UNIVERSITÉ DE NICE–SOPHIA ANTIPOLIS – UFR Sciences

École Doctorale Sciences Fondamentales et Appliquées

THÈSE

pour obtenir le titre de

Docteur en Sciences

de l'UNIVERSITÉ de Nice–Sophia Antipolis

Spécialité : Mathématiques

présentée et soutenue par

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ESPACES DE MODULES DE FIBRÉS ORTHOGONAUX SUR UNE COURBE ALGÉBRIQUE

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soutenue le 11 décembre 2007

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Je tiens à remercier en premier lieu mon directeur de thèse Arnaud Beauville pour le passionnant sujet qu'il m'a suggéré, pour sa patience, pour l'encourageant enthousiasme qu'il a manifesté aux premiers résultats que j'ai rédigés, et bien sûr pour tous les précieux conseils qu'il aura su me donner tout au long de la préparation de cette thèse.

Je suis très honoré que Christian Pauly et Christoph Sorger aient accepté de rapporter cette thèse. Je suis par ailleurs très flatté de l'attention qu'ils avaient déjà portée à ce travail : je remercie Christian de m'avoir invité à Montpellier, ce qui m'a donné l'occasion d'améliorer grandement la présentation de la première partie de cette thèse ; et je veux remercier Christoph pour sa disponibilité, sa générosité, et plus particulièrement pour les conseils et les encouragements qu'il m'a prodigués à Luminy à l'automne 2005 (ce sont ces encouragements, joints à ceux de Joseph Le Potier, qui m'ont incité à poursuivre le calcul d'invariants dans lequel j'hésitais à me plonger).

Je garde un très bon souvenir du cours de géométrie algébrique dispensé par Yves Laszlo que j'ai eu l'opportunité de suivre il y a quelques années. Je suis très heureux qu'il ait pris le temps de venir assister à ma thèse.

C'est un plaisir de remercier André Hirschowitz et Carlos Simpson d'avoir accepté de compléter ce jury de thèse. Je remercie aussi Carlos de l'intérêt qu'il a montré pour mon travail, y compris pour la partie la plus technique.

J'ai eu la chance de rencontrer C. S. Seshadri et Mihnea Popa, dont la disponibilité et la gentillesse m'ont impressioné. Ces trop rares rencontres ont été pour moi une excellente source de motivation.

Je voudrais aussi exprimer ici ma profonde gratitude envers Luc Illusie : c'est son cours de maîtrise qui m'a conduit à étudier la géométrie algébrique, et les nombreux conseils qu'il m'a donnés par la suite m'ont tous été très profitables.

Je ne manquerais pas de remercier également Samuel Boissière, ne serait-ce que pour tout le temps qu'il a du me consacrer. J'ai ainsi maintes fois profité de son écoute, de ses remarques, ainsi que de son aide quant à l'utilisation de logiciels de calcul qui m'étaient alors totalement inconnus.

Je remercie l'ensemble du laboratoire pour son accueil et sa gentillesse, et plus spécialement Cécile, Fernande, Isabelle, Janine, Jean-Louis et Marie-Christine. Sans oublier Jean-Marc qui se sera montré bien patient avec moi.

Merci aussi aux thésards (et post-docs) que j'ai rencontrés durant ces années passées à Nice : Alessandro, Angela, Baohua, Delphine, Jérémy, Marc, Marie, Patrick, Pierre, Stéphane, Xavier... et en particulier à Marcello avec qui j'ai agréablement partagé le bureau depuis septembre 2004 et à Fabien pour son inébranlable bonne humeur et tous ses bons conseils.

Un grand merci à mes parents, ma sœur, et mes grands-parents, pour m'avoir toujours soutenu après m'avoir offert des conditions plus que favorables pour me lancer dans les études que j'ai poursuivies. Merci aussi à ma belle-famille, tout aussi prompte à m'encourager.

Tout ceci doit beaucoup à la présence à mes côtés de Fanny, sans qui ces neuf dernières années auraient été à bien des moments autrement plus difficiles.

Olivier Serman

ESPACES DE MODULES DE FIBRÉS ORTHOGONAUX SUR UNE COURBE ALGÉBRIQUE

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***Classification mathématique par sujets* (2000). —** Primaire 14H60 ; Secondaire 14F05,14L30,16G20,14K25.

***Mots clefs.* —** Schémas de modules, fibrés principaux, fibrés orthogonaux, fonctions thêta généralisées.

ESPACES DE MODULES DE FIBRÉS ORTHOGONAUX SUR UNE COURBE ALGÉBRIQUE

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Résumé. — On étudie dans cette thèse les espaces de modules de fibrés orthogonaux sur une courbe algébrique lisse.

On montre dans un premier temps que le morphisme d'oubli associant à un fibré orthogonal le fibré vectoriel sous-jacent est une immersion fermée : ce résultat repose sur un calcul d'invariants sur les espaces de représentations de certains carquois.

On présente ensuite, pour les fibrés orthogonaux de rang 3 et 4, des résultats plus concrets sur la géométrie de ces espaces, en accordant une attention particulière à l'application thêta.

Abstract **(Moduli schemes of orthogonal bundles over an algebraic curve)**

We study in this thesis the moduli schemes of orthogonal bundles over an algebraic smooth curve.

We first show that the forgetful morphism from the moduli space of orthogonal bundles to the moduli space of all vector bundles is a closed immersion: this relies on an explicit description of a set of generators for the invariants on the representation spaces of some quivers.

We then give, for orthogonal bundles of rank 3 and 4, some more concrete results about the geometry of these varieties, with a special attention towards the theta map.

TABLE DES MATIÈRES

INTRODUCTION

L'objet de cette thèse est l'étude des espaces de modules de fibrés orthogonaux définis sur une courbe algébrique complexe C. Ces espaces ont été construits en 1976 dans sa thèse par Ramanathan, qui a en fait plus généralement établi l'existence d'espaces de modules grossier $\mathcal{M}_G$ paramétrant les fibrés G-principaux sur C pour tout groupe algébrique G semi-simple connexe.

0.1. Espaces de modules de fibrés vectoriels

Dans le cas $G = \mathbf{SL}_r$, on retrouve l'espace de modules $\mathcal{SU}_C(r)$ des fibrés vectoriels de rang r et de déterminant trivial, dont la construction (menée à bien grâce à des résultats fondateurs de Mumford, Narasimhan et Seshadri, pour ne citer qu'eux) a constitué l'une des premières grandes réalisations de la théorie de la géométrie invariante (qu'elle a d'ailleurs motivée). La géométrie de ces espaces (qui fournissent des exemples non triviaux de variétés de grande dimension) a fait depuis lors l'objet de nombreux travaux.

L'un des plus remarquables concerne sans aucun doute l'étude des sections de leurs fibrés en droites, qui a tout récemment culminé avec la vérification par Marian et Oprea de la *dualité étrange* (signalons au passage la preuve obtenue indépendamment par Belkale, cf. [**Bel**]). Rappelons rapidement de quoi il s'agit (on renvoie à l'excellent et toujours d'actualité [**Bea95**] pour les détails) : Drézet et Narasimhan ont montré que le groupe de Picard $\mathrm{Pic}(\mathcal{SU}_C(r))$ admet pour générateur ample le *fibré déterminant* $\mathcal{L}$, défini par le diviseur thêta généralisé

$$\Theta_L = \{E | h^0(C, E \otimes L) \neq 0\}$$

associé à un fibré $L \in J^{g-1}$. L'étude des plongements projectifs de $\mathcal{SU}_C(r)$ passe donc par une bonne description des sections des puissances $\mathcal{L}^k$ de ce fibré déterminant. La dimension de ces espaces a été prédite par la *formule de Verlinde* (prouvée indépendamment par Beauville–Laszlo et par Faltings).

Le cas $k = 1$ avait été obtenu plus tôt par Beauville, Narasimhan et Ramanan, comme conséquence de l'existence d'une dualité naturelle entre $\mathrm{H}^0(\mathcal{SU}_C(r), \mathcal{L})$ et $\mathrm{H}^0(J^{g-1}, \mathcal{O}(r\Theta))$, où Θ désigne le diviseur thêta canonique sur J^{g-1}. Cette dualité se réalise comme cas particulier de la construction suivante : considérons l'application induite par le produit tensoriel

$$\mathcal{SU}_C(r) \times \mathcal{U}_C(k, k(g-1)) \longrightarrow \mathcal{U}_C(rk, rk(g-1)),$$

où $\mathcal{U}_C(k, d)$ désigne l'espace des fibrés vectoriels de rang k et degré d. L'image inverse par ce morphisme du diviseur

$$\Theta_{rk} = \{E \in \mathcal{U}_C(rk, rk(g-1)) | h^0(C, E) \neq 0\}$$

définit un élément de $\mathrm{H}^0(\mathcal{SU}_C(r), \mathcal{L}^k) \otimes \mathrm{H}^0(\mathcal{U}_C(k, k(g-1)), \mathcal{O}(r\Theta_k))$. La dualité étrange affirme que la dualité correspondante est parfaite (les deux preuves citées plus haut utilisent la formule de Verlinde).

Cette dualité vient avec une description particulièrement intéressante des applications $\mathcal{SU}_C(r) \dashrightarrow |\mathcal{L}^k|^*$; en particulier, on obtient pour $k = 1$ une identification naturelle entre $\varphi_{\mathcal{L}} \colon \mathcal{SU}_C(r) \dashrightarrow |\mathcal{L}|^*$ et l'*application thêta*

$$\begin{aligned} \mathcal{SU}_C(r) &\dashrightarrow |r\Theta| \\ E &\longmapsto \Theta_E \end{aligned}$$

où $\Theta_E = \{L \in J^{g-1} | h^0(C, E \otimes L) \neq 0\}$ est, pour E générique, un diviseur linéairement équivalent à $r\Theta$.

Cette description de $\varphi_{\mathcal{L}}$ fournit, comme l'avaient déjà remarqué Narasimhan et Ramanan, une approche privilégiée pour l'étude explicite de ces espaces en petit rang et petit genre. On peut citer ici (chronologiquement) les résultats suivants, qui font apparaître dans ce contexte des variétés connues des géomètres classiques :

– en genre 2, $\theta \colon \mathcal{SU}_C(2) \longrightarrow |2\Theta| \simeq \mathbb{P}^3$ est un isomorphisme (voir [**NR69**]),

– en genre 3, $\theta \colon \mathcal{SU}_C(2) \longrightarrow |2\Theta| \simeq \mathbb{P}^7$ est, pour une courbe non hyperelliptique, une immersion fermée sur une quartique, qui se trouve être la *quartique de Coble*, i.e. l'unique quartique $\mathcal{Q} \subset |2\Theta|$ singulière le long de l'image de l'application de Kummer $\alpha \in J \longmapsto \Theta_\alpha + \Theta_{-\alpha} \in |2\Theta|$ (voir [**NR87**]),

– pour toute courbe non hyperelliptique, $\theta \colon \mathcal{SU}_C(2) \longrightarrow |2\Theta|$ est une immersion fermée (voir [**BV96**] et [**vGI01**]),

– en genre 2, $\theta \colon \mathcal{SU}_C(3) \longrightarrow |3\Theta| \simeq \mathbb{P}^8$ est un morphisme de degré 2, ramifié le long d'une sextique $\mathcal{S}$, dont l'hypersurface duale dans $|3\Theta|^*$ est la *cubique de Coble*, i.e. l'unique cubique $\mathcal{C}$ singulière le long de l'image du plongement $J^1 \longrightarrow |3\Theta|^*$ (voir [**Ort05**], ou [**Ngu07**]).

Il semble à l'heure actuelle difficile d'obtenir de nouveaux résultats dans ce sens, même lorsque θ est un morphisme. Aussi est-il naturel d'entreprendre l'étude des espaces de modules de fibrés G-principaux, en confrontant les résultats obtenus aux résultats connus relatifs au cas des fibrés vectoriels.

0.2. Espaces de modules de fibrés G-principaux

On dispose maintenant de nombreux résultats relatifs aux espaces de modules $\mathcal{M}_G$ de fibrés G-principaux. Les groupes de Picard de leurs composantes connexes ont été décrits par Beauville, Laszlo et Sorger, ainsi que par Kumar et Narasimhan : ces groupes sont, au moins lorsque G est presque simple, encore cycliques, engendrés par un fibré en droites ample $\mathcal{L}_G$, et ce générateur peut en fait s'obtenir comme image inverse par un morphisme $\mathcal{M}_G \longrightarrow \mathcal{M}_{\mathbf{SL}_r}$ (associé à une représentation $G \longrightarrow \mathbf{SL}_r$) du fibré déterminant sur $\mathcal{M}_{\mathbf{SL}_r}$ (il faut tout de même mentionner ici une différence majeure : ces variétés ne sont en général plus localement factorielles). On est ainsi à nouveau amené à étudier les espaces de sections des puissances $\mathcal{L}_G^k$. Leur dimension est encore donnée par la formule de Verlinde.

Les groupes classiques ont naturellement fait l'objet d'une attention particulière. On peut notamment citer les travaux sur $\mathcal{M}_{\mathbf{Spin}_r}$ menés par Oxbury, Pauly et Ramanan, qui mettent en dualité l'espace des sections de $\mathcal{L}_{\mathbf{Spin}_r}$ et la somme directe des espaces de fonctions thêta paires de niveau r sur toutes les variétés de Prym associées à C. Les espaces de modules de fibrés symplectiques ont quant à eux été récemment étudiés par Hitching dans sa thèse.

Les fibrés orthogonaux ont en fait été les premiers à être considérés, ce qui peut au premier abord paraître (un peu) étonnant, dans la mesure où $\mathbf{SO}_r$ n'est ni simplement connexe, ni spécial au sens de Serre. C'est en réponse à la question de savoir si la variété M_d des sous-espaces linéaires de $\mathbb{P}^{2g+1}$ de dimension d contenues dans le lieu de base d'un pinceau générique de quadriques admettait une interprétation naturelle en termes de fibrés sur la courbe hyperelliptique C associée à ce pinceau[(1)] que Ramanan a été conduit à décrire dans [**Ram81**] l'espace de modules des fibrés orthogonaux sur une courbe hyperelliptique munis d'un relèvement de l'involution hyperelliptique (voir aussi [**Bho84**]).

L'étude algébrique des thêta-caractéristiques sur une courbe lisse C a été menée par Mumford en partant de l'étude des familles de fibrés vectoriels sur C munis d'une forme quadratique à valeurs dans le fibré canonique K_C. Dans sa thèse, Sorger a généralisé cette étude aux courbes (non nécessairement lisses)

[(1)]On savait déjà alors que M_{g-1} s'identifie à la jacobienne de la courbe, et que M_{g-2} est isomorphe à l'espace $\mathcal{SU}_C(2,1)$ des fibrés vectoriels sur C de rang 2 et de déterminant fixé de degré 1.

tracées sur une surface, en construisant au passage, pour toute courbe de Gorenstein, des espaces de modules de faisceaux quadratiques (semi-stables) de multiplicité r.

Plus récemment, Beauville a étendu au cas des fibrés orthogonaux le résultat de [**BNR89**] : plus précisement, la restriction de l'application thêta $\mathcal{M}_{\mathbf{SL}_r} \dashrightarrow |r\Theta|$ définie plus haut à l'espace de modules des fibrés orthogonaux est exactement l'application rationnelle $\mathcal{M}_{\mathbf{SO}_r} \dashrightarrow |\mathcal{L}_{\mathbf{SO}_r}|^*$ définie par le générateur ample du groupe de Picard de $\mathcal{M}_{\mathbf{SO}_r}$.

0.3. Organisation de la thèse

Le premier chapitre rassemble quelques rappels (et donc, au passage, les notations utilisées tout au long du texte) concernant les espaces de modules de fibrés principaux sur une courbe.

On étudie dans la deuxième partie l'application naturelle reliant $\mathcal{M}_{\mathbf{SO}_r}$ et $\mathcal{SU}_C(r)$. On y établit le résultat suivant :

Théorème. — *Le morphisme d'oubli $\mathcal{M}_{\mathbf{O}_r} \longrightarrow \mathcal{M}_{\mathbf{GL}_r}$ est une immersion fermée.*

Ce résultat, évident sur l'ouvert de stabilité régulière, requiert aux autres points un calcul d'invariants relativement sophistiqué, mené en 2.2 : il s'agit de comprendre les fonctions polynomiales sur l'espace des représentations d'un carquois invariantes sous l'action d'un produit de groupes classiques. On obtient ce résultat en adaptant la démarche adoptée par Le Bruyn et Procesi pour résoudre le cas de l'action d'un produit de groupes linéaires.

Bien entendu, la même preuve s'applique au morphisme $\mathcal{M}_{\mathbf{Sp}_{2r}} \longrightarrow \mathcal{M}_{\mathbf{SL}_{2r}}$, qui est ainsi lui aussi une immersion fermée.

Pour décrire l'application $\mathcal{M}_{\mathbf{SO}_r} \longrightarrow \mathcal{M}_{\mathbf{SL}_r}$ il ne reste plus qu'à étudier l'application $\mathcal{M}_{\mathbf{SO}_r} \longrightarrow \mathcal{M}_{\mathbf{O}_r}$ oubliant l'orientation. Lorsque r est impair, c'est tout simplement un isomorphisme sur son image. En revanche, lorsque r est pair, c'est un morphisme de degré 2 (sur son image) : un fibré orthogonal (unitaire) de rang pair général admet deux orientations non-équivalentes.

La troisième partie rassemble quelques éléments relatifs à la structure locale des espaces de modules de fibrés sur C : la traduction du problème d'invariant donné par le théorème des slices étales de Luna en termes de représentations de carquois suggère en effet de chercher du côté des travaux concernant les algèbres de fonctions invariantes sur les espaces de représentations de certains carquois. Néanmoins, on ne dispose de descriptions complètes que dans très peu de cas, et de tels résultats semblent de plus en plus difficiles à obtenir. On décrit ainsi dans cette partie la structure locale de $\mathcal{M}_{\mathbf{O}_r}$ aux points où c'est le plus faisable, puis on complète l'étude de $\mathcal{SU}_C(3)$ pour une courbe de genre 2 commencée par Laszlo : c'est, en rang supérieur à 3, le seul cas où l'on peut aujourd'hui obtenir, par cette méthode, une description locale exhaustive de $\mathcal{SU}_C(r)$ en termes de générateurs et relations d'un modèle affine[(2)]. Cette étude point par point donne le résultat suivant :

Théorème. — *L'espace* $\mathcal{SU}_C(3)$ *des fibrés vectoriels de rang* 3 *sur une courbe de genre* 2 *est localement intersection complète.*

Les quatrième et cinquième parties sont consacrées à l'étude explicite des espaces de modules de fibrés orthogonaux en rang 3 et 4, en accordant une attention particulière à l'application thêta. On utilise constamment les isomorphismes exceptionnels $\mathbf{Spin}_3 \simeq \mathbf{SL}_2$ et $\mathbf{Spin}_4 \simeq \mathbf{SL}_2 \times \mathbf{SL}_2$.

Il résulte de [**Ray82**] que l'application thêta $\mathcal{M}_{\mathbf{SL}_3} \dashrightarrow |3\Theta|$ est un morphisme pour une courbe générique. On vérifie dans la quatrième partie que sa restriction à $\mathcal{M}_{\mathbf{SO}_3}$ est toujours un morphisme. Pour une courbe de genre 2, on obtient une description assez précise de chacune des composantes connexes de $\mathcal{M}_{\mathbf{SO}_3}$ (l'essentiel de cette étude étant une reformulation des résultats contenus dans [**NR03**]).

Le résultat le plus inattendu de la cinquième partie concerne la composante connexe $\mathcal{M}^+_{\mathbf{SO}_4}$ de $\mathcal{M}_{\mathbf{SO}_4}$ constituée des fibrés topologiquement triviaux.

Théorème. — *En rang* 4*, l'application thêta* $\mathcal{M}^+_{\mathbf{SO}_4} \longrightarrow |4\Theta|$ *n'a pas de point de base.*

Au contraire, un argument (dû à Beauville) montre que, en genre 2, dix des seize points de base dans $\mathcal{M}_{\mathbf{SL}_4}$ construits par Raynaud sont naturellement munis d'une structure quadratique (tandis que les six autres portent une structure symplectique). Pauly a récemment montré qu'il n'y avait pas d'autre fibré de rang 4 (de déterminant trivial) sans diviseur thêta. Ce résultat permet en particulier de déterminer le lieu de base de $\theta_4\colon \mathcal{M}_{\mathbf{SO}_4} \dashrightarrow |4\Theta|$. On retrouve ici ce résultat, en donnant une description directe du lieu de base de la restriction de l'application thêta à la composante connexe $\mathcal{M}^-_{\mathbf{SO}_4}$.

Le théorème précédent implique que l'application thêta $\mathcal{M}^+_{\mathbf{SO}_4} \longrightarrow |4\Theta|$ est un morphisme fini.

(2) La récente prébublication Drensky-La Scala (`arXiv :0708.3583`) montre que l'on ne comprend à l'heure actuelle que les éléments de degré minimal de l'idéal de relations sur lequel repose l'étude de la structure locale de $\mathcal{SU}_C(4)$ pour une courbe de genre 2 au voisinage du fibré trivial, tandis que Benanti-Drensky, C. R. Acad. Bulg. Sci., **60** (2007), indique que l'on est confronté au même problème lorsque l'on considère le voisinage du fibré trivial dans $\mathcal{SU}_C(3)$ pour une courbe de genre 3.

Théorème. — *Si C est une courbe de genre* 2*, l'application thêta* $\mathcal{M}^{+}_{\mathbf{SO}_4} \longrightarrow |4\Theta|$ *est un morphisme fini sur son image de degré générique* 2*. Son image est une sous-variété de dimension* 6 *et de degré* 40*.*

L'appendice A présente la description des espaces de modules $\mathcal{M}_{\mathbf{SO}_r}$ pour une courbe elliptique. L'appendice B contient l'essentiel de ce dont on a eu besoin au sujet des variétés abéliennes et des représentations de Heisenberg. L'appendice C rappelle quant à lui les quelques éléments de cohomologie non-abélienne dont on a eu l'usage au cours du texte. Si l'on s'appuie bien sûr sur la référence [**Gir71**], ces résultats sont en fait essentiellement contenus dans le mémoire de Frenkel, Bull. Soc. Math. France **85** (ou encore dans le cours de Serre « Cohomologie galoisienne »).

CHAPTER 1

PRELIMINARIES

Throughout this thesis we fix a smooth projective curve C of genus g ($\geqslant 2$, except in Appendix A) defined over an algebraically closed field k of characteristic zero. All varieties are defined over k.

We will denote by J^d the Jacobian variety parametrizing line bundles of degree d on C. Two values of d deserve a special attention: when $d = 0$, we will write J instead of J^0; when $d = g-1$, the variety J^{g-1} contains the canonical theta divisor Θ, whose support consists of all line bundles having non-zero sections. Some other notations and facts about Jacobian varieties (used in Chapter 4 and Chapter 5) have been compiled in Appendix B.

1.1. Principal G-bundles

This section is aimed to collect rather quickly some old and well-known material. Let us begin with some basic facts about G-bundles: the best reference for the notion of principal bundles remains Serre's seminar report [**Ser**]. We briefly recall from it the definitions and basic constructions.

Let G be a reductive algebraic group.

Definition 1.1.1 **(Principal G-bundle)**. — A principal G-bundle $P \to S$ over a variety S is a variety with a right action of G locally trivial in the étale topology: S admits an étale cover $f\colon Y \to S$ such that f^*P is isomorphic to the trivial G-bundle $Y \times G$.

There is a natural way to associate to any G-bundle P on S and any quasi-projective variety F acted on by G a fiber bundle over S:

***Definition 1.1.2* (Associated fiber bundle).** — If G acts (from the left) on a quasi-projective variety F, we define $P(F) = P \times^G F$ to be the quotient $(P \times F)/G$, where G acts on $P \times F$ by $(p, f) \cdot g = (p \cdot g, g^{-1} \cdot f)$ (it is the unique variety Q over which $P \times F$ is a G-bundle).

In particular, if $\rho\colon G \to G'$ is a morphism, the associated bundle $P(G')$ is naturally a G'-bundle: this is the *extension of structure group* of P from G to G', which will be also denoted by $\rho_* P$.

Conversely, if H is a subgroup G, we call *reduction of structure group* of P from G to H a H-bundle Q together with an (G-)isomorphism $Q(H) \to P$.

Lemma 1.1.3. — *If P is a G-bundle, reductions of structure group of P from G to a subgroup H are in a one-to-one correspondence with sections over X of the associated fiber bundle $P/H = P \times^G G/H$. Two sections give isomorphic reductions of structure group if and only if they differ by a G-automorphism of P.*

The H-bundle associated to a G-bundle P and a section $\sigma\colon X \longrightarrow P/H$ is the pull-back (via σ) of P viewed as an H-bundle over P/H. We will thus denote it by $\sigma^* P$.

Example 1.1.4. — We review here what principal bundles are for classical groups.

(i) Giving a $\mathbf{GL}_r$-bundle P is exactly giving the associated rank r vector bundle $E = P(k^r)$, and $\mathbf{SL}_r$-bundles correspond to vector bundles with trivial determinant. A $\mathbf{PGL}_r$-bundle corresponds to a Severi-Brauer variety (this follows from the fact that $\mathbf{PGL}_r$ is the group of automorphisms of the projective space $\mathbb{P}^{r-1}$).

(ii) In the case we are studying, we find that $\mathbf{O}_r$-bundles are precisely vector bundles E over S of rank r endowed with a non degenerate quadratic form $q\colon E \to \mathcal{O}_S$ (which may advantageously be thought of as a symmetric isomorphism $E \to E^*$); we will therefore call them *orthogonal bundles.* An $\mathbf{SO}_r$-bundle P is then an *oriented orthogonal bundle* $P = (E, q, \omega)$, the orientation ω coming as a section of $H^0(S, \det E)$ whose square is equal to 1 (which means that $\tilde{q}(\omega) = 1$, where $\tilde{q}$ is the quadratic form on $\det E$ induced by q).

(iii) In the same way, for even r, $\mathbf{Sp}_r$-bundles are *symplectic bundles*, that is vector bundles E with a symplectic form $E \otimes E \longrightarrow \mathcal{O}_S$ (or, equivalently, with an antisymmetric isomorphism $E \longrightarrow E^*$).

1.2. The moduli schemes $\mathcal{M}_G$ over a curve

The problem of classifying principal G-bundles on the curve C has been solved by Ramanathan in his thesis (published twenty years later in [**Ram96**]). We recall here the precise definition of the moduli spaces, and the basic properties of these schemes.

1.2.1. As for vector bundles, the set of isomorphism classes of G-bundles over C is not bounded. To get a moduli space for G-bundles, we thus need to exclude some G-bundles. Ramanathan gave the right notion of semi-stability. This is done by considerating the reductions of structure group of P to the parabolic subgroups of G.

Definition 1.2.2 **(Ramanathan)**. — A G-bundle P over C is *semi-stable* (resp. *stable*) if, for every parabolic subgroup $H \subset G$, for every non trivial character χ of H, and for every H-bundle P' whose associated G-bundle $P'(G)$ is isomorphic to P, the line bundle $\chi_* P'$ satisfies $\deg(\chi_* P') \leqslant 0$ (resp. < 0).

When $G = \mathbf{SL}_r$, this gives the classical notion of semi-stability for vector bundles: a rank r vector bundle E of degree d is semi-stable (resp. stable) if and only if

$$\mu(F) = \deg(F)/\operatorname{rk}(F) \leqslant \mu = d/r$$

(resp. $<$) for every proper subbundle $F \subset E$. When G is the orthogonal group, the situation remains as simple: an (oriented) orthogonal bundle is semi-stable if and only if the preceding slope inequality holds for all *isotropic* subbundle F (see [**Ram75**, Remark 3.1]). The correspondence between semi-stability of an orthogonal bundle and semi-stability of its underlying vector bundle is as good as we can hope:

Proposition 1.2.3 **(Ramanan)**. — *An orthogonal bundle $P = (E, \sigma)$ is semi-stable if and only if its underlying vector bundle E is semi-stable. It is stable if and only if E is the direct sum of some mutually non-isomorphic stable bundles.*

Note that, in the general case, Ramanathan has proved that a G-bundle P is semi-stable if and only if its adjoint (vector) bundle $P \times^G \mathfrak{g}$ is semi-stable.

1.2.4. We can now define what the coarse moduli scheme $\mathcal{M}_G$ is. Let us consider the functor F_G which associates to any scheme S the set of isomorphism classes of G-bundles $\mathcal{P}$ over $S \times C$ such that, for every $s \in S$, the fiber $\mathcal{P}_s$ is a semi-stable G-bundle over C.

Ramanathan has shown how to construct a coarse moduli scheme for this functor. Recall first that in the vector bundle case, there is a coarse moduli scheme (which is not projective when rank and degree are not coprime) for stable vector bundle. To get a moduli scheme for all semi-stable vector bundles (which is a compactification of the previous one), we need to identify two semi-stable vector bundles when their Jordan-Hölder filtrations have the same associated graded objects. Let us see how Ramanathan generalized this equivalence relation to the case of G-bundles.

***Definition 1.2.5* (Ramanathan).** — If P is a G-bundle on C, a reduction of structure group P' to a parabolic subgroup $H \subset G$ is *admissible* if, for any character χ on H which is trivial on the (neutral component of the) center of G, the line bundle $\chi_* P'$ has degree zero.

***Example 1.2.6*.** — For $G = \mathbf{GL}_r$, admissible reductions of structure group of a vector bundle E correspond to filtrations $0 \subset E_1 \subset \cdots \subset E_l = E$ with $\mu(E_i/E_{i-1}) = \mu(E)$ (see [**Ram96**, Remark 3.4]). Jordan-Hölder filtrations of E are thus "maximal" admissible reductions of structure group of E (note that this means that the corresponding parabolic subgroup is minimal among parabolic subgroups admitting an admissible reduction).

The corresponding assertion for orthogonal bundles will be explained in the next chapter.

