|\xi_{\boldsymbol{n}}(\boldsymbol{u'}) - \xi_{\boldsymbol{n}}\Big(\frac{[n_1 t']}{n_1}, \boldsymbol{s}\Big)\right\| + \left\|\xi_{\boldsymbol{n}}\Big(\frac{[n_1 t']}{n_1}, \boldsymbol{s}\Big) - \xi_{\boldsymbol{n}}(\boldsymbol{u})\right\| \\ &\le 3^{d-1}(n_1 t' - [n_1 t'])Z_{\boldsymbol{n}} + 3^{d-1}([n_1 t'] - n_1 t)Z_{\boldsymbol{n}} \\ &= 3^{d-1} n_1 (t' - t) Z_{\boldsymbol{n}}. \end{aligned} \tag{59}$$

Case 2. $t' - t \ge 1/n_1$. Then $[n_1 t] \le n_1 t < [n_1 t] + 1 \le [n_1 t'] \le n_1 t'$ and putting

$$t_1 := \frac{[n_1 t]}{n_1},\ t_1' := \frac{[n_1 t']}{n_1}, \quad \boldsymbol{v} := (t_1, \boldsymbol{s}),\ \boldsymbol{v'} := (t_1', \boldsymbol{s}),$$

we get the upper bound

$$\|\xi_{\boldsymbol{n}}(\boldsymbol{u'}) - \xi_{\boldsymbol{n}}(\boldsymbol{u})\| \le \|\xi_{\boldsymbol{n}}(\boldsymbol{u'}) - \xi_{\boldsymbol{n}}(\boldsymbol{v'})\| + \|\xi_{\boldsymbol{n}}(\boldsymbol{v'}) - \xi_{\boldsymbol{n}}(\boldsymbol{v})\| + \|\xi_{\boldsymbol{n}}(\boldsymbol{v}) - \xi_{\boldsymbol{n}}(\boldsymbol{u})\|,$$

where the first and third terms fall within the case 1 since $t' - t_1' < 1/n_1$ and $t - t_1 < 1/n_1$. As $n_1 v_1 = n_1 t_1 = [n_1 t]$, we have

$$[n\boldsymbol{v}] = ([n_1 t_1], [\boldsymbol{n}_{2:d}\boldsymbol{s}]) = [n\boldsymbol{u}] \quad \text{and} \quad \{n_1 v_1\} = \{[n_1 t]\} = 0,$$

so the representation (40) for $\xi_{\boldsymbol{n}}(\boldsymbol{v})$ may be recast as

$$\xi_{\boldsymbol{n}}(\boldsymbol{v}) = S_{[\boldsymbol{n}\boldsymbol{u}]} + \sum_{l=1}^{d-1} \sum_{2 \le i_1 < i_2 < \cdots < i_l \le d} \Big(\prod_{k=1}^{l} \{n_{i_k} v_{i_k}\}\Big)\Big(\prod_{k=1}^{l} \Delta^{(i_k)}_{[n_{i_k} v_{i_k}]+1}\Big) S_{[\boldsymbol{n}\boldsymbol{u}]}.$$

Clearly the same representation holds for $\xi_{\boldsymbol{n}}(\boldsymbol{v'})$, by just replacing $\boldsymbol{u}$ by $\boldsymbol{u'}$. Now since Δ's are interchangable and

$$S_{[\boldsymbol{n}\boldsymbol{u'}]} - S_{[\boldsymbol{n}\boldsymbol{u}]} = \sum_{i=[nt]+1}^{[nt']} \Delta^{(1)}_i S_{(i, [\boldsymbol{n}_{2:d}\boldsymbol{s}])},$$

we get

$$\|\xi_{\boldsymbol{n}}(\boldsymbol{v'}) - \xi_{\boldsymbol{n}}(\boldsymbol{v})\| \le \psi_{\boldsymbol{n}}(t', t) \sum_{l=0}^{d-1} \binom{d-1}{l} 2^l = 3^{d-1}\psi_{\boldsymbol{n}}(t', t),$$

with $\psi_{\boldsymbol{n}}(t', t)$ defined by (55). Using case 1 to bound $\|\xi_{\boldsymbol{n}}(\boldsymbol{u'}) - \xi_{\boldsymbol{n}}(\boldsymbol{v'})\|$ and $\|\xi_{\boldsymbol{n}}(\boldsymbol{v}) - \xi_{\boldsymbol{n}}(\boldsymbol{u})\|$, we obtain

$$\begin{aligned}\|\xi_{\boldsymbol{n}}(t', s) - \xi_{\boldsymbol{n}}(t, s)\| &\le 3^{d-1}\{n_1 t'\} Z_{\boldsymbol{n}} + 3^{d-1}\psi_{\boldsymbol{n}}(t', t) + 3^{d-1}\{n_1 t\} Z_{\boldsymbol{n}} \\ &\le 3^{d-1}\psi_{\boldsymbol{n}}(t', t) + 2 \cdot 3^{d-1} Z_{\boldsymbol{n}}. \end{aligned} \tag{60}$$

