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ESTIMATION OF CHANGE POINTS OF INFINITE DIMENSIONAL PARAMETERS IN SHORT EPIDEMICS

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Abstract. We consider the epidemic change of the distribution of a real-valued sample and of the mean of Banach-space-valued random elements. For these two models, we propose consistent procedures for estimating the location and length of epidemic change.

Keywords: change-point location, empirical process, epidemic model, partial sums process in Hölder space.

1. INTRODUCTION

Let $X_1, \dots, X_n$ be independent observations in a measurable space with the corresponding parameters of interest $\theta_1, \dots, \theta_n$. The parameters are said to have an epidemic change at points $1 < k^* < m^* < n$ if

$$\theta_1 = \cdots = \theta_{k^*} = \theta_{m^*+1} = \cdots = \theta_n = \theta'$$

and

$$\theta_k = \theta'' \neq \theta' \quad \text{for } k^* + 1 \leqslant k \leqslant m^*.$$

In the case where $m^* = n$, we have a model with a single abrupt change. They are intensively studied in the literature (see, e.g., Csörgő and Horváth [4] for a comprehensive review). Models with epidemic changes of parameters appear in neurophysiology (see Commenges *et al.* [3] and references therein), in the desoxyribonucleic acid (DNA) sequences context (Avery and Henderson [1]), in econometrics via bubbles phenomenon (see Kirman and Teyssière [5] and references therein), etc. In all these applications, testing for epidemic change and estimating

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its length $\ell^* = m^* - k^*$ as well as the location $(k^*, m^*]$ undoubtedly are very important problems. Several tests are discussed in the above-mentioned papers and in, e.g., Yao [11], Csörgő and Horváth [4], and Račkauskas and Suquet [7, 8]. In this paper, we propose procedures to estimate the location and length of epidemic change. Our results apply to a very short epidemic duration which, roughly speaking, can be of order $\ln^\beta n$ with some $\beta > 1$. Two models are investigated. In Section 2, we deal with real-valued observations and distribution functions as parameters. In Section 3, we consider infinite dimensional observations (taking values in a separable Banach space) and the mean as parameter of interest. For both models, we prove the consistency of estimators of the length and location of epidemic change.

2. EPIDEMIC OF DISTRIBUTION FUNCTIONS

Let $X_1, \dots, X_n$ be real-valued random variables with continuous distribution functions $F_1, \dots, F_n$. Consider the epidemic model

$$\begin{aligned}(H_E)\colon\quad & \text{there are integers } \ 1 < k^* < m^* < n \ \text{ such that} \\ & F_1 = F_2 = \cdots = F_{k^*} = F_{m^*+1} = \cdots = F_n = F, \\ & F_{k^*+1} = \cdots = F_{m^*} = G, \quad \text{and} \quad F \neq G.\end{aligned}$$

We denote by $\ell^* = m^* - k^*$ the length of epidemic. In this section, we consider estimators for the length ℓ^* as well as for the locations k^*, m^*.

For simplicity, we denote by $(j, k]$ the set of integers $\{j+1, \dots, k\}$. For $i \in (0, n]$ we set

$$V_i := F(X_i)$$

and

$$V_i' := \begin{cases} F(X_i) & \text{if } \ i \notin (k^*, k^* + \ell^*], \\ G(X_i) & \text{if } \ i \in (k^*, k^* + \ell^*]. \end{cases}$$

So, $V_1', \dots, V_n'$ are uniformly on $[0,1]$ distributed independent random variables. Introduce the partial sums processes

$$\nu(k, j) := \sum_{i=k+1}^{k+j} Y_i, \qquad \nu'(k, j) := \sum_{i=k+1}^{k+j} Y_i',$$

where

$$Y_i(t) := \mathbf{1}\{V_i \leqslant t\} - t, \qquad Y_i'(t) := \mathbf{1}\{V_i' \leqslant t\} - t, \quad t \in [0,1], \ i = 1, \dots, n.$$

For some weight function $\rho\colon [0,1] \to [0, \infty)$, we define

$$T_\rho(j) := \frac{1}{\rho(j/n)} \max_{0 \leqslant k \leqslant n-j} \|\nu(k, j)\| \tag{1}$$

and similarly, $T_\rho'(j)$ with ν' instead of ν. For a function $f\colon [0,1] \to \mathbb{R}$, we denote by $\|f\|$ either the uniform norm

$$\|f\| = \|f\|_\infty = \sup_{0 \leqslant t \leqslant 1} |f(t)|$$

or the L_2 norm

$$\|f\| = \|f\|_2 = \left(\int_0^1 f^2(t)\, \mathrm{d}t \right)^{1/2}.$$

Depending on the norm $\|\cdot\|$ considered in (1), we use different distances between distributions F and G. By $d(F,G)$ we denote the Kolmogorov–Smirnov distance

$$d(F,G) = d_\infty(F,G) = \sup_{x\in\mathbb{R}} |F(x) - G(x)|$$

if the uniform norm $\|\cdot\| = \|\cdot\|_\infty$ is used in (1), whereas

$$d(F,G) = d_2(F,G) = \left(\int_{-\infty}^{\infty} |F(x) - G(x)|^2 \, \mathrm{d}F(x)\right)^{1/2}$$

if $\|\cdot\| = \|\cdot\|_2$. Throughout, we allow F and G to depend on n and $d(F,G)$ to converge to zero as $n \to \infty$.

The weight function ρ in (1) aims to control the duration of epidemic change. Our assumptions on ρ are summarized by its belonging to the following class $\mathcal{R}$.

Definition 1. Let $\mathcal{R}$ be the class of nondecreasing functions $\rho : [0,1] \to [0,\infty)$ satisfying the following conditions:

(i) for some $0 < \alpha \leqslant 1/2$ and some function L, positive on $[1,\infty)$ and normalized slowly varying at infinity,

$$\rho(h) = h^\alpha L(1/h), \quad 0 < h \leqslant 1;$$

(ii) $t^{1/2}\rho(1/t)$ is C^1 on $[1,\infty)$;

(iii) there exist $\beta > 1/2$ and $a > 0$ such that $t \to t^{1/2}\rho(1/t)\ln^{-\beta}(t)$ is nondecreasing on $[a,\infty)$.

Let us recall that L is normalized slowly varying at infinity if and only if, for every $\delta > 0$, $t^\delta L(t)$ is ultimately increasing and $t^{-\delta}L(t)$ is ultimately decreasing ([2], Th. 1.5.5). The main practical examples we have in mind can be parametrized by

$$\rho(h) = \rho(h,\alpha,\beta) := h^\alpha \ln^\beta(c/h),$$

where $\beta \in \mathbb{R}$ if $0 < \alpha < 1/2$ and $\beta > 1/2$ if $\alpha = 1/2$.

Remark 1. If $\rho \in \mathcal{R}$, then $f(h) := h^{1/2}/\rho(h)$ is nondecreasing on some interval $(0,b]$. Indeed,

$$f\left(\frac{1}{t}\right) = \left(t^\delta L(t)\right)^{-1}$$

with $\delta = 1/2 - \alpha$ and $t^\delta L(t)$ is ultimately increasing by slow variation of L when $\alpha < 1/2$ or because of (iii) in Definition 1. When $\alpha = 1/2$, $L(t) = t^{1/2}\rho(1/t)$. The function $g(h) := h/\rho(h)$ is *a fortiori* increasing in a neighborhood of zero.