***Proposition 1.2.7* (Ramanathan).** — *Let P be a semi-stable G-bundle on C. There exists an admissible reduction of structure group σ of P to a parabolic subgroup H such that, if $p\colon H \longrightarrow L$ is the projection onto a maximal reductive subgroup L of H, the L-bundle $p_*\sigma^*P$ is stable. The G-bundle $j_*p_*\sigma^*P$ (where j is the inclusion $L \subset G$) depends only on P: we denote it* $\mathrm{gr}P$.

1.2.8. Let us check that, when $G = \mathbf{GL}_r$, we indeed recover the Jordan-Hölder graded object. We have just recalled that an admissible reduction σ of structure group to a parabolic subgroup H associated to a flag $0 \subset V_1 \subset \cdots \subset V_l = k^r$ is admissible if and only if the vector bundles E_i/E_{i-1} have slope $\mu(E)$ (where $E_i = \sigma^*(E)(V_i)$). What does the stability of $p_*\sigma^*P$ mean? If we choose for the projection onto a maximal reductive subgroup the morphism $\mathbf{GL}_r \longrightarrow L = \prod_i \mathbf{GL}(V_i/V_{i-1})$, the L-bundle $p_*\sigma^*P$ is the product of the $\mathbf{GL}(V_i/V_{i-1})$-bundles E_i/E_{i-1}. The next lemma shows that this L-bundle is stable if and only if every E_i/E_{i-1} is a stable vector bundle, which occurs exactly for a Jordan-Hölder filtration (and gr$E$ is therefore the usual Jordan-Hölder graded object, as expected).

Lemma 1.2.9. — *Let $(E_i)_{i=1,2}$ be two semi-stable (resp. stable) G_i-bundles. Then the $G_1\times G_2$-bundle $E_1\times_C E_2$ is also semi-stable (resp. stable). Moreover, if E_1 or E_2 is strictly semi-stable, then the product $E_1\times_C E_2$ is strictly semi-stable.*

According to [**Bor91**, 11.14 (1)] any parabolic subgroup of $G_1\times G_2$ is a product $\Gamma=\Gamma_1\times\Gamma_2$ where Γ_i is a parabolic subgroup of G_i. It follows that maximal proper parabolic subgroups may be written $\Gamma_1\times G_2$ or $G_1\times\Gamma_2$, Γ_i being a maximal proper parabolic subgroup of G_i. Therefore the associated bundle E/Γ is isomorphic to E_1/Γ_1 or E_2/Γ_2, and the lemma is a consequence of the very definition of stability.

1.2.10. It can be shown that P and gr$P$ both appear in a one-parameter family $\mathcal{P}$ over $\mathbb{A}^1\times C$ such that $\mathcal{P}_t$ is equal to P when $t\neq 0$ and to gr$P$ when $t=0$. In order to construct a moduli scheme for semi-stable G-bundles, we therefore need to identify these two bundles: Ramanathan defined two G-bundles to be equivalent if they share the same associated graded bundle. The main result of [**Ram96**] can now be recalled.

Theorem 1.2.11 **(Ramanathan)**. — *The functor F_G has a coarse moduli space $\mathcal{M}_G$, whose closed points correspond bijectively to equivalence classes of semi-stable G-bundles over C.*

Remark 1.2.12. — The existence of these moduli spaces has been since then proved for higher dimensional base varieties in arbitrary characteristic (see [**GLSS05**]).

1.2.13. Let us briefly recall the main lines of the construction of $\mathcal{M}_G$ (following [**BLS98**] or [**BS02**]). We fix a faithful representation $\rho\colon G\longrightarrow \mathbf{SL}_N$ and an integer M such that, for every G-bundle P, the rank N vector bundle $\rho_*P\otimes\mathcal{O}_C(M)$ is generated by its global sections and satisfies $\mathrm{H}^1(C,\rho_*P\otimes\mathcal{O}_C(M))=0$. This allows us to consider the functor $\underline{R}_G$ which associate to a scheme S the set of classes of pairs $(\mathcal{P},\alpha)$ consisting of a G-bundle $\mathcal{P}$ over $S\times C$ (with semi-stable fibers) together with an isomorphism $\alpha\colon\mathcal{O}_S^\chi\xrightarrow{\sim} p_{S*}(\rho_*P\otimes\mathcal{O}_C(M))$ (where $\chi=N(M+1-g)$). This functor, which is introduced to relate G-bundles to vector bundles, is representable by a *smooth* scheme R_G, which will be refered to as a *parameter scheme*: $\underline{R}_{\mathbf{SL}_N}$ is indeed represented by a locally closed subscheme of the Hilbert scheme $\mathbf{Quot}^{N,NM}_{\mathcal{O}_C^\chi}$, and, if $(\mathcal{U}, u)$ denotes the universal pair on $R_{\mathbf{SL}_N}$, we see (using Lemma 1.1.3) that R_G is the $R_{\mathbf{SL}_N}$-scheme representing the functor of global sections of $\mathcal{U}/G$.

Simpson's construction presents $\mathcal{M}_{\mathbf{SL}_N}$ as a (good) quotient $R_{\mathbf{SL}_N}/\!\!/\Gamma$ (for sufficiently high M) by the natural action of $\Gamma = \mathbf{GL}_\chi$. This action automatically lifts to R_G, and the structural morphism $R_G \longrightarrow R_{\mathbf{SL}_N}$ is Γ-equivariant. A good quotient $R_G/\!\!/\Gamma$, if it exists, provides the desired coarse moduli space for F_G. According to [**Ram96**, Lemma 5.1], its existence follows from the one of $R_{\mathbf{SL}_N}/\!\!/\Gamma$.

Remark 1.2.14. — As Balaji and Seshadri have noticed, the morphism $\mathcal{M}_G \longrightarrow \mathcal{M}_{\mathbf{SL}_N}$ is finite onto its image. It is therefore natural to ask for more fine information about this morphism. In the next chapter, we answer this question for the standard representations of $\mathbf{SO}_r$ and $\mathbf{Sp}_r$.

1.2.15. As in the case of vector bundles we know a great deal about the properties of these schemes. The first fact to notice is that these schemes are not necessarily connected: any G-bundle P has a topological type determined by its degree $\delta(P) \in \pi_1(G)$ (which is defined as the image in $\mathrm{H}^2_{\text{ét}}(C, \pi_1(G))$ of the class of $P \in \mathrm{H}^1_{\text{ét}}(C, G)$ via the connecting homomorphism deduced from the exact sequence of groups $1 \to \pi_1(G) \to \widetilde{G} \to G \to 1$), and $\mathcal{M}_G$ is the disjoint union of the moduli spaces $\mathcal{M}^\delta_G$ parametrizing equivalence classes of G-bundles of given degree $\delta \in \pi_1(G)$.

It follows from their construction as a GIT quotient of a smooth scheme that each of these components is an irreducible, normal (even Cohen-Macaulay), projective variety of dimension $(g-1)\dim G + \dim Z(G)$. We also know that, in general, they are not locally factorial.

We know from [**BLS98**, Proposition 7.4] that the Picard group $\mathrm{Pic}(\mathcal{M}^\delta_G)$ is infinite cyclic *when G is almost simple.* In the sequel, we will denote by $\mathcal{L}_{\mathcal{M}^\delta_G}$ (or $\mathcal{L}^\delta_G$) its ample generator. The question of giving a natural description of this line bundle has also been studied in (*loc. cit.*). Recall first that the generator of the Picard group of $\mathcal{M}_{\mathbf{SL}_r}$ had been previously described in [**DN89**]. This is the *determinant line bundle* $\mathcal{L}_{\mathbf{SL}_r}$, which is associated to the *generalized theta divisor* $\Theta_L = \{E \in \mathcal{M}_{\mathbf{SL}_r} | \mathrm{H}^0(C, E \otimes L) \neq 0\}$ for any $L \in J^{g-1}$. In the general case we first observe that every representation $\rho \colon G \longrightarrow \mathbf{SL}_N$ induces a morphism $\mathcal{M}^\delta_G \longrightarrow \mathcal{M}_{\mathbf{SL}_N}$, so that the pull-back of $\mathcal{L}_{\mathbf{SL}_r}$ defines a line bundle on $\mathcal{M}^\delta_G$ (called a determinantal line bundle). For a classical group, the ample generator of the Picard group is such a determinantal bundle (see [**BLS98**, Proposition 12.4]).

1.3. Moduli spaces of orthogonal bundles

In this thesis, we will focus on the case $G = \mathbf{SO}_r$, which amounts to considering the moduli space of semi-stable orthogonal bundles of rank r with an orientation. Let us now specialize to this case what has just been said for any structure group G.

1.3.1. As we have just said, it is a normal projective variety, composed of two connected components which are both irreducible and of dimension $\frac{r(r-1)}{2}(g-1)$. The topological invariant distinguishing these two components has been known for long: this is the *second Stiefel-Whitney class* $w_2 \colon \mathrm{H}^1_{\text{ét}}(C, \mathbf{SO}_r) \longrightarrow \mathbb{Z}/2\mathbb{Z}$ (originally defined for real orthogonal bundles), which satisfies the following property (see e.g. [**Ser90**]): for all orthogonal bundle (E, q) and all theta-characteristic $\kappa \in J^{g-1}$ on C, we have

$$w_2(E) \equiv h^0(C, E \otimes \kappa) + rh^0(C, \kappa) \mod 2. \tag{1.3.1.1}$$

We will denote by $\mathcal{M}^+_{\mathbf{SO}_r}$ and $\mathcal{M}^-_{\mathbf{SO}_r}$ the two components of $\mathcal{M}_{\mathbf{SO}_r}$. The first one is a quotient of the moduli $\mathcal{M}_{\mathbf{Spin}_r}$ of $\mathbf{Spin}_r$-bundles, while the second one is a quotient of the twisted moduli space $\mathcal{M}^-_{\mathbf{Spin}_r}$ (see [**BLS98**, 2.3.c]).

1.3.2. These two varieties are not locally factorial: this has been shown first in [**LS97**] (by constructing some *pfaffian divisors* which are not Cartier), and it was then proved in [**BLS98**] that this is the case for every non simply connected group G.

Moreover, the determinantal bundle induced by the standard representation $\mathbf{SO}_r \longrightarrow \mathbf{GL}(k^r)$ generates the Picard group, except when $r = 4$: when $r \geqslant 7$, this is [**BLS98**, Proposition 12.4], and when r is equal to 3, 5 or 6 this is a consequence of the exceptional isomorphisms $\mathbf{Spin}_3 \simeq \mathbf{SL}_2$, $\mathbf{Spin}_5 \simeq \mathbf{Sp}_4$ and $\mathbf{Spin}_6 \simeq \mathbf{SL}_4$. When $r = 4$, the group $\mathbf{SO}_4$ is no longer almost simple, and the Picard group of $\mathcal{M}^{\pm}_{\mathbf{SO}_4}$ has rank 2. We will go back to this more precisely in Chapter 5.

1.3.3. We have recalled in the introduction how the theta map $\mathcal{M}_{\mathbf{SL}_r} \dashrightarrow |r\Theta|$ provides a somehow concrete description of the map $\mathcal{M}_{\mathbf{SL}_r} \dashrightarrow |\mathcal{L}_{\mathbf{SL}_r}|^*$.

We have also pointed out that this has been extended to the case of $\mathcal{M}_{\mathbf{SO}_r}$ in [**Bea06a**]. Let us give now more details on this result.

We consider the restriction, still denoted $\theta\colon \mathcal{M}_{\mathbf{SO}_r} \dashrightarrow |r\Theta|$, of the theta map through the morphism[(1)] $\mathcal{M}_{\mathbf{SO}_r} \longrightarrow \mathcal{M}_{\mathbf{GL}_r}$ associated to the standard representation of $\mathbf{SO}_r$.

Since the theta divisor $\Theta_E \subset J^{g-1}$ associated to a self-dual bundle E is obviously fixed by the natural involution $\iota\colon L \mapsto K_C \otimes L^{-1}$ of J^{g-1}, we see that θ maps $\mathcal{M}_{\mathbf{SO}_r}$ into the fixed locus of ι^* in $|r\Theta|$. This fixed locus is the union $|r\Theta|^+ \cup |r\Theta|^-$ of the two eigenspaces of ι^*, and θ sends $\mathcal{M}^+_{\mathbf{SO}_r}$ in $|r\Theta|^+$ and $\mathcal{M}^-_{\mathbf{SO}_r}$ into $|r\Theta|^-$.

Theorem 1.3.4 **(Beauville)**. — *The theta map induces two canonical isomorphisms* $(\theta^+)^*\colon \big(\mathrm{H}^0(J^{g-1},\mathcal{O}(r\Theta))^+\big)^* \longrightarrow \mathrm{H}^0(\mathcal{M}^+_{\mathbf{SO}_r},\mathcal{L}^+_{\mathbf{SO}_r})$ *and* $(\theta^-)^*\colon \big(\mathrm{H}^0(J^{g-1},\mathcal{O}(r\Theta))^-\big)^* \longrightarrow \mathrm{H}^0(\mathcal{M}^-_{\mathbf{SO}_r},\mathcal{L}^-_{\mathbf{SO}_r})$, *which make the two following diagrams commutative:*

$$\begin{tikzcd} & {|\mathcal{L}^+_{\mathbf{SO}_r}|^*} \arrow[dd, "(\theta^+)^*"] \\ \mathcal{M}^+_{\mathbf{SO}_r} \arrow[ur, "\varphi_{\mathcal{L}^+_{\mathbf{SO}_r}}"] \arrow[dr, "\theta^+"'] & \\ & {|r\Theta|^+} \end{tikzcd} \qquad \begin{tikzcd} & {|\mathcal{L}^-_{\mathbf{SO}_r}|^*} \arrow[dd, "(\theta^-)^*"] \\ \mathcal{M}^-_{\mathbf{SO}_r} \arrow[ur, "\varphi_{\mathcal{L}^-_{\mathbf{SO}_r}}"] \arrow[dr, "\theta^-"'] & \\ & {|r\Theta|^-.} \end{tikzcd}$$

(1) This morphism sends an oriented orthogonal bundle $P = (E, q, \omega)$ to its underlying vector bundle E: this is the forgetful morphism, which will be investigated in the next chapter.

CHAPTER 2

THE FORGETFUL MORPHISM

The variety $\mathcal{M}_{\mathbf{SO}_r}$ is related to the moduli space $\mathcal{M}_{\mathbf{SL}_r}$ of vector bundles of rank r and trivial determinant on C through the forgetful map $\mathcal{M}_{\mathbf{SO}_r} \longrightarrow \mathcal{M}_{\mathbf{SL}_r}$ which sends any $\mathbf{SO}_r$-bundle P to its underlying vector bundle $E = P(\mathbf{SL}_r)$. It is natural to ask whether this map is a closed embedding. In fact, when r is even, it even fails to be injective, and it is therefore more convenient to ask the same question about $\mathcal{M}_{\mathbf{O}_r} \longrightarrow \mathcal{M}_{\mathbf{GL}_r}$.

In the same way we consider the forgetful morphism $\mathcal{M}_{\mathbf{Sp}_{2r}} \longrightarrow \mathcal{M}_{\mathbf{SL}_{2r}}$ from the variety of symplectic bundles of rank $2r$ to the variety of all vector bundles of the same rank with trivial determinant.

The main result of this chapter may be stated as follows:

Theorem. — (i) *The forgetful map $\mathcal{M}_{\mathbf{O}_r} \longrightarrow \mathcal{M}_{\mathbf{GL}_r}$ is an embedding.*

(ii) *When r is odd, $\mathcal{M}_{\mathbf{SO}_r} \longrightarrow \mathcal{M}_{\mathbf{SL}_r}$ is again an embedding, while, when r is even, it is a 2-sheeted cover onto its image.*

(iii) *The forgetful map $\mathcal{M}_{\mathbf{Sp}_{2r}} \longrightarrow \mathcal{M}_{\mathbf{SL}_{2r}}$ is also an embedding.*

We give the full proof for the orthogonal case, and sketch the obvious modifications required by the symplectic one.

We consider in the first section the injectivity of $\mathcal{M}_{\mathbf{SO}_r} \longrightarrow \mathcal{M}_{\mathbf{SL}_r}$: this comes down to an easy comparison of the equivalence relations between $\mathbf{SO}_r$-bundles and vector bundles which define the closed points of the corresponding moduli spaces. We then check that the tangent maps of $\mathcal{M}_{\mathbf{O}_r} \longrightarrow \mathcal{M}_{\mathbf{GL}_r}$ are injective. This differential point of view is much more involved: it relies on Luna's étale slice theorem, which naturally leads to the consideration of representations of quivers. To carry our discussion to its end we need an auxiliary result relative to the invariant theory of these representations for the action of a product of classical groups: this is the aim of the second section. In the third one we show how this computation results in our main theorem.

2.1. About the injectivity of $\mathcal{M}_{\mathbf{SO}_r} \longrightarrow \mathcal{M}_{\mathbf{SL}_r}$

In this section we study the injectivity of the forgetful map $\mathcal{M}_{\mathbf{SO}_r} \longrightarrow \mathcal{M}_{\mathbf{SL}_r}$. The closed points of $\mathcal{M}_G$ are in a one-to-one correspondence with the set of equivalence classes of semi-stable G-bundles (cf. [**Ram96**]). When $G = \mathbf{SL}_r$ one easily recovers from this notion Seshadri's definition of S-equivalence for vector bundles.

The natural manner to study the injectivity of the forgetful morphism is to proceed in the same way to link together equivalence between $\mathbf{SO}_r$-bundles and S-equivalence between their underlying vector bundles. This will be carried out in 2.1.2. Before this, we give another proof, which is far shorter, but has the disadvantage to miss a precise description of what closed points of $\mathcal{M}_{\mathbf{SO}_r}$ are.

2.1.1. First proof, using Narasimhan-Seshadri's theorem. —

2.1.1.1. Let us recall from [**Ram96**, 3.15] that the theorem of Narasimhan and Seshadri remains true for principal G-bundles; more precisely, Ramanathan proved that any equivalence classe of semi-stable principal G-bundles is characterized by a *unitary* G-bundle, unique up to isomorphism. Recall that a unitary G-bundle is a bundle associated to a representation of the fundamental group $\pi_1(C)$ of the curve in a maximal compact real subgroup K of G in the following way (we only need to assume G to be semi-simple): we associate to a representation $\rho\colon \pi_1(C) \to K \subset G$ the semi-stable G-bundle over C defined as $\widetilde{C} \times^{\pi_1(C)} G$, where $\widetilde{C} \to C$ is the universal cover of the curve C.

When $G = \mathbf{SL}_r$, this is exactly the celebrated theorem of Narasimhan and Seshadri (in this case, one may choose K to be the unitary group $\mathbf{SU}_r$, whence the terminology). In this case we also know that unitary vector bundles are *polystable* bundles, that is vector bundles which splits as the direct sum of some stable vector bundles (having all the same slope).

In the orthogonal case, this theorem says that closed points of the moduli space $\mathcal{M}_{\mathbf{SO}_r}$ correspond to isomorphism classes of representations $\pi_1(C) \to \mathbf{SO}_r(\mathbb{R})$. Since $\mathbf{SO}_r(\mathbb{R}) \subset \mathbf{SU}_r$, we see that any closed point corresponds (up to isomorphism) to a *unitary* $\mathbf{SO}_r$-bundle P whose underlying vector bundle $P(\mathbf{SL}_r)$ is a polystable vector bundle.

2.1.1.2. Two unitary $\mathbf{SO}_r$-bundles P and P' are sent to the same point of $\mathcal{M}_{\mathbf{SL}_r}$ if and only if they are both obtained from reduction of structure group to $\mathbf{SO}_r$ of the same polystable vector bundle E. Such a reduction amounts to a section of $E/\mathbf{SO}_r \to C$, and two of them give isomorphic $\mathbf{SO}_r$-bundles if and only if they are conjugated by the action of $\mathrm{Aut}_{\mathbf{SL}_r}(E)$ on $\Gamma(C, E/\mathbf{SO}_r)$. Elements of $\Gamma(C, E/\mathbf{SO}_r)$ correspond to isomorphisms $\iota\colon E \xrightarrow{\sim} E^*$ such that $\iota^* = \iota$ and $\det \iota$ is the square of the trivialisation of $\det E$ inherited from the $\mathbf{SL}_r$-torsor structure. The action of $\mathrm{Aut}_{\mathbf{SL}_r}(E)$ simply is

$$(f, \iota) \in \mathrm{Aut}_{\mathbf{SL}_r}(E) \times \Gamma(C, E/\mathbf{SO}_r) \mapsto f^* \iota f.$$

Since E is polystable, $\mathrm{Aut}_{\mathbf{GL}_r}(E)$ acts transitively on the set $\Gamma(C, E/\mathbf{O}_r)$ of all symmetric isomorphisms from E onto E^*: indeed the Jordan-Hölder filtration allows us to split E as

$$E = \bigoplus_i \Big(F_i^{(1)} \otimes V_i^{(1)}\Big) \oplus \bigoplus_j \Big(F_j^{(2)} \otimes V_j^{(2)}\Big) \oplus \bigoplus_k \Big((F_k^{(3)} \oplus F_k^{(3)*}) \otimes V_k^{(3)}\Big),$$

where $V_i^{(l)}$ are finite-dimensional vector spaces and the $F_i^{(1)}$ (resp. $F_j^{(2)}$, resp. $F_k^{(3)}$) are orthogonal (resp. symplectic, resp. non isomorphic to their dual) mutually non isomorphic stable vector bundles, in such a way that any symmetric isomorphism $E \to E^*$ is equivalent to the data of orthogonal (resp. symplectic, resp. non degenerate) forms on each one of the $V_i^{(1)}$ (resp. $V_j^{(2)}$, resp. $V_k^{(3)}$). The action of

$$\mathrm{Aut}_{\mathbf{GL}_r}(E) = \prod_i \mathbf{GL}(V_i^{(1)}) \times \prod_j \mathbf{GL}(V_j^{(2)}) \times \Big(\prod_k \mathbf{GL}(V_k^{(3)}) \times \mathbf{GL}(V_k^{(3)})\Big)$$

on the set of these collections is obviously transitive.

Any two elements σ and σ' of $\Gamma(C, E/\mathbf{SO}_r)$ are then conjugate under the action of $\mathrm{Aut}_{\mathbf{GL}_r}(E)$, by an automorphism whose determinant equals to ± 1. When r is odd $-\mathrm{id}_E$ is an $\mathbf{O}_r$-isomorphism which exchanges the orientation, and the action of $\mathrm{Aut}_{\mathbf{SL}_r}(E)$ on $\Gamma(C, E/\mathbf{SO}_r)$ is transitive too. On the contrary when r is even this action fails to remain transitive. For example let F be a vector bundle of rank $r/2$, non isomorphic to its dual, and consider the two *oriented* orthogonal bundles $F \oplus F^*$ and $F^* \oplus F$, equipped with the standard hyperbolic pairing: these bundles cannot be $\mathbf{SO}_r$-isomorphic (in fact any orthogonal automorphism of $F \oplus F^*$ must preserve the orientation). We have proven so far:

Proposition 2.1.1.3. — *When r is odd the map $\mathcal{M}_{\mathbf{SO}_r}(k) \longrightarrow \mathcal{M}_{\mathbf{SL}_r}(k)$ is injective; when r is even this is a finite map of degree* 2.

2.1.2. Closed points of $\mathcal{M}_{\mathbf{SO}_r}$. — We now give a more complete description of the closed points of $\mathcal{M}_{\mathbf{SO}_r}$.

2.1.2.1. Recall first the vector bundle case: every semi-stable vector bundle E admits a Jordan-Hölder filtration, whose associated graded object gives a polystable vector bundle, which characterizes the S-equivalence class of E. Ramanathan explained in [**Ram96**] what this notion becomes in the general case: he associated to any semi-stable G-bundle P a *graded object* $\operatorname{gr} P$, obtained from a suitable reduction of structure group of P to a parabolic subgroup of G. Let us expose now with some details what it gives in the orthogonal case (note that this has been known for long; for example see [**Bho84**]).

2.1.2.2. Parabolic subgroups of $\mathbf{SO}_r$. — If Q is a quadratic form on a r-dimensional vector space V, any parabolic subgroup Γ of $\mathbf{SO}(Q)$ is the stabilizer of an isotropic flag of V. If $0 = N_0 \subset N_1 \subset \cdots \subset N_l$ is such a flag, then its stabilizer Γ stabilizes $0 = N_0 \subset N_1 \subset \cdots \subset N_l \subset N_l^{\perp} \subset \cdots \subset N_1^{\perp} \subset V$ too. In a basis adapted to the first flag, in which Q is represented by a matrix $\mathrm{Mat}_Q = \begin{pmatrix} 0 & 0 & \mathrm{I}_{\dim N_l} \\ 0 & \mathrm{Mat}_{Q'} & 0 \\ \mathrm{I}_{\dim N_l} & 0 & 0 \end{pmatrix}$ where $\mathrm{Mat}_{Q'}$ is the matrix of the non degenerate quadratic form Q' induced on $N_l^{\perp}/N_l$ by Q, Γ must be a subgroup of the group of matrices of the form

$$M = \left(\begin{array}{ccccccccc} A_1 & * & \cdots & * & * & & & & \\ 0 & A_2 & \ddots & \vdots & \vdots & & & & \\ \vdots & \ddots & \ddots & * & \vdots & & \huge{*} & & \\ 0 & \cdots & 0 & A_l & * & & & & \\ 0 & \cdots & \cdots & 0 & B & * & \cdots & \cdots & * \\ & & & & 0 & {}^tA_1^{-1} & 0 & \cdots & 0 \\ & & \huge{0} & & \vdots & * & {}^tA_2^{-1} & \ddots & \vdots \\ & & & & \vdots & \vdots & \ddots & \ddots & 0 \\ & & & & 0 & * & \cdots & * & {}^tA_l^{-1} \end{array}\right),$$

where $A_i \in \mathbf{GL}_{r_i - r_{i-1}}$, $B \in \mathbf{SO}(Q')$ (and $r_i = \dim N_i$).

Hence, if P is a semi-stable $\mathbf{SO}_r$-bundle and $E = P(\mathbf{SL}_r)$ its associated vector bundle, a reduction of structure group σ of P to a parabolic subgroup Γ defined by an isotropic flag $0 = N_0 \subset N_1 \subset \cdots \subset N_l$ induces isotropic subbundles $E_i = \sigma^* E(N_i) \subset E$ of rank r_i giving a filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_l \subset E_l^{\perp} \subset \cdots \subset E_1^{\perp} \subset E$. But one knows how to construct such a filtration, with the extra conditions that E_i/E_{i-1} is a stable bundle of degree 0 and $E_l^\perp/E_l$ a stable orthogonal bundle. By [**Ram81**, 4.5] the latter splits as a direct orthogonal sum of mutually non-isomorphic stable bundles. The graded object $\mathrm{gr}E_\bullet$ is then precisely the Jordan-Hölder one, which is known to characterize the point of $\mathcal{M}_{\mathbf{SL}_r}$ corresponding to E.

2.1.2.3. A representative of the equivalence class of P as an orthogonal bundle is given by the $\mathbf{SO}_r$-bundle $\mathrm{gr}P$ obtained from a suitable reduction of structure group of P to a parabolic subgroup of $\mathbf{SO}_r$ (cf. [**Ram96**, 3.12]). Let us check that the reduction σ attached to the above filtration satisfies the conditions of (*loc. cit.*). Since any character χ on Γ is of the form $M \mapsto \prod \det(A_i)^{\alpha_i}$, we see that $\deg(\chi_*\sigma^*E(k)) = 0$ for every χ if and only if $\mu(E_i/E_{i-1}) = 0$ for $i = 1, \dots, l$: this means that σ definitely is an admissible reduction.

On the other hand, the unipotent radical $\mathrm{R}_u(\Gamma)$ of Γ is the unipotent part of the (neutral component of the) intersection of all of its Borel subgroups. Since these are the stabilizers of the flags adapted to the one giving Γ whose length is maximal, $\mathrm{R}_u(\Gamma)$ is a subgroup consisting of matrices M with $A_i = \mathrm{id}$ for all i and $B = \mathrm{id}$. We deduce from the preceding a Levi decomposition of Γ, the Levi component L being isomorphic to the product $\prod \mathbf{GL}_{r_i - r_{i-1}} \times \mathbf{SO}_{r-2r_l}$. Let $p\colon \Gamma \to L$ be the projection on this Levi component. The required stability of the L-bundle $p_*\sigma^*E$ is then a consequence of the stability of the successive quotients E_i/E_{i-1} together with Lemma 1.2.9

2.1.2.4. The class of P is therefore defined by the $\mathbf{SO}_r$-bundle $\mathrm{gr}P = p_*\sigma^*P(\mathbf{SO}_r)$, and $(\mathrm{gr}P)(\mathbf{SL}_r)$ is again the Jordan-Hölder graded object. This gives another proof of the fact (proved in 2.1.1.1) that the underlying vector bundle of a unitary $\mathbf{SO}_r$-bundle must be polystable. But, above all, this provides the following explicit description of closed points of $\mathcal{M}_{\mathbf{SO}_r}$ (which has already been noticed and used, see for example [**Bho84**]): as we have just seen, any semi-stable (oriented) orthogonal bundle $P = (E, q)$ admits a filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_l \subset E_l^\perp \subset \cdots \subset E_1^\perp \subset E$ such that E_i/E_{i-1} is a stable bundle of degree 0 and $E_l^\perp/E_l$ a stable (oriented) orthogonal bundle.