Combining (58), (59) and (60) we complete the proof of lemma 11. □

Now we continue the proof of the sufficiency of condition (11) by introducing truncated variables and finding estimates for their moments. Let $\delta \in (0,1)$ be an arbitrary number. Define

$$\widetilde{X}_{\boldsymbol{j}} := X_{\boldsymbol{j}} \chi\{\|X_{\boldsymbol{j}}\| \le \delta n_1^{1/p} (n_2 \dots n_d)^{1/2}\}, \tag{61}$$

$$X'_{\boldsymbol{j}} := \widetilde{X}_{\boldsymbol{j}} - \mathbf{E}\, \widetilde{X}_{\boldsymbol{j}}, \quad \mathbf{1} \le \boldsymbol{j} \le \boldsymbol{n}. \tag{62}$$

Denote for $m \ge 0$

$$c(m) := \sup_{u \ge m} \sup_{\boldsymbol{v}_{2:d} \ge m} u v_2 \dots v_d P(\|X_{\mathbf{1}}\| > u^{1/p} (v_2 \dots v_d)^{1/2})$$
$$c_p := \sup_{t \ge 0} t^{d/(d/2-\alpha)} P(\|X_{\mathbf{1}}\| > t).$$

Evidently condition (11) yields $c(m) \to 0$ as $m \to \infty$ and $c_p < \infty$. Set

$$c_{p,m} := \max\{c_p;\ c(m)\}.$$

Lemma 12 *With $m = \mathrm{m}(\boldsymbol{n})$ and any $q > p$*

$$\|\mathbf{E}\, \widetilde{X}_{\mathbf{1}}\| \le 2\delta^{1-p} c_{p,m} n_1^{1/p-1} (n_2 \dots n_d)^{-1/2}; \tag{63}$$

$$\mathbf{E}\, \|\widetilde{X}_{\mathbf{1}}\|^q \le \frac{2 c_{p,m}}{q-p} \delta^{q-p} n_1^{q/p-1} (n_2 \dots n_d)^{q/2-1}; \tag{64}$$

$$\mathbf{E}\, \|X'_{\mathbf{1}}\|^2 \le \mathbf{E}\, \|X_{\mathbf{1}}\|^2; \tag{65}$$

$$\mathbf{E}\, \|X'_{\mathbf{1}}\|^q \le \frac{2^{q+1} c_{p,m}}{q-p} \delta^{q-p} n_1^{q/p-1} (n_2 \dots n_d)^{q/2-1}. \tag{66}$$

Proof. To check (63), we observe first that since $\mathbf{E}\, X_{\mathbf{1}} = 0$,

$$\begin{aligned}
\|\mathbf{E}\, \widetilde{X}_{\mathbf{1}}\| &= \big\|\mathbf{E}\, X_{\mathbf{1}} - \mathbf{E}\, X_{\mathbf{1}} \chi\{\|X_{\mathbf{1}}\| > \delta n_1^{1/p} (n_2 \dots n_d)^{1/2}\}\big\| \\
&\le \int_{\delta n_1^{1/p} (n_2 \dots n_d)^{1/2}}^{\infty} P(\|X_{\mathbf{1}}\| > t)\, \mathrm{d}t \\
&\quad + \delta n_1^{1/p} (n_2 \dots n_d)^{1/2} P\Big(\|X_{\mathbf{1}}\| > \delta n_1^{1/p} (n_2 \dots n_d\Big)^{1/2}).
\end{aligned}$$

