ON THETA FUNCTIONS OF ORDER FOUR

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ABSTRACT. We prove that the fourth powers of theta functions with even characteristics form a basis of the space $H^0(A, \mathcal{O}_A(4\vartheta))_+$ of even theta functions of order four on a principally polarized Abelian variety (A, ϑ) without vanishing theta-null.

1. INTRODUCTION

Let (A, ϑ) be a principally polarized Abelian variety of dimension g defined over an algebraically closed field k of characteristic different from two. By Mumford's algebraic theory of theta functions [M1] (see also the appendix of [B1]) there is a canonical bijection between the set of symmetric effective divisors $\Theta_\kappa \subset A$ representing the principal polarization and the set $\mathcal{K}$ of theta-characteristics κ of A. We denote by L the unique symmetric line bundle over A representing 2ϑ and such that the linear system $|L|$ contains the divisors $2\Theta_\kappa$ for all $\kappa \in \mathcal{K}$. The space $H^0(A, L^2)$ of theta functions of order four decomposes under the natural involution of A into $\pm$-eigenspaces $H^0(A, L^2)_+$ and $H^0(A, L^2)_-$ of dimensions

$$d_+ = 2^{g-1}(2^g + 1) \qquad \text{and} \qquad d_- = 2^{g-1}(2^g - 1).$$

We note that d_+ and d_- are equal to the number of even and odd theta-characteristics of A. Moreover any even theta-characteristic $\kappa_0 \in \mathcal{K}_+$ decomposes according to the values ± 1 taken by κ_0 the set of 2-torsion points $A[2]$ into a union of two subsets $A[2]_+$ and $A[2]_-$ of cardinality d_+ and d_-. Note that $0 \in A[2]_+$. We denote by

$$\varphi : A \longrightarrow |L^2|_+^* = \mathbb{P}^{d_+ - 1}$$

the morphism induced by the linear system $|L^2|_+$. Then our main result is the following

Theorem 1.1. *For any even theta-characteristic $\kappa_0 \in \mathcal{K}_+$, the d_+ points*

$$\varphi(a) \in |L^2|_+^* = \mathbb{P}^{d_+ - 1} \qquad \textit{with} \qquad a \in A[2]_+$$

form a projective basis if and only if A has no vanishing theta-null.

There is also a "dual" version of this result.

Theorem 1.2. *The d_+ divisors*

$$4\Theta_\kappa \in |L^2|_+ \qquad \textit{with} \qquad \kappa \in \mathcal{K}_+$$

form a projective basis (or, in other words, the fourth powers of theta functions with even characteristics form a basis of $H^0(A, L^2)_+$) if and only if A has no vanishing theta-null.

Although of independent interest, these results also have some consequences on *generalized* theta functions (see [S] for details): let $\mathcal{M}^+_{\mathrm{SO}_r}$ denote the moduli space of topologically trivial oriented orthogonal bundles of rank r over a smooth projective complex curve C of genus $g \geq 2$ and let $\mathcal{L}^+_{\mathrm{SO}_r}$ denote the determinant line bundle over $\mathcal{M}^+_{\mathrm{SO}_r}$. It was shown in [B2] that the

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linear system $|\mathcal{L}^+_{\mathrm{SO}_r}|$ identifies canonically with the dual $|r\Theta|^*_+$ of even theta functions of order r over the Picard variety parametrizing degree $g-1$ line bundles over C. In particular, for $r=4$ we observe that $\dim H^0(\mathcal{M}^+_{\mathrm{SO}_4}, \mathcal{L}^+_{\mathrm{SO}_4}) = d_+$. Moreover we can associate to any even theta-characteristic $\kappa \in \mathcal{K}_+$ a proper effective divisor $\Delta_\kappa \in |\mathcal{L}^+_{\mathrm{SO}_4}|$ with support

$$\{E \in \mathcal{M}^+_{\mathrm{SO}_4} \mid \dim H^0(E \otimes \kappa) \neq 0\}.$$

By pulling back the d_+ divisors Δ_κ to the Jacobian $\mathrm{Jac}(C)$ under the map $\mathrm{Jac}(C) \to \mathcal{M}^+_{\mathrm{SO}_4}$, $L \mapsto L \oplus L^{-1} \oplus L \oplus L^{-1}$, we deduce from Theorem 1.2 the following