Whereas such a filtration is not unique, the associated graded *(oriented) orthogonal* object

$$\mathrm{gr}(P) = \bigoplus_{i=1}^{l} \big((E_i/E_{i-1}) \oplus (E_{i-1}^\perp/E_i^\perp)\big) \oplus E_l^\perp/E_l$$

is uniquely defined (note that the summands $(E_i/E_{i-1}) \oplus (E_{i-1}^{\perp}/E_i^{\perp})$ are endowed with the hyperbolic (oriented) structure deduced from the isomorphisms $E_{i-1}^{\perp}/E_i^{\perp} \longrightarrow (E_i/E_{i-1})^*$). We then have:

Proposition 2.1.2.5. — *Two semi-stable (oriented) orthogonal bundles are equivalent if and only their associated graded objects are isomorphic (as (oriented) orthogonal bundles).*

Since the corresponding moduli spaces are known to parametrize equivalence classes for this relation, this gives a precise description of the image of a semi-stable orthogonal bundle.

2.1.3. A few words about non-abelian cohomology. — The distinction between the odd and even cases relies on the fact that the semi-direct product $1 \to \mathbf{SO}_r \to \mathbf{O}_r \to \mathbb{Z}/2\mathbb{Z} \to 0$ may be direct or not. Indeed, we have recalled in C.3 a general way to compute the fibers of the map (of pointed sets) $H^1_{\text{ét}}(C, \mathbf{SO}_r) \longrightarrow H^1_{\text{ét}}(C, \mathbf{O}_r)$ (which forgets the orientation). These are exactly the orbits under a natural action of $H^0(C, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ on $H^1_{\text{ét}}(C, \mathbf{SO}_r)$, for which stabilizers are easy to describe: the stabilizer of an $\mathbf{SO}_r$-bundle P is the image of the map $\mathrm{Aut}_{\mathbf{O}_r}(P) \to \mathbb{Z}/2\mathbb{Z}$ obtained by *twisting* the quotient morphism $\mathbf{O}_r \to \mathbb{Z}/2\mathbb{Z}$ by P (with respect to the action of $\mathbf{SO}_r$ by inner automorphisms) and taking global sections.

When r is odd, the product $1 \to \mathbf{SO}_r \to \mathbf{O}_r \to \mathbb{Z}/2\mathbb{Z} \to 0$ is direct, and the map $H^1_{\text{ét}}(C, \mathbf{SO}_r) \longrightarrow H^1_{\text{ét}}(C, \mathbf{O}_r)$ is automatically injective. But, as soon as r is even, its section is no longer compatible with the action of $\mathbf{SO}_r$ by inner automorphisms; we have just chosen a bundle $E = F \oplus F^*$ such that $\mathrm{Aut}_{\mathbf{O}_r}(E) = \{\mathrm{id}\}$, whence the lack of injectivity.

In view of 2.1.1.2 we can give the following complete description of $\mathcal{M}_{\mathbf{SO}_r}(k) \longrightarrow \mathcal{M}_{\mathbf{O}_r}(k)$:

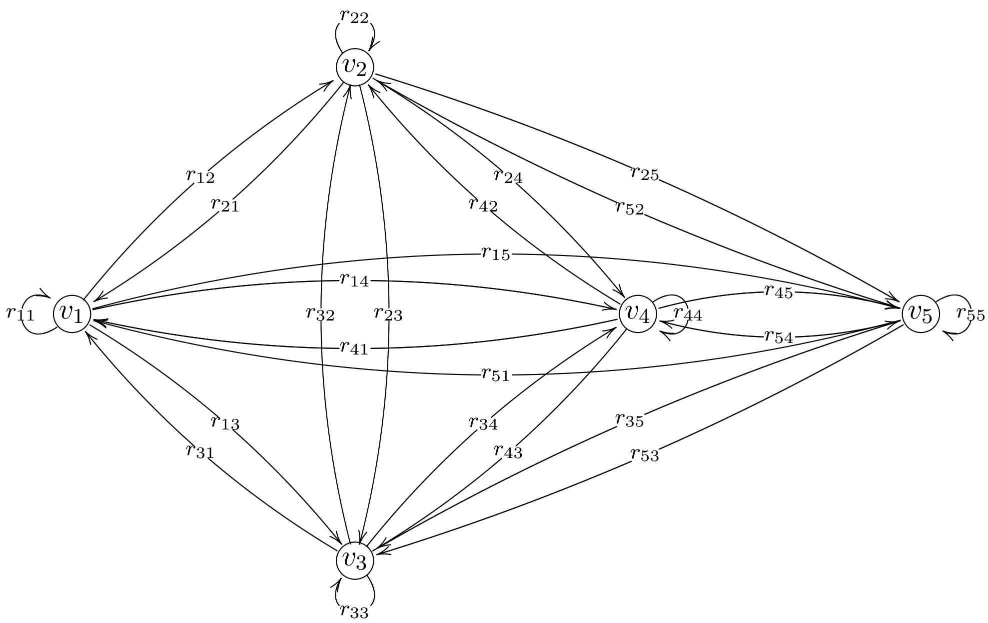
Proposition 2.1.3.1. — *A unitary orthogonal bundle P with trivial determinant has two antecedents in $\mathcal{M}_{\mathbf{SO}_r}$ if and only if every orthogonal bundle $F_i^{(1)}$ appearing in the splitting of its underlying vector bundle $P(\mathbf{GL}_r)$ has even rank.*

This immediately results from the explicit description of the twisted map $\mathrm{Aut}_{\mathbf{O}_r}(P) \to \mathbb{Z}/2\mathbb{Z}$, which is nothing else than the determinant.

Note that such a discussion may be applied to any morphism $\mathcal{M}_H \longrightarrow \mathcal{M}_G$ deduced from an inclusion $H \subset G$, especially when H is a normal subgroup.

2.2. Invariant theory of representations of quivers

Let me recall from [**LBP90**] the standard setting of *representations of quivers*: a quiver is just an oriented graph $Q = (Q_0, Q_1)$ (by which we mean that Q_0 is a set of vertices $\{v_1, \ldots, v_n\}$, and Q_1 a collection of oriented arrows $a\colon v_{t(a)} \to v_{h(a)}$ between these vertices). Here is a quiver having five vertices (a number r on an arrow means that this arrow stands for r parallel arrows):



A representation V of Q associates to any vertex v_i a vector space V_i, and to any arrow $a\colon v_i \to v_j$ a linear map $V_i \to V_j$ between the corresponding vector spaces. The vector $\alpha = (\dim V_i)_i$ is called the dimension vector of V. If $\alpha \in \mathbb{N}^n$ is fixed, we denote by $R(Q, \alpha)$ the space of all representations of Q of dimension α with $V_i \simeq k^{\alpha_i}$. This carries a natural action of the group $\mathbf{GL}(\alpha) = \prod \mathbf{GL}_{\alpha_i}$: $(g_i)_i$ sends a representation $(f_a)_{a \in Q_1}$ to the representation $(g_{h(a)} f_a g_{t(a)}^{-1})$.

We now turn our attention to some special quivers: let Q stand for a quiver consisting of $n = n_1 + n_2 + 2n_3$ vertices

$$s_1, \ldots, s_{n_1}, t_1, \ldots, t_{n_2}, u_1, u_1^*, \ldots, u_{n_3}, u_{n_3}^*,$$

and $\alpha \in \mathbb{N}^n$ be an admissible dimension vector (that is a vector such that α_{t_j} is even and $\alpha_{u_k} = \alpha_{u_k^*}$). We define Γ_α to be the group

$$\Gamma_\alpha = \prod \mathbf{O}_{\alpha_{s_i}} \times \prod \mathbf{Sp}_{\alpha_{t_j}} \times \prod \mathbf{GL}_{\alpha_{u_k}},$$

which is actually thought here as a subgroup of $\mathbf{GL}(\alpha) = \prod_{i=1}^{n} \mathbf{GL}_{\alpha_i}$ via the inclusions $P \in \mathbf{GL}_{\alpha_{u_k}} \mapsto (P, {}^tP^{-1}) \in \mathbf{GL}_{\alpha_{u_k}} \times \mathbf{GL}_{\alpha_{u_k^*}}$. The natural action of $\mathbf{GL}(\alpha)$ on the space $R(Q, \alpha)$ restricts to an action of Γ_α on $R(Q, \alpha)$.

Le Bruyn and Procesi have shown in (*loc. cit.*) that the algebra of polynomial invariants $k[R(Q, \alpha)]^{\mathbf{GL}(\alpha)}$ is generated by traces along oriented cycles in the quiver Q. By considering some universal algebras, they reduce the general case to the one of the quivers with one vertex and m arrows, which had been solved by Procesi twenty-five years before. Following their path, we produce here a set of generators for the algebra of invariants under the action of Γ_α. The local study of the map $\mathcal{M}_{\mathbf{O}_r} \to \mathcal{M}_{\mathbf{GL}_r}$ made later heavily rests on this description.

To achieve this goal, we need to adapt the main result of [**Pro87**] in our setting (this is done in 2.2.2), which in turn requires to find first the $\mathbf{O}_N \times \mathbf{Sp}_{N'}$ invariants of m matrices, which is the aim of 2.2.1. Note that it means that we now deduce the general case from the case of the quivers having two vertices, respectively acted upon by $\mathbf{O}_N$ and $\mathbf{Sp}_{N'}$, with exactly m arrows between any two of them.

2.2.1. First fundamental theorem for $\mathbf{O}_N \times \mathbf{Sp}_{N'}$. — In this section, we first adapt the argument of [**ABP73**, Appendix 1], and then show how it allows us to find a set of generators for the $\mathbf{O}_N \times \mathbf{Sp}_{N'}$-invariants of m matrices.

2.2.1.1. We will denote by M_n the space of $n \times n$ matrices. Let M be the matrix $\begin{pmatrix} I_N & 0 \\ 0 & J_{N'} \end{pmatrix}$, with $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$; it represents a bilinear pairing, given as the standard orthogonal sum of a quadratic form of rank N and a symplectic form of rank N'.

To prove the first main theorem of invariant theory for $\mathbf{O}_N \times \mathbf{Sp}_{N'}$, we follow the proof of [**ABP73**]: Weyl's original proof in the orthogonal (resp. symplectic) case relies on the so-called *Capelli's identities*, while the proof of (*loc. cit.*) avoids such considerations. The key lemma becomes (note that $\mathrm{M}_N \times \mathrm{M}_{N'}$ is identified with its image in $\mathrm{M}_{N+N'}$):

Lemma 2.2.1.2. — *Any polynomial function $f\colon (\mathrm{M}_N \times \mathrm{M}_{N'})(k) \to k$ such that $f(BA) = f(A)$ for all $B \in \mathbf{O}_N \times \mathbf{Sp}_{N'}$ may be written $f\colon A \mapsto F({}^tAMA)$ with F a polynomial map on $(\mathrm{M}_N \times \mathrm{M}_{N'})(k) \subset \mathrm{M}_{N+N'}(k)$.*

In other words f factors through the application

$$\pi\colon A \in (\mathrm{M}_N \times \mathrm{M}_{N'})(k) \mapsto {}^tAMA \in (\mathrm{M}_N \times \mathrm{M}_{N'})(k).$$

Let $\Psi_{N,N'}$ be its image, which is nothing else than the product of the space of symmetric $N \times N$ matrices and the space of antisymmetric $N' \times N'$ matrices. The restriction of π to $\mathbf{GL}_N \times \mathbf{GL}_{N'}$ identifies the open subset $\Psi^\circ_{N,N'}$ consisting of non-degenerate forms with the geometric quotient $(\mathbf{GL}_N \times \mathbf{GL}_{N'})/\!\!/(\mathbf{O}_N \times \mathbf{Sp}_{N'})$ (say by [**MFK94**, Proposition 0.2]). The lemma then follows from the commutative diagram

$$\begin{array}{ccc} (\mathrm{M}_N \times \mathrm{M}_{N'})/\!\!/(\mathbf{O}_N \times \mathbf{Sp}_{N'}) & \xrightarrow[\pi]{} & \Psi_{N,N'} \\ \cup & & \cup \\ (\mathbf{GL}_N \times \mathbf{GL}_{N'})/\!\!/(\mathbf{O}_N \times \mathbf{Sp}_{N'}) & \xrightarrow[\pi]{\sim} & \Psi^\circ_{N,N'}\,; \end{array}$$

the restriction to $(\mathbf{GL}_N \times \mathbf{GL}_{N'})/\!\!/(\mathbf{O}_N \times \mathbf{Sp}_{N'})$ of a map f defined on the good quotient $(\mathrm{M}_N \times \mathrm{M}_{N'})/\!\!/(\mathbf{O}_N \times \mathbf{Sp}_{N'})$ must indeed be induced by a function of the form $A \in \mathbf{GL}_N \times \mathbf{GL}_{N'} \mapsto F({}^tAMA)/H({}^tAMA)$ with F and H two coprime polynomials defined on the *affine space* $\Psi_{N,N'}$. The equality $F({}^tAMA) = f(A)H({}^tAMA)$ finally ensures that H is invertible[(1)].

Remark 2.2.1.3. — We can give a quicker proof of the preceding lemma: indeed, since we are considering here a product group acting on a product factor by factor, we deduce that the quotient of the product $(\mathrm{M}_N \times \mathrm{M}_{N'})/\!\!/(\mathbf{O}_N \times \mathbf{Sp}_{N'})$ is the product of the quotient $(\mathrm{M}_N/\!\!/\mathbf{O}_N) \times (\mathrm{M}_{N'}/\!\!/\mathbf{Sp}_{N'})$, which is isomorphic (by [**ABP73**], for example) isomorphic to the space denoted by $\Psi_{N,N'}$ in the proof. However, the first proof gives a direct proof of the general case, avoiding the explicit construction of a local section used in (*loc. cit.*).

2.2.1.4. We are now in a position to establish the first main theorem of invariant theory for $\mathbf{O}_N \times \mathbf{Sp}_{N'}$. Let us start with the case of multilinear invariants, again after [**ABP73**]. Let V be a vector space of dimension $N + N'$ endowed with the non-degenerate bilinear form $\langle \cdot\,, \cdot \rangle$ given by the matrix M. So $V = V_1 \overset{\perp}{\oplus} V_2$, V_1 being a quadratic space of dimension N and V_2 a symplectic space of dimension N'.

Theorem 2.2.1.5. — *Any linear $\mathbf{O}_N \times \mathbf{Sp}_{N'}$-invariant morphism $V^{\otimes 2i} \to k$ is a linear combination of products of the contractions $v_1 \otimes \cdots \otimes v_{2i} \longmapsto \langle v_l, v_{l'} \rangle$.*

---

(1) The fact that the function ring of $\Psi_{N,N'}$ is factorial is crucial here: this explains why we had to restrict ourselves to functions $f\colon \mathrm{M}_N \times \mathrm{M}_{N'} \to k$ rather than functions $\mathrm{M}_{N+N'} \to k$, which would have require something like the factoriality of the ring of the quotient $\mathrm{M}_{N+N'}/\!\!/\mathbf{O}_N \times \mathbf{Sp}_{N'}$.

We give two proofs. Let $\varphi\colon V^{\otimes 2i} \to k$ be any linear $\mathbf{O}_N \times \mathbf{Sp}_{N'}$-invariant map, and consider the following polynomial function:

$$f\colon (A,\omega) \in (\operatorname{End} V_1 \oplus \operatorname{End} V_2) \times V^{\otimes 2i} \mapsto \varphi(A\omega) \in k.$$

By the previous lemma there exists a polynomial F on $(\mathbf{S}^2 V_1^* \oplus \mathbf{\Lambda}^2 V_2^*) \times V^{\otimes 2i}$, linear in the second variable, such that $f(A,\omega) = F({}^tAMA, \omega)$. This polynomial certainly is invariant for the natural action of $\mathbf{GL}(V_1) \times \mathbf{GL}(V_2)$ on $(\mathbf{S}^2 V_1^* \oplus \mathbf{\Lambda}^2 V_2^*) \times V^{\otimes 2i}$: for any $\Gamma \in \mathbf{GL}(V_1) \times \mathbf{GL}(V_2)$, we have $F({}^t\Gamma^{-1}\,{}^tAMA\Gamma^{-1}, \Gamma\omega) = F({}^tAMA, \omega)$.

The assertion results, by polarization, from the description of linear forms on $V_1^{*\otimes a_1} \otimes V_1^{\otimes b_1} \otimes V_2^{*\otimes a_2} \otimes V_2^{\otimes b_2}$ which are invariant for the action of $\mathbf{GL}(V_1) \times \mathbf{GL}(V_2)$: F is an homogeneous function of degree i in its first variable, which therefore arises from linear forms on $(\mathbf{S}^2 V_1^*)^{\otimes k} \otimes V_1^{\otimes 2k} \otimes (\mathbf{\Lambda}^2 V_2^*)^{\otimes i-k} \otimes V_2^{\otimes 2i-2k}$ (via the projections $(\mathbf{S}^2 V_1^* \oplus \mathbf{\Lambda}^2 V_2^*)^{\otimes i} \times V^{\otimes 2i} \to (\mathbf{S}^2 V_1^*)^{\otimes k} \otimes V_1^{\otimes 2k} \otimes (\mathbf{\Lambda}^2 V_2^*)^{\otimes i-k} \otimes V_2^{\otimes 2i-2k}$, and, of course, the diagonal embedding $\mathbf{S}^2 V_1^* \oplus \mathbf{\Lambda}^2 V_2^* \to (\mathbf{S}^2 V_1^* \oplus \mathbf{\Lambda}^2 V_2^*)^{\otimes i}$). Since $\varphi(\omega) = F(M, \omega)$, one then just has to evaluate F on M.

Here is a second proof. We decompose $V^{\otimes 2i}$ as the direct sum

$$\bigoplus_{(a_l)_l \in \{1,2\}^{2i}} \Big(V_{a_1} \otimes V_{a_2} \otimes \cdots \otimes V_{a_{2i}} \Big)$$

and remark that the linear invariants for the action of $\mathbf{O}_N \times \mathbf{Sp}_{N'}$ on any summand isomorphic to $V_1^{\otimes k} \otimes V_2^{\otimes 2i-k}$ are sums of products of linear invariants for the action of $\mathbf{O}_N$ on $V_1^{\otimes k}$ with linear invariants for the action of $\mathbf{Sp}_{N'}$ on $V_2^{\otimes 2i-k}$. The result is then a consequence of the first fundamental theorems for the orthogonal and symplectic groups, which are for example recalled from Weyl's book in [**Pro76**].

2.2.1.6. One easily deduces from the foregoing a family of generators for the algebra of polynomial invariants under the diagonal action (by conjugation) of $\mathbf{O}_N \times \mathbf{Sp}_{N'}$ on $\mathrm{M}_{N+N'}(k)^m$: according to [**Pro76**, §7] it is enough to work out the behaviour of the composition, the trace and the (right) adjunction (denoted by $A \mapsto A^* = M^{-1}\,{}^tAM$) via the identification $\operatorname{End} V \simeq V \otimes V$ induced by the bilinear pairing. More precisely we choose here the identification $v \otimes v' \mapsto \langle v', \cdot \rangle v$. If $v = v_1 + v_2 \in V = V_1 \oplus V_2$ (cf. 2.2.1.4), we call σ the involution of V defined by $v_1 \oplus v_2 \mapsto v_1 - v_2$, which is the unique automorphism(2)

(2) This morphism, whose existence and uniqueness are obvious, may already have a name (like left/right commutation map?); however, I did not find a trace of it in Bourbaki, Algèbre IX.

of V verifying $\langle v, v'\rangle = \langle \sigma(v'), v\rangle = \langle v', \sigma(v)\rangle$. We have then the following equalities

- $(v \otimes w) \circ (u \otimes t) = \langle w, u\rangle v \otimes t$,
- $\operatorname{tr}(v \otimes w) = \langle w, v\rangle$,
- $(v \otimes w)^* = \sigma(w) \otimes v$.

This relations allow us to translate the functions given in theorem 2.2.1.5 in a way leading to the following statement:

Theorem 2.2.1.7. — (i) *Any $\mathbf{O}_N \times \mathbf{Sp}_{N'}$-invariant function defined on $\mathrm{M}_{N+N'}(k)^m$ is a polynomial in the*

$$(A_1, \ldots, A_m) \mapsto \operatorname{tr}(U_{j_1} M^{2\varepsilon_1} U_{j_2} M^{2\varepsilon_2} \cdots U_{j_k} M^{2\varepsilon_k}),$$

with $U_j \in \{A_j, A_j^*\}$, and $\varepsilon_j \in \{0, 1\}$.

(ii) *The ring of $\mathbf{O}_N \times \mathbf{Sp}_{N'}$-equivariant morphisms from $\mathrm{M}_{N+N'}(k)^m$ to $\mathrm{M}_{N+N'}(k)$ is generated as an algebra over $k[\mathrm{M}_{N+N'}(k)^m]^{\mathbf{O}_N \times \mathbf{Sp}_{N'}}$ by the constant function M^2 and the elements $(A_1, \ldots, A_m) \mapsto A_j$ and $(A_1, \ldots, A_m) \mapsto A_j^*$.*

We follow Procesi's proof. Let P be a polynomial on the space of m matrices, homogeneous of degree d. We may fully polarize it to obtain a multilinear $\mathbf{O}_N \times \mathbf{Sp}_{N'}$-invariant function $\tilde{P}$ defined on the space $(\mathrm{M}_{N+N'}(k)^m)^{\oplus d}$ such that $P = (1/m!)\tilde{P} \circ \Delta$ (this is known as *Aronhold's method*). Now we know from 2.2.1.5 that $\tilde{P}$ is a linear combination of functions of the form (where we have identified $(\mathrm{M}_{N+N'}(k)^m)^{\oplus d}$ with $(V \otimes V)^{md}$)

$$\varphi_\tau \colon v_1 \otimes \cdots \otimes v_{2md} \mapsto \langle v_{\tau(1)}, v_{\tau(2)}\rangle \cdots \langle v_{\tau(2md-1)}, v_{\tau(2md)}\rangle$$

where $\tau \in \mathfrak{S}_{2md}$. Let us now rewrite this basic invariant as a map $(A_1, \ldots, A_{md}) \in (\mathrm{M}_{N+N'})^{md} \mapsto \varphi_\tau((A_i)_i) \in k$.

By linearity we restrict ourselves to decomposable matrices $A_i = u_i \otimes v_i$ (with our choice about $V \otimes V \xrightarrow{\sim} \operatorname{End} V$ this means $A_i = u_i {}^t v_i M$). We may write $\varphi_\tau((u_i \otimes v_i)_i)$ as

$$\langle w_{i_1}^{(1)}, \bar{w}_{i_2}^{(1)}\rangle \langle \sigma^{\varepsilon_2^1}(w_{i_2}^{(1)}), \bar{w}_{i_3}^{(1)}\rangle \cdots \langle \sigma^{\varepsilon_l^1}(w_{i_l}^{(1)}), \bar{w}_{i_1}^{(1)}\rangle \langle w_{i_1}^{(2)}, \bar{w}_{i_2}^{(2)}\rangle \cdots$$

where $\{w_{i_j}^{(a)}, \bar{w}_{i_j}^{(a)}\} = \{u_r, v_r\}$ for exactly one $r = r_{a,j}$, and ε_j^a is equal to 0 or 1, depending on whether we have to switch $w_{i_j}^{(a)}$ and $\bar{w}_{i_{j+1}}^{(a)}$. We now just have to focus on the expression $\langle w_{i_1}, \bar{w}_{i_2}\rangle \langle \sigma^{\varepsilon_2} w_{i_2}, \bar{w}_{i_3}\rangle \cdots \langle \sigma^{\varepsilon_l} w_{i_l}, \bar{w}_{i_1}\rangle$. Using the equalities recalled earlier, we successively obtain

$$
\begin{aligned}
&\langle w_{i_1}, \bar{w}_{i_2}\rangle\langle\sigma^{\varepsilon_2} w_{i_2}, \bar{w}_{i_3}\rangle\cdots\langle\sigma^{\varepsilon_l} w_{i_l}, \bar{w}_{i_1}\rangle \\
&\qquad = \langle w_{i_1}, \bar{w}_{i_2}\rangle\cdots\langle\sigma^{\varepsilon_{l-1}} w_{i_{l-1}}, \bar{w}_{i_l}\rangle\operatorname{tr}(\bar{w}_{i_1}\otimes\sigma^{\varepsilon_l} w_{i_l}) \\
&\qquad = \langle w_{i_1}, \bar{w}_{i_2}\rangle\cdots\langle\sigma^{\varepsilon_{l-2}} w_{i_{l-2}}, \bar{w}_{i_{l-1}}\rangle\operatorname{tr}\Big((\bar{w}_{i_1}\otimes\sigma^{\varepsilon_{l-1}} w_{i_{l-1}})\circ(\bar{w}_{i_l}\otimes\sigma^{\varepsilon_l} w_{i_l})\Big) \\
&\qquad = \operatorname{tr}\Big((\bar{w}_{i_1}\otimes w_{i_1})\circ(\bar{w}_{i_2}\otimes\sigma^{\varepsilon_2} w_{i_2})\circ\cdots\circ(\bar{w}_{i_l}\otimes\sigma^{\varepsilon_l} w_{i_l})\Big).
\end{aligned}
$$

Using the identity $v_i \otimes u_i = (u_i \otimes \sigma(v_i))^*$, we see that φ_τ is a product of (multilinear) maps of the form

$$(A_1, \ldots, A_{dm}) \longmapsto \operatorname{tr}(U_{i_1} \cdots U_{i_l})$$

where U_i is either equal to A_i, $M^2 A_i^*$, $A_i M^2$ or $M^2 A_i^* M^2$.

Finally, $\tilde{P}$ is a map $(A_1, \ldots, A_{dm}) \mapsto \sum \lambda_\tau \prod \operatorname{tr}(U_{i_1^{(a)}} \cdots U_{i_l^{(a)}})$, and the first part of the theorem is proved.

The second assertion follows exactly as in [**Pro76**]: we introduce a new matrix variable X and consider the function $\operatorname{tr}(f(\cdot)X)$ deduced from any $\mathbf{O}_N \times \mathbf{Sp}_{N'}$-equivariant morphism $\mathrm{M}_{N+N'}(k)^m$ to $\mathrm{M}_{N+N'}(k)$. The first part of the theorem expresses this invariant of $m+1$ matrices as a polynomial in the $\operatorname{tr}(M^{2\varepsilon_X} U_{i_1} M^{2\varepsilon_1} \cdots U_{i_l} M^{2\varepsilon_l} X)$ and the $\operatorname{tr}(M^{2\varepsilon_l} U_{i_l}^* \cdots M^{2\varepsilon_1} U_{i_1}^* M^{2\varepsilon_X} X)$ with coefficients in $k[\mathrm{M}_{N+N'}(k)^m]^{\mathbf{O}_N \times \mathbf{Sp}_{N'}}$. The non degeneracy of the trace implies that f is a polynomial in the A_i, A_i^* and M^2. Since such a morphism is equivariant, the theorem is proved.

Remark 2.2.1.8. — The fact that the only constant and equivariant functions are linear combinations of Id and M^2 is just a reformulation of the trivial description of the centralizer of $\mathbf{O}_N \times \mathbf{Sp}_{N'}$ in $\mathrm{M}_{N+N'}(k)$ (in particular, we cannot avoid those M^2 in the theorem).

Example 2.2.1.9. — In order to illustrate this statement, let me present some decomposition of some invariant functions in terms of the generating sets given in the theorem.

In the simplest case $m = 1$, we see that the $\mathbf{O}_N \times \mathbf{Sp}_{N'}$-equivariant morphism sending A to $\begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix}$ is obtained as $\frac{1}{4}(A + AM^2 + M^2 A + M^2 A M^2)$; the polynomial $A \mapsto \operatorname{tr}(A_{12}A_{21})$ is given by $\frac{1}{4}\big(\operatorname{tr}(A^2) - \operatorname{tr}(M^2 A^* M^2 A^*)\big)$.

When $m = 2$, we can check that the polynomial $(A, B) \mapsto \operatorname{tr}(B_{12}A_{21})$ may be written as $\frac{1}{4}\big(\operatorname{tr}(BA) + \operatorname{tr}(M^2 BA) - \operatorname{tr}(BM^2 A) - \operatorname{tr}(M^2 B M^2 A)\big)$.

2.2.2. Extension of the main result of [Pro87]. — The main result of [**Pro87**] asserts that, if R is a k-algebra with trace satisfying the n-th

Cayley-Hamilton identities, then there exists a universal map $R \to \mathrm{M}_n(A)$ which induces an isomorphism $R \to \mathrm{M}_n(A)^{\mathbf{GL}_n}$. In this section we state a similar result dealing with algebras with a trace and an antimorphism of order dividing 4.

Definition 2.2.2.1 (**[Pro87]**). — A *k-algebra with trace* is an algebra R equipped with a k-linear endomorphism $\mathrm{tr}\colon R \to R$ verifying the following identities

- $\mathrm{tr}(a)b = b\,\mathrm{tr}(b)$,
- $\mathrm{tr}(ab) = \mathrm{tr}(ba)$,
- $\mathrm{tr}(\mathrm{tr}(a)b) = \mathrm{tr}(a)\mathrm{tr}(b)$,

for all $a, b \in R$.

We will then call *k-algebra with trace and antimorphism of order* 4 an algebra with trace equipped with an antimorphism $\tau\colon R \to R$ of order 4 (from now on we will write "of order 4" instead of "of order dividing 4"). When R is the algebra $\mathrm{M}_{N+N'}(B)$ of all matrices with coefficients in a commutative ring B we choose τ to be the adjunction map (for the considered bilinear form) $\iota\colon A \in \mathrm{M}_{N+N'}(k) \mapsto M^{-1\,t}AM$ (and tr is the "natural" trace $A \mapsto \mathrm{Trace}(A) I_{N+N'}$). As soon as N or N' is zero, ι is in fact of order 2, and we could restrict ourselves to algebras with anti-involution.