We introduce the following conditions:

$$\ell^* \xrightarrow[n\to+\infty]{} +\infty, \qquad \frac{\ell^*}{n} \xrightarrow[n\to+\infty]{} 0, \tag{2}$$

$$\frac{\ell^* d(F,G)^2}{\ln \ell^*} \xrightarrow[n\to+\infty]{} +\infty, \tag{3}$$

$$\frac{\ell^* d(F,G)}{\sqrt{n}\,\rho(\ell^*/n)} \xrightarrow[n\to+\infty]{} +\infty. \tag{4}$$

Remark 2. Under the mild assumption $\ell^* = \mathrm{O}(n^a)$ for some $0 < a < 1$, (3) follows from (4). Indeed, we can write

$$\frac{\ell^* d(F,G)^2}{\ln \ell^*} = \left(\frac{\ell^* d(F,G)}{\sqrt{n}\rho(\ell^*/n)}\right)^2 \frac{\rho(\ell^*/n)^2}{(\ell^*/n)\ln(n/\ell^*)} \frac{\ln(n/\ell^*)}{\ln \ell^*},$$

where the first two factors on the right-hand side tend to infinity by (4) and (iii) of Definition 1, respectively, while $\ell^* = \mathrm{O}(n^a)$ gives

$$\liminf_{n\to\infty} \frac{\ln(n/\ell^*)}{\ln \ell^*} > 0.$$

Set

$$\Delta_n := \max_{1<j<n} T_\rho(j), \qquad \Delta'_n := \max_{1<j<n} T'_\rho(j).$$

The estimation of epidemic length is based on the following result.

THEOREM 1. *Let $\rho \in \mathcal{R}$. Then*

$$n^{-1/2}\Delta'_n = \mathrm{O}_P(1). \tag{5}$$

If $\ell^* \to \infty$ and (4) *is satisfied, then*

$$n^{-1/2}\Delta_n \xrightarrow[n\to+\infty]{\mathrm{P}} \infty. \tag{6}$$

Proof. The stochastic boundedness of $n^{-1/2}\Delta'_n$ follows from Proposition 10 given in Appendix, applied to the processes $Y'_k(t), t \in [0,1]$, $k = 1, 2, \dots,$ and from the Dvoretzky–Kiefer–Wolfowitz inequality (see, e.g., Shorack and Wellner [10]).

To check (6) we note that

$$\Delta_n \geqslant \frac{\|\nu(k^*, \ell^*)\|}{\rho(\ell^*/n)}$$

and

$$\nu(k^*, \ell^*) = \sum_{k^*<i\leqslant m^*} (Y_i - Y'_i) + \nu'(k^*, \ell^*).$$

In view of (5), this leads to

$$n^{-1/2}\Delta_n \geqslant \frac{\ell^*}{n^{1/2}\rho(\ell^*/n)} \left\| \frac{1}{\ell^*} \sum_{k^*<i\leqslant m^*} (Y_i - Y'_i) \right\| - \mathrm{O}_P(1). \tag{7}$$

Put

$$\widetilde{Y}_i := Y_i - \mathbf{E}\, Y_i \quad \text{and} \quad \widetilde{Y}'_i := Y'_i - \mathbf{E}\, Y'_i.$$

Computing $\mathbf{E}\,(Y_i - Y'_i)$ as a Pettis integral, we easily find that

$$\mathbf{E}\,(Y_i - Y'_i) = G \circ F^{-1} - \mathrm{Id},$$

where Id denotes the identity function on $[0, 1]$. Moreover,

$$\left\|\mathbf{E}\,(Y_i - Y'_i)\right\| = d(F, G).$$

Now we get (noting that $\|\cdot\| \leqslant \|\cdot\|_\infty$)

$$\left\| \frac{1}{\ell^*} \sum_{k^*<i\leqslant m^*} (Y_i - Y_i') \right\| = \left\| \mathbf{E}(Y_i - Y_i') + \frac{1}{\ell^*} \sum_{k^*<i\leqslant m^*} \big(\widetilde{Y}_i - \widetilde{Y}_i'\big) \right\|$$
$$\geqslant d(F,G) - \left\| \frac{1}{\ell^*} \sum_{k^*<i\leqslant m^*} \widetilde{Y}_i \right\|_\infty - \left\| \frac{1}{\ell^*} \sum_{k^*<i\leqslant m^*} \widetilde{Y}_i' \right\|_\infty$$

The last term has the same distribution as that of $(\ell^*)^{-1/2}\|\varepsilon_{\ell^*}\|_\infty$, where ε_{ℓ^*} denotes the empirical process built on independent $[0,1]$-uniformly distributed U_i's. The previous term has the same distribution as that of $(\ell^*)^{-1/2}\|\varepsilon_{\ell^*}(G\circ F^{-1})\|_\infty$. Due to the weak convergence of ε_{ℓ^*} to the Brownian bridge, this leads to the estimate

$$\left\| \frac{1}{\ell^*} \sum_{k^*<i\leqslant m^*} (Y_i - Y_i') \right\| \geqslant d(F,G) - \mathrm{O}_P\Big(\frac{1}{\sqrt{\ell^*}}\Big). \tag{8}$$

By (7) this gives

$$n^{-1/2}\Delta_n \geqslant \frac{\ell^* d(F,G)}{n^{1/2}\rho(\ell^*/n)}\left[1 - \mathrm{O}_P\Big(\frac{1}{\sqrt{\ell^*}d(F,G)}\Big)\right] - \mathrm{O}_P(1).$$

To finish the proof, we only have to check that, under condition (4),

$$\sqrt{\ell^*}d(F,G) \xrightarrow[n\to+\infty]{} \infty. \tag{9}$$

We reformulate (4) as

$$\frac{\sqrt{\ell^*}d(F,G)}{(n/\ell^*)^{1/2}\rho(\ell^*/n)} \xrightarrow[n\to+\infty]{} +\infty. \tag{10}$$

Since $t^{1/2}\rho(1/t)$ tends to infinity as t does and $t\mapsto t^{1/2}\rho(1/t)$ is continuous positive on $[1,\infty)$, we have $\inf\{t^{1/2}\rho(1/t);\ t\geqslant 1\} > 0$. Hence, (9) follows from (10).

