BLOW-ANALYTIC EQUIVALENCE OF TWO VARIABLE REAL ANALYTIC FUNCTION GERMS

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We give a complete characterization of blow-analytic equivalence classes of two variable real analytic function germs.

Theorem 0.1. *Let* $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ *and* $g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ *be real analytic function germs. Then the following conditions are equivalent*

1. f *and* g *are blow-analytically equivalent.*
2. f *and* g *have isomorphic minimal resolutions.*
3. *The real tree models of* f *and* g *coincide.*

In the above we consider for simplicity only the orientation preserving case. Thus, for instance, in (1) we require that f and g are blow-analytically equivalent by the orientation preserving blow-analytic homeomorphism. A similar theorem can be also stated in the non-oriented case.

0.1. **Blow-analytic equivalence.** In a search for a "right" equivalence relation of real analytic function germs, that should play a similar role to the topological equivalence in the complex analytic set-up, Tzee-Char Kuo introduced at the end of 1970's, [11], [12], [13], the notion of blow-analytic equivalence. Kuo also established its basic properties in particular the local finiteness of blow-analytic types for a family of real analytic function-germs with isolated singularities.

One says that two real analytic function germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ and $g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are *blow-analytically equivalent* if there exist real modifications $\mu : (M, \mu^{-1}(0)) \to (\mathbb{R}^n, 0)$, $\mu' : (M', \mu'^{-1}(0)) \to (\mathbb{R}^n, 0)$ and an analytic isomorphism $\Phi : (M, \mu^{-1}(0)) \to (M', \mu'^{-1}(0))$ which induces a homeomorphism $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $f = g \circ h$:

$$\begin{array}{ccccc} (M, \mu^{-1}(0)) & \xrightarrow{\mu} & (\mathbb{R}^n, 0) & \xrightarrow{f} & (\mathbb{R}, 0) \\ \Phi \big\downarrow & & h \big\downarrow & & \big\| \\ (\tilde{M}, \tilde{\mu}^{-1}(0)) & \xrightarrow{\tilde{\mu}} & (\mathbb{R}^n, 0) & \xrightarrow{g} & (\mathbb{R}, 0) \end{array} \tag{0.1}$$

Let X, Y be a real analytic connected manifolds of pure dimension n. We say that a proper real analytic mapping $\sigma : X \to Y$ is a *real modification* if there exist complexifications $X_{\mathbb{C}}, Y_{\mathbb{C}}$ of X and Y, respectively, and a holomorphic extension $\sigma_{\mathbb{C}} : X_{\mathbb{C}} \to Y_{\mathbb{C}}$ of σ, such that:

$\sigma_{\mathbb{C}}$ is an isomorphism in the complement of a closed nowhere dense subset B of $X_{\mathbb{C}}$. (that is $\sigma_{\mathbb{C}}$ resticted to $X_{\mathbb{C}} \setminus B$ is open and an isomorphism onto its image.)

The simplest example of a real modification is the blowing-up with a nonsingular center. A real analytic map that is an isomorphism in the complement of a closed nowhere dense subset of X is not necessarily a real modification, for instance $\sigma : \mathbb{R} \to \mathbb{R}$ given by $\sigma(x) = x^3$. For two variables, as we show, the notion of a real modification is as simple as expected.

Theorem 0.2. *Let X, Y be connected nonsingular real analytic surfaces and let $\sigma : X \to Y$ be a proper surjective real analytic map. Then σ is a real modification if and only if it is a composition of point blowings-up.*

Most of the positive results that prove that two function germs are blow-analytically equivalent were obtained using, mostly toric, equiresolutions by Kuo, Fukui-Yoshinaga [5], Fukui-Paunescu[8], Abderrahmane[1], and others. Several invariants allowing to distinguish different blow-analytic types were constructed, using the geometry of arc-spaces and motivic integration, by Fukui [6], Fichou[4], and the authors of this note [10]. For mor on the blow-analytic equivalence see also [7].

0.2. **Proof of theorem 0.1.** Theorem 0.2 can be proved by an argument similar to the elimination of indeterminacy of the rational maps between algebraic surfaces, see e.g. [3] Proposition II.8.

Suppose that $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ and $g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ are blow-analytically equivalent. That is, afteer Theorem 0.2, there exists a commutative diagram

$$
\begin{array}{ccccccc}
(M, \mu'^{-1}(E_1)) & \xrightarrow{\mu'} & (\widetilde{\mathbb{R}^2}, E_1) & \xrightarrow{\pi} & (\mathbb{R}^2, 0) & \xrightarrow{f} & \mathbb{R} \\
{\scriptstyle\Phi}\downarrow & & {\scriptstyle\exists h_1 ?}\downarrow & & {\scriptstyle h}\downarrow & & \| \\
(\tilde{M}, \tilde{\mu}'^{-1}(\tilde{E}_1)) & \xrightarrow{\tilde{\mu}'} & (\widetilde{\mathbb{R}^2}, \tilde{E}_1) & \xrightarrow{\tilde{\pi}} & (\mathbb{R}^2, 0) & \xrightarrow{g} & \mathbb{R}
\end{array}
\tag{0.2}
$$

where π and $\tilde{\pi}$ are the blowing-up of the origin, μ' and $\tilde{\mu}'$ are compositions of point blowings-up, Φ is an analytic isomorphism and h is a homeomorphism such that $f = g \circ h$.