Corollary 1.3. *Assume that C has no vanishing theta-null. Then the d_+ divisors $\Delta_\kappa \in |\mathcal{L}^+_{\mathrm{SO}_4}|$ for $\kappa \in \mathcal{K}_+$ form a projective basis.*

Finally, we mention that for general r the divisors $\Delta_\kappa \in |\mathcal{L}^+_{\mathrm{SO}_r}|$ — more precisely, some relatives of Δ_κ on the moduli stack of Spin_r-bundles over C — have recently been studied in connection with the strange duality of generalized theta functions [Be].

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2. Review of algebraic theory of theta functions

In this section we recall the results on theta functions of order four, which we need in the proof of the two main theorems. We refer to [M1] and [B1] for details and proofs.

We recall that a theta-characteristic is a map $\kappa : A[2] \to \{\pm 1\}$ such that $\kappa(a+b) = \kappa(a)\kappa(b)\langle a, b\rangle$, where $\langle \ , \ \rangle$ is the symplectic Weil form on $A[2]$. The group $A[2]$ acts transitively on the set $\mathcal{K}$ of theta-characteristics by the formula

$$(a \cdot \kappa)(b) = \langle a, b\rangle \kappa(b). \tag{1}$$

We say that κ is even if κ takes $d_\pm$ times the value ± 1. The set of even theta-characteristics is denoted by $\mathcal{K}_+$.

The Heisenberg group H associated to the line bundle L consists of all pairs $\alpha = (a, \varphi)$ with $a \in A[2]$ and $\varphi : T_a^* L \xrightarrow{\sim} L$ an isomorphism (where $T_a \colon A \to A$ is the translation by a), and fits into an exact sequence

$$0 \longrightarrow k^* \longrightarrow H \stackrel{p}{\longrightarrow} A[2] \longrightarrow 0, \qquad \text{with } p(\alpha) = a.$$

For any $\kappa \in \mathcal{K}$ we consider the character $\chi_\kappa : H \to k^*$ of weight 2 defined by $\chi_\kappa(\alpha) = ||\alpha||\kappa(a)$, with $||\alpha|| = \alpha^2 \in k^*$. The main result we need in this note is

Proposition 2.1 ([B1] Proposition A.8)**.** *The vector space $H^0(A, L^2)_+$ decomposes as an H-module into a direct sum $\bigoplus_{\kappa \in \mathcal{K}_+} W_\kappa$ with $\dim W_\kappa = 1$ and W_κ is the character space associated to the character χ_κ. The zero divisor of any nonzero section $s_\kappa \in W_\kappa$ equals $2_A^*\Theta_\kappa$, where 2_A is the duplication map of A.*

3. The matrix M

We say that a pair $(a_1, a_2) \in (\mathbb{Z}/2\mathbb{Z})^g \times (\mathbb{Z}/2\mathbb{Z})^g$ is even if $a_1 \cdot a_2 = 0$. Let M denote the $d_+ \times d_+$ matrix with lines and columns indexed by even pairs (a_1, a_2) and (b_1, b_2) and with entries

$$(-1)^{a_1 \cdot b_2 + a_2 \cdot b_1}.$$

We will use the fact that M is invertible, which has already been proved in [F] Lemma 1.1. For the convenience of the reader, and for later use, we recall here the computations.

Lemma 3.1. *For fixed $(a_1, a_2) \in (\mathbb{Z}/2\mathbb{Z})^g \times (\mathbb{Z}/2\mathbb{Z})^g$, we have the equality*

$$\sum_{\substack{b_1, b_2 \in (\mathbb{Z}/2\mathbb{Z})^g, \\ b_1 \cdot b_2 = 0}} (-1)^{a_1 \cdot b_2 + a_2 \cdot b_1} = \begin{cases} d_+ & \textit{if } \ a_1 = a_2 = 0, \\ (-1)^{a_1 \cdot a_2} 2^{g-1} & \textit{else.} \end{cases}$$