As an estimator of the epidemic length ℓ^*, we consider

$$\widehat{\ell^*} = \widehat{\ell^*}(\rho) = \min\Big\{\ell : T_\rho(\ell) = \max_{1<j<n} T_\rho(j)\Big\}. \tag{11}$$

THEOREM 2. *Let $\rho\in\mathcal{R}$. For model (H_E), assume that* (2), (3), *and* (4) *are satisfied. Then*

$$\frac{\widehat{\ell^*}}{\ell^*} \xrightarrow[n\to+\infty]{\mathrm{P}} 1.$$

Proof. The conclusion of Theorem 2 is equivalent to

$$\mathbf{P}\big(\widehat{\ell^*} \leqslant (1-\varepsilon)\ell^*\big) \xrightarrow[n\to+\infty]{} 0 \quad\text{and}\quad \mathbf{P}\big(\widehat{\ell^*} \geqslant (1+\varepsilon)\ell^*\big) \xrightarrow[n\to+\infty]{} 0, \quad \forall\varepsilon>0.$$

We shall detail only the first convergence, the proof of the second one being completely similar. The structure of the proof is given by the following elementary argument. We first note that, *on the event* $\{\widehat{\ell^*}\leqslant(1-\varepsilon)\ell^*\}$,

$$\max_{\ell\leqslant(1-\varepsilon)\ell^*} n^{-1/2}T_\rho(\ell) = \max_{\ell\leqslant\ell^*} n^{-1/2}T_\rho(\ell). \tag{12}$$

It follows that if Mj is an upper bound for the left-hand side of (12) and Mn a lower bound for its right-hand side, then

$$\mathbf{P}\big(\widehat{\ell^*} \leqslant (1-\varepsilon)\ell^*\big) \leqslant \mathbf{P}(\mathrm{Mj} \geqslant \mathrm{Mn}). \tag{13}$$

Next, we find Mj and Mn good enough to obtain the convergence to zero of $\mathbf{P}(\mathrm{Mj} \geqslant \mathrm{Mn})$. Before going into details, we need to introduce some more notation. Put

$$A(k,\ell) := (k, k+\ell] \cap (k^*, k^*+\ell^*] \tag{14}$$

and write $|A(k,\ell)|$ for the number of elements of this set. Now split the processes $\nu(k,\ell) = \{\nu(k,\ell;t), t \in [0,1]\}$ into

$$\nu(k,\ell) = \nu'(k,\ell) + \nu''(k,\ell), \tag{15}$$

where

$$\nu''(k,\ell;t) = \sum_{i \in A(k,\ell)} \Big(\mathbf{1}\big\{F(X_i) \leqslant t\big\} - \mathbf{1}\big\{G(X_i) \leqslant t\big\}\Big).$$

For $i \in (k^*, k^*+\ell^*]$, we have

$$\mathbf{E}\Big(\mathbf{1}\big\{F(X_i) \leqslant t\big\} - \mathbf{1}\big\{G(X_i) \leqslant t\big\}\Big) = G(F^{-1}(t)) - t$$

and, hence,

$$\nu''(k,\ell) = \big|A(k,\ell)\big|(G \circ F^{-1} - \mathrm{Id}) + \sum_{i \in A(k,\ell)} \eta_i, \tag{16}$$

where

$$\eta_i(t) := \mathbf{1}\big\{F(X_i) \leqslant t\big\} - \mathbf{1}\big\{G(X_i) \leqslant t\big\} - \mathbf{E}\Big(\mathbf{1}\big\{F(X_i) \leqslant t\big\} - \mathbf{1}\big\{G(X_i) \leqslant t\big\}\Big).$$

Evidently,

$$\sup_{t \in [0,1]} \big|G\big(F^{-1}(t)\big) - t\big| = d_\infty(F,G)$$

and

$$\int_0^1 \big(G\big(F^{-1}(t)\big) - t\big)^2 \, \mathrm{d}t = d_2^2(F,G).$$

Clearly,

$$\big\|\nu''(k,\ell) - |A(k,\ell)|(G \circ F^{-1} - \mathrm{Id})\big\| \leqslant \delta(k,\ell), \tag{17}$$

where

$$\delta(k,\ell) := \bigg\| \sum_{i \in A(k,\ell)} \eta_i \bigg\|.$$

Since $A(k^*,\ell^*) = (k^*, k^*+\ell^*]$, we deduce by the triangular inequality that

$$\big\|\nu''(k^*,\ell^*)\big\| \geqslant \ell^* d(F,G) - \delta(k^*,\ell^*). \tag{18}$$

Now the desired lower bound Mn for $\max_{\ell \leqslant \ell^*} T_\rho(\ell)$ can be obtained as follows:

$$\max_{\ell \leqslant \ell^*} T_\rho(\ell) \geqslant T_\rho(\ell^*)$$

$$
\begin{aligned}
&= \frac{1}{\rho(\ell^*/n)} \max_{0\leqslant k\leqslant n-\ell^*} \big\|\nu(k,\ell^*)\big\| \\
&\geqslant \frac{1}{\rho(\ell^*/n)} \big\|\nu(k^*,\ell^*)\big\| \qquad (19)\\
&\geqslant \frac{1}{\rho(\ell^*/n)} \big\|\nu''(k^*,\ell^*)\big\| - \frac{1}{\rho(\ell^*/n)} \big\|\nu'(k^*,\ell^*)\big\| \\
&\geqslant \frac{1}{\rho(\ell^*/n)} \big\|\nu''(k^*,\ell^*)\big\| - \Delta_n'.
\end{aligned}
$$

Hence, from (18) and (20) we get

$$
\max_{\ell\leqslant\ell^*} T_\rho(\ell) \geqslant \frac{\ell^*}{\rho(\ell^*/n)} d(F,G) - \frac{\delta(k^*,\ell^*)}{\rho(\ell^*/n)} - \Delta_n'. \tag{20}
$$

Let us now look for an upper bound Mj for $\max_{\ell\leqslant(1-\varepsilon)\ell^*} T_\rho(\ell)$. We have

$$
\begin{aligned}
T_\rho(\ell) &\leqslant \frac{1}{\rho(\ell/n)} \max_{1<k<n-\ell} \big(\|\nu'(k,\ell;t)\| + \|\nu''(k,\ell;t)\|\big) \\
&\leqslant \Delta_n' + \frac{1}{\rho(\ell/n)} \max_{1<k<n-\ell} \delta(k,\ell) + \frac{1}{\rho(\ell/n)} \max_{1\leqslant k\leqslant n-\ell} |A(k,\ell)| d(F,G).
\end{aligned}
$$

For any $\ell \leqslant \ell^*$, we evidently have $|A(k,\ell)| \leqslant \ell$, so, in view of Remark 1, we arrive at

$$
\max_{\ell\leqslant(1-\varepsilon)\ell^*} T_\rho(\ell) \leqslant \Delta_n' + \frac{(1-\varepsilon)\ell^* d(F,G)}{\rho((1-\varepsilon)\ell^*/n)} + Z_n, \tag{21}
$$

where

$$
Z_n := \max_{\ell\leqslant\ell^*} \frac{1}{\rho(\ell/n)} \max_{1<k<n-\ell} \delta(k,\ell).
$$

Now applying (13) with the lower bound (20) and upper bound (21), we obtain

$$
\begin{aligned}
\mathbf{P}\big\{\widehat{\ell^*} \leqslant (1-\varepsilon)\ell^*\big\} &\leqslant \mathbf{P}\left\{\Delta_n' + Z_n \geqslant \frac{d(F,G)}{2}\left(\frac{\ell^*}{\rho(\ell^*/n)} - \frac{(1-\varepsilon)\ell^*}{\rho((1-\varepsilon)\ell^*/n)}\right)\right\} \\
&\leqslant \mathbf{P}\{Z_n \geqslant A_n\} + \mathbf{P}\big\{n^{-1/2}\Delta_n' \geqslant C\big\},
\end{aligned}
$$

where $C > 0$ is a constant to be specified later and

$$
\begin{aligned}
A_n &:= \frac{d(F,G)}{2}\left(\frac{\ell^*}{\rho(\ell^*/n)} - \frac{(1-\varepsilon)\ell^*}{\rho((1-\varepsilon)\ell^*/n)}\right) - Cn^{1/2} \\
&= \frac{d(F,G)}{2} \frac{\ell^*}{\rho(\ell^*/n)}\left(1 - \frac{\rho(\ell^*/n)(1-\varepsilon)}{\rho((1-\varepsilon)\ell^*/n)} - \frac{2Cn^{1/2}\rho(\ell^*/n)}{\ell^* d(F,G)}\right).
\end{aligned} \tag{22}
$$