Main Point: *Using the combinatorial properties of dual graphs of real resolutions we show the existence a homeomorphism $h_1 : (\widetilde{\mathbb{R}^2}, E_1) \to (\widetilde{\mathbb{R}^2}, \tilde{E}_1)$ closing the above diagram.*

Thus, by induction, we show that f and g are *cascade blow-analytic equivalent*

$$
\begin{array}{ccccccccccccc}
M & \xrightarrow{b_k} & M_{k-1} & \xrightarrow{b_{k-1}} & \cdots & \xrightarrow{b_2} & M_1 & \xrightarrow{b_1} & \mathbb{R}^2 & \xrightarrow{f} & \mathbb{R} \\
{\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle h_{k-1}} & & & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h} & & \| \\
\tilde{M} & \xrightarrow{\tilde{b}_k} & \tilde{M}_{k-1} & \xrightarrow{\tilde{b}_{k-1}} & \cdots & \xrightarrow{\tilde{b}_2} & \tilde{M}_1 & \xrightarrow{b_1} & \mathbb{R}^2 & \xrightarrow{g} & \mathbb{R},
\end{array}
\tag{0.3}
$$

where $b_i, \tilde{b}_i$ are point blowings-up and h_i are homeomorphisms. A function blow-analytically equivalent to normal crossings is normal crossings (we show it for two variable function germs, the general case is open) and hence we may get rid of unnecessary blowings-up and assume that both $b_1 \circ \cdots \circ b_k$ and $\tilde{b}_1 \circ \cdots \circ \tilde{b}_k$ are the minimal resolutions of f and g resp.. This shows the most difficult implication (1) $\Longrightarrow$ (2) of Theorem 0.1. The implication (2) $\Longrightarrow$ (1) is fairly standard.

We also establish the following basic properties of *cascade blow-analytic homeomorphisms* (i.e. the homeomorphisms like h in (0.3)), that are used in the proof of (1)$\Longleftrightarrow$ (3) of theorem 0.1.

Theorem 0.3. *Let $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a cascade blow-analytic homeomorphism. We suppose for simplicity that h does not change the orientation. Then h preserves:*

(b) *The Puiseux characteristics sequence of real analytic demi-branches $\gamma : [0, \varepsilon) \to \mathbb{R}^2$ and the signs of coefficients at the Puiseux characteristic exponents.*

(c) *The order of contact between two real analytic demi-branches and the algebraic intersection number between two real analytic 1-cycles at 0.*

(1) *The magnitude of the distance to the origin: there exists a constant $C > 0$ such that for all (x, y) close to the origin*

$$C^{-1}\|(x,y)\| \le \|h(x,y)\| \le C\|(x,y)\|.$$

(2) *There exist constants $C, c > 0$ such that the jacobian determinant $Jac(h)$ of h, that is defined in the complement of the origin, satisfies*

$$c \le Jac(h)(x,y) \le C.$$

Remark 0.4. Kobayashi and Kuo [9] constructed an example of a blow-analytic homeomorphism that sends a smooth curve to a singular one and vice versa. As follows from Theorem 0.3 such a homeomorphism cannot be a cascade one and hence cannot establish the blow-analytic equivalence between two real analytic function germs.

Similarly to the complex case the real tree model of $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ contains information about the order contact and the Puiseux characteristic exponents of the Newton-Puiseux roots of f. For a complex root

$$x = \lambda(y) = a_1 y^{n_1/N} + a_2 y^{n_2/N} + \cdots . \tag{0.4}$$

we consider only its real part up to the first non-real coefficient. The real tree model contains also the information about the signs of coefficients at the Puiseux characteristic exponents. Thanks to Theorem 0.3 the proof of (1) $\Longleftrightarrow$ (3) of Theorem 0.1 is based on a fairly straightforward computation of the tree model of the blown-up singularity in terms of the original tree model.

0.3. **Examples.** Abderrahmane [2] showed that blow-analytically equivalent weighted homogeneous singular $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ and $g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ have the same weights. The blow-analytic classification of Brieskorn two variable singularities $\pm x^p \pm y^q$ was obtained in [10]. It coincides with $(x, y) \to (\pm x, \pm y)$ classification except the case p odd and $q = pm$ with m even. Thus, for instance, $f(x, y) = x^3 - y^6$ and $g(x, y) = x^3 + y^6$ are blow-analytically equivalent even if there is no bi-lipschitz homeomorphism $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $f = g \circ h$.

All these results can be easily verified by Theorem 0.1. Here are the simplest examples of different blow-analytic classes that cannot be distinguish by the previously known invariants

(1) $x^3 + y^5$ and $x^3 - y^5$ are not blow-analytically equivalent by an orientation preserving homeomorphism.
(2) $x(x^3 - y^5)(x^3 + y^5)$, $x(x^3 - y^5)(x^3 - 2y^5)$ are not blow-analytically equivalent.

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