2.2.2.2. Let R be a k-algebra with trace and antimorphism of order 4, and let $j\colon R \to \mathrm{M}_{N+N'}(A)$ be the universal map corresponding to the functor of trace preserving representations of R

$$X_{R,N+N'}\colon B \mapsto \mathrm{Hom}_k(R, \mathrm{M}_{N+N'}(B))$$

(cf. [**DCPRR05**, §2.2]; note that here the morphisms between algebras with trace are supposed to preserve the traces[(3)]). The existence of this morphism easily leads to the representability of the functor $\widetilde{X}_{R,N+N'}$ which associates to any commutative algebra B the set of all morphisms of k-algebras from R to $\mathrm{M}_{N+N'}(B)$ preserving the antimorphisms. This functor is actually represented

[(3)]It seems useful to stress the following point, which remains quite vague in [**Pro87**]: when dealing with an algebra with trace R, the universal couple (A, j) is not the same as the universal couple corresponding to the algebra R without its trace (just consider the case of the free algebra with trace built on exactly one variable). One way to construct the universal morphism preserving traces is to present R as the quotient of a free algebra F with trace built on $x_s, s \in \Sigma$ (*without* imposing any Cayley-Hamilton identity) and to repeat the begining of the proof of Theorem 2.6 in (*loc. cit.*): this has been more carefully explained recently in [**DCPRR05**].

by a closed subscheme of $X_{R,N+N'} = \mathrm{Spec}(A)$, still called $\widetilde{X}_{R,N+N'}$: the map $r \in R \mapsto \iota j \tau^3(r) \in \mathrm{M}_{N+N'}(A)$ comes from a morphism $t\colon A \to A$ of order 4, and the induced map $\tilde{j}\colon R \to \mathrm{M}_{N+N'}(\widetilde{A})$ (where $\widetilde{A}$ is the quotient of A by the action of t) is universal among the morphisms $R \to \mathrm{M}_{N+N'}(B)$ commuting with τ and ι (we use here the fact the *rational* antimorphism ι commutes with any application $\mathrm{M}_{N+N'}(f)$ induced by a morphism $f\colon B \to B'$).

The group $\mathbf{O}_N \times \mathbf{Sp}_{N'}$ acts by conjugation on $\mathrm{M}_{N+N'}(B)$, inducing a right action on $\widetilde{A}$, hence an action of $\mathbf{O}_N \times \mathbf{Sp}_{N'}$ on $\mathrm{M}_{N+N'}(\widetilde{A})$. The universal map $\tilde{j}$ maps R to the algebra $\mathrm{M}_{N+N'}(\widetilde{A})^{\mathbf{O}_N \times \mathbf{Sp}_{N'}}$ of $\mathbf{O}_N \times \mathbf{Sp}_{N'}$-equivariant morphisms from $\widetilde{X}_{R,N+N'}$ to $\mathrm{M}_{N+N'}(k)$ (cf. [**Pro87**, 1.2]).

2.2.2.3. Our purpose is to generalize the main theorem of (*loc. cit.*) to algebras with trace and antimorphism of order 4. As usual, we start by studying the free algebras in this category. Let us denote by $\widetilde{T}^{\Sigma}_{N,N'}$ the free algebra built on the generators $(x_s)_{s\in\Sigma}$, for any non necessarily finite set Σ: it is generated (as an algebra) by some elements $x_s, x_s^{\tau}, x_s^{\tau^2}, x_s^{\tau^3}$ and the traces of all their products. The corresponding universal morphism $\tilde{j}\colon \widetilde{T}^{\Sigma}_{N,N'} \to \mathrm{M}_{N+N'}(\widetilde{A})$ sends $x_s^{\tau^a}$ to $\iota^a(\xi^{(s)}_{l,l'})_{l,l'}$ (note that the universal ring $\widetilde{A}$ is, once again, the ring of generic $(N+N') \times (N+N')$ matrices $k[\xi^{(s)}_{l,l'}, s \in \Sigma, l, l' = 1, \ldots, N+N']$). Since $\mathrm{M}_{N+N'}(\widetilde{A})$ is naturally identified with the ring of polynomial functions $\mathrm{M}_{N+N'}(k)^I \to \mathrm{M}_{N+N'}(k)$, Theorem 2.2.1.7 provides a set of generators for the $\mathbf{O}_N \times \mathbf{Sp}_{N'}$-invariant subalgebra: if we denote by $\Lambda = \widetilde{T}^{\Sigma}_{N,N'} \overset{\mathrm{tr}}{*} \langle y\rangle$ the free product (in the category of algebras with trace and antimorphism of order 4) obtained by adding an indeterminate fixed by the antimorphism[(4)], then the morphism $\Lambda \to \mathrm{M}_{N+N'}(\widetilde{A})^{\mathbf{O}_N \times \mathbf{Sp}_{N'}}$ sending y to the matrix M^2 is onto. This allows us to state the following result:

Proposition 2.2.2.4. — *If R is a k-algebra with trace and antimorphism of order 4, then the map $\tilde{j} * (M^2)\colon R * \langle y\rangle \to \mathrm{M}_{N+N'}(\widetilde{A})^{\mathbf{O}_N \times \mathbf{Sp}_{N'}}$ is onto.*

It is an immediate adaptation of Procesi's proof: let me repeat here his argument (this mainly amounts to adding some $\widetilde{\ }$ here and there). We present R as a quotient of $\widetilde{T} = \widetilde{T}^{\Sigma}_{N,N'}$ by an ideal $\widetilde{I}$. Consider the ideal $\widetilde{B}\widetilde{I}\widetilde{B} \subset \widetilde{B} = \mathrm{M}_{N+N'}(\widetilde{A})$, where $\widetilde{A}$ is the universal object associated to $\widetilde{T}$. There must exist an ideal $\widetilde{J} \subset \widetilde{A}$ such that $\widetilde{B}\widetilde{I}\widetilde{B} = \mathrm{M}_{N+N'}(\widetilde{J})$, and the morphism $R \longrightarrow \widetilde{B}/\widetilde{B}\widetilde{I}\widetilde{B} = \mathrm{M}_{N+N'}(\widetilde{A}/\widetilde{J})$ (induced by the universal morphism

(4) In fact we can avoid this last condition if we allow Λ to be only an algebra with trace.

$\tilde{j}\colon \widetilde{T} \to \mathrm{M}_{N+N'}(\widetilde{A})$ associated to the free algebra) is the universal morphism corresponding to R. Since $\mathbf{O}_N \times \mathbf{Sp}_{N'}$ is *linearly reductive*, the diagram

$$\begin{tikzcd}
\widetilde{T} * \langle y \rangle \arrow[r, two heads, "\tilde{j}*(M^2)"] \arrow[d, two heads] & \mathrm{M}_{N+N'}(\widetilde{A})^{\mathbf{O}_N \times \mathbf{Sp}_{N'}} \arrow[d, two heads] \\
R * \langle y \rangle \arrow[r, "\tilde{j}_R*(M^2)"] & \mathrm{M}_{N+N'}(\widetilde{A}/\widetilde{J})^{\mathbf{O}_N \times \mathbf{Sp}_{N'}}
\end{tikzcd}$$

implies that map $\tilde{j}_R * (M^2)$ is surjective onto the invariant algebra $\mathrm{M}_{N+N'}(\widetilde{A}/\widetilde{J})^{\mathbf{O}_N \times \mathbf{Sp}_{N'}}$.

2.2.2.5. Let us now say a word about injectivity: since we do not have (as far as I know) any nice description of the second main theorem of invariant theory for $\mathbf{O}_N \times \mathbf{Sp}_{N'}$, there is no comfortable way to obtain a statement as neat as Procesi's result. We give here a proposition involving bijectivity of $\tilde{j}$, but the condition on R is too badly expressed. Define $\widetilde{U}^{\Sigma}_{N,N'}$ to be the quotient of $\Lambda = \widetilde{T}^{\Sigma}_{N,N'} * \langle y \rangle$ by the kernel of the surjective map $\Lambda \to \mathrm{M}_{N+N'}(\widetilde{A})^{\mathbf{O}_N \times \mathbf{Sp}_{N'}}$ (this exactly amounts to imposing in Λ any polynomial identity given by the second main theorem of invariant theory: in particular, even if N or N' is zero, it seems that we may have to content ourselves with this kind of assertion). The algebra $\widetilde{U}^{\Sigma}_{N,N'}$ is thus exactly the algebra of $\mathbf{O}_N \times \mathbf{Sp}_{N'}$-equivariant polynomial maps from $\mathrm{M}_{N+N'}(k)^{\Sigma} \to \mathrm{M}_{N+N'}(k)$.

Proposition. — *If R is a quotient of $\widetilde{U}^{\Sigma}_{N,N'}$, then the universal map $\tilde{j}\colon R \to \mathrm{M}_{N+N'}(\widetilde{A})^{\mathbf{O}_N \times \mathbf{Sp}_{N'}}$ is an isomorphism.*

We continue to recopy Procesi's proof. The algebra $\widetilde{T}$ appearing in the proof of surjectivity must of course be replaced now by the algebra $\widetilde{U}^{\Sigma'}_{N,N'}$, where Σ' is a set containing Σ such that $\Sigma' \setminus \Sigma$ is infinite (and, of course, the ideal $\widetilde{I}$ has to be modified consequently). Consider an element $a \in \widetilde{T} \cap \widetilde{B}\widetilde{I}\widetilde{B}$, and write it $a = \sum b_k i_k b'_k$ (with of course $b_k, b'_k \in \widetilde{B}$ and $i_k \in \widetilde{I}$). Choose a variable x which do not appear in a. Let us denote by r the Reynolds operator. Recall from (*loc. cit.*) that it commutes with the trace. We have then $\mathrm{tr}(ax) = \mathrm{tr}(\sum b_k i_k b'_k x) = \mathrm{tr}(\sum b'_k x b_k i_k)$. Applying r (and recalling that i_k is $\mathbf{O}_N \times \mathbf{Sp}_{N'}$-invariant), we get $\mathrm{tr}(ax) = \mathrm{tr}(\sum r(b'_k x b_k) i_k)$. By definition, $r(b'_k x b_k)$ is an linear polynomial in x invariant under the action of $\mathbf{O}_N \times \mathbf{Sp}_{N'}$, which implies according to 2.2.1.7 (identifying elements of Λ with their images in $\{\mathrm{M}_{N+N'}(k)^{\Sigma'} \to \mathrm{M}_n(k)\}$) that

$$r(b'_k x b_k) = \sum_l s_{kl} x t_{kl} + \sum_l s'_{kl} \tau(x) t'_{kl} + \sum_l \operatorname{tr}(m_{kl} x) n_{kl}$$

with s_{kl}, t_{kl}, s'_{kl}, t'_{kl}, m_{kl} and n_{kl} independent of x.

Thus $\operatorname{tr}(ax) = \operatorname{tr}(\sum_{k,l} t_{kl} i_k s_{kl} x + \sum_{k,l} \tau^3(s'_{kl})\tau^3(i_k)\tau^3(t'_{kl})x + \sum_{k,l} \operatorname{tr}(n_{kl} i_k) m_{kl} x)$, and, since $\operatorname{tr}(ax) = \operatorname{tr}(bx)$ implies $a = b$ for all a and b in $\widetilde{T}$ independent of x,

$$a = \sum_{k,l} t_{kl} i_k s_{kl} + \sum_{k,l} \tau^3(s'_{kl})\tau^3(i_k)\tau^3(t'_{kl}) + \sum_{k,l} \operatorname{tr}(n_{kl} i_k) m_{kl} \in \widetilde{I},$$

as was to be proved.

Remark 2.2.2.6. — Note that this proof is not suited for positive characteristic: even if we only need the surjectivity of $\tilde{j}$ to prove the forthcoming main theorem 2.2.3.6, the characteristic zero assumption is already essential.

2.2.2.7. Carlos Simpson has suggested to me that this result could be improved by replacing the matrix algebra by something more natural in this picture. It seems that a somehow finer way to adapt Procesi's work is to consider the *ortho-symplectic Lie superalgebra* $\mathfrak{osp}(N, N')$ defined by Kac, and to introduce a notion of *formal supertrace*, on $\mathbb{Z}/2\mathbb{Z}$-graded algebras (the supertrace on $\mathrm{M}_{N+N'}$ would then be str: $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} \operatorname{tr}(A) & 0 \\ 0 & -\operatorname{tr}(D) \end{pmatrix}$). If R is an algebra with supertrace, we can consider the functor of supertrace preserving homomorphisms from R to $\mathfrak{osp}(N, N')(B)$.

This would avoid those antimorphisms of order 4 and this extra variable y; but this would eventually give a weakener result (since $\mathfrak{osp}(N, N')$ is in fact a subalgebra of $\mathrm{M}_{N+N'}$), which would be insufficient for our purpose (we will have to include the case of the orthogonal group operating on the space of symmetric matrices: in a group-theoretic setting, we are concerned here with a problem regarding $(G \times \cdots \times G)/H$, with H a strict subgroup of G). Nevertheless, we could consider the supertrace on $\mathfrak{gl}(N, N')$ equipped with our antimorphism of order 4 (a "supertranspose", or rather "superadjoint", since supertranspose already exists), and consider the functor of representations of $\mathbb{Z}/2\mathbb{Z}$-algebras with supertrace and supertranspose respecting these data.

2.2.3. Generators for $k[R(Q,\alpha)]^{\Gamma_\alpha}$. —

2.2.3.1. Let us go back to the quiver Q, and the action of Γ_α on its representation space $R(Q,\alpha)$. We need here to adapt the map ι previously defined on $\mathrm{M}_{N+N'}(k)$: ι still associates to a map its adjoint, but the bilinear pairing has to be replaced by the one represented by a matrix of the form

$$\Phi = \left(\begin{array}{ccc} I_{N_1} & 0 & 0 \\ 0 & J_{N_2} & 0 \\ 0 & 0 & \begin{array}{cc} 0 & I_{N_3} \\ I_{N_3} & 0 \end{array} \end{array} \right),$$

where $N_1 = \sum_{i=1}^{n_1} \alpha_{s_i}$, $N_2 = \sum_{j=1}^{n_2} \alpha_{t_j}$, $N_3 = \sum_{k=1}^{n_3} \alpha_{u_k}$. We put $N = N_1 + N_2 + N_3$, and consider the decomposition of k^N into pairwise orthogonal subspaces $k^{N_1} \oplus k^{N_2} \oplus k^{2N_3}$ given by Φ. We are interested in representations of Q adaptated to this decomposition (to be precise we mean here that the *fixed* decomposition $k^N = \bigoplus V_{s_i} \oplus \bigoplus V_{t_j} \oplus \bigoplus (V_{u_k} \oplus V_{u_k^*})$ is adaptated to the previous one).

Consider the quiver $\widetilde{Q}$ obtained from Q by adding one new arrow $a^*\colon \sigma(v') \to \sigma(v)$ for any arrow $a\colon v \to v'$, where we called σ the involution of the set of vertices fixing the s_i and t_j, and permuting u_k and u_k^*. Let R be the path algebra of the opposite quiver Q^{op}, and $\widetilde{R}$ be the algebra deduced from the path algebra of $\widetilde{Q}^{\mathrm{op}}$ *by adding traces*. There is quite a natural way to endow $\widetilde{R}$ with an antimorphism τ of order 4 whose action on the idempotents (who are associated to the constant paths) comes from the one of σ: τ fixes the e_{s_i} and e_{t_j}, permutes e_{u_k} and $e_{u_k^*}$, while it sends an arrow a to εa^*, where ε equals -1 if a starts from s_i, u_k or u_k^* and ends at t_j, and 1 otherwise (we will see in a second why we need these weird signs).

Note that representations of $\widetilde{R}$ of dimension α commuting with τ and ι adapted to the previous decomposition correspond bijectively to representations of R of the same dimension adapted to this decomposition: a representation of $\widetilde{R}$ respecting the antimorphisms must associate a dual arrow a^* to the adjoint morphism corresponding to a (up to a sign). This allows us to identify $R(Q,\alpha)$ with the subspace of $R(\widetilde{Q},\alpha)$ consisting of all representations which preserve the preceding antimorphisms.

Remark 2.2.3.2. — If we had omitted the minus signs in the definition of τ, we would have found that morphisms $\widetilde{R} \to \mathrm{M}_N(k)$ commuting with τ and ι should map $\widetilde{R}$ into the ι^2-fixed subspace of $\mathrm{M}_N(k)$, which consits of all matrices of the form $\begin{pmatrix} M_{11} & 0 & M_{13} \\ 0 & M_{22} & 0 \\ M_{31} & 0 & M_{33} \end{pmatrix}$, so that we would have missed any arrow between a symplectic vertex and the non-symplectic one. The definition of τ had been guided by the behaviour of ι^2. In particular, we could replace τ by the antimorphism sending a to εa^*, with $\varepsilon = \sqrt{-1}$ if and only if a is an arrow between a symplectic vertex t_j and a non-symplectic one.

2.2.3.3. We apply now the argument of [**LBP90**, §3]: the strategy is to establish a convenient relation between the subspace of $R(\widetilde{Q},\alpha)$ consisting of all representations preserving the antimorphisms and a subspace of $\widetilde{X}_{\widetilde{R},N}(k)$.

To do this, let us consider first the algebra $\widetilde{S}_n$ defined as the quotient of $k[e_{s_i}, e_{t_j}, e_{u_k}, e_{u_k^*}]$ by the ideal generated by the relations $e_v^2 = e_v$, $e_v e_{v'} = 0$ if $v \neq v'$, $\sum_v e_v = 1$. This algebra is contained in $\widetilde{R}$. The restriction of τ to this algebra is exactly the antiinvolution described in 2.2.3.1, and we have a fairly nice description of $\widetilde{X}_{\widetilde{S}_n,N}$: its closed points correspond to "complete families of projectors" $(p_v)_{v\in Q_1}$, such that p_{s_i} and p_{t_j} are selfadjoint, while the adjoint of p_{u_k} is $p_{u_k^*}$. We see (say as in 2.2.3.2) that these projectors split as some sums $p_v = p_v^{or} + p_v^{sy}$ of two projectors $p_v^{or}\colon k^N \to k^{N_1+2N_3}$ and $p^{sy}\colon k^N \to k^{N_2}$. This description (and an easy exercice using Witt's theorem) results in the following decomposition:

$$\widetilde{X}_{\widetilde{S}_n,N} = \bigcup_{\sigma,\omega} \widetilde{X}_{\sigma,\omega}\,,$$

where σ and ω range over pairs of admissible vectors in $\mathbb{N}^n$ such that $\sum \sigma_j = N_1 + 2N_3$ and $\sum \omega_j = N_2$, the component $\widetilde{X}_{\sigma,\omega}$ being isomorphic to

$$(\mathbf{O}_{N_1+2N_3}\times\mathbf{Sp}_{N_2})/\Big(\prod(\mathbf{O}_{\sigma_{s_i}}\times\mathbf{Sp}_{\omega_{s_i}})\times\prod(\mathbf{O}_{\sigma_{t_j}}\times\mathbf{Sp}_{\omega_{t_j}})\times\prod(\mathbf{GL}_{\sigma_{u_k}}\times\mathbf{GL}_{\omega_{u_k}})\Big).$$

It induces a decomposition of $\widetilde{X}_{\widetilde{R},N}$ as the union $\bigcup_{\sigma,\omega}\varpi^{-1}\widetilde{X}_{\sigma,\omega}$, where $\varpi\colon \widetilde{X}_{\widetilde{R},N} \to \widetilde{X}_{\widetilde{S}_n,N}$ is the map induced by the inclusion $\widetilde{S}_n \subset \widetilde{R}$. If we call $\widetilde{A}_{\sigma,\omega}$ the affine ring of the component corresponding to a pair of admissible vectors like above, we get a decomposition of the universal algebra $\widetilde{A} = \prod \widetilde{A}_{\sigma,\omega}$ associated to $\widetilde{R}$. Since $\mathbf{O}_{N_1+2N_3}\times\mathbf{Sp}_{N_2}$ acts separately on each summand of $\mathrm{M}_N(\widetilde{A}) = \prod \mathrm{M}_N(\widetilde{A}_{\sigma,\omega})$, there is a splitting $\widetilde{R}/\ker\tilde{j} = \prod \widetilde{R}_{\sigma,\omega}$, where $\tilde{j}\colon \widetilde{R} \to \mathrm{M}_N(\widetilde{A})$ is the universal map[(5)]. Now, Proposition 2.2.2.4 tells us that $\tilde{j}$ and $y \mapsto \Phi^2 = \begin{pmatrix} \mathrm{I}_{N_1} & 0 & 0 \\ 0 & -\mathrm{I}_{N_2} & 0 \\ 0 & 0 & \mathrm{I}_{2N_3} \end{pmatrix}$ give a surjective morphism from $\widetilde{R}_{\sigma,\omega}$ onto the invariant algebra $\mathrm{M}_N(\widetilde{A}_{\sigma,\omega})^{\mathbf{O}_{N_1+2N_3}\times\mathbf{Sp}_{N_2}}$. Note also that

(5) At his point we could have avoided this quotient by $\ker\tilde{j}$ by having imposed at the begining some relations (which are $\ker\tilde{j}$!) in $\widetilde{R}$.

the projection on $\widetilde{R}_{\sigma,\omega}$ of an idempotent e_v has trace $\sigma_v + \omega_v$ (we use that $\tilde{j}$ commutes with the traces, and that the image of e_v in $\mathrm{M}_N(\widetilde{A}_{\sigma,\omega})$ by $\tilde{j}$ is a projector onto a subspace of the given dimension).

Let us concentrate on the component $\widetilde{R}_{\sigma,\omega}$ corresponding to the dimension vectors σ and ω whose coordinate are $\sigma_{s_i} = \alpha_{s_i}$, $\sigma_{t_j} = 0$, $\sigma_{u_k} = \alpha_{u_k}$, $\omega_{t_j} = \alpha_{t_j}$ and $\omega_{s_i} = \omega_{u_k} = 0$. For this component, the following lemma holds:

Lemma 2.2.3.4. — *The morphism $\widetilde{R}_{\sigma,\omega} \to \mathrm{M}_N(\widetilde{A}_{\sigma,\omega})^{\mathbf{O}_{N_1+2N_3}\times\mathbf{Sp}_{N_2}}$ induced by $\tilde{j}$ is an isomorphism.*

Indeed, our choice about σ and ω implies that $\Phi^2 \cdot \tilde{j}(e_v) = \pm\tilde{j}(e_v)(= \tilde{j}(e_v) \cdot \Phi^2)$ for all v (in fact the sign is negative if and only if v is a symplectic vertex). Since any element $r \in \widetilde{R}_{\sigma,\omega}$ may decomposed as $r = \sum_{v,v'} e_v r e_{v'}$, we see that the image of $\widetilde{R}_{\sigma,\omega} * \langle y \rangle$ is already contained in $\tilde{j}(\widetilde{R}_{\sigma,\omega})$, which proves the lemma.

2.2.3.5. We decompose the identity matrix as the sum $\sum_v u_v$, where u_v is the diagonal matrix with 1 in the block corresponding to v and 0 elsewhere. We introduce the subfunctor $\widetilde{X}^*_{\widetilde{R}_{\sigma,\omega}} : B \mapsto \{f \colon \widetilde{R}_{\sigma,\omega} \to \mathrm{M}_N(B) | f(e_v) = u_v\}$, which is represented by a closed subscheme $\widetilde{X}^*_{\widetilde{R}_{\sigma,\omega}}$ of $\widetilde{X}_{\widetilde{R}_{\sigma,\omega},N}$, acted on by the centralizer of the u_v, which is exactly Γ_α. Let us remark that this subscheme is in fact precisely the fiber of ϖ over the point $p \in \widetilde{X}_{\sigma,\omega}(k)$ defined by the u_v, and that the stabilizer of this point is Γ_α. This, together with the following commutative diagram summing up the situation

$$\begin{tikzcd}
\widetilde{X}^*_{\widetilde{R}_{\sigma,\omega}} \arrow[r, hook] \arrow[d, "\circlearrowleft\,\Gamma_\alpha"] & \widetilde{X}_{\widetilde{R}_{\sigma,\omega},N} \arrow[r, hook] \arrow[d, "\varpi"', "\circlearrowleft\,\mathbf{O}_{N_1+2N_3}\times\mathbf{Sp}_{N_2}"] & \widetilde{X}_{\widetilde{R},N} \arrow[d, "\varpi"', "\circlearrowleft\,\mathbf{O}_{N_1+2N_3}\times\mathbf{Sp}_{N_2}"] \\
\{p\} \arrow[r, hook] & \widetilde{X}_{\sigma,\omega} \arrow[r, hook] & \widetilde{X}_{\widetilde{S}_n,N},
\end{tikzcd}$$

allows us to identify $\widetilde{R}_{\sigma,\omega}$ with the ring of Γ_α-equivariant maps $\varpi^{-1}(p) = \widetilde{X}^*_{\widetilde{R}_{\sigma,\omega}} \to \mathrm{M}_N(k)$: indeed, we deduce from the diagram that the latter is exactly the ring of $\mathbf{O}_{N_1+2N_3}\times\mathbf{Sp}_{N_2}$-equivariant maps $\widetilde{X}_{\widetilde{R}_{\sigma,\omega},N} \to \mathrm{M}_N(k)$, while Lemma 2.2.3.4 tells us that $\tilde{j}$ induces an isomorphism from the former onto this same ring.

It is now great time to put in evidence the (trivial but crucial) fact that the fiber $\varpi^{-1}(p)$ is naturaly isomorphic to the subset of $R(\widetilde{Q}, \alpha)$ consisting of all the representations which preserve the antimorphisms, which has been previously identified with $R(Q,\alpha)$. The precedent discussion therefore leads to an isomorphism between $\widetilde{R}_{\sigma,\omega}$ and the ring of Γ_α-equivariant morphisms $R(Q,\alpha) \to \mathrm{M}_N(k)$ (Le Bruyn and Procesi had obtained an isomorphism between the path algebra R (rather than its component R_α) endowed with traces and Cayley-Hamilton relations and the ring of $\mathbf{GL}(\alpha)$-morphisms $R(Q,\alpha) \to \mathrm{M}_N(k)$ by imposing to R the extra (and apparently arbitrary) relations $\mathrm{tr}(e_i) = \alpha_i$ in R, which implies the emptyness of the components $\varpi^{-1}(X_\beta)$, $\beta \neq \alpha$; we do not feel necessary to keep here this detail of their proof). Taking traces, we get the expected result:

Theorem 2.2.3.6. — *The algebra of polynomials on $R(Q,\alpha)$ invariant under the action of $\prod \mathbf{O}_{\alpha_{s_i}} \times \prod \mathbf{Sp}_{\alpha_{t_j}} \times \prod \mathbf{GL}_{\alpha_{u_k}}$ is generated by the functions*

$$(f_a)_a \mapsto \mathrm{tr}(f_{\tilde{a}_p} \cdots f_{\tilde{a}_1}),$$

where $\tilde{a}_i$ is an arrow in the associated quiver $\widetilde{Q}$ equals to either a_i or ${a_i}^*$, in such a way that $(\tilde{a}_1, \dots, \tilde{a}_p)$ forms an oriented path in that quiver and $f_{\tilde{a}_i}$ means f_{a_i} or its adjoint according to whether $\tilde{a}_i$ is a_i or ${a_i}^*$.

Remark 2.2.3.7. — (i) Donkin has given a characteristic free proof of the result of Le Bruyn and Procesi, using the notions of *good filtration* and *good pairs*. In the preprint [**Lop06**], Lopatin has followed this different approach to obtain a characteristic-free proof of the preceding theorem.

(ii) The second main theorem of representation theory for this situation seems to constitute a very difficult problem (finding a finite set of generators for $k[R(Q,\alpha)]^{\Gamma_\alpha}$ is easy, but not really relevant); in fact, invariants for the action of $\mathbf{GL}_\alpha$ are already quite uneasy to understand: Le Bruyn and Procesi's proof certainly gives some informations, but their assertion ("all relations among the previously defined invariants and covariants can be deduced from the Cayley-Hamilton polynomails for N by N matrices") does not seem to lend itself to an effective description of the ideal of relations, even for very small quivers (see 3.2.4).

2.2.3.8. It is now easy to deal with the case where we let the whole linear group act (by conjugation) above some of the unpaired vertices. If we call $r_1, \dots, r_{n_4}$ these vertices, we deduce from the quiver Q a new quiver Q' as follows: Q' has n_4 new vertices r_l^*, and one new arrow $a' \colon r_l^* \to \sigma(v)$ (resp. $a' \colon \sigma(v) \to r_l^*$) for any arrow $a \colon v \to r_l$ (resp. $a \colon r_l \to v$), where σ is the involution of the set of vertices defined in 2.2.3.1. Let us also denote $\alpha' \in \mathbb{N}^{n_1+n_2+2(n_3+n_4)}$ the admissible vector naturally deduced from any given admissible dimension vector $\alpha \in \mathbb{N}^{n_1+n_2+2n_3+n_4}$. The group

$$\Gamma_\alpha = \prod \mathbf{O}_{\alpha_{s_i}} \times \prod \mathbf{Sp}_{\alpha_{t_j}} \times \prod \mathbf{GL}_{\alpha_{u_k}} \times \prod \mathbf{GL}_{\alpha_{r_l}}$$

acts on $R(Q, \alpha)$ and $R(Q', \alpha')$ (the action of $g \in \mathbf{GL}_{\alpha_{r_l}}$ on r_l^* being $f \mapsto {}^t g^{-1} f {}^t g$), and the morphism $R(Q, \alpha) \longrightarrow R(Q', \alpha')$ which sends a representation $(f_a)_a$ of Q to the extended representation of Q' which associates to a new arrow a' the adjoint of the map corresponding to a induces a Γ_α-equivariant projection $k[R(Q', \alpha')] \longrightarrow k[R(Q, \alpha)]$. Now, our previous theorem provides a generating family for the algebra of Γ_α-invariants of $k[R(Q', \alpha)]$ in terms of the associated quiver $\widetilde{Q'}$ deduced from Q'. The group Γ_α being linearly reductive, this has the following consequence:

Corollary 2.2.3.9. — *The algebra of polynomials on $R(Q, \alpha)$ invariant under the action of $\prod \mathbf{O}_{\alpha_{s_i}} \times \prod \mathbf{Sp}_{\alpha_{t_j}} \times \prod \mathbf{GL}_{\alpha_{u_k}} \times \prod \mathbf{GL}_{\alpha_{r_l}}$ is generated by the functions*

$$(f_a)_a \mapsto \mathrm{tr}(f_{\tilde{a}_p} \cdots f_{\tilde{a}_1}),$$

where $\tilde{a}_i$ is an arrow in the associated quiver $\widetilde{Q'}$ equals to either a_i or ${a_i}^*$, in such a way that $(\tilde{a}_1, \ldots, \tilde{a}_p)$ forms an oriented path in that quiver and $f_{\tilde{a}_i}$ means f_{a_i} or its adjoint according to whether $\tilde{a}_i$ is a_i or ${a_i}^*$.