In view of (5) in Theorem 1, for all $\varepsilon_1 > 0$, we can find $C = C(\varepsilon_1)$ such that

$$
\mathbf{P}\big\{n^{-1/2}\Delta_n' \geqslant C\big\} < \varepsilon_1 \quad \text{for } n \geqslant n_1 = n_1(\varepsilon_1).
$$

Hence, it only remains to prove the convergence to zero of $\mathbf{P}(Z_n \geqslant A_n)$ for any fixed value of C. From (i) of Definition 1 we get

$$
\lim_{n\to\infty} \frac{\rho(\ell^*/n)(1-\varepsilon)}{\rho((1-\varepsilon)\ell^*/n)} = (1-\varepsilon)^{1-\alpha}. \tag{23}
$$

Since by (4) we also have that

$$\lim_{n\to\infty}\frac{2Cn^{1/2}\rho(\ell^*/n)}{\ell^* d(F,G)}=0,$$

we find n_2 such that, for $n\geqslant n_2$, the quantity in the parenthesis in (23) is not less than $\gamma:=(1-(1-\varepsilon)^{1-\alpha})/2$. Now we have

$$\mathbf{P}\{Z_n\geqslant A_n\}\leqslant\sum_{\ell\leqslant\ell^*}\sum_{1<k<n-\ell}\mathbf{P}\left\{\frac{\delta(k,\ell)}{\rho(\ell/n)}\geqslant\frac{\gamma\ell^* d(F,G)}{2\rho(\ell^*/n)}\right\}. \tag{24}$$

Due to the definition of $\delta(k,\ell)$, only the terms for which $A(k,\ell)$ is nonempty are to be accounted in the above sum. In view of the inequality $\|\nu(k,\ell)\|_2\leqslant\|\nu(k,\ell)\|_\infty$, it suffices to consider the case where $\|.\|_\infty$ is used to compute $\delta(k,\ell)$ in the following inequalities (but still allowing $d(F,G)$ to be interpreted as $d_2(F,G)$ or $d_\infty(F,G)$). Then the Dvoretsky–Kiefer–Wolfowitz inequality for the uniform empirical process leads to

$$\mathbf{P}\left\{\delta(k,\ell)\geqslant\frac{\gamma\rho(\ell/n)\ell^* d(F,G)}{2\rho(\ell^*/n)}\right\}\leqslant c_1\exp\left(-c_2\frac{\rho(\ell/n)^2(\ell^*)^2 d(F,G)^2}{\rho(\ell^*/n)^2|A(k,\ell)|}\right), \tag{25}$$

where c_1 and c_2 are positive constants. Now writing

$$\frac{\rho(\ell/n)^2(\ell^*)^2}{\rho(\ell^*/n)^2}=\frac{\rho(\ell/n)^2}{(\ell/n)}\frac{(\ell^*/n)}{\rho(\ell^*/n)^2}\ell\ell^*$$

and recalling Remark 1 and (2), we see that if n is large enough, then

$$\frac{\rho(\ell/n)^2}{(\ell/n)}\frac{(\ell^*/n)}{\rho(\ell^*/n)^2}\geqslant 1,\quad 1\leqslant\ell\leqslant\ell^*,$$

whence we have

$$\mathbf{P}\left\{\delta(k,\ell)\geqslant\frac{\gamma\rho(\ell/n)\ell^* d(F,G)}{2\rho(\ell^*/n)}\right\}\leqslant c_1\exp\left(-c_2\frac{\ell\ell^* d(F,G)^2}{|A(k,l)|}\right). \tag{26}$$

Noting that, for $\ell\leqslant\ell^*$, there are at most $2\ell^*$ indices k for which $A(k,\ell)$ is nonempty, by (24) we obtain that, for n large enough,

$$\mathbf{P}\{Z_n\geqslant A_n\}\leqslant c_1\ell^*\sum_{\ell=1}^{\ell^*}\exp\left(-c_2\frac{\ell\ell^* d(F,G)^2}{|A(k,l)|}\right).$$

Since $\ell\geqslant|A(k,l)|$, we finally arrive at

$$\mathbf{P}\{Z_n\geqslant A_n\}\leqslant c_1(\ell^*)^2\exp\big(-c_2\ell^* d(F,G)^2\big),$$

and this upper bound tends to zero by condition (3).

Next we consider estimation of the location of epidemics. Set

$$\widehat{k}^*=\widehat{k}^*(\rho)=\min\left\{k\colon\ \big\|\nu(k,\widehat{\ell}^*)\big\|=\max_{0\leqslant i\leqslant n-\widehat{\ell}^*}\big\|\nu(i,\widehat{\ell}^*)\big\|\right\} \tag{27}$$

and

$$\widehat{m}^*=\widehat{m}^*(\rho)=\widehat{k}^*+\widehat{\ell}^*,$$

where the length ℓ^* is estimated by (11), which explains the dependence of $\widehat{k}^*$ and $\widehat{m}^*$ on ρ. When $\rho(h)=\rho(h;\alpha,\beta)$, we denote $\widehat{\ell}^*(\rho)$, $\widehat{k}^*(\rho)$, and $\widehat{m}^*(\rho)$ by $\widehat{\ell}^*(\alpha,\beta)$, $\widehat{k}^*(\alpha,\beta)$, and $\widehat{m}^*(\alpha,\beta)$, respectively.

THEOREM 3. *Under the conditions of Theorem 2, we have*

$$\frac{|\widehat{k}^* - k^*|}{\ell^*} \xrightarrow[n\to+\infty]{\mathrm{P}} 0 \quad\text{and}\quad \frac{|\widehat{m}^* - m^*|}{\ell^*} \xrightarrow[n\to+\infty]{\mathrm{P}} 0.$$

Proof. We shall prove the first statement only, since the second statement follows from the first one and Theorem 2.

From the definition of k^* by (27) we get

$$\left\|\nu(\widehat{k}^*, \widehat{\ell}^*)\right\| - \left\|\nu(k^*, \widehat{\ell}^*)\right\| \geqslant 0. \tag{28}$$

Combining (28) with (15) and (17) and noting that $|A(k^*, \widehat{\ell}^*)| = \widehat{\ell}^* \wedge \ell^*$, we obtain

$$0 \leqslant \left(\left|A(\widehat{k}^*, \widehat{\ell}^*)\right| - \widehat{\ell}^* \wedge \ell^*\right) d(F, G) + J_n, \tag{29}$$

where

$$J_n := \delta(\widehat{k}^*, \widehat{\ell}^*) + \delta(k^*, \widehat{\ell}^*) + \left\|\nu'(\widehat{k}^*, \widehat{\ell}^*)\right\| + \left\|\nu'(k^*, \widehat{\ell}^*)\right\|.$$

To exploit (29) we need some relationship between $|A(\widehat{k}^*, \widehat{\ell}^*)|$ and $|\widehat{k}^* - k^*|$. The following array presents all possible configurations and the corresponding results.