Next we have

$$\begin{aligned}
&\int_{\delta n_1^{1/p} (n_2 \dots n_d)^{1/2}}^{\infty} P(\|X_{\mathbf{1}}\| > t)\, \mathrm{d}t \\
&= \delta n_1^{1/p-1} (n_3 \dots n_d)^{-1/2} \int_{n_2^{1/2}}^{\infty} v^2 n_1 n_3 \dots n_d P(\|X_{\mathbf{1}}\| > \delta v n_1^{1/p} (n_3 \dots n_d)^{1/2}) \frac{\mathrm{d}v}{v^2} \\
&\le \delta n_1^{1/p-1} (n_3 \dots n_d)^{-1/2} b(m, \delta) \int_{n_2^{1/2}}^{\infty} v^{-2}\, \mathrm{d}v \\
&\le \delta b(m, \delta) n_1^{1/p-1} (n_2 \dots n_d)^{-1/2},
\end{aligned}$$

where

$$b(m,\delta) := \sup_{u\geq m} \sup_{\boldsymbol{v}_{2:d}\geq m} uv_2\dots v_d P(\|X_{\mathbf{1}}\| > \delta u^{1/p}(v_2\dots v_d)^{1/2}).$$

We complete the proof of (63) noting that

$$\begin{aligned}
b(m;\delta) &= \delta^{-p} \sup_{u\geq \delta^p m} \sup_{\boldsymbol{v}_{2:d}\geq m} uv_2\dots v_d P(\|X_{\mathbf{1}}\| > u^{1/p}(v_2\dots v_d)^{1/2}) \\
&= \delta^{-p} \max\Big\{ \sup_{m\geq u\geq \delta^p m} \sup_{\boldsymbol{v}_{2:d}\geq m} uv_2\dots v_d P(\|X_{\mathbf{1}}\| > u^{1/p}(v_2\dots v_d)^{1/2}); \\
&\qquad \sup_{u\geq m} \sup_{\boldsymbol{v}_{2:d}\geq m} uv_2\dots v_d P(\|X_{\mathbf{1}}\| > u^{1/p}(v_2\dots v_d)^{1/2})\Big\} \\
&\leq \delta^{-p} c_{p,m}, \qquad (67)
\end{aligned}$$

since

$$\begin{aligned}
&\sup_{u\leq m} \sup_{\boldsymbol{v}_{2:d}\geq m} uv_2\dots v_d P(\|X_{\mathbf{1}}\| > u^{1/p}(v_2\dots v_d)^{1/2}) \\
&\quad \leq \sup_{u\leq m} \sup_{\boldsymbol{v}_{2:d}\geq m} uv_2\dots v_d c_p (u^{1/p}(v_2\dots v_d)^{1/2})^{-d/(d/2-\alpha)} \\
&\quad = c_p \sup_{u\leq m} u^{2\alpha(d-1)/(d-2\alpha)} \sup_{\boldsymbol{v}_{2:d}\geq m} (v_2\dots v_d)^{-2\alpha/(d-2\alpha)} = c_p.
\end{aligned}$$

Next we have

$$\begin{aligned}
\mathbf{E}\,\|\widetilde{X_{\mathbf{1}}}\|^q &\leq \int_0^{\delta n_1^{1/p}(n_2\dots n_d)^{1/2}} t^{q-1} P(\|X_{\mathbf{1}}\| > t)\,\mathrm{d}t \\
&= \int_0^{\delta(n_2\dots n_d)^{1/2}} t^{q-1} P(\|X_{\mathbf{1}}\| > t)\,\mathrm{d}t \\
&\quad + \int_{\delta(n_2\dots n_d)^{1/2}}^{\delta n_1^{1/p}(n_2\dots n_d)^{1/2}} t^{q-1} P(\|X_{\mathbf{1}}\| > t)\,\mathrm{d}t.
\end{aligned}$$

By Chebyshev inequality $P(\|X_{\mathbf{1}}\| > t) \leq t^{-2}$, hence the first integral does not exceed $(q-2)^{-1}\delta^{q-2}(n_2\dots n_d)^{q/2-1}$. As $\int_1^{n_1^{1/p}} \leq n_1^{q/p-1}$, the second integral does not exceed

$$\begin{aligned}
&\delta^q (n_2\dots n_d)^{q/2-1} \int_1^{n_1^{1/p}} n_2\dots n_d u^p P(\|X_{\mathbf{1}}\| > \delta u(n_2\dots n_d)^{1/2}) u^{q-p-1}\,\mathrm{d}u \\
&\leq \delta^q (n_2\dots n_d)^{q/2-1} \sup_{\boldsymbol{v}_{2:d}\geq m} \sup_{1\leq u\leq n_1} uv_2\dots v_d P(\|X_{\mathbf{1}}\| > \delta u^{1/p}(v_2\dots v_d)^{1/2}) n_1^{q/p-1} \\
&\leq \frac{1}{q-p} \max\{b'(m,\delta);\ b(m;\delta)\} \delta^q n_1^{q/p-1} (n_2\dots n_d)^{q/2-1},
\end{aligned}$$

where

$$b'(m,\delta) := \sup_{\boldsymbol{v}_{2:d}\geq m} \sup_{1\leq u\leq m} uv_2\dots v_d P(\|X_{\mathbf{1}}\| > \delta u^{1/p}(v_2\dots v_d)^{1/2})$$
$$\leq \delta^{-2d/(d/2-\alpha)}c_p \leq \delta^{-p}c_p,$$

recalling that $0<\delta<1$ and $p=(1/2-\alpha)^{-1}$. Accounting (67) inequality (64) now follows.