Proof. If $a_1 = a_2 = 0$ there is nothing to prove. Otherwise we may assume that $a_1 \neq 0$, and the result follows from the easy equality

$$\sum_{\substack{b_1 \in (\mathbb{Z}/2\mathbb{Z})^g, \\ b_1 \cdot b_2 = 0}} (-1)^{a_1 \cdot b_1} = \begin{cases} 2^{g-1} & \text{if } b_2 = a_1 \\ 0 & \text{else.} \end{cases}$$

□

Proposition 3.2. *The matrix M is invertible.*

Proof. Using Lemma 3.1, we easily check that the inverse of M is given by

$$M^{-1} = \frac{1}{2^{2g-1}}(M - 2^{g-1} I).$$

□

4. Proof of Theorem 1.1

Consider the linear map given by evaluating the d_+ theta functions $s_\kappa \in H^0(A, L^2)_+$ at the d_+ points $a \in A[2]_+$

$$\mathrm{ev} : H^0(A, L^2)_+ \longrightarrow \bigoplus_{a \in A[2]_+} L_a^2.$$

It is clear that the assertion of Theorem 1.1 is equivalent to ev being an isomorphism. First note that saying that A has a vanishing theta-null means that there exists a section s_κ, with $\kappa \in \mathcal{K}_+$, which vanishes at every point $a \in A[2]$, i.e. $s_\kappa \in \ker \mathrm{ev}$. On the other hand we will show that, if A has no vanishing theta-null, ev is given after suitable normalization by the matrix M. The theorem then follows from Proposition 3.2.

We consider for each $\kappa \in \mathcal{K}_+$ a nonzero section $s_\kappa \in H^0(A, L^2)_+$ in the one-dimensional χ_κ-character space W_κ, and for each $a \in A[2]$ an isomorphism $\phi_a : L_a^2 \xrightarrow{\sim} k$. Since we assume that A has no vanishing theta-null, we have $\phi_a(s_\kappa(a)) \neq 0$ for all $\kappa \in \mathcal{K}_+$ and all $a \in A[2]$. The quotient

$$\mu(a, \kappa, \kappa') = \frac{\phi_0(s_\kappa(0)) \cdot \phi_a(s_{\kappa'}(a))}{\phi_a(s_\kappa(a)) \cdot \phi_0(s_{\kappa'}(0))}, \qquad a \in A[2], \ \kappa, \kappa' \in \mathcal{K}_+$$

does not depend on the choice of the sections $s_\kappa, s_{\kappa'}$ and the isomorphisms ϕ_a. Given $a \in A[2]$ we choose an $\alpha = (a, \varphi) \in H$ such that $\varphi : T_a^* L \xrightarrow{\sim} L$ preserves the isomorphisms ϕ_0 and ϕ_a, which is equivalent to the equality $\phi_0((\alpha.s_\kappa)(0)) = \phi_a(s_\kappa(a))$ for all $\kappa \in \mathcal{K}_+$. On the other hand $\alpha.s_\kappa = \chi_\kappa(\alpha)s_\kappa = ||\alpha||\kappa(a)s_\kappa$, hence $\frac{\phi_a(s_\kappa(a))}{\phi_0(s_\kappa(0))} = ||\alpha||\kappa(a)$. Therefore we obtain the formula

$$\mu(a, \kappa, \kappa') = \kappa(a)\kappa'(a). \tag{2}$$

In order to obtain the matrix M we normalize as follows: consider a section s_{κ_0} corresponding to the fixed even theta-characteristic κ_0, and choose the isomorphisms ϕ_a such that $\phi_a(s_{\kappa_0}(a)) = 1$ for all $a \in A[2]_+$. Then we choose the sections s_κ such that $\phi_0(s_\kappa(0)) = 1$. Any $\kappa \in \mathcal{K}$ can be written $\kappa = b \cdot \kappa_0$ for a unique $b \in A[2]$, and $\kappa \in \mathcal{K}_+$ if and only if $b \in A[2]_+$ by [B1] formula (1) page 279. Using formulae (2) and (1) we obtain the equalities

$$\phi_a(s_{b\cdot\kappa_0}(a)) = \mu(a, \kappa_0, b \cdot \kappa_0) = \kappa_0(a)(b \cdot \kappa_0)(a) = \langle a, b\rangle.$$