Case	Configuration	$\lvert A(\widehat{k}^*, \widehat{\ell}^*)\rvert =$
1	$\widehat{k}^* + \widehat{\ell}^* \leqslant k^*$	0
2	$\widehat{k}^* < k^* < \widehat{k}^* + \widehat{\ell}^* \leqslant k^* + \ell^*$	$\widehat{\ell}^* - \lvert\widehat{k}^* - k^*\rvert$
3	$\widehat{k}^* < k^* < k^* + \ell^* < \widehat{k}^* + \widehat{\ell}^*$	ℓ^*
4	$k^* \leqslant \widehat{k}^* < k^* + \ell^* < \widehat{k}^* + \widehat{\ell}^*$	$\widehat{\ell}^* - \lvert\widehat{k}^* - k^*\rvert$
5	$k^* + \ell^* \leqslant \widehat{k}^*$	0

Moreover, in Case 3, $k^* + \ell^* < \widehat{k}^* + \widehat{\ell}^*$, whence

$$\ell^* < \widehat{k}^* - k^* + \widehat{\ell}^* = \widehat{\ell}^* - |\widehat{k}^* - k^*|,$$

which enables us to unify Cases 2, 3, and 4 under the common estimate

$$|A(\widehat{k}^*, \widehat{\ell}^*)| \leqslant \widehat{\ell}^* - |\widehat{k}^* - k^*| \quad \text{on } E_{2,3,4}, \tag{30}$$

where $E_{2,3,4}$ is the event which corresponds to the union of Cases 2, 3, and 4. Using (30) with the lower bound for $|A(\widehat{k}^*, \widehat{\ell}^*)|$ provided by (29) leads to

$$|\widehat{k}^* - k^*| \leqslant \widehat{\ell}^* - \widehat{\ell}^* \wedge \ell^* + \frac{J_n}{d(F,G)} \leqslant |\widehat{\ell}^* - \ell^*| + \frac{J_n}{d(F,G)} \quad \text{on } E_{2,3,4}.$$

On the other hand, denoting by $E_{1,5}$ the complementary event of $E_{2,3,4}$, from (29) and Cases 1 and 5 above we see that

$$\widehat{\ell}^* \wedge \ell^* \leqslant \frac{J_n}{d(F,G)} \quad \text{on } E_{1,5}.$$

This implies that

$$\mathbf{P}(E_{1,5}) \leqslant \mathbf{P}\left\{\frac{\widehat{\ell}^* \wedge \ell^*}{\ell^*} \leqslant \frac{J_n}{\ell^* d(F,G)}\right\}.$$

By Theorem 2, $(\widehat{\ell^*} \wedge \ell^*)/\ell^*$ tends to one in probability. Thus, if we prove that

$$\frac{J_n}{\ell^* d(F,G)} \xrightarrow[n\to+\infty]{\mathrm{P}} 0, \tag{31}$$

this will give the convergence to 1 of $\mathbf{P}(E_{2,3,4}) = 1 - \mathbf{P}(E_{1,5})$. Since

$$\frac{|\widehat{k^*} - k^*|}{\ell^*} \leqslant \frac{|\widehat{\ell^*} - \ell^*|}{\ell^*} + \frac{J_n}{\ell^* d(F,G)} \quad \text{on } E_{2,3,4},$$

the conclusion will follow by a new invocation of Theorem 2.

To check (31), it is easy to verify that

$$\delta(k^*, \widehat{\ell^*}) = \mathrm{O}_P(\sqrt{\ell^*}) \quad \text{and} \quad \delta(\widehat{k^*}, \widehat{\ell^*}) = \mathrm{O}_P(\sqrt{\ell^*}).$$

Since $\sqrt{\ell^*} d(F,G) \to \infty$ by (3), this gives

$$\frac{\delta(\widehat{k^*}, \widehat{\ell^*}) + \delta(k^*, \widehat{\ell^*})}{\ell^* d(F,G)} \xrightarrow[n\to+\infty]{\mathrm{P}} 0.$$

Finally, bounding

$$\left\|\nu'(\widehat{k^*}, \widehat{\ell^*})\right\| + \left\|\nu'(k^*, \widehat{\ell^*})\right\| \quad \text{by } 2\rho(\widehat{\ell^*}/n)\Delta_n',$$

we can write

$$\frac{\|\nu'(\widehat{k^*}, \widehat{\ell^*})\| + \|\nu'(k^*, \widehat{\ell^*})\|}{\ell^* d(F,G)} \leqslant \frac{2\sqrt{n}\rho(\ell^*/n)}{\ell^* d(F,G)} \times \frac{\Delta_n'}{\sqrt{n}} \times \frac{\rho(\widehat{\ell^*}/n)}{\rho(\ell^*/n)}.$$

In this upper bound, the first factor tends to zero by condition (4), the second one is $\mathrm{O}_P(1)$ by Theorem 1, and the last one converges to 1 in probability by Theorem 2 and the regular variation of ρ near zero. The proof is complete.

When the weight function ρ is of the form $\rho(h, \alpha, \beta)$, Theorems 2 and 3 have the following corollaries.

COROLLARY 4. *For model* (H_E), *assume that conditions* (2) *and* (3) *are satisfied and that*

$$\frac{{\ell^*}^{1-\alpha} d(F,G)}{n^{1/2-\alpha}} \to \infty \quad \textit{for some } 0 < \alpha < 1/2.$$

Then

$$\frac{\widehat{\ell^*}(\alpha,0)}{\ell^*} \xrightarrow[n\to+\infty]{\mathrm{P}} 1,$$

$$\frac{|\widehat{k^*}(\alpha,0) - k^*|}{\ell^*} \xrightarrow[n\to+\infty]{\mathrm{P}} 0,$$

$$\frac{|\widehat{m^*}(\alpha,0) - m^*|}{\ell^*} \xrightarrow[n\to+\infty]{\mathrm{P}} 0.$$

COROLLARY 5. *For model* (H_E), *assume that conditions* (2) *and* (3) *are satisfied and that*

$$\frac{\ell^* d^2(F,G)}{\ln^{2\beta} n} \to \infty \quad \textit{for some } \beta > 1/2.$$

Then

$$\frac{\widehat{\ell^*}(1/2,\beta)}{\ell^*} \xrightarrow[n\to+\infty]{\mathrm{P}} 1,$$

$$\frac{|\widehat{k^*}(1/2,\beta)-k^*|}{\ell^*} \xrightarrow[n\to+\infty]{\mathrm{P}} 0,$$

$$\frac{|\widehat{m^*}(1/2,\beta)-m^*|}{\ell^*} \xrightarrow[n\to+\infty]{\mathrm{P}} 0.$$

3. EPIDEMIC OF THE MEAN

Let B be a separable Banach space with norm $\|x\|$. Let $\varepsilon_1,\dots,\varepsilon_n$ be i.i.d. B-valued random elements with mean zero and let $(\mu_n)\subset B$. Consider the model

$$X_k = \begin{cases} \varepsilon_k & \text{for } k=1,\dots,k^*,\ m^*+1,\dots,n,\\ \mu_n+\varepsilon_k & \text{for } k=k^*+1,\dots,m^*. \end{cases} \tag{32}$$

For $0\leqslant k<m\leqslant n$, define

$$S_n(k,m) := \sum_{k<i\leqslant m}(X_i-\overline{X}) \quad\text{and}\quad S_n'(k,m) := \sum_{k<i\leqslant m}(\varepsilon_i-\overline{\varepsilon}),$$

where

$$\overline{X} = n^{-1}(X_1+\dots+X_n) \quad\text{and}\quad \overline{\varepsilon}=n^{-1}(\varepsilon_1+\dots+\varepsilon_n).$$

Let $\rho:[0,1]\to\mathbb{R}_+$ be a weight function. Define, for $1<j<n$,

$$U(j) = \max_{0\leqslant k\leqslant n-j}\left\|S_n(k,k+j)\right\|, \qquad V_\rho(j)=\frac{1}{\rho(j/n)}U(j)$$

and, similarly, $U'(j)$, $V_\rho'(j)$.