To check (65), let us denote by $(e_k,\ k\in\mathbb{N})$ some orthonormal basis of the separable Hilbert space $\mathbb{H}$. Then we have

$$\|X'_{\mathbf{1}}\|^2 = \sum_{k=0}^{\infty}\left|\langle \widetilde{X}_{\mathbf{1}} - \mathbf{E}\,\widetilde{X}_{\mathbf{1}}, e_k\rangle\right|^2 = \sum_{k=0}^{\infty}\left|\langle \widetilde{X}_{\mathbf{1}}, e_k\rangle - \mathbf{E}\,\langle \widetilde{X}_{\mathbf{1}}, e_k\rangle\right|^2,$$

whence

$$\mathbf{E}\,\|X'_{\mathbf{1}}\|^2 = \sum_{k=0}^{\infty}\mathrm{Var}(\langle \widetilde{X}_{\mathbf{1}}, e_k\rangle) \leq \sum_{k=0}^{\infty}\mathbf{E}\,\left|\langle \widetilde{X}_{\mathbf{1}}, e_k\rangle\right|^2$$
$$= \mathbf{E}\,\sum_{k=0}^{\infty}\left|\langle \widetilde{X}_{\mathbf{1}}, e_k\rangle\right|^2 = \mathbf{E}\,\|\widetilde{X}_{\mathbf{1}}\|^2 \leq \mathbf{E}\,\|X_{\mathbf{1}}\|^2,$$

which gives (65).

Finally we note that (66) is obviously obtained from (64) since the convexity inequality $\|X'_{\mathbf{1}}\|^q \leq 2^{q-1}\|\widetilde{X}_{\mathbf{1}}\|^q + 2^{q-1}\|\mathbf{E}\,\widetilde{X}_{\mathbf{1}}\|^q$ together with $\mathbf{E}\,\|\widetilde{X}_1\| \leq (\mathbf{E}\,\|\widetilde{X}_1\|^q)^{1/q}$ gives $\mathbf{E}\,\|X'_1\|^q \leq 2^q\mathbf{E}\,\|\widetilde{X}_1\|^q$. □

Lemma 13 *If condition* (11) *is satisfied, then*

$$n_1^{-1/p}(n_2\dots n_d)^{-1/2}Z_{\boldsymbol{n}} \xrightarrow[\mathrm{m}(\boldsymbol{n})\to\infty]{\mathrm{Pr}} 0. \tag{68}$$

Proof. First note that really

$$Z_{\boldsymbol{n}} = \max_{\mathbf{1}\leq\boldsymbol{k}\leq\boldsymbol{n}}\left\|\sum_{i_2=1}^{k_2}\cdots\sum_{i_d=1}^{k_d}X_{(k_1,i_2,\dots,i_d)}\right\|.$$

Fix $\varepsilon>0$ and associate to any $\delta\in(0,1)$ the truncated random variables $\widetilde{X}_k$ and X'_k defined by (61), (62). Substituting X_k by $\widetilde{X}_k$, respectively X'_k, in the definition of $Z_{\boldsymbol{n}}$ we obtain $\widetilde{Z}_{\boldsymbol{n}}$, respectively $Z'_{\boldsymbol{n}}$. Introducing the complementary events

$$E_{\boldsymbol{n}} := \left\{\forall\boldsymbol{k}\leq\boldsymbol{n},\ \|X_{\boldsymbol{k}}\|\leq\delta n_1^{1/p}(n_2\dots n_d)^{1/2}\right\},\quad E^c_{\boldsymbol{n}} := \Omega\setminus E_{\boldsymbol{n}},$$

we have

$$P(Z_{\boldsymbol{n}} > \varepsilon n_1^{1/p}(n_2\dots n_d)^{1/2}) \le P(\{Z_{\boldsymbol{n}} > \varepsilon n_1^{1/p}(n_2\dots n_d)^{1/2}\} \cap E_{\boldsymbol{n}}) + P(E_{\boldsymbol{n}}^c).$$