Note that two vertices r_l and $r_{l'}^*$ are never connected by a single arrow in $\widetilde{Q'}$.

2.3. Local study of the forgetful map

In order to simplify the local study of $\mathcal{M}_{\mathbf{SO}_r} \to \mathcal{M}_{\mathbf{SL}_r}$ it is convenient to investigate separately the injective morphism $\mathcal{M}_{\mathbf{O}_r} \to \mathcal{M}_{\mathbf{GL}_r}$ and the natural map from $\mathcal{M}_{\mathbf{SO}_r}$ to the subscheme $\mathcal{M}_{\mathbf{O}_r}^{\mathcal{O}} \subset \mathcal{M}_{\mathbf{O}_r}$ consisting of all orthogonal bundles with trivial determinant. This distinction seems to be quite valuable since the direct differential study of $\mathcal{M}_{\mathbf{SO}_r}$ would involve invariant theory for special orthogonal groups, which is far more difficult to deal with (see 2.3.2). We show here that the former is an embedding, while the later is an isomorphism (resp. a 2-sheeted cover) when r is odd (resp. even).

2.3.1. Differential behaviour of $\mathcal{M}_{\mathbf{O}_r} \to \mathcal{M}_{\mathbf{GL}_r}$. —

2.3.1.1. Let us now briefly point out the classical way to analyse the local behaviour of $\mathcal{M}_{\mathbf{O}_r} \to \mathcal{M}_{\mathbf{GL}_r}$. We have recalled in 1.2.13 that this application arises as a quotient by the general linear group $\Gamma = \mathbf{GL}_\chi$ of an equivariant map between two smooth parameter schemes $R_{\mathbf{O}_r} \to R_{\mathbf{GL}_r}$. Luna's étale slice theorem and deformation theory then allow us to grasp the local structure of these good quotients (cf. [**KLS06**, 2.5]): at any polystable vector bundle E, $\mathcal{M}_{\mathbf{GL}_r}$ is étale locally isomorphic to an étale neighbourhood of the origin in the good quotient

$$\mathrm{Ext}^1(E,E)/\!\!/\mathrm{Aut}_{\mathbf{GL}_r}(E),$$

while $\mathcal{M}_{\mathbf{O}_r}$ is, at any polystable orthogonal bundle P, étale locally isomorphic to an étale neighbourhood of the origin in

$$H^1(C,\mathrm{Ad}(P))/\!\!/\mathrm{Aut}_{\mathbf{O}_r}(P),$$

where $\mathrm{Ad}(P)$ stands for the vector bundle $P\times^{\mathbf{O}_r}\mathfrak{so}_r$ associated to the adjoint representation of $\mathbf{O}_r$, which is nothing else than the vector bundle of germs of endomorphisms f of E such that $\sigma f + f^*\sigma = 0$, where $\sigma\colon E \to E^*$ is the symmetric isomorphism given by the quadratic structure on E; in other words the adjoint vector bundle $\mathrm{Ad}(P)$ is canonically isomorphic to $\mathbf{\Lambda}^2 E^*$.

Then, if $P \in \mathcal{M}_{\mathbf{O}_r}$ is a polystable orthogonal bundle with associated vector bundle $E \in \mathcal{M}_{\mathbf{GL}_r}$, the application $\mathcal{M}_{\mathbf{O}_r} \to \mathcal{M}_{\mathbf{GL}_r}$ coincides at P, through the preceding local isomorphisms (in the étale topology), with the natural map

$$H^1(C,\mathrm{Ad}(P))/\!\!/\mathrm{Aut}_{\mathbf{O}_r}(P) \to \mathrm{Ext}^1(E,E)/\!\!/\mathrm{Aut}_{\mathbf{GL}_r}(E)$$

at the origin. In particular the corresponding tangent maps are identified.

2.3.1.2. A more explicit description of the vector spaces $H^1(C,\mathrm{Ad}(P))$ and $\mathrm{Ext}^1(E,E)$ is then strongly needed in order to understand their quotients; we show here that $H^1(C,\mathrm{Ad}(P))/\!\!/\mathrm{Aut}_{\mathbf{O}_r}(P)$ is a closed subscheme of $\mathrm{Ext}^1(E,E)/\!\!/\mathrm{Aut}_{\mathbf{GL}_r}(E)$, which implies our main theorem.

According to 2.1.1.2 the orthogonal structure on the polystable vector bundle E associated to any point $q \in \mathcal{M}_{\mathbf{O}_r}$ gives rise to a splitting of E as a direct orthogonal sum of the form

$$E = \bigoplus_{i=1}^{n_1} E_i^{(1)} \oplus \bigoplus_{j=1}^{n_2} E_j^{(2)} \oplus \bigoplus_{k=1}^{n_3} E_k^{(3)}, \tag{2.3.1.2.1}$$

where each direct summand $E_l^{(a)}$ may be written as

– $E_i^{(1)} = F_i^{(1)} \otimes V_i^{(1)}$, where $(F_i^{(1)})_i$ are mutually non isomorphic stable orthogonal bundles and $(V_i^{(1)})_i$ some quadratic vector spaces,

– $E_j^{(2)} = F_j^{(2)} \otimes V_j^{(2)}$, where $(F_j^{(2)})_j$ are mutually non isomorphic stable symplectic bundles and $(V_j^{(2)})_j$ some symplectic vector spaces,

– $E_k^{(3)} = (F_k^{(3)} \oplus F_k^{(3)*}) \otimes V_k^{(3)}$, where $(F_k^{(3)})_k$ are mutually non isomorphic stable bundles such that $F_k^{(3)} \not\simeq F_{k'}^{(3)*}$ and $(V_k^{(3)})_k$ some vector spaces carrying a non degenerate bilinear form.

Let us denote by $\psi_l^{(a)} \colon F_l^{(a)} \to F_l^{(a)*}$ the duality isomorphism (when $a = 1, 2$), and $\sigma_l^{(a)} \colon E_l^{(a)} \to E_l^{(a)*}$ the symmetric isomorphism defined on $E_l^{(a)}$ (note that $F_k^{(3)} \oplus F_k^{(3)*}$ has been tacitly endowed with the hyperbolic form).

The two isotropy groups are then easily identified: we find that

$$\mathrm{Aut}_{\mathbf{GL}_r}(E) \simeq \prod_{i=1}^{n_1} \mathbf{GL}(V_i^{(1)}) \times \prod_{j=1}^{n_2} \mathbf{GL}(V_j^{(2)}) \times \prod_{k=1}^{n_3} \Big(\mathbf{GL}(V_k^{(3)}) \times \mathbf{GL}(V_k^{(3)})\Big),$$

while $\mathrm{Aut}_{\mathbf{O}_r}(P) \subset \mathrm{Aut}_{\mathbf{GL}_r}(E)$ is isomorphic to the subgroup

$$\prod_{i=1}^{n_1} \mathbf{O}(V_i^{(1)}) \times \prod_{j=1}^{n_2} \mathbf{Sp}(V_j^{(2)}) \times \prod_{k=1}^{n_3} \mathbf{GL}(V_k^{(3)}),$$

where $\mathbf{GL}(V_k^{(3)})$ stands for its image in $\mathbf{GL}(V_k^{(3)}) \times \mathbf{GL}(V_k^{(3)})$ by the morphism $g \mapsto (g, {}^t g^{-1})$.

The space $\mathrm{Ext}^1(E, E)$ splits into a direct sum of the spaces $\mathrm{Ext}^1(E_i^{(k)}, E_j^{(l)})$, and each of these summands is isomorphic to $\mathrm{Ext}^1(F_i^{(k)}, F_j^{(l)}) \otimes \mathrm{Hom}(V_i^{(k)}, V_j^{(l)})$ when neither k nor l equals 3, or to a sum of summands of this form otherwise. The isotropy groups act on each of those spaces via the natural actions of $\mathbf{GL}(V) \times \mathbf{GL}(V')$ on $\mathrm{Hom}(V, V')$.

An element $\omega = \sum \omega_{i,j}^{(k,l)} \in \mathrm{Ext}^1(E, E) \simeq \bigoplus \mathrm{Ext}^1(E_i^{(k)}, E_j^{(l)})$ belongs to the space $H^1(C, \mathrm{Ad}(P))$ if and only if $\omega_{i,i}^{(k,k)} \in H^1(C, \mathbf{\Lambda}^2 E_i^{(k)*}) \subset \mathrm{Ext}^1(E_i^{(k)}, E_i^{(k)})$ and, for $(i,k) \neq (j,l)$, $\sigma_j^{(l)} \omega_{i,j}^{(k,l)} + \omega_{j,i}^{(l,k)*} \sigma_i^{(k)} = 0$. So, identifying $\mathrm{Ext}^1(E_i^{(k)}, E_j^{(l)})$ with its image in $\mathrm{Ext}^1(E_i^{(k)}, E_j^{(l)}) \oplus \mathrm{Ext}^1(E_j^{(l)}, E_i^{(k)})$ by the application $\omega_{i,j}^{(k,l)} \mapsto \omega_{i,j}^{(k.l)} - \sigma_i^{(k)-1} \omega_{i,j}^{(k,l)*} \sigma_j^{(l)}$, it appears that $H^1(C, \mathrm{Ad}(P))$ is the subspace of $\mathrm{Ext}^1(E, E)$ isomorphic to the direct sum

$$\bigoplus_k \left(\bigoplus_i H^1(C, \mathbf{\Lambda}^2 E_i^{(k)*}) \oplus \bigoplus_{i<j} \mathrm{Ext}^1(E_i^{(k)}, E_j^{(k)}) \right) \oplus \bigoplus_{k<l} \bigoplus_{i,j} \mathrm{Ext}^1(E_i^{(k)}, E_j^{(l)}), \tag{2.3.1.2.2}$$

each one of the diagonal summand being more precisely expressed as:

$$(2.3.1.2.3)\quad H^1(C,\mathbf{\Lambda}^2E_i^{(1)*}) = \Big(H^1(C,\mathbf{S}^2F_i^{(1)*})\otimes\mathfrak{so}(V_i^{(1)})\Big)\oplus \Big(H^1(C,\mathbf{\Lambda}^2F_i^{(1)*})\otimes\mathbf{S}^2V_i^{(1)*}\Big),$$

$$(2.3.1.2.4)\quad H^1(C,\mathbf{\Lambda}^2E_j^{(2)*}) = \Big(H^1(C,\mathbf{\Lambda}^2F_j^{(2)*})\otimes\mathfrak{sp}(V_j^{(2)})\Big)\oplus \Big(H^1(C,\mathbf{S}^2F_j^{(2)*})\otimes\mathbf{\Lambda}^2V_j^{(2)*}\Big),$$

$$(2.3.1.2.5)\quad H^1(C,\mathbf{\Lambda}^2E_k^{(3)*}) = \Big(\mathrm{Ext}^1(F_k^{(3)},F_k^{(3)})\otimes\mathfrak{gl}(V_k^{(3)})\Big)\oplus \Big(H^1(C,\mathbf{S}^2F_k^{(3)*})\otimes\mathbf{\Lambda}^2V_k^{(3)*}\Big)\oplus\Big(H^1(C,\mathbf{\Lambda}^2F_k^{(3)*})\otimes\mathbf{S}^2(V_k^{(3)*})\Big)\oplus \Big(H^1(C,\mathbf{S}^2F_k^{(3)})\otimes\mathbf{\Lambda}^2V_k^{(3)*}\Big)\oplus\Big(H^1(C,\mathbf{\Lambda}^2F_k^{(3)})\otimes\mathbf{S}^2V_k^{(3)*}\Big),$$

where $\mathrm{Ext}^1(F_k^{(3)},F_k^{(3)})$ has been identified with its image in $\mathrm{Ext}^1(F_k^{(3)},F_k^{(3)})\oplus\mathrm{Ext}^1(F_k^{(3)*},F_k^{(3)*})$ by the map $\omega\mapsto\omega-\omega^*$. Note that the dimensions of all the extension spaces under consideration are trivially available.

Example 2.3.1.3. — In order to clarify a bit this discussion, let us focus a moment on the case of the trivial orthogonal bundle $E=\mathcal{O}\otimes V$ ($V=k^r$ quadratic space). In this situation, we have to understand the algebra of $\mathrm{Aut}_{\mathbf{O}_r}(E)=\mathbf{O}_r$-invariant polynomials defined on $\mathrm{Ext}^1_{\mathrm{asym}}(E,E)\subset\mathrm{Ext}^1(E,E)\simeq(\mathrm{M}_r(\mathbb{C}))^g$. We easily check that this is the subspace of g-tuples of antisymmetric matrices. On the other hand, the invariant algebra $k[(\mathrm{M}_r(\mathbb{C}))^g]^{\mathbf{O}_r}$ is already described in [**Pro76**]: it is generated by the functions

$$(M_1,\dots,M_g)\longmapsto\mathrm{tr}(A_{i_1}\cdots A_{i_l})$$

where $A_{i_k}\in\{M_{i_k},{}^tM_{i_k}\}$. The restriction of such a function to the set of g-tuples of antisymmetric matrices is thus clearly $\mathbf{GL}_r$-invariant. This proves the injectivity of the differential map at the trivial bundle.

In the same way we could see that the case of an orthogonal bundle $(F^{(1)}\otimes V)\oplus(F^{(2)}\otimes W)$ already results from Theorem 2.2.1.7.

2.3.1.4. Let us come back to the general case. The pretty intricate situation described in 2.3.1.2 suitably expresses itself in terms of representations of quivers. Indeed let us consider the quiver $\mathcal{Q}$ whose set of vertices is

$$\mathcal{Q}_0=\{s_1^{(1)},\dots,s_{n_1}^{(1)},s_1^{(2)},\dots,s_{n_2}^{(2)},s_1^{(3)},s_1^{(3^*)},\dots,s_{n_3}^{(3)},s_{n_3}^{(3^*)}\},$$

these vertices being connected by $\dim \operatorname{Ext}^1(F_i^{(k)}, F_j^{(l)})$ arrows from $s_i^{(k)}$ to $s_j^{(l)}$ (where we have set $F_i^{(3^*)} = F_i^{(3)*}$). Next define $\alpha \in \mathbb{N}^{n_1+n_2+2n_3}$ according to the dimensions of the corresponding vector spaces $V_l^{(a)}$. Therefore the $\operatorname{Aut}_{\mathbf{GL}_r}(E)$-module $\operatorname{Ext}^1(E,E)$ is exactly the $\mathbf{GL}(\alpha)$-module $R(\mathcal{Q},\alpha)$ composed of all the representations of $\mathcal{Q}$ of dimension α, and the result of [**LBP90**] recalled earlier provides us with a description of the algebra $k[\operatorname{Ext}^1(E,E)]^{\operatorname{Aut}_{\mathbf{GL}_r}(E)}$.

The inclusion $H^1(C,\operatorname{Ad}(P)) \hookrightarrow \operatorname{Ext}^1(E,E)$ is an $\operatorname{Aut}_{\mathbf{O}_r}(P) = \prod_{i=1}^{n_1} \mathbf{O}(V_i^{(1)}) \times \prod_{j=1}^{n_2} \mathbf{Sp}(V_j^{(2)}) \times \prod_{k=1}^{n_3} \mathbf{GL}(V_k^{(3)})$-equivariant application, so that we have an exact sequence

$$k[\operatorname{Ext}^1(E,E)]^{\operatorname{Aut}_{\mathbf{O}_r}(P)} \to k[H^1(C,\operatorname{Ad}(P))]^{\operatorname{Aut}_{\mathbf{O}_r}(P)} \to 0.$$

This sequence and the theorem 2.2.3.6 result in a set of generators for the algebra $k[H^1(C,\operatorname{Ad}(P))]^{\operatorname{Aut}_{\mathbf{O}_r}(P)}$, namely the $(f_a)_a \mapsto \operatorname{tr}(f_{\tilde{a}_p}\cdots f_{\tilde{a}_1})$, where $f_{\tilde{a}_i}$ stands for either f_{a_i} or its adjoint map. Now, according to (2.3.1.2.2), $H^1(C,\operatorname{Ad}(P))$ is a subspace of $R(\mathcal{Q},\alpha)$ made up of representations having the following property: if $f_a\colon V_v \to V_{v'}$ denotes the map associated to an arrow $a\colon v \to v'$, then its adjoint map $f_a^*\colon V_{v'}^* \to V_v^*$ is, up to the sign, the map associated to one of the arrows from v' to v. So the algebra $k[H^1(C,\operatorname{Ad}(P))]^{\operatorname{Aut}_{\mathbf{O}_r}(P)}$ is generated by traces along oriented cycles in the quiver $\mathcal{Q}$. This exactly means that the application $k[\operatorname{Ext}^1(E,E)]^{\operatorname{Aut}_{\mathbf{GL}_r}(E)} \to k[H^1(C,\operatorname{Ad}(P))]^{\operatorname{Aut}_{\mathbf{O}_r}(P)}$ is onto.

In view of what has been discussed in 2.3.1.1 this proves the injectivity of the tangent map of $\mathcal{M}_{\mathbf{O}_r} \to \mathcal{M}_{\mathbf{GL}_r}$ at $q=[P]$ (note that no hypothesis on the determinant of the orthogonal bundle P is needed in the previous discussion: injectivity of the tangent map thus holds at any point of any component of $\mathcal{M}_{\mathbf{O}_r}$). But we have shown in 2.1.1.2 that the map $\mathcal{M}_{\mathbf{O}_r}(k) \to \mathcal{M}_{\mathbf{GL}_r}(k)$ is injective. This implies the following:

Theorem 2.3.1.5. — *The forgetful map $(E,q) \in \mathcal{M}_{\mathbf{O}_r} \mapsto E \in \mathcal{M}_{\mathbf{GL}_r}$ is an embedding.*

One easily gets in the very same way the corresponding assertion relative to the moduli of symplectic bundles:

Theorem 2.3.1.6. — *The forgetful map $\mathcal{M}_{\mathbf{Sp}_{2r}} \to \mathcal{M}_{\mathbf{SL}_{2r}}$ is an embedding.*

The point is that any closed point of $\mathcal{M}_{\mathbf{Sp}_{2r}}$ represents a polystable vector bundle of the form

$$E = \bigoplus_i \Big(F_i^{(1)} \otimes V_i^{(1)} \Big) \oplus \bigoplus_j \Big(F_j^{(2)} \otimes V_j^{(2)} \Big) \oplus \bigoplus_k \Big((F_k^{(3)} \oplus F_k^{(3)\,*}) \otimes V_k^{(3)} \Big),$$

where $(F_i^{(1)})_i$ (resp. $(F_j^{(2)})_j$, resp. $(F_k^{(3)})_k$) is a family of mutually non isomorphic symplectic (resp. orthogonal, resp. not self-dual) bundles (which are stable as vector bundles), and $(V_i^{(1)})_i$ (resp. $(V_j^{(2)})_j$, resp. $(V_k^{(3)})_k$) a family of quadratic (resp. symplectic, resp. endowed with a non-degenerate bilinear pairing) vector spaces ($F_k^{(3)} \oplus F_k^{(3)\,*}$ being now equipped with the standard symplectic form). Let us denote by $\sigma\colon E \to E^*$ the resulting symplectic form on E. The bundle $\mathrm{Ad}(P)$ is now isomorphic to the bundle of germs of symmetric endomorphisms of E (that is endomorphisms verifying $\sigma f + f^*\sigma = 0$), and both the space $H^1(C, \mathrm{Ad}(P))$ and the considered isotropy groups can be described in a manner analogous to that of 2.3.1.2 (one only has to switch the factors $\mathbf{\Lambda}^2 F_l^{(a)*}$ and $\mathbf{S}^2 F_l^{(a)*}$, and of course to redefine in the obvious way every map of the form $\mathrm{Ext}^1(F, F') \to \mathrm{Ext}^1(F, F') \oplus \mathrm{Ext}^1(F'^*, F^*)$). The theorem 2.2.3.6 then allows us to conclude again.

Remark 2.3.1.7. — It would be interesting to try to generalize this study to the moduli of principal bundles over a singular curve.

2.3.2. About $\mathcal{M}_{\mathbf{SO}_r} \longrightarrow \mathcal{M}_{\mathbf{O}_r}$. —

2.3.2.1. We have recalled in 2.1.3 how to compute the fibers of the finite morphism from $\mathcal{M}_{\mathbf{SO}_r}$ onto $\mathcal{M}_{\mathbf{O}_r}^{\mathcal{O}} = \det^{-1}(\mathcal{O}_C)$. A point $[P] \in \mathcal{M}_{\mathbf{O}_r}$ in its image has two pre-images if and only if $\mathrm{Aut}_{\mathbf{SO}_r}(P) \hookrightarrow \mathrm{Aut}_{\mathbf{O}_r}(P)$ is an isomorphism, that is if and only if every orthogonal bundle $F_i^{(1)}$ appearing in the splitting (2.3.1.2.1) of E has even rank.

Luna's theorem reduces once again the differential study of this application to an invariant calculus: the tangent map of $\mathcal{M}_{\mathbf{SO}_r} \to \mathcal{M}_{\mathbf{O}_r}$ at $[P] \in \mathcal{M}_{\mathbf{SO}_r}$ is indeed identified with that of $H^1(C, \mathrm{Ad}(P)) /\!\!/ \mathrm{Aut}_{\mathbf{SO}_r}(P) \to H^1(C, \mathrm{Ad}(P)) /\!\!/ \mathrm{Aut}_{\mathbf{O}_r}(P)$ (at the origin).

Therefore, if r is odd, $\mathcal{M}_{\mathbf{SO}_r} \to \mathcal{M}_{\mathbf{O}_r}^{\mathcal{O}}$ is an isomorphism.

2.3.2.2. Let us consider now the even case. The morphism $\mathcal{M}_{\mathbf{SO}_r} \to \mathcal{M}_{\mathbf{O}_r}$ is then a 2-sheeted cover, which is étale above the locus of points having two antecedents. A branched point corresponds to an orthogonal polystable bundle E containing at least one subbundle isomorphic to $F_i^{(1)} \otimes V_i^{(1)}$ where $F_i^{(1)}$ is an orthogonal bundle of odd rank: we then have to understand the inclusion

$$k[H^1(C, \mathrm{Ad}(P))]^{\mathrm{Aut}_{\mathbf{O}_r}(P)} \hookrightarrow k[H^1(C, \mathrm{Ad}(P))]^{\mathrm{Aut}_{\mathbf{SO}_r}(P)}.$$

It is easy to produce a primitive element for the generic extension (which is of degree 2). First note that the vector space W obtained as the direct sum of the $V_i^{(1)}$ corresponding to the orthogonal bundles $F_i^{(1)}$ of odd rank has even dimension, and has an orthogonal structure inherited from the ones of the $V_i^{(1)}$. The space composed of all the antisymmetric endomorphisms of W may be identified with a direct summand of $H^1(C, \mathrm{Ad}(P))$, and mapping any element $\omega \in H^1(C, \mathrm{Ad}(P))$ to the pfaffian of the endomorphism of W induced by ω then defines a function belonging to $k[H^1(C, \mathrm{Ad}(P))]^{\mathrm{Aut}_{\mathbf{SO}_r}(P)}$ which is not $\mathrm{Aut}_{\mathbf{O}_r}(P)$-invariant; this function certainly generates the generic extension.

It is more difficult to give a convenient description of this algebra: in the (simplest) case where P is isomorphic to $\mathcal{O} \otimes V$ with V a quadratic vector space of even dimension, we have to understand the action of $\mathrm{Aut}_{\mathbf{SO}_r}(P) \simeq \mathbf{SO}_r$ on $H^1(C, \mathrm{Ad}(P)) \simeq H^1(C, \mathcal{O}) \otimes \mathfrak{so}(V)$. This can be solved again thanks to Procesi's trick (cf. 2.2.1.6): the computation has been carried out in [**ATZ95**], and provides a set of generators for the $k[H^1(C, \mathrm{Ad}(P))]^{\mathrm{Aut}_{\mathbf{O}_r}(P)}$-algebra $k[H^1(C, \mathrm{Ad}(P))]^{\mathrm{Aut}_{\mathbf{SO}_r}(P)}$ in terms of *polarized pfaffians*.

Let us finally mention that in the general case we can easily infer from the main result of [**Lop06**] a family of generators for $k[H^1(C, \mathrm{Ad}(P))]^{\mathrm{Aut}_{\mathbf{SO}_r}(P)}$ which are also obtained as polarized pfaffians.

CHAPTER 3

MORE ABOUT THE LOCAL STRUCTURE OF $\mathcal{M}_{\mathbf{SO}_r}$ AND $\mathcal{M}_{\mathbf{SL}_r}$

In this chapter we present a few results concerning the local structure of the moduli spaces of orthogonal bundles, and then use again the translation into the quiver setting to get additional results regarding the study of moduli of rank 3 vector bundles over a curve of genus 2 (as we have already noticed in the introduction, this seems to be the only case where we can obtain, with this method, a precise description at all points).

3.1. Informations about $\mathcal{M}_{\mathbf{SO}_r}$

3.1.1. The discussion held in 2.3.1.1 contains in fact a more precise statement, related to the completed local rings of $\mathcal{M}_{\mathbf{O}_r}$ and $\mathcal{M}_{\mathbf{GL}_r}$. Indeed, if q is a point of $\mathcal{M}_{\mathbf{O}_r}$ representing a polystable bundle P whose image in $\mathcal{M}_{\mathbf{GL}_r}$ is a point $s = [E]$, we have the following commutative diagram, where the rings of the second row are the completions of the local rings (of the involved algebras of invariants) at the origin,

$$\begin{tikzcd} \widehat{\mathcal{O}}_{\mathcal{M}_{\mathbf{GL}_r},s} \arrow[r] \arrow[d, "\wr"] & \widehat{\mathcal{O}}_{\mathcal{M}_{\mathbf{O}_r},q} \arrow[d, "\wr"] \\ \left(k[\mathrm{Ext}^1(E,E)]^{\mathrm{Aut}_{\mathbf{GL}_r}(E)}\right)^{\widehat{}} \arrow[r] & \left(k[H^1(C,\mathrm{Ad}(P))]^{\mathrm{Aut}_{\mathbf{O}_r}(P)}\right)^{\widehat{}}. \end{tikzcd} \tag{3.1.1.1}$$

This description of the completed local rings of $\mathcal{M}_{\mathbf{O}_r}$ provides us with additional informations about the local structure of $\mathcal{M}_{\mathbf{O}_r}$, at least at the points where the situation is not too bad (see [**Las96**] for the case of $\mathcal{M}_{\mathbf{GL}_r}$): the more we know about the second main theorem of invariant theory for the isotropy group of P, the easier our calculations will be.

3.1.2. Let P be an orthogonal bundle whose underlying vector bundle is of the form $E = E_1 \oplus E_2$, with E_1 and E_2 two non-isomorphic $\mathbf{GL}$-stable orthogonal bundles. The description of the inclusion $H^1(C, \mathrm{Ad}(P)) \hookrightarrow \mathrm{Ext}^1(E, E)$ given in (2.3.1.2.2) here reduces to

$$\begin{array}{ccccccccc} H^1(C,\mathrm{Ad}(P)) & = & H^1(C,\mathbf{\Lambda}^2 E_1{}^*) & \oplus & & \mathrm{Ext}^1(E_1,E_2) & & \oplus & H^1(C,\mathbf{\Lambda}^2 E_2{}^*) \\ \cap & & \cap & & & \cap & & & \cap \\ \mathrm{Ext}^1(E,E) & = & \mathrm{Ext}^1(E_1,E_1) & \oplus & \mathrm{Ext}^1(E_1,E_2) & \oplus & \mathrm{Ext}^1(E_2,E_1) & \oplus & \mathrm{Ext}^1(E_2,E_2); \end{array}$$

the isotropy subgroup $\mathrm{Aut}_{\mathbf{O}_r}(P)$, isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, acts trivially on $H^1(C, \mathbf{\Lambda}^2 E_i{}^*)$ and by multiplication by ± 1 on $\mathrm{Ext}^1(E_1, E_2)$ (while $(\alpha_1, \alpha_2) \in \mathrm{Aut}_{\mathbf{GL}_r}(E) \simeq \mathbb{G}_m \times \mathbb{G}_m$ acts on $\mathrm{Ext}^1(E_i, E_j)$ by $\alpha_j \alpha_i^{-1}$).

Let $(X_k^{(i)})_k$ (resp. $(Y_l)_l$) be a basis of $H^1(C, \mathbf{\Lambda}^2 E_i{}^*)^*$ (resp. $\mathrm{Ext}^1(E_1, E_2)^* \subset H^1(C, \mathrm{Ad}(P))^*$). Then $k[H^1(C, \mathrm{Ad}(P))]^{\mathrm{Aut}_{\mathbf{O}_r}(P)}$ is the subring of $k[X_k^{(i)}, Y_l]$ generated by all the $X_k^{(i)}$ and the products $Y_l Y_{l'}$; if $\mathcal{V}$ denotes the affine cone over the Veronese variety $\mathbb{P}(\mathrm{Ext}^1(E_1, E_2)) \subset \mathbb{P}\left(\mathbf{S}^2\mathrm{Ext}^1(E_1, E_2)\right)$ we get the following isomorphism:

$$\mathrm{Spec}(k[H^1(C, \mathrm{Ad}(P))]^{\mathrm{Aut}_{\mathbf{O}_r}(P)}) \xrightarrow{\sim} \Big(H^1(C, \mathrm{Ad}(P_1)) \oplus H^1(C, \mathrm{Ad}(P_2))\Big) \times \mathcal{V}.$$

Using the identification $\widehat{\mathcal{O}}_{\mathcal{M}_{\mathbf{O}_r}, q} \simeq \Big(k[H^1(C, \mathrm{Ad}(P))]^{\mathrm{Aut}_{\mathbf{O}_r}(P)}\Big)^{\widehat{}}$ we have the following result:

Proposition 3.1.3. — *The tangent space to $\mathcal{M}_{\mathbf{O}_r}$ at a point $[P]$ given as the direct sum $E_1 \oplus E_2$ of two non-isomorphic stable orthogonal bundles is isomorphic to*

$$H^1(C, \mathbf{\Lambda}^2 E_1{}^*) \oplus H^1(C, \mathbf{\Lambda}^2 E_2{}^*) \oplus (\mathbf{S}^2\mathrm{Ext}^1(E_1, E_2)),$$

and the multiplicity of $\mathcal{M}_{\mathbf{O}_r}$ at this point is equal to $2^{r_1 r_2(g-1)-1}$, where r_i is the rank of E_i.