As an estimator of length $\ell^*=m^*-k^*$ for model (32), consider

$$\widehat{\ell^*}=\widehat{\ell^*}(\rho)=\min\Big\{\ell\colon\ V_\rho(\ell)=\max_{1<j<n}V_\rho(j)\Big\}. \tag{33}$$

When $\rho(h)=\rho(h;\alpha,\beta)$, we write $\widehat{\ell^*}(\alpha,\beta)$ for $\widehat{\ell^*}(\rho)$.

The random element ε is said to satisfy the central limit theorem in B (denoted $\varepsilon\in\mathrm{CLT}(B)$) if the sequence $n^{-1/2}(\varepsilon_1+\dots+\varepsilon_n)$ converges in distribution in B, where $\varepsilon_1,\dots,\varepsilon_n$ are independent copies of ε. It is known (see [6]) that, in general, the central limit theorem for ε cannot be characterized only in terms of the integrability of ε, since the geometry of B is involved in the problem.

THEOREM 6. *Let $\rho\in\mathcal{R}$. For model* (32), *assume that*

$$\varepsilon_1\in\mathrm{CLT}(B), \tag{34}$$

$$\lim_{t\to\infty} t\mathbf{P}\big(\|\varepsilon_1\|\geqslant At^{1/2}\rho(1/t)\big)=0 \quad \textit{for each } A>0, \tag{35}$$

$$\frac{\ell^*}{n}\to 0 \quad \textit{as } n\to\infty, \tag{36}$$

and

$$n^{1/2}\|\mu_n\|\frac{\ell^*/n}{\rho(\ell^*/n)} \to \infty \quad \text{as } n \to \infty. \tag{37}$$

Then

$$\frac{\widehat{\ell^*}}{\ell^*} \xrightarrow[n\to+\infty]{\mathrm{P}} 1. \tag{38}$$

Proof. The method of the proof is the same as that of Theorem 2 and relies on the inequality

$$\mathbf{P}\big(\widehat{\ell^*} \leqslant (1-\varepsilon)\ell^*\big) \leqslant \mathbf{P}(\mathrm{Mj} \geqslant \mathrm{Mn}),$$

where Mj is an upper bound for $\max_{\ell\leqslant(1-\varepsilon)\ell^*} n^{-1/2}V_\rho(\ell)$, and Mn a lower bound for $\max_{\ell\leqslant\ell^*} n^{-1/2}V_\rho(\ell)$. As a preliminary step, we establish the stochastic boundedness of $n^{-1/2}\Delta'_n$, where

$$\Delta'_n := \max_{1<j<n} V'_\rho(j).$$

Define the process $\xi_n^{\mathrm{sr}} := n^{-1/2}\xi_n$, where ξ_n is the polygonal line $([0,1] \to B)$ with vertices $(k/n, \varepsilon_1 + \cdots + \varepsilon_k)$. Next, introduce the separable Hölder space

$$\mathrm{H}_\rho^o(B) := \Big\{x \in \mathrm{C}(B);\ \lim_{\delta\to 0}\omega_\rho(x,\delta) = 0\Big\}$$

equiped with the norm

$$\|x\|_\rho := \|x(0)\| + \omega_\rho(x,1),$$

where

$$\omega_\rho(x,\delta) := \sup_{\substack{s,t\in[0,1],\\ 0<t-s<\delta}} \frac{\|x(t)-x(s)\|}{\rho(t-s)}.$$

One easily sees that

$$n^{-1/2}\Delta'_n \leqslant \big\|\xi_n^{\mathrm{sr}}\big\| + \sup_{0<h\leqslant 1}\frac{h}{\rho(h)}\big\|\xi_n^{\mathrm{sr}}(1)\big\|.$$

Since $\rho \in \mathcal{R}$, $h/\rho(h)$ can be extended to a continuous function on $[0,1]$. Hence, $\sup_{0<h\leqslant 1} h/\rho(h)$ is finite. Due to (34), $\xi_n^{\mathrm{sr}}(1)$ converges in distribution in B. By (34) and (35), ξ_n^{sr} converges in distribution in the Hölder space $(\mathrm{H}_\rho^o(B), \|.\|_\rho)$ (see [9], Theorem 8). Therefore, we have

$$n^{-1/2}\Delta'_n = \mathrm{O}_P(1) \quad \text{as } n \to \infty. \tag{39}$$

Next, let us note that

$$\overline{X} = \frac{\ell^*}{n}\mu_n + \overline{\varepsilon}$$

and, for any $1 \leqslant \ell \leqslant n$ and $0 \leqslant k \leqslant n-\ell$,

$$S_n(k,k+\ell) = \Big(\big|A(k,\ell)\big| - \frac{\ell\ell^*}{n}\Big)\mu_n + S'_n(k,k+\ell), \tag{40}$$

where, as before, $|A(k,\ell)|$ denotes the number of elements in the intersection

$$\{k+1,\dots,k+\ell\} \cap \{k^*+1,\dots,k^*+\ell^*\}.$$

The special choice $k=k^*,\ \ell=\ell^*$ in (40) gives

$$S_n(k^*,k^*+\ell^*)=S_n'(k^*,k^*+\ell^*)+\ell^*\mu_n(1-\ell^*/n).$$

Since

$$U(\ell^*)\geqslant \big\|S_n(k^*,k^*+\ell^*)\big\|,$$

we get

$$U(\ell^*)\geqslant \ell^*\Big(1-\frac{\ell^*}{n}\Big)\|\mu_n\|-\big\|S_n'(k^*,k^*+\ell^*)\big\|\geqslant \ell^*\Big(1-\frac{\ell^*}{n}\Big)\|\mu_n\|-U'(\ell^*),$$

whence

$$\max_{\ell\leqslant\ell^*}V_\rho(\ell)\geqslant\|\mu_n\|\frac{\ell^*(1-\ell^*/n)}{\rho(\ell^*/n)}-\Delta_n'. \tag{41}$$

In view of (39), this leads to

$$n^{-1/2}\max_{\ell\leqslant\ell^*}V_\rho(\ell)\geqslant n^{1/2}\|\mu_n\|\frac{(\ell^*/n)(1-\ell^*/n)}{\rho(\ell^*/n)}-\mathrm{O}_P(1)=:\mathrm{Mn}. \tag{42}$$