Clearly $Z_{\boldsymbol{n}} = \widetilde{Z}_{\boldsymbol{n}}$ on the event $E_{\boldsymbol{n}}$. By identical distribution of the $X_{\boldsymbol{k}}$'s,

$$P(E_{\boldsymbol{n}}^c) \le n_1 \dots n_d P(\|X_1\| > \delta n_1^{1/p}(n_2\dots n_d)^{1/2})$$

and this upper bound goes to zero when $\mathrm{m}(\boldsymbol{n})$ goes to infinity by condition (11). This leads to

$$\limsup_{\mathrm{m}(\boldsymbol{n})\to\infty} P(Z_{\boldsymbol{n}} > \varepsilon n_1^{1/p}(n_2\dots n_d)^{1/2}) \le \limsup_{\mathrm{m}(\boldsymbol{n})\to\infty} P(\widetilde{Z}_{\boldsymbol{n}} > \varepsilon n_1^{1/p}(n_2\dots n_d)^{1/2}). \tag{69}$$

Because $n_1^{-1/p}(n_2\dots n_d)^{1/2}\|\mathbf{E}\,\widetilde{X}_{\mathbf{1}}\| \to 0$ as $\mathrm{m}(\boldsymbol{n}) \to \infty$ by lemma 12, the right-hand side of (69) does not exceed

$$\limsup_{\mathrm{m}(\boldsymbol{n})\to\infty} P(n_1^{-1/p}(n_2\dots n_d)^{-1/2} Z'_{\boldsymbol{n}} > \varepsilon).$$

Using the extension of Doob inequality (47), we obtain with $q > p$

$$\begin{aligned}
&P(n_1^{-1/p}(n_2\dots n_d)^{-1/2} Z'_{\boldsymbol{n}} > \varepsilon)\\
&\le n_1\, P\left(\max_{\mathbf{1}_{2:d}\le \boldsymbol{k}_{2:d}\le \boldsymbol{n}_{2:d}} \left\| \sum_{\boldsymbol{i}_{2:d}=\mathbf{1}_{2:d}}^{\boldsymbol{k}_{2:d}} X'_{(1,i_2,\dots,i_d)} \right\| > \varepsilon n_1^{1/p}(n_2\dots n_d)^{1/2}\right)\\
&\le \varepsilon^{-q} n_1^{1-q/p}(n_2\dots n_d)^{-q/2}\mathbf{E}\left\| \sum_{\boldsymbol{i}_{2:d}=\mathbf{1}_{2:d}}^{\boldsymbol{n}_{2:d}} X'_{(1,i_2,\dots,i_d)} \right\|^q.
\end{aligned}$$

Applying Rosenthal inequality (46) together with the estimates (65), (66), we obtain

$$\begin{aligned}
&P(n_1^{-1/p}(n_2\dots n_d)^{-1/2} Z'_{\boldsymbol{n}} > \varepsilon)\\
&\le \varepsilon^{-q} n_1^{1-q/p}(n_2\dots n_d)^{-q/2} C_q\Big((n_2\dots n_d)^{q/2}(\mathbf{E}\,\|X'_{\mathbf{1}}\|^2)^{q/2} + n_2\dots n_d \mathbf{E}\,\|X'_{\mathbf{1}}\|^q\Big)\\
&\le C_q \varepsilon^{-q}\left(n_1^{1-q/p}(\mathbf{E}\,\|X_{\mathbf{1}}\|^2)^{q/2} + \frac{2^{q+1}c_{p,m}}{q-p}\delta^{q-p}\right).
\end{aligned}$$

Combined with (69) this gives

$$\limsup_{\mathrm{m}(\boldsymbol{n})\to\infty} P(n_1^{-1/p}(n_2\dots n_d)^{-1/2} Z_{\boldsymbol{n}} > \varepsilon) \le c\delta^{q-p},$$

where the constant c depends on ε, p and q. Since $q > p$ and δ may be choosen arbitrarily small in $(0,1)$, the convergence (68) follows. □