Remark 3.1.4. — (i) The general case of a stable point $q \in \mathcal{M}_{\mathbf{O}_r}$ is more difficult: such a bundle corresponds to a vector bundle which splits as a direct sum of n mutually non-isomorphic $\mathbf{GL}_{r_i}$-stable orthogonal vector bundles. We can use 2.2.3.6 to try to get some additional informations about the local structure at q, for instance by computing the multiplicity at this point. One can easily check that, if $n = 3$ (resp. 4) this multiplicity is equal to $2\prod_{i<j} 2^{r_i r_j (g-1)-1}$ (resp. $8\prod_{i<j} 2^{r_i r_j (g-1)-1}$).

(ii) It is not hard to deal with the case of an orthogonal (non stable) bundle of the form $F \oplus F^*$ with $F \not\simeq F^*$: we see that $\mathcal{M}_{\mathbf{O}_r}$ is, at such a point, étale locally isomorphic to $\mathrm{Ext}^1(F,F) \oplus \mathcal{S}$, where $\mathcal{S}$ is the affine cone over the Segre variety

$$\mathbb{P}(H^1(C, \mathbf{\Lambda}^2 F^*)) \times \mathbb{P}(H^1(C, \mathbf{\Lambda}^2 F)) \subset \mathbb{P}\left(H^1(C, \mathbf{\Lambda}^2 F^*) \otimes H^1(C, \mathbf{\Lambda}^2 F)\right).$$

(iii) Of course, we can give some similar results in the symplectic case.

3.2. Local description of $\mathcal{SU}_C(3)$ for a curve of genus 2

Let C be a smooth irreducible projective curve of genus 2 over an algebraically closed field k of characteristic zero, and let $\mathcal{SU}_C(3)$ be the moduli space of rank 3 vector bundles over C with trivial determinant. Laszlo began to investigate the local structure of this moduli space in [**Las96**, V]. We compute here the local structure at any point of $\mathcal{SU}_C(3)$. We will in particular prove the following result:

Theorem 3.2.1. — *The moduli space of rank* 3 *vector bundles over a curve of genus* 2 *is a local complete intersection.*

The notion of representations of quivers appears to be really helpful to understand the quotients given by Luna's result. Although it may not be clear in this section, where we could have given direct proofs avoiding such considerations, this quiver setting was the very basic point which led to generating sets for the coordinate rings of the quotients.

3.2.2. We know that, at a closed point representing a polystable bundle E, the moduli space $\mathcal{SU}_C(r)$ of rank r vector bundles with trivial determinant is étale locally isomorphic to the quotient $\mathrm{Ext}^1(E,E)_0 /\!\!/ \mathrm{Aut}(E)$ at the origin, where $\mathrm{Ext}^1(E,E)_0$ denotes the kernel of $\mathrm{tr}\colon \mathrm{Ext}^1(E,E) \to H^1(C, \mathcal{O}_C)$. We thus have to understand the ring of invariants of the polynomial algebra $k[\mathrm{Ext}^1(E,E)_0] = \mathrm{Sym}(\mathrm{Ext}^1(E,E)_0{}^*)$ under the action of $\mathrm{Aut}(E)$. Once again, we decompose the polystable bundle E as the direct sum

$$E = \bigoplus_{i=1}^{s} E_i \otimes V_i, \tag{3.2.2.1}$$

where the E_i's are mutually non-isomorphic stable bundles (of rank r_i and degree 0), and the V_i's are vector spaces (of dimension ρ_i). Through this splitting, our data become

$$\mathrm{Ext}^1(E,E) = \bigoplus_{i,j} \mathrm{Ext}^1(E_i, E_j) \otimes \mathrm{Hom}(V_i, V_j), \tag{3.2.2.2}$$

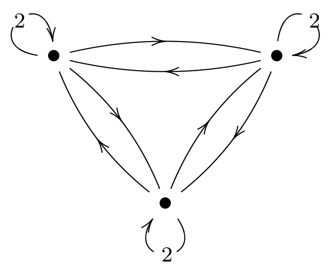
endowed with the natural operation of $\mathrm{Aut}(E) = \prod_i \mathbf{GL}(V_i)$, which is exactly the $\mathbf{GL}(\alpha)$-module $R(Q, \alpha)$ consisting of all representations of Q of dimension α, where Q is the quiver with s vertices $1, \ldots, s$, and $\dim \mathrm{Ext}^1(E_i, E_j)$ arrows from i to j, and $\alpha = (\rho_i)_{i=1,\ldots,s}$. This identifies the quotient $\mathrm{Ext}^1(E, E)_0 /\!\!/ \mathrm{Aut}(E)$ we have in mind with a closed subscheme of $R(Q, \alpha) /\!\!/ \mathbf{GL}(\alpha)$. We have recalled in the Chapter 2 the fine description found by Le Bruyn and Procesi of a family of generators for this algebra. But we also need a precise description of the relations between these generators (the *second main theorem for invariant theory*). Once we have a convenient enough statement about these relations we can describe the completed local ring of $\mathcal{SU}_C(r)$ at E.

3.2.3. When $r = 3$ the decomposition (3.2.2.1) ensures that there are only five cases to deal with, according to the values of the r_i's and ρ_i's. We will now consider these five cases one by one.

The case of a stable bundle is obvious, and the case $r_1 = 2, r_2 = 1$ is a special case of the situation studied in [**Las96**, III]: $\mathcal{SU}_C(3)$ is étale isomorphic at E to a rank 4 quadric in $\mathbb{A}^9$. Here quivers do not provide a shorter proof.

3.2.4. Let us look at the three other cases, where every E_i in (3.2.2.1) is invertible. The generic case consists of bundles E which are direct sum of 3 distinct line bundles. It has already been performed in [**Las96**, V], but may also be recovered in a more convenient fashion as an easy consequence of [**LBP90**]: the generators of [**Las96**, Lemma V.1] then arise nicely as traces along closed cycles in the quiver

(3.2.4.1)



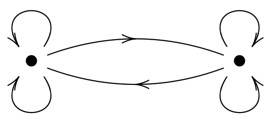
(note that a number r on an arrow means that there are r arrows with the same tail and head). Since the dimension vector is $\alpha = (1, 1, 1)$, we can in fact restrict ourselves to the quiver where the six loops have been removed, and a family of generators is obtained by the three 2-cycles and the two 3-cycles. It is easy too to infer from (3.2.4.1) the relation found by Laszlo; but, although

[**LBP90**] gives a way to produce all the relations, this description turns out to be quite inefficient even in the present case (note however that, in order to conclude here, it is enough to remind that we know a priori the dimension of $\mathrm{Ext}^1(E,E)/\!\!/\mathrm{Aut}(E)$).

In the remaining two cases we already know that the tangent cone at E must be a quadric (in $\mathbb{A}^9$) of rank $\leqslant 2$ (see [**Las96**, V]). We give now more precise statements.

3.2.5. Suppose that $\rho_1 = 2$, i.e. that $E = (L \otimes V) \oplus L^{-2}$ where L is a line bundle of degree 0 with $L^3 \not\simeq \mathcal{O}$ and V a vector space of dimension 2. We have to consider here the ring of invariant polynomials on the representation space $R(Q,(2,1))$ of the quiver Q

(3.2.5.1)



under the action of $\mathbf{GL}(V) \times \mathbb{G}_m$. Since the second vertex corresponds to a 1-dimensional vector space it is enough to consider the quiver obtained by deleting the two loops on the right, and in fact we are brought to the action of $\mathbf{GL}(V)$ on $\mathrm{End}(V) \oplus \mathrm{End}(V) \oplus \mathrm{End}(V)^{\leqslant 1} \subset \mathrm{End}(V)^{\oplus 3}$, where $\mathrm{End}(V)^{\leqslant 1}$ denotes the space of endomorphisms of V of rank at most 1: this simply means that

$$k[R(Q,(2,1))]^{\mathbf{GL}(V)\times\mathbb{G}_m} \simeq \Big(k[R(Q,(2,1))]^{\mathbb{G}_m}\Big)^{\mathbf{GL}(V)},$$

and that $k[R(Q,(2,1))]^{\mathbb{G}_m}$ gets naturally identified (as a $\mathbf{GL}(V)$-module) with $\mathrm{End}(V) \oplus \mathrm{End}(V) \oplus \mathrm{End}(V)^{\leqslant 1} \oplus k \oplus k$, the last two summands being fixed under the induced operation of $\mathbf{GL}(V)$.

Let us now translate this discussion in a more geometric setting. Since (3.2.2.2) identifies here $\mathrm{Ext}^1(E,E)$ with

$$\big(H^1(C,\mathcal{O})\otimes\mathrm{End}(V)\big)\oplus\big(H^1(C,L^{-3})\otimes V^*\big)\oplus\big(H^1(C,L^3)\otimes V\big)\oplus\big(H^1(C,\mathcal{O})\otimes k\big),$$

we can identify the $\mathrm{Aut}(E)$-module $\mathrm{Ext}^1(E,E)_0$ with the $\mathbf{GL}(V)\times\mathbb{G}_m$-module

$$\big(H^1(C,\mathcal{O})\otimes\mathrm{End}(V)\big)\oplus\big(H^1(C,L^{-3})\otimes V^*\big)\oplus\big(H^1(C,L^3)\otimes V\big),$$

so that, up to the choices of some basis of the different cohomology spaces, any element of $\mathrm{Ext}^1(E,E)_0$ can be written $(a_1,a_2,\lambda,v) \in \mathrm{End}(V)\oplus\mathrm{End}(V)\oplus V^*\oplus V$. The map $(a_1,a_2,\lambda,v) \mapsto (a_1,a_2,a_3=\lambda\otimes v) \in \mathrm{End}(V)^{\oplus 3}$ identifies the quotient $\mathrm{Ext}^1(E,E)_0/\!\!/\mathrm{Aut}(E)$ with the closed subscheme of $\mathrm{End}(V)^{\oplus 3}/\!\!/\mathbf{GL}(V)$

defined by the equation $\det a_3 = 0$. A presentation of the invariant algebra $k[\mathrm{End}(V)^{\oplus 3}]^{\mathbf{GL}(V)}$ can be found in [**Dre03**] (note that another presentation of this ring had been previously given in [**For84**]): if we let b_i denote the traceless endomorphism $a_i - \frac{1}{2}\mathrm{tr}(a_i)\mathrm{id}$, this invariant ring is generated by the following ten functions

$$u_i = \mathrm{tr}(a_i) \text{ with } 1 \leqslant i \leqslant 3,\ v_{ij} = \mathrm{tr}(b_i b_j) \text{ with } 1 \leqslant i \leqslant j \leqslant 3,$$

$$w = \sum_{\sigma \in \mathfrak{S}_3} \varepsilon(\sigma)\mathrm{tr}(b_{\sigma(1)} b_{\sigma(2)} b_{\sigma(3)}), \tag{3.2.5.2}$$

subject to the single relation $w^2 + 18\det(v_{ij}) = 0$. We have thus obtained the following result:

Proposition 3.2.6. — *If $E = (L \otimes V) \oplus L^{-2}$ with $L^3 \not\simeq \mathcal{O}$, then $\mathcal{SU}_C(3)$ is étale locally isomorphic at E with the subscheme of $\mathbb{A}^{10}$ defined by the two equations*

$$X_{10}^2 + 18(X_4X_5X_6 + 2X_7X_8X_9 - X_6X_7^2 - X_5X_8^2 - X_4X_9^2) = 0$$

$$\textit{and } X_3^2 - 2X_6 = 0$$

at the origin. Its tangent cone is a double hyperplane in $\mathbb{A}^9$.

3.2.7. Suppose now that $\rho_1 = 3$, i.e. that $E = L \otimes V$ where V is a vector space of dimension 3 (and L a line bundle of order 3). By the same argument as in [**Las96**, Proposition V.4] we know that the tangent cone at such a point is a rank 1 quadric. But an explicit description of an étale neighbourhood is available, thanks to [**ADS06**]. The space $\mathrm{Ext}^1(E,E)_0$ is isomorphic to $H^1(C,\mathcal{O}) \otimes \mathrm{End}_0(V)$ and, if we fix a basis of $H^1(C,\mathcal{O})$, any of its element can be written $(x,y) \in \mathrm{End}_0(V) \oplus \mathrm{End}_0(V)$. The ring of invariants $k[H^1(C,\mathcal{O}) \otimes \mathrm{End}_0(V)]^{\mathbf{GL}(V)}$ is then generated by the nine functions $\mathrm{tr}(x^2)$,$\mathrm{tr}(xy)$, $\mathrm{tr}(y^2)$, $\mathrm{tr}(x^3)$, $\mathrm{tr}(x^2y)$, $\mathrm{tr}(xy^2)$, $\mathrm{tr}(y^3)$, $v = \mathrm{tr}(x^2y^2) - \mathrm{tr}(xyxy)$ and $w = \mathrm{tr}(x^2y^2xy) - \mathrm{tr}(y^2x^2yx)$; moreover the ideal of relations is principal, generated by an explicit equation written in (*loc. cit.*):

$$w^2 = -\frac{4}{27}v^3 + \text{ higher degree terms.} \tag{3.2.7.1}$$

As a result of this case-by-case analysis, we conclude that $\mathcal{SU}_C(3)$ is a local complete intersection, which is the expected Theorem 3.2.1.

3.2.8. This detailed study of the local structure of $\mathcal{M}_{\mathbf{SL}_3}$ could allow us to express the tangent space at a point E in terms of the stable bundles given by the Jordan-Hölder filtration. Such a description can in turn be used to find the smallest *very ample* power of the determinant bundle $\mathcal{L}_{\mathbf{SL}_3}$, in the spirit of what has been done by Esteves and Popa in [**EP04**] on the smooth locus. However, there is in this case an easier way to answer this question: since the theta map is a finite morphism onto $\mathbb{P}^8$ such that $\theta_*\mathcal{O}_{\mathcal{M}_{\mathbf{SL}_3}} \simeq \mathcal{O}_{\mathbb{P}^8} \oplus \mathcal{O}_{\mathbb{P}^8}(-3)$, we easily see that $\mathcal{L}^d_{\mathbf{SL}_3}$ is very ample if and only if $d \geqslant 3$.

3.3. Application: local structure of $\mathcal{S}$

Let now Θ be the canonical Theta divisor on the variety J^1 which parametrizes line bundles of degree 1 on C. It is known for long that the theta map $\theta\colon \mathcal{SU}_C(3) \longrightarrow |3\Theta|$ is a double covering. As we have already recalled in the introduction, Angela Ortega has shown in [**Ort05**] that its branch locus $\mathcal{S} \subset |3\Theta|$ is a sextic hypersurface which is the dual of the Coble cubic $\mathcal{C} \subset |3\Theta|^*$, where the Coble cubic is the unique cubic in $|3\Theta|^*$ which is singular along $J^1 \xrightarrow{|3\Theta|} |3\Theta|^*$ (note that a different proof of this statement has been given by Nguyễn in [**Ngu07**]). The last part of this chapter is devoted to the study of the local structure of the sextic $\mathcal{S}$.

We know from [**Ort05**] that the involution σ associated to the double covering given by the theta map

$$\theta\colon \mathcal{SU}_C(3) \longrightarrow |3\Theta|$$

acts by $E \mapsto \iota^* E^*$, where ι stands for the hyperelliptic involution. The local study of its ramification locus thus reduces to an explicit analysis of the behaviour of σ through the étale morphisms resulting from Luna's theorem.

Proposition 3.3.1. — *The tangent cone at any singular point P of the sextic $\mathcal{S}$ is described in the following table:*

Type of the point P	Tangent cone in $\mathbb{A}^8$
$F \oplus L$	Rank 3 quadric
$L_1 \oplus L_2 \oplus L_3$	Rank 1 quadric
$L \oplus L \oplus L^{-2}$	Cubic hypersurface
$L \oplus L \oplus L$	Triple hyperplane

Once again it comes down to a case-by-case investigation.

3.3.2. When E is stable there is nothing to say. If $E = F \oplus L$ (with F a stable bundle of rank 2 and $L = (\det F)^{-1}$) we have to understand the action of the linearization of σ on

$$\mathrm{Ext}^1(E,E)_0 \simeq \mathrm{Ext}^1(F,F) \oplus \mathrm{Ext}^1(F,L) \oplus \mathrm{Ext}^1(L,F)$$

(note that we tacitly identify $\mathrm{Ext}^1(F,F)$ with its image in $\mathrm{Ext}^1(F,F) \oplus H^1(C,\mathcal{O}) \subset \mathrm{Ext}^1(E,E)$ by the map $\omega \mapsto (\omega, -\mathrm{tr}(\omega))$).

Since $\sigma(E) = E$, ι^*F^* must be isomorphic to F, and σ identifies $\mathrm{Ext}^1(F,L)$ and $\mathrm{Ext}^1(L,F)$; let us choose a basis X_1, X_2 of $\mathrm{Ext}^1(F,L)^*$, and call Y_1, Y_2 the corresponding basis of $\mathrm{Ext}^1(L,F)$. We need here to recall precisely from [**Las96**] the explicit description of the coordinate ring of $\mathrm{Ext}^1(E,E)_0/\!\!/\mathrm{Aut}(E)$ mentionned in 3.2.3: it is generated by $k[\mathrm{Ext}^1(F,F)]$ and the four functions $u_{ij} = X_iY_j$, subject to the relation $u_{11}u_{22} - u_{12}u_{21} = 0$.

It follows from our choice that σ maps u_{ij} to u_{ji}. Furthermore we claim that σ acts identically on $\mathrm{Ext}^1(F,F)$: as a stable bundle, F corresponds to a point of the moduli space $\mathcal{U}(2,0)$, whose tangent space is precisely isomorphic to $\mathrm{Ext}^1(F,F)$. The action of σ on this vector space is the linearization of the one of $F \in \mathcal{U}(2,0) \mapsto \iota^*F^*$. Using that $\mathcal{U}(2,0)$ is a Galois quotient of $J_C \times \mathcal{SU}_C(2)$, our claim comes from the fact that σ is trivial on both J_C and $\mathcal{SU}_C(2)$.

Since the coordinate ring of the fixed locus of σ in $\mathrm{Ext}^1(E,E)_0/\!\!/\mathrm{Aut}(E)$ is the quotient of the one of $\mathrm{Ext}^1(E,E)_0/\!\!/\mathrm{Aut}(E)$ by the involution induced by σ we may conclude that $\mathcal{S}$ is étale locally isomorphic at E to the quadric cone in $\mathbb{A}^8$ defined by $X_3^2 - X_1X_2 = 0$.

3.3.3. Consider now the situation of 3.2.4: let us write $E = L_1 \oplus L_2 \oplus L_3$ with $L_i \not\simeq L_j$ if $i \neq j$. We have $\mathrm{Ext}^1(E,E) \simeq \bigoplus_{i,j} \mathrm{Ext}^1(L_i,L_j)$; let us choose for $i \neq j$ a non-zero element X_{ij} of $\mathrm{Ext}^1(L_i,L_j)^*$ such that X_{ji} corresponds to X_{ij} through the isomorphism $\mathrm{Ext}^1(L_i,L_j) \simeq \mathrm{Ext}^1(L_j,L_i)$ induced by σ and the natural isomorphisms $\iota^*L_i^* \simeq L_i$. It then follows from 3.2.4 (see [**Las96**] for a complete proof) that the ring $k[\mathrm{Ext}^1(E,E)_0]^{\mathrm{Aut}(E)}$ is generated by $k[\ker(\bigoplus_i \mathrm{Ext}^1(L_i,L_i) \to H^1(C,\mathcal{O}))]$ and the five functions $Y_1 = X_{23}X_{32}$, $Y_2 = X_{13}X_{31}$, $Y_3 = X_{12}X_{21}$, $Y_4 = X_{12}X_{23}X_{31}$, $Y_5 = X_{13}X_{32}X_{21}$, subject to the relation $Y_4Y_5 - Y_1Y_2Y_3 = 0$. One easily checks that the involution σ fixes $k[\ker(\bigoplus_i \mathrm{Ext}^1(L_i,L_i) \to H^1(C,\mathcal{O}))]$, Y_1, Y_2 and Y_3, while it sends Y_4 to Y_5. The fixed locus $\mathrm{Fix}(\sigma)$ is then defined by the equation $Y_4 - Y_5 = 0$, so that $\mathcal{S}$ is étale locally isomorphic to the hypersurface in $\mathbb{A}^8$ defined by $Z_4^2 - Z_1Z_2Z_3 = 0$. Its tangent cone is a double hyperplane.

3.3.4. In the situation of 3.2.5 we have to make a more precise choice of the non-zero elements of $\mathrm{Ext}^1(L^{-2}, L)$ and $\mathrm{Ext}^1(L, L^{-2})$, so as to make them correspond through σ and the natural isomorphism $\iota^* L^* \simeq L$; such a choice ensures that σ operates on $\mathrm{Ext}^1(E, E)_0$ in the following way:

$$(x, y, \lambda, v) \in \mathrm{End}(V)^{\oplus 2} \oplus V^* \oplus V \mapsto ({}^t x, {}^t y, {}^t v, {}^t \lambda),$$

so that we know how it acts on the generators of $k[\mathrm{Ext}^1(E, E)_0 /\!\!/ \mathrm{Aut}(E)]$ given in (3.2.5.2): σ fixes u_i, v_{ij}, and sends w to $-w$. This implies that the fixed locus is defined by the equation $w = 0$. The sextic $\mathcal{S}$ is étale locally isomorphic to the subscheme of $\mathbb{A}^9$ whose ideal is generated by the two equations

$$X_4 X_5 X_6 + 2X_7 X_8 X_9 - X_6 X_7^2 - X_5 X_8^2 - X_4 X_9^2 = 0 \text{ and } X_3^2 - 2X_6 = 0;$$

its tangent cone is therefore the cubic hypersurface of $\mathbb{A}^8$ defined by $2X_7 X_8 X_9 - X_5 X_8^2 - X_4 X_9^2 = 0$.

3.3.5. We are now left with the last case, where E is of the form $L \otimes V$ (with $L^3 = \mathcal{O}$): $\mathrm{Ext}^1(E, E)_0$ is then isomorphic to $H^1(C, \mathcal{O}) \otimes \mathrm{End}_0(V)$, and σ acts by $\omega \otimes a \in H^1(C, \mathcal{O}) \otimes \mathrm{End}_0(V) \longmapsto \omega \otimes {}^t a$. This induces an action on $k[H^1(C, \mathcal{O}) \otimes \mathrm{End}_0(V)]^{\mathrm{Aut}(E)}$ which fixes the first eight generators of 3.2.7, and acts by -1 on the last one, namely w; the fixed locus is thus defined in $\mathrm{Ext}^1(E, E)_0 /\!\!/ \mathrm{Aut}(E)$ by the linear equation $w = 0$.

The sextic $\mathcal{S}$ is then étale locally isomorphic to an hypersurface in $\mathbb{A}^8$ defined by an explicit equation: (3.2.7.1) shows that its tangent cone is a triple hyperplane.

CHAPTER 4

ORTHOGONAL BUNDLES OF RANK 3 OVER A GENUS 2 CURVE

In this chapter, we give elements of description of the moduli spaces of rank 3 orthogonal bundles over curves of genus 2. As we will see, this will be done in a somehow computational way, which is not well suited in higher genus.

We also show that the theta map has no base point in $\mathcal{M}^{\pm}_{\mathbf{SO}_3}$, for every curve, which gives a new evidence towards [**Bea06b**, Conjecture 6.2] (Raynaud had shown that there is no base point in the whole $\mathcal{M}_{\mathbf{SL}_3}$ only for a generic curve).

4.1. Preliminaries

4.1.1. The quotient maps. — The classical isomorphism between $\mathbf{SO}_3$ and $\mathbf{PGL}_2$ gives rise to a morphism

$$E \in \mathcal{M}_{\mathbf{GL}_2} \longmapsto (\mathcal{E}nd_0(E), \det) \in \mathcal{M}_{\mathbf{SO}_3}$$

which actually provides two J_2-quotients $\pi\colon \mathcal{M}^0_{\mathbf{SL}_2} \longrightarrow \mathcal{M}^+_{\mathbf{SO}_3}$ and $\pi\colon \mathcal{M}^1_{\mathbf{SL}_2} \longrightarrow \mathcal{M}^-_{\mathbf{SO}_3}$, where $\mathcal{M}^d_{\mathbf{SL}_2} \simeq \mathcal{SU}(2,d)$: this follows for instance from [**MFK94**, Proposition 0.2], and the assertion about the target spaces results from the formula for the Stiefel-Whitney class recalled in (1.3.1.1) and the equality $h^0(\mathcal{E}nd_0(E)\otimes\kappa) = \deg(E) + h^0(\kappa) \mod 2$ ([**Bea91**, 1.1.a)]).

We will therefore have to analyze the quotients of $\mathcal{M}^d_{\mathbf{SL}_2}$ by the natural action of J_2 by tensorisation.

4.1.2. The Picard groups. — We recall from [**BLS98**] some facts about the Picard groups of the involved spaces, and the behaviour of the quotient maps.

We have already said that the Picard group of each component $\mathcal{M}^{\pm}_{\mathbf{SO}_3}$ is infinite cyclic, generated by the determinant bundle $\mathcal{L}^{\pm}_{\mathbf{SO}_3}$. The key point is that their pull-back to $\mathcal{M}^d_{\mathbf{SL}_2}$ are $\pi^*\mathcal{L}^+_{\mathbf{SO}_3} = \mathcal{L}^{\otimes 4}_{\mathbf{SL}_2}$ and $\pi^*\mathcal{L}^-_{\mathbf{SO}_3} = \mathcal{L}^{\otimes 2}_{\mathcal{M}^1_{\mathbf{SL}_2}}$: this is [**BLS98**, Proposition 10.1].

4.2. The theta map $\mathcal{M}^+_{\mathbf{SO}_3} \longrightarrow |3\Theta|^+$ is a morphism

The aim of this section is to study the base locus of the theta map. [**Ray82**] implies that the theta map $\mathcal{M}_{\mathbf{SL}_3} \dashrightarrow |3\Theta|$ is a morphism for a generic curve of any genus. For orthogonal bundles, this result can be improved:

Theorem 4.2.1. — *The theta map $\theta_3^+ \colon \mathcal{M}^+_{\mathbf{SO}_3} \longrightarrow |3\Theta|^+$ is a morphism for every curve C.*

We have to find a family of hyperplanes $(H_i)_i$ on $|3\Theta|^+$ such that the intersection of their pull-back ${\theta_3^+}^*H_i$ is empty (according to [**Bea06a**], it is equivalent to produce divisors belonging to $|\mathcal{L}^+_{\mathbf{SO}_3}|$ with no common point). This will be done with the help of [**Bea91**, §1], using the commutative diagram

$$\begin{tikzcd} \mathcal{M}_{\mathbf{SL}_2} \arrow[d, "\pi"] \arrow[dr, dashed, "\varphi"] & \\ \mathcal{M}^+_{\mathbf{SO}_3} \arrow[r, dashed, "\theta_3^+"] & {|3\Theta|^+} \end{tikzcd}$$

Consider, for any *even* theta-characteristic κ, the hyperplane[(1)] $H_\kappa = \{D \in |3\Theta|^+ | \kappa \in D\}$. Its pull-back to $\mathcal{M}^+_{\mathbf{SO}_3}$ is the determinant divisor $\Theta_\kappa = \{(E,q) | h^0(C, E\otimes\kappa) \geqslant 1\}$ (which is a divisor in view of [**Bea06a**], or else in view of what follows). The quotient π (or rather its restriction to the stable locus $\mathcal{M}^{\mathrm{s}}_{\mathbf{SL}_2} \subset \mathcal{M}_{\mathbf{SL}_2}$) corresponds to the rank 3 vector bundle $\mathcal{E}nd_0$ on $\mathcal{M}^{\mathrm{s}}_{\mathbf{SL}_2} \times C$, which is a universal family for traceless endomorphisms of stable bundles (recall that its existence follows from Kempf's descent lemma [**DN89**, Théorème 2.3], applied to the rank 3 vector bundle $\mathcal{E}nd_0(\mathcal{U})$, where $\mathcal{U}$ denotes the universal rank 2 vector bundle defined on $R_{\mathbf{SL}_2} \times C$, see 1.2.13). The pull-back of Θ_κ to $\mathcal{M}_{\mathbf{SL}_2}$ therefore is the determinant divisor associated to the vector bundle $\mathcal{E}nd_0 \otimes p_C^*\kappa$.