We now choose a level-2 structure $\lambda : A[2] \xrightarrow{\sim} (\mathbb{Z}/2\mathbb{Z})^g \times (\mathbb{Z}/2\mathbb{Z})^g$ which maps the set $A[2]_+$ bijectively onto the set of even pairs — this is possible, since $\mathrm{Sp}(2g, \mathbb{Z}/2\mathbb{Z})$ acts transitively on the set of even theta-characteristics. Then we can write $\langle a, b\rangle = (-1)^{a_1\cdot b_2 + a_2 \cdot b_1}$ with $\lambda(a) = (a_1, a_2)$ and $\lambda(b) = (b_1, b_2)$. This finishes the proof.

5. Proof of Theorem 1.2

Let κ_0 be a theta-characteristic of A. Recall from [B1] that the morphism $\delta_{\kappa_0} : A \longrightarrow |L|$ which sends $a \in A$ to the divisor $T_a^*\Theta_{\kappa_0} + T_{-a}^*\Theta_{\kappa_0}$ fits into the commutative diagram

$$\begin{tikzcd} & {|L|^*} \arrow[dd, "\wr"] \\ A \arrow[ur, "\varphi_L"] \arrow[dr, "\delta_{\kappa_0}"'] & \\ & {|L|,} \end{tikzcd}$$

where φ_L is the morphism defined by the complete linear system $|L|$. The isomorphism between $|L|^*$ and $|L|$ is given by any nonzero element in the one-dimensional character space in $H^0(A, L) \otimes H^0(A, L)$ associated to χ_{κ_0}. Now, if A has no vanishing theta-null, we know from [B1] Proposition A.9 that the multiplication map

$$\mathrm{Sym}^2 H^0(A, L) \longrightarrow H^0(A, L^2)_+$$

is bijective. It follows that the duplication morphism $|L| \longrightarrow |L^2|_+, D \longmapsto 2D$ is identified, through the isomorphism $|L|^* \xrightarrow{\sim} |L|$, with the 2-uple embedding of $|L|^*$, and that the morphism $\varphi : A \longrightarrow |L^2|_+^*$ is the composite of φ_L with this 2-uple embedding.

But we know from Theorem 1.1 that the images $\varphi(a)$ of the d_+ points $a \in A[2]_+$ form a projective basis of $|L^2|_+^*$. This implies that the points $2\delta_{\kappa_0}(a) = 4\Theta_{a\cdot\kappa_0}$ form a projective basis of $|L^2|_+$, which finishes the proof.

Remark 5.1. The previous considerations show that our result is also equivalent to the following one: if A has no vanishing theta-null, there is no quadric hypersurface in $|L|$ containing the d_+ points defined by $2\Theta_\kappa$, with $\kappa \in \mathcal{K}_+$.

Remark 5.2. It easily follows from the previous proof that the codimension of the linear span of the fourth powers of even theta functions in $H^0(A, L^2)_+$ equals the number of vanishing theta-nulls of A.

6. An analytic proof of Theorem 1.2

When $k=\mathbb{C}$, we can give a short analytic proof of Theorem 1.2 by using Riemann's quartic addition theorem to express the fourth powers of the theta functions θ_κ in terms of the functions $2_A^*\theta_\kappa$.

In this case, the Abelian variety A is a quotient $\mathbb{C}^g/\Gamma_\tau$ of a g-dimensional vector space by a lattice $\Gamma_\tau=\mathbb{Z}^g\oplus\tau\mathbb{Z}^g$ for some τ in the Siegel upper half-plane. Let us choose a level-2 structure to identify the set of theta-characteristics with $(\mathbb{Z}/2\mathbb{Z})^g\times(\mathbb{Z}/2\mathbb{Z})^g$, in such a way that even theta-characteristics κ correspond to even pairs (a_1,a_2) (see [B1] A.6). We want to prove that the space $H^0(A,L^2)_+$ of even theta functions of order four is spanned by the fourth powers of the theta functions with even characteristics $a_1,a_2\in(\mathbb{Z}/2\mathbb{Z})^g$

$$\theta\begin{bmatrix}a_1\\a_2\end{bmatrix}(z)=\sum_{m\in\mathbb{Z}^g}\exp\pi i\big({}^t(m+\frac{1}{2}\tilde{a_1})\tau(m+\frac{1}{2}\tilde{a_1})+2\,{}^t(m+\frac{1}{2}\tilde{a_1})(z+\frac{1}{2}\tilde{a_2})\big),$$

where $\tilde{a_1}$ and $\tilde{a_2}$ denote some representatives in $\mathbb{Z}^g$ of a_1 and a_2 respectively.