Next we continue proving (iii) of Theorem 6. Due to the definition of $\lambda_{j,\boldsymbol{v}}(\xi_{\boldsymbol{n}})$

it is easy to check that (iii) holds provided one proves for every $\varepsilon > 0$

$$\lim_{J\to\infty} \limsup_{\boldsymbol{n}\to\infty} \Pi(J, \boldsymbol{n}; \varepsilon) = 0, \tag{70}$$

where

$$\Pi(J, \boldsymbol{n}; \varepsilon) := P\Bigg(\sup_{j\ge J} 2^{\alpha j} (n_1 \dots n_d)^{-1/2} \max_{\substack{0\le k<2^j \\ \boldsymbol{0}\le \boldsymbol{\ell} \le \boldsymbol{2}^j}} \Delta_{\boldsymbol{n}}(t_{k+1}, t_k; \boldsymbol{s}_{\boldsymbol{\ell}}) > \varepsilon \Bigg) = 0, \tag{71}$$

with $t_k = k2^{-j}$, $\boldsymbol{\ell} = (l_2, \dots, l_d)$, $\boldsymbol{2}^j = (2^j, \dots, 2^j)$ (vector of dimension $d-1$) and $\boldsymbol{s}_\ell = \boldsymbol{\ell}\boldsymbol{2}^{-j}$.

By lemma 11 the probability $\Pi(J, \boldsymbol{n}; \varepsilon)$ does not exceed

$$P\Bigg(\sup_{j\ge J} 2^{\alpha j} (n_1 \dots n_d)^{-1/2} \max_{0\le k<2^j} \Big[3^d \chi\{t_{k+1} - t_k \ge 1/n_1\} \psi_{\boldsymbol{n}}(t_{k+1}, t_k) \\ + 3^d \min\{1, n_1(t_{k+1} - t_k)\} Z_{\boldsymbol{n}} \Big] > \varepsilon \Bigg).$$

In what follows, we denote by "log" the logarithm *with basis* 2 ($\log 2 = 1$). For notational simplification, let us agree to denote by ε' the successive splittings of ε, i.e. $\varepsilon' = c\varepsilon$ where the constant $c \in (0, 1)$ may decrease from one formula to following one. For $j > \log n_1$, we have $2^j > n_1$, whence $(t_{k+1} - t_k) = 2^{-j} < 1/n_1$ and noting that $1 - \alpha = 1/2 + 1/p$,

$$2^{\alpha j} n_1^{-1/2} n_1(t_{k+1} - t_k) \le n_1^{1/2} 2^{-j(1-\alpha)} = n_1^{1/2} 2^{-j(1/2+1/p)} \le n_1^{-1/p}.$$

This gives

$$\sup_{j>\log n_1} 2^{\alpha j} (n_1 \dots n_d)^{-1/2} \max_{0\le k<2^j} n_1(t_{k+1} - t_k) Z_{\boldsymbol{n}} \le n_1^{-1/p} (n_2 \dots n_d)^{-1/2} Z_{\boldsymbol{n}}.$$

On the other hand, for $J \le j \le \log n_1$, we have $2^{\alpha j} n_1^{-1/2} \le n_1^{\alpha - 1/2} = n_1^{-1/p}$, whence

$$\max_{J\le j\le \log n_1} 2^{\alpha j} (n_1 \dots n_d)^{-1/2} Z_{\boldsymbol{n}} \le n_1^{-1/p} (n_2 \dots n_d)^{-1/2} Z_{\boldsymbol{n}}.$$

Now, applying lemma 13 twice, we reduce (70) to

$$\lim_{J\to\infty} \limsup_{\mathrm{m}(\boldsymbol{n})\to\infty} P(J, \boldsymbol{n}; \varepsilon') = 0, \tag{72}$$

where

$$P(J, \boldsymbol{n}; \varepsilon') = P\Bigg(\max_{J\le j\le \log n_1} 2^{\alpha j} (n_1 \dots n_d)^{-1/2} \max_{0\le k<2^j} \psi_{\boldsymbol{n}}(t_{k+1}, t_k) > \varepsilon' \Bigg).$$

Notations $\widetilde{\psi}_{\boldsymbol{n}}(t_{k+1},t_k)$ and $\psi'_{\boldsymbol{n}}(t_{k+1},t_k)$ mean that $X_{\boldsymbol{j}}$ are substituted by $\widetilde{X}_{\boldsymbol{j}}$ and $X'_{\boldsymbol{j}}$ respectively in the definition of $\psi_{\boldsymbol{n}}(t_{k+1},t_k)$. Accordingly we introduce the notations $\widetilde{P}(J,\boldsymbol{n};\varepsilon')$ and $P'(J,\boldsymbol{n};\varepsilon')$. Splitting Ω in $E_{\boldsymbol{n}}$ and $E^c_{\boldsymbol{n}}$ like in the proof of lemma 13, we obtain

$$P(J,\boldsymbol{n};\varepsilon') \le \widetilde{P}(J,\boldsymbol{n};\varepsilon) + n_1\dots n_d P(\|X_{\mathbf{1}}\| \ge \delta n_1^{1/p}(n_2\dots n_d)^{1/2}).$$