This divisor has been considered in [**Bea91**, 1.6], where it is denoted D'_κ: it admits a square root $D_\kappa \in |\mathcal{L}^2_{\mathcal{M}_{\mathbf{SL}_2}}|$ (which is in fact the *pfaffian divisor* associated to κ constructed in [**LS97**]). This square root is the divisor of zeros

(1) This is indeed an hyperplane because κ must belong to every odd divisor linearly equivalent to 3Θ, so that it cannot also belong to every even divisor in $|3\Theta|$.

of a section d_κ of $\mathcal{L}^2_{\mathbf{SL}_2}$, and [**Bea91**, Théorème 1.2] ensures that $(d_\kappa)_{\kappa \text{ even}}$ forms a basis of $\mathrm{H}^0(\mathcal{M}_{\mathbf{SL}_2}, \mathcal{L}^2_{\mathbf{SL}_2})$. Now, since it follows from [**Ray82**] that $\mathcal{L}_{\mathbf{SL}_2}$ already is globally generated, the intersection of the determinant divisors $(\pi^*\Theta_\kappa)_{\kappa \text{ even}}$ must be empty: φ is a morphism, and so does θ_3.

4.2.2. The description of $\mathcal{M}^+_{\mathbf{SO}_3}$ as a quotient of $\mathcal{M}_{\mathbf{SL}_2}$ naturally leads us to consider the theta map

$$\theta_2 \colon \mathcal{M}_{\mathbf{SL}_2} \longrightarrow |2\Theta|.$$

When C is non hyperelliptic, this is an embedding (see [**BV96**] and [**vGI01**]). When C is hyperelliptic, θ_2 factors through the action of the hyperelliptic involution ι: the induced map $\mathcal{M}_{\mathbf{SL}_2}/\iota^* \longrightarrow |2\Theta|$ is an embedding by [**Bea88**], whose image admits an explicit description (see [**DR77**] and [**vG88**]).

The natural representation of the theta group $\mathcal{G}(\mathcal{O}_{J^{g-1}}(2\Theta))$ in the space of sections $\mathrm{H}^0(J^{g-1}, \mathcal{O}_{J^{g-1}}(2\Theta))$ induces an action of the 2-torsion subgroup $J_2 \subset J$ on $|2\Theta|$, such that the theta map is J_2-equivariant. From [**BNR89**], we know that the following diagram

$$\begin{tikzcd}
 & \mathbb{P}\mathrm{H}^0(\mathcal{M}_{\mathbf{SL}_2}, \mathcal{L}_{\mathbf{SL}_2}) \arrow[d, "\wr"] \\
\mathcal{M}_{\mathbf{SL}_2} \arrow[ur, "\varphi_{\mathcal{L}}"] \arrow[r, "\theta_2"'] & |2\Theta|
\end{tikzcd}$$

is commutative, which implies that the vertical isomorphism is J_2 equivariant (in fact this also results from the duality between $\mathrm{H}^0(\mathcal{M}_{\mathbf{SL}_2}, \mathcal{L}_{\mathbf{SL}_2})$ and $\mathrm{H}^0(J^{g-1}, 2\Theta)$ described in [**Bea88**]).

Remark 4.2.3. — The fact that the same observations hold for any r enables us to hope to get something about the geometry of the moduli space of $\mathbf{PGL}_r$-bundles $\mathcal{M}^0_{\mathbf{PGl}_r} \simeq \mathcal{M}_{\mathbf{SL}_r}/J_r$ as soon as we have precise enough informations about the corresponding theta map. For example in genus 2 rank 3 we get

$$\begin{tikzcd}
\mathcal{M}_{\mathbf{SL}_3} \arrow[r, "\theta_3"] \arrow[d, "\pi_3"', "J_3"] & |3\Theta| \arrow[d, "J_3"] \\
\mathcal{M}^0_{\mathbf{PGL}_3} \arrow[r, "2:1"] & |3\Theta|/J_3,
\end{tikzcd}$$

where the quotient $|3\Theta|/J_3$ comes from the Heisenberg representation. Note that we know that $\pi_3^*\mathcal{L}_{\mathcal{M}_{\mathbf{PGL}_3}}$ is $\mathcal{L}^3_{\mathbf{SL}_3}$.

4.2.4. When $r = 2$, the morphism $L \in J \longmapsto L \oplus L^{-1} \in \mathcal{M}_{\mathbf{SL}_2}$ induces an isomorphism between $\mathrm{H}^0(\mathcal{M}_{\mathbf{SL}_2}, \mathcal{L}_{\mathbf{SL}_2})$ and $\mathrm{H}^0(J, 2\vartheta)$. We will identify these two spaces via this isomorphism. In particular, we may view $\varphi_{\mathcal{L}}$ as a morphism $\mathcal{M}_{\mathbf{SL}_2} \longrightarrow |2\vartheta|^*$.

We thus have at our disposal the explicit realization of the preceding action given by the Heisenberg representation, as recalled in Appendix B: choosing a theta-structure $\mathcal{G}(\mathcal{O}(2\vartheta)) \simeq H_2$ for $(J, 2\vartheta)$, induces a basis $(X_b)_{b\in\mathbb{F}_2^g}$ of $\mathrm{H}^0(J, \mathcal{O}(2\vartheta))$ such that $(t, a, \alpha) \in H_2$ sends X_b to

$$(t, a, \alpha) \cdot X_b = t\alpha(a + b)X_{a+b}.$$

It follows from B.2 that the fixed locus of $\eta \in J_2 \setminus \{\mathcal{O}\}$ consists of two linear subspaces of dimension $2^{g-1} - 1$.

We can also give a precise description of the fixed locus of η in $\mathcal{M}_{\mathbf{SL}_2}$ (see [**NR75**]): if $\mathrm{Nm}_\eta \colon J_{C_\eta} \longrightarrow J$ is the norm map associated to the 2-sheeted cover $\pi_\eta \colon C_\eta = \mathrm{Spec}(\mathcal{O}_C \oplus \eta) \longrightarrow C$, this fixed locus is the image of $\mathrm{Nm}_\eta^{-1}(\eta)$ by the push forward $\pi_{\eta*} \colon \mathrm{Nm}_\eta^{-1}(\eta) \longrightarrow \mathcal{SU}(2)$. Since $\mathrm{Nm}^{-1}(\eta)$ consists of two connected components, both isomorphic to the Prym variety P_η, its image in $\mathcal{SU}(2)$ is the union of two copies of the Kummer variety $K(P_\eta) = P_\eta/\pm 1$ associated to η.

The orthogonal bundle obtained from such a fixed point is of the form $\eta \oplus F$, with F an orthogonal bundle of rank 2 with determinant η: indeed, the two maps $P_\eta \longrightarrow \mathcal{SU}(2) \longrightarrow \mathcal{M}^+_{\mathbf{SO}_3}$ send a line bundle L (with $\mathrm{Nm}_\eta(L) = \mathcal{O}$) defined on C_η to $\mathcal{E}nd_0(\pi_{\eta*}L)$, and we have (for example by [**Bea91**, 1.3]) an exact sequence $0 \to \pi_{\eta*}\mathcal{O}_{C_\eta} \to \mathcal{E}nd(\pi_{\eta*}L) \to \pi_{\eta*}(L \otimes \sigma_\eta^* L^{-1}) \to 0$, where σ_η is the involution of the 2-sheeted cover. In fact, since $L \otimes \sigma_\eta^* L^{-1} = L^2$ for $L \in \ker(\mathrm{Nm}_\eta)$, this means that the following diagram

$$\begin{tikzcd} \mathrm{Nm}_\eta^{-1}(\eta) \arrow[r, "\pi_{\eta*}"] \arrow[d, "2_{P_\eta}"] & \mathcal{SU}(2) \arrow[d] \\ P_\eta \arrow[r] & \mathcal{M}^+_{\mathbf{SO}_3} \end{tikzcd} \tag{4.2.4.1}$$

is commutative (the bottom arrow is $L \longmapsto \eta \oplus \pi_{\eta*}L$).

Proposition 4.2.5. — *If C has genus $\geqslant 3$, the singular locus of $\mathcal{M}^{+}_{\mathbf{SO}_3}$ is the union of the Kummer variety of J and of all the $2^{2g}-1$ Prym varieties associated to C.*

Since the ramification locus $Z \subset \mathcal{M}_{\mathbf{SL}_2}$ of the quotient morphism $\mathcal{M}_{\mathbf{SL}_2} \longrightarrow \mathcal{M}^{+}_{\mathbf{SO}_3}$ has pure codimension $2g-2$, the purity theorem [**SGA71**, Théorème X.3.1] implies that a point in the quotient $\mathcal{M}^{+}_{\mathbf{SO}_3} = \mathcal{M}_{\mathbf{SL}_2}/J_2$ is smooth only if it does not belong to Z. Conversely, a point outside this fixed locus is smooth if and only if it comes from a smooth point in $\mathcal{M}_{\mathbf{SL}_2}$. The conclusion now follows from the preceding discussion, together with the well-known description of the singular locus of $\mathcal{M}_{\mathbf{SL}_2}$.

Remark 4.2.6. — The proof of the last Proposition shows that the two copies of the Prym variety P_η fixed by $\eta \in J_2$ are conjugated by the action of J_2.

4.3. Description of $\mathcal{M}^{+}_{\mathbf{SO}_3}$ for a curve of genus 2

We concentrate now on the case of a curve of genus 2. The majority of what follows has already been noticed in [**NR03**].

4.3.1. The moduli space $\mathcal{M}^{+}_{\mathbf{SO}_3}$ is the quotient of $\mathcal{SU}(2,0) \simeq \mathbb{P}^3$ under the action of $J_2 \simeq (\mathbb{Z}/2\mathbb{Z})^4$ given as the projectivization of the Heisenberg representation. It has been known for long that this quotient may be identified with the Satake compactification of the moduli space $\mathcal{A}_2(2)$ of principally polarized abelian surfaces with level 2 structure, and that this quotient is isomorphic to a quartic $\mathcal{Q}$ in $\mathbb{P}^4$, which is dual to the Segre cubic (see [**vdG82**] or [**DO88**, IX.4]). The commutative diagram

$$\begin{tikzcd}
\mathcal{SU}_C(2) \arrow[r, "\sim"] \arrow[d] & {|2\Theta| \simeq \mathbb{P}^3} \arrow[d] & \mathcal{A}_2(2,4) \arrow[l] \arrow[d] \\
\mathcal{M}^{+}_{\mathbf{SO}_3} \arrow[r] & \mathcal{Q} \subset \mathbb{P}^4 & \mathcal{A}_2(2), \arrow[l]
\end{tikzcd}$$

where $\mathcal{A}_2(2,4)$ parametrizes isomorphism classes of principally polarized abelian surfaces with (level 2) theta structure, sums up the situation: the square on the right is the object of the example in [**DO88**, IX.4], its bottom arrow identifying the Satake compactification of $\mathcal{A}_2(2)$ to $\mathcal{Q}$ by [**vdG82**, 5.2] (or [**DO88**, VIII.7]).

4.3.2. Another way to see this is to look at the following commutative diagram

$$\begin{tikzcd} {|2\Theta|} \arrow[d, "\pi"'] \arrow[dr, "\varphi"] & \\ \mathcal{M}^+_{\mathbf{SO}_3} \arrow[r, "\theta_3^+"'] & {|3\Theta|^+}. \end{tikzcd}$$

Let us remark that $\mathcal{M}^+_{\mathbf{SO}_3}$ is contained in the sextic along which $\mathcal{M}_{\mathbf{SL}_3} \longrightarrow |3\Theta|$ is ramified: according to [**Ort05**], the involution associated to the theta map is the morphism $E \mapsto \iota^* E^*$ (where ι denotes the hyperelliptic involution), and, for a curve of genus 2, ι^* acts trivially on $\mathcal{SU}(2, \mathcal{O})$, and hence on $\mathcal{M}^+_{\mathbf{SO}_3}$. The main result of Chapter 2 ensures that $\mathcal{M}^+_{\mathbf{SO}_3} \longrightarrow \mathcal{M}_{\mathbf{SL}_3}$ is an embedding, which implies that $\theta_3^+ : \mathcal{M}^+_{\mathbf{SO}_3} \longrightarrow |3\Theta|^+$ is also an embedding. The morphism φ is thus a finite map of degree 16 onto its image. The equality $\varphi^* \mathcal{O}_{|3\Theta|^+}(1) = \mathcal{O}_{|2\Theta|}(4)$ then shows that this image must be a quartic in $|3\Theta|^+ \simeq \mathbb{P}^4$.

Remark 4.3.3. — We can then recover some of the properties of this quartic: according to (for example) [**Hun96**, §3.3], this hypersurface is singular along 15 lines. But we have seen in the proof of Proposition 4.2.5 that a singular point in the quotient by J_2 of the *smooth* variety $\mathcal{M}_{\mathbf{SL}_2} \simeq \mathbb{P}^3$ comes from a point fixed by a non trivial element of J_2. The singular locus is thus the union of the 15 Kummer varieties $P_\eta / \pm 1$. Since any of these Prym has genus 1, we obtain in this way a bundle-theoretic description of the expected 15 rational curves of the singular locus of $\mathcal{M}^+_{\mathbf{SO}_3}$. That they are lines follows for example from 4.2.4.1, by noticing that the determinant bundle $\mathcal{L}^+_{\mathbf{SO}_4}$ pulls back to $\mathcal{O}(2\vartheta_\eta)$ via the 2-to-1 morphism $P_\eta \longrightarrow \mathcal{M}^+_{\mathbf{SO}_3}$.

4.3.4. One can directly obtain the quartic $\mathcal{M}^+_{\mathbf{SO}_3} \subset |3\Theta|^+$ (as well as its equation) from an explicit enough description of the ring of polynomial functions on $\mathrm{H}^0(J, 2\vartheta)$ invariant for the action of the *finite* Heisenberg group $\widetilde{H_2}$: the quotient $\mathcal{M}^+_{\mathbf{SO}_3}$ of $\mathcal{M}_{\mathbf{SL}_2}$ (identified to $|2\vartheta| = \mathrm{Proj}(k[\mathrm{H}^0(J, 2\vartheta)])$ via $\varphi_{\mathcal{L}}$, see 4.2.4) by the action of $\widetilde{H_2}$ is automatically the homogeneous spectrum associated to the corresponding invariant algebra.

Mollien's formula shows that the Poincaré series is

$$\frac{1}{64}\Big(\frac{1}{(1-t)^4}+\frac{1}{(1+t)^4}+\frac{1}{(1-it)^4}+\frac{1}{(1+it)^4}+\frac{30}{(1-t^2)^2}+\frac{30}{(1+t^2)^2}\Big)=\frac{1+t^4+t^8+t^{12}}{(1-t^4)^4},$$

from which we easily deduce that $k[\mathrm{H}^0(J,2\vartheta)]^{\widetilde{H_2}}$ is can be generated by five elements of degree 4, subject to a single quartic relation. Indeed, let us consider the following five elements Y_i, which generates the space of invariants of degree 4:

$$Y_0=\prod_{b\in\mathbb{F}_2^2}X_b,\ Y_1=X^2_{\binom{0}{0}}X^2_{\binom{1}{0}}+X^2_{\binom{0}{1}}X^2_{\binom{1}{1}},\ Y_2=X^2_{\binom{0}{0}}X^2_{\binom{0}{1}}+X^2_{\binom{1}{0}}X^2_{\binom{1}{1}},$$

$$Y_3=X^2_{\binom{0}{0}}X^2_{\binom{1}{1}}+X^2_{\binom{1}{0}}X^2_{\binom{0}{1}}\text{ and }Y_4=X^4_{\binom{0}{0}}+X^4_{\binom{1}{0}}+X^4_{\binom{0}{1}}+X^4_{\binom{1}{1}}.$$

Elimination theory (and a computer) tells us that the kernel of the corresponding morphism $k[Z_0,\ldots,Z_4]\to k[\mathrm{H}^0(J,2\vartheta)]^{\widetilde{H_2}}$ is a principal ideal. Since the Poincaré series looks like the one of a quartic hypersurface in $\mathbb{P}^4$ (up to the graduation), we conclude that the Y_i's generate the ring of invariants, and that the equation of the quartic hypersurface $\mathcal{M}^+_{\mathbf{SO}_3}\subset|3\Theta|^+\simeq\mathrm{Proj}(k[Z_0,\ldots,Z_4])$ is

$$16Z_0^4-4Z_0^2Z_1^2-4Z_0^2Z_2^2-4Z_0^2Z_3^2+Z_1^2Z_2^2+Z_1^2Z_3^2+Z_2^2Z_3^2-Z_1Z_2Z_3Z_4+Z_0^2Z_4^2=0.$$

Note that general results of Stanley about Poincaré series ([**Sta79**]) show that, in order to prove that the invariant ring is generated by the Y_i, it is enough to check that $Y_1,\ldots,Y_4$ is a regular sequence: this implies that this invariant ring is a free $k[Y_1,\ldots,Y_4]$–module generated by 1 and three elements of degree 4, 8 and 12, which must of course be Y_0 and its successive powers (but checking that a given sequence is regular is not really easier than the previous elimination computation).

Remark 4.3.5. — We can of coure easily find the 15 singular lines in terms of this explicit description of $\mathcal{M}^+_{\mathbf{SO}_3}$. For example, the line associated to an element of $\widetilde{H_2}$ lying over $(0,\alpha)\in J_2\simeq\mathbb{F}_2^2\times\widehat{\mathbb{F}_2^2}$ with α non trivial is the line defined by the 3 equations $Z_0=Z_i=Z_j=0$.

Let us sum up now the main observations of this section:

Theorem 4.3.6. — *If C is a curve of genus 2, the moduli space $\mathcal{M}^+_{\mathbf{SO}_3}$ of topologically trivial rank 3 orthogonal bundles is isomorphic to the Satake compactification of the moduli space of principally polarized abelian surfaces with* *level* 2 *structure. Moreover, the theta map is an isomorphism from this moduli space onto the quartic hypersurface in* $|3\Theta|^+$ *defined (in suitable coordinates) by the equation*

$$16Z_0^4-4Z_0^2Z_1^2-4Z_0^2Z_2^2-4Z_0^2Z_3^2+Z_1^2Z_2^2+Z_1^2Z_3^2+Z_2^2Z_3^2-Z_1Z_2Z_3Z_4+Z_0^2Z_4^2=0.$$

4.3.7. Let us recall now how [**vdG82**, §6] allows us, via the modular interpretation, to recover the Kummer variety of the curve from $\mathcal{M}^+_{\mathbf{SO}_3}$. The orthogonal bundle $\mathcal{O}^{\oplus 3}$ is, exceptionally in genus 2, a smooth point in the moduli space[(2)]) and thus defines a principally polarized abelian surface endowed with a level 2 structure. According to [**vdG82**, 6.1], its Kummer variety is exactly the intersection of $\mathcal{M}^+_{\mathbf{SO}_3}$ (imbedded in $|3\Theta|^+$) with its tangent space at the corresponding point, and it is easy to verify that the tangent space at $\mathcal{O}^{\oplus 3}$ is exactly $\{\mathcal{O}\oplus F, F\in\mathcal{SU}(2,\mathcal{O})\}$ (see [**NR03**]), whose intersection with $\mathcal{M}^+_{\mathbf{SO}_3}$ is the image of the Kummer surface of C. We conclude from the preceding that the point $\mathcal{O}^{\oplus 3}$ defines (as a point in $\mathcal{A}_2(2)$) the jacobian of C (with a certain level 2 structure).

Remark 4.3.8. — As explained in [**vdG82**], $\mathcal{A}_2(2)$ contains an *Humbert surface*, consisting of 10 surfaces, which all contains exactly 6 of the 15 lines of the singular locus. They are in fact tangent hyperplane sections, isomorphic to a quadric surface. On the other hand, we have recalled in 4.2 that we can associate to every even theta-characteristic a (reduced) divisor D_κ on $\mathcal{M}_{\mathbf{SL}_2}$ such that the pull-back of the theta divisor Θ_κ satisfies $\pi^*\Theta_\kappa = 2D_\kappa$.

There should be a correspondence between the 10 components of the Humbert section in $\mathcal{M}^+_{\mathbf{SO}_3}\subset\mathbb{P}^4$ and the 10 divisors Θ_κ (which could give some informations on the rational map $J^1 \dashrightarrow |\mathcal{L}^+_{\mathbf{SO}_3}|$, whose base locus contains the six odd theta-characteristic).

4.4. The theta map $\mathcal{M}^-_{\mathbf{SO}_3}\longrightarrow|3\Theta|^-$ is a morphism

In this section, we explain briefly why the theta map is, again, always a morphism.

Proposition 4.4.1. — *The theta map* $\theta_3^-:\mathcal{M}^-_{\mathbf{SO}_3}\longrightarrow|3\Theta|^-$ *is a morphism for every curve.*

(2) This results from the description of the singular locus of the quotient $|2\Theta|/\!\!/J_2$, or else from a direct computation of the ring of an étale neighborhood $k[H^1(\mathcal{O})\otimes\mathfrak{so}_3]^{\mathbf{SO}_3}$.

The proof goes exactly as the one given for the other connected component of $\mathcal{M}_{\mathbf{SO}_3}$, based on the following diagram:

(4.4.1.1)
$$\begin{tikzcd} \mathcal{SU}(2,1) \arrow[d, "\pi"'] \arrow[dr, "\varphi"] & \\ \mathcal{M}^-_{\mathbf{SO}_3} \arrow[r, "\theta_3^-"'] & {|3\Theta|^-.} \end{tikzcd}$$

If κ is an odd theta characteristic, Θ_κ defines a divisor(3) in $\mathcal{M}^-_{\mathbf{SO}_3}$. Its pullback to $\mathcal{M}^1_{\mathbf{SL}_2}$ is the determinant divisor associated to $\mathcal{E}nd_0 \otimes \kappa$ (note that in this case $\mathcal{E}nd_0$ is the sheaf of traceless endomorphisms of the universal rank 2 bundle on $\mathcal{M}^1_{\mathbf{SL}_2}$): it therefore admits as a square root the pfaffian divisor $D_\kappa \in |\mathcal{L}_{\mathcal{M}^1_{\mathbf{SL}_2}}|$, which is defined by a section d_κ of $\mathcal{L}_{\mathcal{M}^1_{\mathbf{SL}_2}}$. For every point $p \in C$, [**Bea91**] constructs a morphism $\varphi_p\colon \mathcal{SU}(2,1) \longrightarrow \mathbb{P}(\mathbf{\Lambda}^2\mathrm{H}^0(J, 2\vartheta))$, which is in fact, by (*loc. cit.*), Proposition 3.12, the projective morphism given by some of the d_κ's. This shows that no point is contained in the intersection of the pfaffian divisors $(D_\kappa)_{\kappa\ \mathrm{odd}}$: φ, and thus θ_3^-, is a morphism.

4.5. Description of $\mathcal{M}^-_{\mathbf{SO}_3}$ in genus 2

4.5.1. We consider now the case of a curve C of genus 2, with hyperelliptic involution ι. The strategy to describe $\mathcal{M}^-_{\mathbf{SO}_3}$ is to use the very explicit description of $\mathcal{M}^1_{\mathbf{SL}_2}$ given by Desale and Ramanan for hyperelliptic curves. Here again computations become too difficult for curves of higher genus.

We first give an elementary way to show that the forgetful morphism is a closed immersion.

Proposition 4.5.2. — *The forgetful morphism $\mathcal{M}^-_{\mathbf{SO}_3} \longrightarrow \mathcal{M}_{\mathbf{SL}_3}$ induces an isomorphism onto $\theta^{-1}(|3\Theta|^-)$.*

Although it is an immediate consequence of the main result of Chapter 2 we give here a direct way to recover this statement. Every scheme appearing in the diagram (4.4.1.1) has dimension 3, and all the morphisms are finite (${\theta_3^-}^*\mathcal{O}(1)$ being ample, θ_3^- is quasi-finite). We also know that $\mathcal{SU}(2,1)$ is isomorphic to the intersection of two quadrics in $\mathbb{P}^5$ (cf. [**New68**]), so that

(3) This is [**LS97**, 7.10] adapted to the odd component.

$\mathrm{Pic}(\mathcal{SU}(2,1))$ is generated by the restriction $\mathcal{O}_{\mathcal{SU}(2,1)}(1)$ of $\mathcal{O}_{\mathbb{P}^5}(1)$, which verifies $c_1(\mathcal{O}_{\mathcal{SU}(2,1)}(1))^3 = 4$. Finally the pull-back of $\mathcal{L}_{\mathcal{M}^-_{\mathbf{SO}_3}}$ by π is given in [**BLS98**]: we have $\pi^*\mathcal{L}_{\mathcal{M}^-_{\mathbf{SO}_3}} = \mathcal{O}_{\mathcal{SU}(2,1)}(2)$.

Since $\varphi^*\mathcal{O}_{|3\Theta|^-}(1) = \mathcal{O}_{\mathcal{SU}(2,1)}(2)$ we know that $\deg\varphi = 2^3.4$, and the equality $16.\deg\theta_3^- = \deg\varphi$ then implies $\deg\theta_3^- = 2$. But $\theta\colon \mathcal{M}_{\mathbf{SL}_3} \to |3\Theta|$ is already of degree 2.

Remark 4.5.3. — Following [**Las96**, V.5] we can find directly $c_1(\mathcal{L}_{\mathcal{M}^-_{\mathbf{SO}_3}})^3$, hence another proof of 4.5.2: indeed it is enough to recall that the dualizing sheaf $\omega_{\mathcal{M}^-_{\mathbf{SO}_3}}$ is, according to [**BLS98**, §13], equal to $\mathcal{L}^{-1}_{\mathcal{M}^-_{\mathbf{SO}_3}}$. Therefore the Hilbert polynomial $n \mapsto P(n) = \chi(\mathcal{M}^-_{\mathbf{SO}_3}, \mathcal{L}^n_{\mathcal{M}^-_{\mathbf{SO}_3}})$ satisfies the identity $P(n) = -P(-n-1)$. The Kodaira vanishing theorem and the main result of [**Bea06a**] then ensures that $P(0) = 1$ and $P(1) = 4$, from what we easily infer the leading coefficient of P.

4.5.4. Let me recall now the explicit description of $\mathcal{M}^1_{\mathbf{SL}_2}$ given in [**DR77**] (we use this description rather than Newstead's for two reasons: [**DR77**] gives a result for all hyperelliptic curves, and in genus 2 it actually provides an explicit description of the embedding in terms of vector bundles, while Newstead constructs a universal family on the intersection of the two quadrics): let ξ be a fixed line bundle on C of degree 5, $V = \sum \xi_w$ the direct sum of the fibers of ξ at the six Weierstrass points. Desale and Ramanan have constructed a pencil of quadratic forms on V^* and a morphism $\varphi\colon \mathcal{SU}(2,\xi) \to \mathbb{P}(V)$ which associates to any vector bundle E of rank 2 and determinant ξ the hyperplane of V defined as the image by the evaluation maps of the (-1)-eigenspace of the involution of $H^0(C, E \otimes \iota^*E)$ induced by ι (see (*loc. cit.*), §5). Since the restriction of every form of the pencil has rank 4 on such an hyperplane, φ factorizes through the base locus of this pencil, and is actually an isomorphism onto this subscheme.

Moreover the product $G = J_2 \times \mathbb{Z}/2\mathbb{Z}$ operates on both $\mathcal{SU}(2,\xi)$ and $\mathbb{P}(V)$: the normal subgroup $J_2 \subset G$ acts on $\mathcal{SU}(2,\xi)$ by tensorisation, while the other elements of G acts by $E \mapsto \iota^*E \otimes \xi \otimes \beta$, with β a line bundle such that $\beta^2 \simeq K_C^{-5}$; on the other hand, G acts on $\mathbb{P}(V)$ in the following way: an element $\alpha \in G$ naturally corresponds to a partition $W = S \cup T$ of the set W of Weierstrass points (so that elements of $J_2 \subset G$ correspond to partition with $|S|$ even), and the involution given by α is the one which fixes exactly $\mathbb{P}(\sum_{w\in S}\xi_w)$ and $\mathbb{P}(\sum_{w\in T}\xi_w)$. The point is that φ is then G-equivariant, which allows us to give a fairly nice description of the quotient $\mathcal{SU}(2,\xi)/J_2 \simeq \mathcal{M}^-_{\mathbf{SO}_3}$ as a subscheme of $\mathbb{P}(V)/J_2$.

Let us choose some coordinates $(X_w)_{w\in W}$ on $\mathbb{P}(V)$ (in accordance with the ξ_w, so that $\mathcal{SU}(2,\xi)$ is the intersection in $\mathbb{P}^5$ of the two quadrics $\sum X_w^2$ and $\sum \lambda_w X_w^2$, where $(\lambda_w)_{w\in W}$ is the ramification locus of $C \longrightarrow \mathbb{P}^1$). The quotient $\mathbb{P}(V)/J_2$ is easily seen to be isomorphic to the sextic hypersurface $\mathcal{S}$ in the weighted projective space $\mathbb{P}(1,1,1,1,1,1,3) = \mathrm{Proj}(k[Y_w,\Upsilon])$ of equation $\Upsilon^2 - \prod_w Y_w = 0$: this follows from the fact that the operation of J_2 on $\mathbb{P}(V)$ comes from the action of a natural extension of J_2 by μ_2 on V. More precisely, this extension is identified with the group of involutions in $\mathbf{SL}(V)$ acting by ± 1 on each ξ_w, and the invariant subalgebra of $k[X_w]$ is generated by the elements $Y_w = X_w^2$ and $\Upsilon = \prod_w X_w$.