Riemann's quartic addition theorem (see [M2] II.6 formula $(\mathrm{R_{ch}})$) implies that, for any pair (a_1,a_2),

$$\big(\theta\begin{bmatrix}a_1\\a_2\end{bmatrix}(z)\big)^4=2^{-g}\sum_{b_1,b_2\in(\mathbb{Z}/2\mathbb{Z})^g}(-1)^{a_1\cdot b_2+a_2\cdot b_1}\big(\theta\begin{bmatrix}b_1\\b_2\end{bmatrix}(0)\big)^3\theta\begin{bmatrix}b_1\\b_2\end{bmatrix}(2z).$$

We now invert this formula as follows. Let us sum over all even pairs (a_1,a_2) the previous relations multiplied by the factor $(-1)^{a_1\cdot c_2+a_2\cdot c_1}$, where (c_1,c_2) is a fixed even pair. Using Lemma 3.1, we obtain

$$2\sum_{\substack{a_1,a_2\in(\mathbb{Z}/2\mathbb{Z})^g,\\ a_1\cdot a_2=0}}(-1)^{a_1\cdot c_2+a_2\cdot c_1}\big(\theta\begin{bmatrix}a_1\\a_2\end{bmatrix}(z)\big)^4=$$

$$2^g\big(\theta\begin{bmatrix}c_1\\c_2\end{bmatrix}(0)\big)^3\theta\begin{bmatrix}c_1\\c_2\end{bmatrix}(2z)+\sum_{b_1,b_2\in(\mathbb{Z}/2\mathbb{Z})^g}(-1)^{(b_2+c_2)\cdot(b_1+c_1)}\big(\theta\begin{bmatrix}b_1\\b_2\end{bmatrix}(0)\big)^3\theta\begin{bmatrix}b_1\\b_2\end{bmatrix}(2z).$$

In the last sum, the terms corresponding to odd pairs (b_1,b_2) vanish, because $\theta\begin{bmatrix}b_1\\b_2\end{bmatrix}(0)=0$. Since (c_1,c_2) is an even pair, this sum is equal to $2^g\big(\theta\begin{bmatrix}c_1\\c_2\end{bmatrix}(z)\big)^4$, again by Riemann's quartic relation. We finally get, for every even pair (c_1,c_2),

$$2^g\big(\theta\begin{bmatrix}c_1\\c_2\end{bmatrix}(0)\big)^3\theta\begin{bmatrix}c_1\\c_2\end{bmatrix}(2z)=-2^g\big(\theta\begin{bmatrix}c_1\\c_2\end{bmatrix}(z)\big)^4+2\sum_{\substack{a_1,a_2\in(\mathbb{Z}/2\mathbb{Z})^g,\\ a_1\cdot a_2=0}}(-1)^{a_1\cdot c_2+a_2\cdot c_1}\big(\theta\begin{bmatrix}a_1\\a_2\end{bmatrix}(z)\big)^4.$$

But we have already recalled in Proposition 2.1 that the functions $z\mapsto\theta\begin{bmatrix}c_1\\c_2\end{bmatrix}(2z)$ with $c_1\cdot c_2=0$ span $H^0(A,L^2)_+$. This shows that the functions $\big(\theta\begin{bmatrix}a_1\\a_2\end{bmatrix}(z)\big)^4$ with $a_1\cdot a_2=0$ span this vector space if and only if A does not have any vanishing theta-null.

Remark 6.1. It is possible to make this analytic proof work in the algebraic set-up [M1] of theta functions over any algebraically closed field of characteristic different from two. However, its algebraic version would be longer and more technical than the proof we gave in section 5.

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