Then (72) is reduced by condition (11) to

$$\lim_{J\to\infty}\limsup_{\mathrm{m}(\boldsymbol{n})\to\infty} \widetilde{P}(J,\boldsymbol{n};\varepsilon') = 0. \tag{73}$$

The number of variables $\widetilde{X}_{\boldsymbol{k}}$ to be centered in the sum $\widetilde{\psi}_{\boldsymbol{n}}(t_{k+1},t_k)$ is at most $n_1(t_{k+1}-t_k)n_2\dots n_d \le n_1 2^{-J} n_2\dots n_d$ and (63) yields

$$\begin{aligned}\max_{J\le j\le \log n_1} 2^{\alpha j}(n_1\dots n_d)^{-1/2}\|\mathbf{E}\,\widetilde{X}_{\mathbf{1}}\| &\le n_1^{\alpha-1/2}(2\delta^{1-p}c_{p,m})n_1^{1/p-1}(n_2\dots n_d)^{-1}\\ &= 2\delta^{1-p}c_{p,m}(n_1\dots n_d)^{-1}.\end{aligned}$$

Therefore

$$\limsup_{\mathrm{m}(\boldsymbol{n})\to\infty}\max_{J\le j\le \log n_1} 2^{\alpha j}(n_1\dots n_d)^{-1/2} n_1 2^{-J} n_2\dots n_d \|\mathbf{E}\,\widetilde{X}_{\mathbf{1}}\| \le \delta^{1-p}c_p 2^{-J+1}.$$

This upper bound going to zero when J goes to infinity, (73) is reduced to

$$\lim_{J\to\infty}\limsup_{\mathrm{m}(\boldsymbol{n})\to\infty} P'(J,\boldsymbol{n};\varepsilon') = 0. \tag{74}$$

We have with $q>p$

$$\begin{aligned}P'(J,\boldsymbol{n};\varepsilon') &\le \sum_{j=J}^{\log n_1} P\Big(2^{\alpha j}(n_1\dots n_d)^{-1/2}\max_{0\le k<2^j}\psi'_{\boldsymbol{n}}(t_{k+1},t_k) > \varepsilon'\Big)\\ &\le \sum_{j=J}^{\log n_1} 2^{q\alpha j}(n_1\dots n_d)^{-q/2}\varepsilon'^{-q}2^j\,\mathbf{E}\,\psi'_{\boldsymbol{n}}(t_{k+1},t_k)^q. \end{aligned}\tag{75}$$

Denote $u_k = [n_1 t_k]$ and observe that $u_{k+1}-u_k \le n_1 2^{-j}$. By (47),

$$\mathbf{E}\,\psi'_{\boldsymbol{n}}(t_{k+1},t_k)^q \le \mathbf{E}\left\|\sum_{i_1=1+u_k}^{u_{k+1}}\sum_{\boldsymbol{i}_{2:d}=\mathbf{1}_{2:d}}^{\boldsymbol{n}_{2:d}} X'_{\boldsymbol{i}}\right\|^q.$$

Estimating this last q-moment by Rosenthal inequality (46) with a number of summands $N \leq (n_1 2^{-j}) n_2 \dots n_d$, we obtain

$$\begin{aligned}
\mathbf{E}\, \psi'_{\boldsymbol{n}}(t_{k+1}, t_k)^q &\leq C_q \Big((n_1 2^{-j})^{q/2} (n_2 \dots n_d)^{q/2} \mathbf{E}\, \|X'_{\mathbf{1}}\|^2 + n_1 2^{-j} n_2 \dots n_d \mathbf{E}\, \|X'_{\mathbf{1}}\|^q \Big) \\
&\leq C_q \mathbf{E}\, \|X_{\mathbf{1}}\|^2 2^{-jq/2} (n_1 \dots n_d)^{q/2} \\
&\quad + \frac{2^{q+1} C_q c_{p,m}}{q-p} \delta^{q-p} 2^{-j} n_1^{q/p} (n_2 \dots n_d)^{q/2}.
\end{aligned}$$