Looking now for some Mj, we note that, due to (40), for any $\ell\leqslant\ell^*$, we have

$$\big\|S_n(k,k+\ell)\big\|\leqslant\|\mu_n\|\ell(1-\ell^*/n)+\big\|S_n'(k,k+\ell)\big\|,$$

whence

$$\max_{\ell\leqslant(1-\varepsilon)\ell^*}V_\rho(\ell)\leqslant\|\mu_n\|\max_{\ell\leqslant(1-\varepsilon)\ell^*}\frac{\ell(1-\ell^*/n)}{\rho(\ell/n)}+\Delta_n' \tag{43}$$

Due to Remark 1 and condition (36), we have that, for n large enough,

$$\max_{\ell\leqslant(1-\varepsilon)\ell^*}\frac{\ell(1-\ell^*/n)}{\rho(\ell/n)}=\frac{(1-\varepsilon)\ell^*(1-\ell^*/n)}{\rho((1-\varepsilon)\ell^*/n)},$$

so by (39) and (43) we obtain

$$n^{-1/2}\max_{\ell\leqslant(1-\varepsilon)\ell^*}V_\rho(\ell)\leqslant n^{1/2}\|\mu_n\|\frac{(1-\varepsilon)(\ell^*/n)(1-\ell^*/n)}{\rho((1-\varepsilon)\ell^*/n)}+\mathrm{O}_P(1)=:\mathrm{Mj}. \tag{44}$$

In view of condition (37), we have that, *on the event* $\{\mathrm{Mj}\geqslant\mathrm{Mn}\}$,

$$\frac{(1-\varepsilon)\rho(\ell^*/n)}{\rho((1-\varepsilon)\ell^*/n)}\geqslant 1-\mathrm{o}_P(1). \tag{45}$$

The limit of the left-hand side in (45) being $(1-\varepsilon)^{1-\alpha}<1$, this implies that $\mathbf{P}(\mathrm{Mj}\geqslant\mathrm{Mn})$ tends to zero as n tends to infinity. Consequently,

$$\lim_{n\to\infty}\mathbf{P}\big(\widehat{\ell^*}<(1-\varepsilon)\ell^*\big)=0.$$

Similarly, one gets

$$\lim_{n\to\infty}\mathbf{P}\big(\widehat{\ell^*}>(1+\varepsilon)\ell^*\big)=0,$$

which completes the proof of the theorem.

Choosing $\rho(h)=\rho(h;\alpha,\beta)$ provides the following corollaries.

COROLLARY 7. *Let $0 \leqslant \alpha < 1/2$. For model* (32), *assume that*

$$\lim_{t\to\infty} t\mathbf{P}\big(\|\varepsilon_1\| \geqslant t^{1/2-\alpha}\big) = 0 \tag{46}$$

and

$$\frac{\ell^*}{n} \to 0, \qquad \|\mu_n\| \frac{\ell^{*(1-\alpha)}}{n^{1/2-\alpha}} \to \infty \quad \text{as } n \to \infty. \tag{47}$$

Then

$$\frac{\widehat{\ell^*}(\alpha, 0)}{\ell^*} \xrightarrow[n\to+\infty]{\mathrm{P}} 1.$$

COROLLARY 8. *Let $\beta > 1/2$. For model* (32), *assume that*

$$\mathbf{E}\exp\big(\lambda\|\varepsilon_1\|^{1/\beta}\big) < \infty \quad \text{for } \lambda > 0 \tag{48}$$

and

$$\frac{\ell^*}{n} \to 0, \qquad \|\mu_n\| \frac{\ell^{*1/2}}{\ln^\beta n} \to \infty \quad \text{as } n \to \infty. \tag{49}$$

Then

$$\frac{\widehat{\ell^*}(1/2, \beta)}{\ell^*} \xrightarrow[n\to+\infty]{\mathrm{P}} 1.$$

As the model allows the convergence $\mu_n \to 0$, condition (37) (as well as either (47) or (49)) impose constraints on the rate of convergence in relation with the length of epidemic change.

Next we consider the estimation of the location of epidemics. Let the length ℓ^* be estimated by (33). Set

$$\widehat{k^*} = \widehat{k^*}(\rho) = \min\Big\{k\colon\ U\big(\widehat{\ell^*}(\rho)\big) = \big\|S_n(k, k + \widehat{\ell^*}(\rho))\big\|\Big\} \tag{50}$$

and

$$\widehat{m^*} = \widehat{m^*}(\rho) = \widehat{k^*}(\rho) + \widehat{\ell^*}. \tag{51}$$

THEOREM 9. *Under the conditions of Theorem* 6, *the estimators $\widehat{k^*}$ and $\widehat{m^*}$ satisfy*

$$\left|\frac{\widehat{k^*}}{\ell^*} - \frac{k^*}{\ell^*}\right| \xrightarrow[n\to+\infty]{\mathrm{P}} 0 \quad \text{and} \quad \left|\frac{\widehat{m^*}}{\ell^*} - \frac{m^*}{\ell^*}\right| \xrightarrow[n\to+\infty]{\mathrm{P}} 0.$$

Proof. Due to Theorem 6, it suffices to prove the convergence result for $\widehat{k^*}$ only. The definition of $\widehat{k^*}$ gives

$$\big\|S_n(k^*, k^* + \widehat{\ell^*})\big\| \leqslant \big\|S_n(\widehat{k^*}, \widehat{k^*} + \widehat{\ell^*})\big\|.$$

From this, using (40) and the fact that

$$\big|A(k^*, \widehat{\ell^*})\big| = \widehat{\ell^*} \wedge \ell^* \geqslant \widehat{\ell^*}\ell^*/n,$$

we obtain

$$0 \leqslant \left|\big|A(\widehat{k^*}, \widehat{\ell^*})\big| - \frac{\widehat{\ell^*}\ell^*}{n}\right| \cdot \|\mu_n\| - \Big(\widehat{\ell^*} \wedge \ell^* - \frac{\widehat{\ell^*}\ell^*}{n}\Big)\|\mu_n\| + 2U'(\widehat{\ell^*}). \tag{52}$$

Introduce the complementary events

$$E' := \Big\{\big|A(\widehat{k^*}, \widehat{\ell^*})\big| < \frac{\widehat{\ell^*}\ell^*}{n}\Big\}, \qquad E'' := \Big\{\big|A(\widehat{k^*}, \widehat{\ell^*})\big| \geqslant \frac{\widehat{\ell^*}\ell^*}{n}\Big\}.$$

On the event E', by (52) we have

$$0 \leqslant -\big|A(\widehat{k}^*, \widehat{\ell}^*)\big| + 2\frac{\widehat{\ell}^*\ell^*}{n} - \widehat{\ell}^* \wedge \ell^* + \frac{2U'(\widehat{\ell}^*)}{\|\mu_n\|},$$

whence

$$\min\Big(\frac{\widehat{\ell}^*}{\ell^*}, 1\Big) \leqslant \frac{2\widehat{\ell}^*}{n} + \frac{2\rho(\widehat{\ell}^*/n)\Delta_n'}{\ell^*\|\mu_n\|}. \tag{53}$$