Reporting this estimate into (75) we obtain

$$P'(J, \boldsymbol{n}; \varepsilon') \leq \Sigma_1(J, \boldsymbol{n}; \varepsilon') + \Sigma_2(J, \boldsymbol{n}; \varepsilon')$$

with Σ_1 and Σ_2 explicited and bounded as follows. First

$$\begin{aligned}
\Sigma_1(J, \boldsymbol{n}; \varepsilon') &:= \frac{C_q}{\varepsilon'^q} \mathbf{E}\, \|X_{\mathbf{1}}\|^2 \sum_{J \leq j \leq \log n_1} 2^{(1+q(\alpha-1/2))j} \\
&\leq \frac{C_q}{\varepsilon'^q} \mathbf{E}\, \|X_{\mathbf{1}}\|^2 \sum_{j=J}^{\infty} 2^{-(q/p-1)j} \\
&= \frac{C_q}{\varepsilon'^q} \mathbf{E}\, \|X_{\mathbf{1}}\|^2 \frac{2^{-(q/p-1)J}}{1 - 2^{-(q/p-1)}}.
\end{aligned}$$

Hence

$$\lim_{J \to \infty} \limsup_{\mathrm{m}(\boldsymbol{n}) \to \infty} \Sigma_1(J, \boldsymbol{n}; \varepsilon') = 0.$$

Next

$$\begin{aligned}
\Sigma_2(J, \boldsymbol{n}; \varepsilon') &:= \frac{2^{q+1} C_q c_{p,m}}{(q-p)\varepsilon'^q} \delta^{q-p} n_1^{-q\alpha} \sum_{J \leq j \leq \log n_1} 2^{jq\alpha} \\
&\leq \frac{2^{q+1} C_q c_{p,m}}{(q-p)\varepsilon'^q} \delta^{q-p} n_1^{-q\alpha} \frac{n_1^{q\alpha}}{2^{q\alpha} - 1}
\end{aligned}$$

Noting that $m = \mathrm{m}(\boldsymbol{n})$ and $\limsup_{m \to \infty} c_{p,m} = c_p$, we obtain

$$\limsup_{\mathrm{m}(\boldsymbol{n}) \to \infty} \Sigma_2(J, \boldsymbol{n}; \varepsilon') \leq \frac{2^{q+1} C_q c_p}{(q-p)(2^{q\alpha} - 1)\varepsilon'^q} \delta^{q-p}.$$

Recalling (71) and summing up all the successive reductions leads to

$$\limsup_{J \to \infty} \limsup_{\mathrm{m}(\boldsymbol{n}) \to \infty} \Pi(J, \boldsymbol{n}; \varepsilon) \leq \frac{2^{q+1} C_q c_p}{(q-p)(2^{q\alpha} - 1)\varepsilon'^q} \delta^{q-p}.$$

Since $\Pi(J, \boldsymbol{n}; \varepsilon)$ does not depend on δ which may be choosen arbirarily small, the left-hand side is null and this gives (70). Consequently the condition (ii) follow and the proof of Theorem 2 is completed.

4.4 Proof of Theorem 3

The necessary and sufficient condition which is technically relevant in the proof of Theorem 2 is (49):

$$n_1 \cdots n_d P\Big(\|X_{\mathbf{1}}\| > n_1^{1/p} n_2^{1/2} \cdots n_d^{1/2}\Big) \xrightarrow[\mathrm{m}(\boldsymbol{n})\to\infty]{} 0.$$

Now looking back at the proof of Theorem 2, having in mind the extra assumption that $n_1 = n_2 = \cdots = n_d = n$, it should be clear that the weak $\mathrm{H}_\alpha^o(\mathbb{H})$ convergence of $n^{-d/2}\xi_{(n,\dots,n)}$ to W is equivalent to the condition obtained by reporting this equality of the n_i's in (49), namely to

$$n^d P\Big(\|X_{\mathbf{1}}\| > n^{1/p+(d-1)/2}\Big) \xrightarrow[n\to\infty]{} 0. \qquad (76)$$

It is easily checked that in (76) the integer n can be replaced by a positive real number s and then puting $t = s^{1/p+(d-1)/2}$, we obtain the equivalence of (76) with

$$\lim_{t\to\infty} t^{\frac{2pd}{2+p(d-1)}} P\Big(\|X_{\mathbf{1}}\| > t\Big) = 0. \qquad (77)$$

Finally recalling that $p = p(\alpha) = 2/(1-2\alpha)$, we get

$$\frac{2pd}{2+p(d-1)} = \frac{2d}{d-2\alpha},$$

which reported in (77) gives (14) and completes the proof.

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