By (36)–(38) the left-hand side of (53) tends to 1 in probability, while its right-hand side tends to 0 in probability. It follows that

$$\lim_{n\to\infty} \mathbf{P}(E_n') = 0.$$

On the event E'', the inequality

$$\big|A(\widehat{k}^*, \widehat{\ell}^*)\big| \geqslant \widehat{\ell}^*\ell^*/n$$

excludes Cases 1 and 5 (cf. the proof of Theorem 3), where $|A(\widehat{k}^*, \widehat{\ell}^*)| = 0$, since both $\widehat{\ell}^*$ and ℓ^* are positive. Hence, E'' is included in $E_{2,3,4}$, and (30) holds on E''. In view of (52), we then have that, on E'',

$$0 \leqslant \widehat{\ell}^* - \widehat{\ell}^* \wedge \ell^* - |\widehat{k}^* - k^*| + \frac{2\rho(\widehat{\ell}^*/n)\Delta_n'}{\|\mu_n\|},$$

whence

$$\frac{|\widehat{k}^* - k^*|}{\ell^*} \leqslant \frac{\widehat{\ell}^*}{\ell^*} - \min\Big(\frac{\widehat{\ell}^*}{\ell^*}, 1\Big) + \frac{2\rho(\widehat{\ell}^*/n)\Delta_n'}{\ell^*\|\mu_n\|}. \tag{54}$$

By (36)–(38) the right-hand side of (54) tends to 0 in probability, which leads to the conclusion, since

$$\lim_{n\to\infty} \mathbf{P}(E_n'') = 1.$$

4. AUXILIARY RESULT

For a stochastic process $X = \{X(t), t \in T\}$ indexed by some arbitrary index set T, define

$$\|X\| := \sup_{t\in T} |X(t)|.$$

This stochastic process is not necessarily measurable map into a Banach space. Independence of stochastic processes is understood in the usual sense. By P^* we denote the outer probability.

PROPOSITION 10. *Let $X_1, \dots, X_n$ be independent identically distributed stochastic processes indexed by an arbitrary set, and let $S_0 := 0$, $S_n := X_1 + \dots + X_n$, $n \geqslant 1$. Assume that there exist constants c and a such that, for each $k \geqslant 1$,*

$$P^*\big(\|k^{-1/2}S_k\| \geqslant t\big) \leqslant c\exp(-at^2) \quad \textit{for } t > 0. \tag{55}$$

Define

$$T_n := n^{-1/2} \max_{0\leqslant i<j\leqslant n} \frac{\|S_j - S_i\|}{\rho((j-i)/n)}, \quad n \geqslant 1. \tag{56}$$

Then the sequence $(T_n)_{n\geqslant 1}$ *is stochastically bounded, provided that*

$$\sum_{j=1}^{+\infty} 2^j \exp(-\tau\theta^2(2^j)) < \infty \tag{57}$$

for each $\tau > 0$.

Proof. Rewriting (56) in the form

$$n^{1/2}T_n = \max_{1\leqslant \ell\leqslant n} \frac{1}{\rho(\ell/n)} \max_{0\leqslant k\leqslant n-\ell} \|S_{k+\ell} - S_k\|,$$

we shall use a dyadic splitting of the ℓ's and k's indexation ranges. Defining the integer J_n by

$$2^{J_n} \leqslant n < 2^{J_n+1},$$

we get

$$\begin{aligned} n^{1/2}T_n &= \max_{1\leqslant j\leqslant J_n+1} \max_{n2^{-j}<\ell\leqslant n2^{-j+1}} \frac{1}{\rho(\ell/n)} \max_{1\leqslant k\leqslant n-\ell} \|S_{k+\ell} - S_k\| \\ &\leqslant \max_{1\leqslant j\leqslant J_n+1} \max_{n2^{-j}<\ell\leqslant n2^{-j+1}} \frac{1}{\rho(2^{-j})} \max_{0\leqslant k<n-n2^{-j}} \|S_{k+\ell} - S_k\| \\ &\leqslant \max_{1\leqslant j\leqslant J_n+1} \max_{n2^{-j}<\ell\leqslant n2^{-j+1}} \frac{1}{\rho(2^{-j})} \max_{1\leqslant i<2^j} \max_{(i-1)n2^{-j}\leqslant k<in2^{-j}} \|S_{k+\ell} - S_k\|. \end{aligned}$$

For $n2^{-j} < \ell \leqslant n2^{-(j-1)}$ and $(i-1)n2^{-j} \leqslant k < in2^{-j}$, we have

$$\begin{aligned} \|S_{k+\ell} - S_k\| &\leqslant \left\|S_{k+\ell} - S_{[in2^{-j}]}\right\| + \left\|S_{[in2^{-j}]} - S_k\right\| \\ &\leqslant \max_{in2^{-j}<u<(i+2)n2^{-j}} \left\|S_u - S_{[in2^{-j}]}\right\| \\ &\quad + \max_{(i-1)n2^{-j}\leqslant k<in2^{-j}} \left\|S_{[in2^{-j}]} - S_k\right\|, \end{aligned}$$

where $[t]$ denotes the integer part of a real number t. Therefore,

$$T_n \leqslant T_n' + T_n'',$$

where

$$T_n' = n^{-1/2} \max_{1\leqslant j\leqslant J_n+1} \frac{1}{\rho(2^{-j})} \max_{1\leqslant i<2^j} \max_{in2^{-j}<u<(i+2)n2^{-j}} \left\|S_u - S_{[in2^{-j}]}\right\|$$

and

$$T_n'' = n^{-1/2} \max_{1\leqslant j\leqslant J_n+1} \frac{1}{\rho(2^{-j})} \max_{1\leqslant i<2^j} \max_{(i-1)n2^{-j}\leqslant k<in2^{-j}} \left\|S_{[in2^{-j}]} - S_k\right\|.$$

Due to the stationarity, for each $\lambda > 0$, we have

$$P^*\{T_n' > \lambda\} \leqslant \sum_{j=1}^{J_n+1} P^*\left\{ \max_{1\leqslant i<2^j} \max_{in2^{-j}<u<(i+2)n2^{-j}} \left\|S_u - S_{[in2^{-j}]}\right\| > \lambda n^{1/2}\rho(2^{-j})\right\}$$

$$\leqslant \sum_{j=1}^{J_n+1} 2^j P^*\Big\{ \max_{u\leqslant 2n2^{-j}} \|S_u\| > \lambda n^{1/2}\rho(2^{-j})\Big\}.$$

Applying Ottaviani's inequality (Ledoux and Talagrand [6], Lemma 6.2) and condition (55), we obtain

$$P^*(T_n' > \lambda) \leqslant \sum_{j=1}^{J_n+1} 2^j \frac{c\exp(-a\lambda^2 2^{j-3}\rho^2(2^{-j}))}{1 - c\exp(-a\lambda^2 2^{j-3}\rho^2(2^{-j}))},$$

provided that the denominator above is positive for all $j \geqslant 1$. This condition is clearly satisfied for λ large enough (independently of n), since $2^{-j}\rho(2^j)$ tends to infinity with j and ρ is positive. Hence, condition (57) gives the stochastic boundedness of $(T_n')_{n\geqslant 1}$ via the dominated convergence theorem for series. The proof of the stochastic boundedness of $(T_n'')_{n\geqslant 1}$ clearly is similar.

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