Here is a commutative diagram which sums up the situation:

$$\begin{tikzcd}
 & \mathcal{SU}(2,\xi) \arrow[r, hook] \arrow[d] & \mathbb{P}(V) \simeq \mathrm{Proj}(k[X_w]) \arrow[d] \\
\mathcal{M}^-_{\mathbf{SO}_3} \arrow[d, "\theta"'] & \mathcal{SU}(2,\xi)/J_2 \arrow[l, "\sim"'] \arrow[r, hook] \arrow[d] & \mathcal{S} \subset \mathrm{Proj}(k[Y_w,\Upsilon]) \arrow[d, dashed] \\
|3\Theta|^- & \mathcal{SU}(2,\xi)/G \arrow[l] \arrow[r, hook] & \mathrm{Proj}(k[Z_w]),
\end{tikzcd}$$

where Υ is an element of degree 3 in the graded algebra $k[Y_w,\Upsilon]$ appearing on the middle row. The vertical arrows on the right come from $Y_w \mapsto X_w^2$ and $\Upsilon \mapsto \prod X_i$, and $Z_w \mapsto Y_w$. Since the choice of the X_w defines $\mathcal{SU}(2,\xi)$ in $\mathbb{P}^5$ as the intersection of the quadrics $\sum X_w^2 = 0$ and $\sum \lambda_w X_w^2 = 0$, we have obtained the expected explicit description:

Theorem 4.5.5. — *The moduli space $\mathcal{M}^-_{\mathbf{SO}_3}$ is isomorphic to the subscheme of the weighted projective space $\mathbb{P}(1,1,1,1,1,1,3)$ defined by the three equations*

$$\sum_{w\in W} Y_w = 0,\ \sum_{w\in W} \lambda_w Y_w = 0,\ \Upsilon^2 - \prod_{w\in W} Y_w = 0.$$

We can of course deduce from these equations a description of $\mathcal{M}^-_{\mathbf{SO}_3}$ as an hypersurface in $\mathbb{P}(1,1,1,1,3)$.

4.5.6. We also get a precise description of the theta map: the action of $G/J_2 \simeq \mathbb{Z}/2\mathbb{Z}$ on $\mathcal{M}^-_{\mathbf{SO}_3} = \mathcal{SU}(2,\xi)/J_2$ is exactly $E \mapsto \iota^* E$; since it commutes with the theta map, it is the involution associated to θ_3. This shows that its ramification locus is defined in $\mathcal{SU}(2,\xi)/J_2 \subset \mathcal{S}$ by the hyperplane section

$\Upsilon = 0$: henceforth the branch locus in $|3\Theta|^-$ is the union of six hyperplane sections Π_w, $w \in W$, which meet in 15 lines $L_{w,w'}$, $w \neq w'$; these lines in turn meet in 20 points $P_{w,w',w''}$ labelled by the 3 points subsets of W, and there are 4 points a line and 3 lines a point (this gives a new presentation of the results of [**NR03**, §6]). Note that $L_{w,w'}$ is precisely the image of the closed subspace of $\mathcal{SU}(2,\xi)$ fixed by the involution corresponding to the partition $W = \{w,w'\} \cup (W \setminus \{w,w'\})$. This subscheme is in fact the elliptic curve $E_{w,w'}$ defined by the two quadrics $\sum_{w''\neq w,w'} X_{w''}^2$ and $\sum_{w''\neq w,w'} \lambda_{w''} X_{w''}^2$.

Let us finally remark that the pullback $\pi^*\Pi_w$ of Π_w to $\mathcal{SU}(2,\xi)$ is defined as the double hyperplane section $X_w^2 = 0$: the associated reduced scheme is thus a Del Pezzo surface of degree 4.

4.5.7. Let us explain how the previous observations can be reinterpreted in terms of orthogonal bundles. Note that we already knew from [**Ort05**] that the involution of the 2-sheeted cover is $E \in \mathcal{M}^-_{\mathbf{SO}_3} \longmapsto \iota^*E$, so that the ramification locus of θ is $\mathcal{R} = \{E \in \mathcal{M}^-_{\mathbf{SO}_3} \mid \iota^*E \sim E\}$: it is the intersection of the Dolgachev sextic $\mathcal{S} \subset |3\Theta|$ with $|3\Theta|^-$, and its singular locus is exactly the one of $\mathcal{M}^-_{\mathbf{SO}_3}$.

This locus cannot contain any strictly semi-stable $\mathbf{SO}_3$-bundle: indeed, such a bundle must be topologically trivial. But we know how stable $\mathbf{SO}_3$-bundles look like: they are either $\mathbf{GL}$-stable (in which case the corresponding point in $\mathcal{M}^-_{\mathbf{SO}_3}$ is smooth), or equal to a sum $\eta \oplus F$, with $\eta \in J_2$ and F an $\mathbf{O}_2$-bundle (with determinant η). According to [**Mum71**], any rank 2 orthogonal bundle of determinant η is the direct image via $\pi_\eta \colon C_\eta = \operatorname{Spec}(\mathcal{O} \oplus \eta) \longrightarrow C$ of a line bundle with trivial norm. The kernel of $\operatorname{Nm}_\eta \colon J_{C_\eta} \longrightarrow J_C$ consists of two connected components of dimension 1, the neutral one being the Prym variety P_η of C_η over C, which is the component where the Stiefel-Whitney $w_2(\eta \oplus \pi_{\eta *}(\cdot))$ vanishes. Let us consider the other component P'_η. The map $L \longmapsto \eta \oplus \pi_{\eta *}(L)$ defines a morphism $P'_\eta \longrightarrow \mathcal{M}^-_{\mathbf{SO}_3}$, which commutes with the involution $L \mapsto L^{-1}$; since P'_η is an elliptic curve, its image in (the singular locus of) $\mathcal{R}$ is a rational curve L_η. We have thus found the 15 singular lines $L_{w,w'}$ lying in the intersection of the ramification locus of θ_3 and the singular locus of $\mathcal{M}^-_{\mathbf{SO}_3}$ (and the 20 points are $\eta_1 \oplus \eta_2 \oplus (\eta_1 \otimes \eta_2)$ with $\langle \eta_1, \eta_2 \rangle = -1$).

Remark 4.5.8. — We have in particular seen that a point of $\mathcal{M}^-_{\mathbf{SO}_3}$ is singular if and only if its underlying vector bundle is not stable, but also if and only if it comes from a rank 2 bundle fixed by a non-zero $\alpha \in J_2$: we retrieve [**Ort05**, Lemma 4.2]. They are bundles E such that $\mathcal{E}nd_0(E) = \ker(E^* \otimes E \to \mathcal{O})$ is unstable, i.e. contains a line bundle.

4.5.9. We now turn our attention to their inverse images in $\mathcal{M}^1_{\mathbf{SL}_2}$: we know that the fixed locus $\mathrm{Fix}(\eta) \subset \mathcal{SU}(2,\xi)$ is the image of $\mathrm{Nm}_\eta^{-1}(\eta \otimes \xi)$ by the morphism $\pi_{\eta*}$. Using the isomorphism between $\mathrm{Nm}_\eta^{-1}(\eta\otimes\xi)$ and $\ker(\mathrm{Nm}_\eta$ given by any line bundle $\tilde{\xi}$ on C_η with $\mathrm{Nm}_\eta(\tilde{\xi}) = \eta\otimes\xi$, we see that the orthogonal bundle obtained from a point $F = \pi_{\eta*}(L\otimes\tilde{\xi})$ (now with $\mathrm{Nm}_\eta(L) = \mathcal{O}$) fixed by η fits in the exact sequence

$$0 \to \eta \to \mathcal{E}nd_0(F) \to \pi_{\eta*}(L\otimes\sigma_\eta^* L^{-1}\otimes\tilde{\xi}\otimes\sigma_\eta^*\tilde{\xi}^{-1}) \to 0,$$

which implies the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Nm}_\eta^{-1}(\eta\otimes\xi) \arrow[r, "\pi_{\eta*}"] \arrow[d] & \mathcal{SU}(2,\xi) \arrow[d] \\
P'_\eta \arrow[r] & \mathcal{M}^-_{\mathbf{SO}_3}
\end{tikzcd}$$

where the vertical arrow on the left is $L \mapsto L^2\otimes\tilde{\xi}^{-1}\otimes\sigma_\eta^*\tilde{\xi}^{-1}$, and the bottom one $L \mapsto \eta\oplus\pi_{\eta*}L$. Since the involution induced by σ_η exchanges the two components of $\mathrm{Nm}_\eta^{-1}(\eta\otimes\xi)$, we see that $\pi_{\eta*}$ is the trivial 2 cover of its image, isomorphic to the elliptic curve P_η: this is the inverse image of the rational curve $L_\eta \subset \mathcal{M}^-_{\mathbf{SO}_3}$.

Proposition 4.5.10. — *If w and w' are two distinct Weierstrass points of C, the subscheme L_η of $\mathcal{M}^-_{\mathbf{SO}_3}$ associated to the line bundle $\eta = \mathcal{O}(w-w')$ is the line $L_{w,w'}$.*

We just have to compare the two descriptions of the singular locus we have given: on the one hand we have described, for each η the fixed locus in $\mathcal{SU}(2,\xi)$ of an element $\eta\in J_2$, while, on the other hand, we have obtained a similar statement for the action of $J_2\subset G$ on $\mathbb{P}^5$ (and hence on the image of $\mathcal{SU}(2,\xi)$ in $\mathbb{P}^5$) constructed in [**DR77**]. Since the embedding $\mathcal{SU}(2,\xi)\longrightarrow\mathbb{P}^5$ is equivariant for these actions, we deduce the precise correspondence between the two descriptions from the identification between J_2 and its image in G constructed in (*loc. cit.*), Lemma 2.1: the order 2 line bundle $\eta = \mathcal{O}(w-w')$ (for $w\neq w'$) corresponds to the involution of $\mathbb{P}^5$ acting by -1 on X_w and $X_{w'}$, and by $+1$ elsewhere, so that the image L_η of P'_η in $\mathcal{M}^-_{\mathbf{SO}_3}$ is covered by the elliptic curve defined by $X_w = X_{w'} = 0$. This is therefore the line $L_{w,w'}\subset\mathcal{M}^-_{\mathbf{SO}_3}$.

4.5.11. Of course, the six hyperplane sections Π_w in $\mathcal{M}^-_{\mathbf{SO}_3}$ correspond to the theta divisors Θ_κ: this follows from the fact that the supports of Θ_κ are obviously contained in the ramification locus of θ_3, which has been shown to be the union $\bigcup \Pi_w$.

We can show that the correspondence between $(\Theta_\kappa)_{\kappa \text{ even}}$ and $(\Pi_w)_{w\in W}$ is the one we have in mind.

Proposition 4.5.12. — *The determinant divisor Θ_κ in $\mathcal{M}^-_{\mathbf{SO}_3}$ associated to the theta characteristic $\kappa = \mathcal{O}(w)$ corresponding to a Weierstrass point w is exactly the hyperplane section Π_w.*

This easily follows from the last Proposition, by comparing the lines contained in the two divisors. Obviously, Π_w contains a line $L_{w',w''}$ if and only if $w \in \{w', w''\}$. On the other hand, [**Bea91**, Lemma 1.5] shows that $\pi^{-1}\Theta_\kappa$ contains the elliptic curve lying over the line L_η if and only if $h^0(C, \kappa\otimes\eta) \neq 0$. This immediately implies that $\Theta_{\mathcal{O}(w)}$ is equal to Π_w.

Remark 4.5.13. — Coming back to the bundle-theoretic definition of the action of G given in [**DR77**], we see that an orthogonal bundle E belongs to Θ_κ if and only if it comes from a rank 2 vector bundle F fixed by the involution $F \longmapsto \iota^*F \otimes \xi \otimes K_C^{-3} \otimes \mathcal{O}(w)$. The preceding result implies that, if F is a rank 2 bundle with determinant $\xi \in J^{-1}$ over a curve of genus 2, then $h^0(C, \mathcal{E}nd_0(F) \otimes \mathcal{O}(w)) \neq 0$ if and only if $F \simeq \iota^*F \otimes \xi \otimes \mathcal{O}(w)$.

Let me finally mention the nice version of the following Torelli theorem, already noticed in [**Ngu07**]:

Corollary 4.5.14. — *A curve of genus 2 is completely determined by its moduli space $\mathcal{M}^-_{\mathbf{SO}_3}$.*

Here is an explicit way to recover the curve from the moduli space $\mathcal{M}^-_{\mathbf{SO}_3}$ (motivated by the fact that the Prym varieties associated to C are 2-covers of the lines contained in the branch locus): the branch locus $\mathcal{R}$ of the morphism $\mathcal{M}^-_{\mathbf{SO}_3} \longrightarrow |\mathcal{L}^-_{\mathbf{SO}_3}|^*$ defined by the ample generator of the Picard group of $\mathcal{M}^-_{\mathbf{SO}_3}$ defines 6 planes, 15 lines and 20 points. Choose first a point P among these points. There are exactly 3 lines L, L' and L'' in $\mathcal{R}$ passing through P, and exactly 3 planes Π_0, Π_1 and Π_∞ in $\mathcal{R}$ which do *not* contain P. Now, the curve C is the hyperelliptic 2-cover of $\mathbb{P}^1$ with branch locus $\{0, 1, \infty, \alpha, \alpha', \alpha''\}$, where $\alpha^{(j)}$ is the image of P by the automorphism of $L^{(j)}$ sending the (unique) intersection point with Π_t to t (for $t \in \{0, 1, \infty\}$).

Remark 4.5.15. — One can directly say that the six planes making the branch locus of the morphism $\mathcal{M}^-_{\mathbf{SO}_3} \longrightarrow |\mathcal{L}^-_{\mathbf{SO}_3}|^*$ correspond to six points in

$|\mathcal{L}^-_{\mathbf{SO}_3}| \simeq \mathbb{P}^3$. These six points determine a unique rational normal curve, and the curve C is isomorphic to the 2-cover of this rational curve branched exactly at these six points. Such a formulation is quite less satisfactory than the one given above, since it misses a right interpretation of the rational map $C \longrightarrow |\mathcal{L}^-_{\mathbf{SO}_3}|$ as the morphism $x \in C \longmapsto \Theta_x = \{E \in \mathcal{M}^-_{\mathbf{SO}_3} | h^0(C, E \otimes \mathcal{O}_C(x)) > 0\}$. The fine investigation of the dual maps held in [**Ngu07**] provides another way to give a geometric description of the Torelli result.

CHAPTER 5

ORTHOGONAL BUNDLES OF RANK 4 AND THETA FUNCTIONS

In this chapter, we describe the moduli spaces of rank 4 orthogonal bundles over C, giving a special attention to the theta map. The main result is that its restriction to the component $\mathcal{M}^+_{\mathbf{SO}_4}$ of topologically trivial bundles is defined everywhere, for *every* curve (it is somehow surprising, since an extra condition, like the non-vanishing of the thetanulls, could have been expected here again). Regarding the behaviour of the restriction to the other component, we only get partial results.

5.1. Preliminaries

5.1.1. The quotient maps. — Our study relies on the basic isomorphism between $\mathbf{SO}_4$ and $(\mathbf{SL}_2 \times \mathbf{SL}_2)/\mu_2$, deduced from the exceptional isomorphism $\mathbf{Spin}_4 \simeq \mathbf{SL}_2 \times \mathbf{SL}_2$. This isomorphism gives two natural morphisms

$$\begin{array}{ccc} \mathcal{M}_{\mathbf{SL}_2} \times \mathcal{M}_{\mathbf{SL}_2} & \mathcal{M}^1_{\mathbf{SL}_2} \times \mathcal{M}^1_{\mathbf{SL}_2} & (E_1, E_2) \\ {\scriptstyle\pi}\downarrow & {\scriptstyle\pi}\downarrow & \downarrow \\ \mathcal{M}^+_{\mathbf{SO}_4} & \mathcal{M}^-_{\mathbf{SO}_4} & (\mathcal{H}om(E_1, E_2), \det) \end{array}$$

which present each component of the moduli space we are investigating as a quotient by J_2 of the self-product of a moduli space of rank 2 vector bundles with fixed determinant (recall that $\mathcal{M}^d_{\mathbf{SL}_2}$ is the moduli space of rank 2 vector bundles with fixed determinant of degree d).

This results for example from some long exact sequence of nonabelian cohomology: if we think rather of $\mathbf{SO}_4$ as the quotient of the group[(1)] $G =$

[(1)] We could also deduce this from the quotient $\mathbf{SL}_2 \times \mathbf{SL}_2 \longrightarrow \mathbf{SO}_4$...

$\{(g_1,g_2) \in \mathbf{GL}_2 \times \mathbf{GL}_2 | \det(g_1) = \det(g_2)\}$ by its *central* subgroup k^*, we get an exact sequence $\mathrm{H}^1_{\text{ét}}(C,G) \to \mathrm{H}^1_{\text{ét}}(C,\mathbf{SO}_4) \to \mathrm{H}^2_{\text{ét}}(C,k^*) = *$ (see C.5). Since the quotient $G \to \mathbf{SO}_4$ is induced by the natural representation of $\mathbf{GL}_2 \times \mathbf{GL}_2$ in $\mathrm{M}_2(k)$, this implies that any $\mathbf{SO}_4$-bundle arises as the sheaf of homomorphisms between two rank 2 vector bundles E_1 and E_2 with $\det(E_1) = \det(E_2)$. It is then enough to recall from [**Bea91**, 1.1.b)] that the parity of $h^0\left(C, \mathcal{H}om(E_1,E_2)\otimes\kappa\right)$ then coincides with the one of $\deg(E_1)$.

Another quotient will come into play: the isomorphism between $\mathbf{PSO}_4$ and $\mathbf{SO}_3 \times \mathbf{SO}_3$ induces two morphisms $\mathcal{M}^{\pm}_{\mathbf{SO}_4} \longrightarrow \mathcal{M}^{\pm}_{\mathbf{SO}_3} \times \mathcal{M}^{\pm}_{\mathbf{SO}_3}$, which are quotients by the action J_2 by tensorisation. We can express the image of an oriented orthogonal bundle (E,q,ω) in terms of the two half-spin representations of $\mathbf{SO}_4$: if $\mathbf{\Lambda}^2 E = E_1 \oplus E_2$ is the adjoint bundle of E, endowed with the quadratic form $\Lambda^2 q$ and the orientation $\Lambda^2\omega$, then (E,q,ω) is sent to (E_1,E_2) with the induced quadratic structures(2).

5.1.2. The Picard groups. — We recall here what the Picard groups of these different spaces are, and explain the behaviour of pull-backs via the different quotient maps. We will denote by(3) $\mathcal{L}^+_{\mathbf{SO}_4}$ (resp. $\mathcal{L}^-_{\mathbf{SO}_4}$) the determinantal bundle obtained as the pull-back of $\mathcal{L}_{\mathbf{SL}_4}$ to $\mathcal{M}^+_{\mathbf{SO}_4}$ (resp. $\mathcal{M}^-_{\mathbf{SO}_4}$) by the forgetful morphism.

Let us recall first how to compute the Picard group of the two components of $\mathcal{M}_{\mathbf{SO}_4}$. An easy proof goes as follows: we compute the pull-back of $\mathcal{L}_{\mathbf{SL}_4}$ to $\mathcal{M}^d_{\mathbf{SL}_2} \times \mathcal{M}^d_{\mathbf{SL}_2}$, which is done by pulling it back to the self product of the parameter scheme $R_{\mathbf{SL}_2} \longrightarrow \mathcal{M}^d_{\mathbf{SL}_2}$ defined in 1.2.13. The morphism $R_{\mathbf{SL}_2} \times R_{\mathbf{SL}_2} \longrightarrow \mathcal{M}_{\mathbf{SL}_4}$ is exactly the morphism associated to the vector bundle $\mathcal{H}om\big(p_1^*(\mathcal{E}_1), p_2^*(\mathcal{E}_2)\big)$ on $R_{\mathbf{SL}_2} \times R_{\mathbf{SL}_2} \times C$, where $\mathcal{E}_i$ is a Poincaré bundle on $R_{\mathbf{SL}_2} \times C$ (and p_i the projections), so that the pull-back of $\mathcal{L}_{\mathbf{SL}_4}$ is the inverse of $\det Rf_*\Big(\mathcal{H}om\big(p_1^*(\mathcal{E}_1), p_2^*(\mathcal{E}_2)\big) \otimes g^*L\Big)$, where L is a line bundle on C of degree $g-1$, and f and g the projections from $R_{\mathbf{SL}_2} \times R_{\mathbf{SL}_2} \times C$ onto, respectively, $R_{\mathbf{SL}_2} \times R_{\mathbf{SL}_2}$ and C. The see-saw principle and the construction of the determinant bundle on $\mathcal{M}^d_{\mathbf{SL}_2}$ implies the following two identities:

– $\pi^*\mathcal{L}^+_{\mathbf{SO}_4} = \mathcal{L}^2_{\mathbf{SL}_2} \boxtimes \mathcal{L}^2_{\mathbf{SL}_2}$ on $\mathcal{M}_{\mathbf{SL}_2} \times \mathcal{M}_{\mathbf{SL}_2}$,

(2) We can also describe it in terms of the two rulings of the quadric surfaces defined by the quadratic form: if $\mathbb{P}E_1 \times \mathbb{P}E_2 \subset \mathbb{P}E$ is the family of rulings of the projective bundle $\mathbb{P}E$, (E,q,ω) is mapped to $(\mathcal{E}nd_0(E_1), \mathcal{E}nd_0(E_2))$ or $(\mathcal{E}nd_0(E_2), \mathcal{E}nd_0(E_1))$ (depending on the orientation ω).

(3) Note that this is not inconsistent with the general definition of $\mathcal{L}^\delta_G$ given p. 6, since that definition suppose that G is almost simple.

– $\pi^*\mathcal{L}^-_{\mathbf{SO}_4} = \mathcal{L}_{\mathbf{SL}_2} \boxtimes \mathcal{L}_{\mathbf{SL}_2}$ on $\mathcal{M}^1_{\mathbf{SL}_2} \times \mathcal{M}^1_{\mathbf{SL}_2}$.

This also results from the identities $c_1\Big(Rf_*\big(\mathcal{H}om\big(p_1^*(\mathcal{E}_1), p_2^*(\mathcal{E}_2)\big) \otimes g^*\kappa\big)\Big) = -2g_*\big(p_1^*c_2(\mathcal{E}_1) + p_2^*c_2(\mathcal{E}_2)\big)$ and $c_1\big(Rf_*(\mathcal{E}_i \otimes g^*\kappa\big) = g_*\big(p_i^*c_2(\mathcal{E}_i)\big)$ given by Grothendieck-Riemann-Roch Theorem, or from the now classical computation of the determinant line bundles based on the Dynkin index.

The Picard groups of $\mathcal{M}^\pm_{\mathbf{SO}_4}$ fit in the following exact sequences, induced by the quotient map π:

$$0 \to \mathrm{Pic}(\mathcal{M}^+_{\mathbf{SO}_4}) \to \mathrm{Pic}(\mathcal{M}_{\mathbf{Spin}_4}) \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \to 0,$$

$$0 \to \mathrm{Pic}(\mathcal{M}^-_{\mathbf{SO}_4}) \to \mathrm{Pic}(\mathcal{M}^-_{\mathbf{Spin}_4}) \to \mathbb{Z}/2\mathbb{Z} \to 0$$

(indeed, we know that $\mathcal{L}^2 \boxtimes \mathcal{L}^2$, $\mathcal{L}^4 \boxtimes \mathcal{O}$ and $\mathcal{O} \boxtimes \mathcal{L}^4$ descend from $\mathcal{M}_{\mathbf{Spin}_4} = \mathcal{M}_{\mathbf{SL}_2} \times \mathcal{M}_{\mathbf{SL}_2}$ to $\mathcal{M}^+_{\mathbf{SO}_4}$, while $\mathcal{L}^2 \boxtimes \mathcal{O}$ cannot descend since it would imply that $\mathcal{L}^2_{\mathbf{SL}_2}$ descend to $\mathcal{M}^+_{\mathbf{SO}_3}$; in the same way we find that $\mathrm{Pic}(\mathcal{M}^-_{\mathbf{SO}_4})$ is generated by $\mathcal{L} \boxtimes \mathcal{L}$, $\mathcal{L}^2 \boxtimes \mathcal{O}$ and $\mathcal{O} \boxtimes \mathcal{L}^2$).

It is also easy to describe the behaviour of the second quotient morphisms $\mathcal{M}^+_{\mathbf{SO}_4} \longrightarrow \mathcal{M}^+_{\mathbf{SO}_3} \times \mathcal{M}^+_{\mathbf{SO}_3}$ and $\mathcal{M}^-_{\mathbf{SO}_4} \longrightarrow \mathcal{M}^-_{\mathbf{SO}_3} \times \mathcal{M}^-_{\mathbf{SO}_3}$: the pullback of $\mathcal{L}^+_{\mathbf{SO}_3} \boxtimes \mathcal{L}^+_{\mathbf{SO}_3}$ to $\mathcal{M}_{\mathbf{SL}_2} \times \mathcal{M}_{\mathbf{SL}_2}$ is $\mathcal{L}^4_{\mathcal{M}_{\mathbf{SL}_2}} \boxtimes \mathcal{L}^4_{\mathcal{M}_{\mathbf{SL}_2}}$, which shows that the square of the determinantal bundle $\mathcal{L}^+_{\mathbf{SO}_4}$ descends, but not the bundle itself; and the pullback of $\mathcal{L}^-_{\mathbf{SO}_3} \boxtimes \mathcal{L}^-_{\mathbf{SO}_3}$ to $\mathcal{M}^1_{\mathbf{SL}_2} \times \mathcal{M}^1_{\mathbf{SL}_2}$ is $\mathcal{L}^2_{\mathcal{M}^1_{\mathbf{SL}_2}} \boxtimes \mathcal{L}^2_{\mathcal{M}^1_{\mathbf{SL}_2}}$, which again shows that the square of the determinantal bundle $\mathcal{L}^-_{\mathbf{SO}_4}$ descends, but not the bundle itself.

5.2. The theta map $\mathcal{M}^+_{\mathbf{SO}_4} \longrightarrow |4\Theta|^+$ is a morphism

The aim of this section is to establish that the theta map $\mathcal{M}^+_{\mathbf{SO}_4} \dashrightarrow |4\Theta|^+$ has no base point (in genus 2, this is contained in the recent preprint [**Pau07**]): using pfaffian divisors, we check that this is indeed a morphism (which means in view of [**Bea06a**] that the line bundle $\mathcal{L}^+_{\mathbf{SO}_4}$ is indeed globally generated).

Let V be the space $\mathrm{H}^0(\mathcal{M}_{\mathbf{SL}_2}, \mathcal{L}_{\mathbf{SL}_2})$ of sections of the determinant bundle on $\mathcal{M}_{\mathbf{SL}_2}$. We will denote by $\varphi_{\mathcal{L}}$ the morphism $\mathcal{M}_{\mathbf{SL}_2} \to \mathbb{P}V$ given by $|\mathcal{L}_{\mathbf{SL}_2}|$. Beauville has proved in [**Bea88**] that this space is naturally isomorphic with $\mathrm{H}^0(J, 2\vartheta)$ via the morphism induced by $j\colon J \longrightarrow \mathcal{M}_{\mathbf{SL}_2}$, $L \longmapsto L \oplus L^{-1}$ (in other words, $\varphi_{\mathcal{L}} \circ j\colon J \longrightarrow \mathbb{P}V$ is identified to the morphism $J \longrightarrow |2\vartheta|^*$ associated to the complete linear system $|2\vartheta|$). We will in the sequel use this isomorphism to identify these two spaces.

5.2.1. Recall from [**LS97**, 7.10] that, for all theta-characteristic κ, there is a (Cartier) divisor $\Theta_\kappa \subset \mathcal{M}^+_{\mathbf{SO}_4}$ on $\mathcal{M}^+_{\mathbf{SO}_4}$, which is the inverse image by θ of the hyperplane $\mathrm{H}_\kappa \subset |4\Theta|^+$ consisting of all divisors in J^{g-1} passing through κ (the point here is to ensure that this does not define the whole space). Its inverse image by the quotient map π is thus an effective divisor D_κ with $\mathcal{O}(D_\kappa) \simeq \mathcal{L}^2_{\mathbf{SL}_2} \boxtimes \mathcal{L}^2_{\mathbf{SL}_2}$.

Since r is even, we can associate to any theta-characteristic κ a *pfaffian divisor* $\widetilde{\Theta}_\kappa$, which provides a square root of the theta divisor Θ_κ on the regularly stable locus $\mathcal{M}^{+,\mathrm{r}}_{\mathbf{SO}_4}$ of $\mathcal{M}^+_{\mathbf{SO}_4}$ (see [**LS97**, 7.10]). More precisely, this divisor is locally defined (on $\mathcal{M}^{+,\mathrm{r}}_{\mathbf{SO}_4}$) by a square root of an equation defining the divisor Θ_κ.

These pfaffian divisors give the easiest way to analyze the base locus of the theta-map: let us denote by D'_κ their pull-back to $\mathcal{M}_{\mathbf{SL}_2} \times \mathcal{M}_{\mathbf{SL}_2}$. The next proposition ensures that they define some Cartier divisors, which are automatically linearly equivalent to $\mathcal{L}_{\mathbf{SL}_2} \boxtimes \mathcal{L}_{\mathbf{SL}_2}$.

Proposition 5.2.2. — *The product $\mathcal{M}_{\mathbf{SL}_2} \times \mathcal{M}_{\mathbf{SL}_2}$ is locally factorial.*

Since $\mathcal{M}_{\mathbf{SL}_2}$ is smooth in genus 2 we can suppose $g \geqslant 3$. Consider the stable locus $\mathcal{M}^{\mathrm{s}}_{\mathbf{SL}_2} \subset \mathcal{M}_{\mathbf{SL}_2}$, which is an open subset whose complementary has codimension at least 2. Since $\mathcal{M}_{\mathbf{SL}_2} \times \mathcal{M}_{\mathbf{SL}_2}$ is normal, the Proposition will follow from the bijectivity of the natural morphism $\mathrm{Pic}(\mathcal{M}_{\mathbf{SL}_2} \times \mathcal{M}_{\mathbf{SL}_2}) \longrightarrow \mathrm{Cl}(\mathcal{M}_{\mathbf{SL}_2} \times \mathcal{M}_{\mathbf{SL}_2})$. The commutative diagram (see [**DN89**])

$$\begin{tikzcd}
\mathrm{Pic}(\mathcal{M}_{\mathbf{SL}_2} \times \mathcal{M}_{\mathbf{SL}_2}) \arrow[r] \arrow[d] & \mathrm{Cl}(\mathcal{M}_{\mathbf{SL}_2} \times \mathcal{M}_{\mathbf{SL}_2}) \arrow[d] \\
\mathrm{Pic}(\mathcal{M}^{\mathrm{s}}_{\mathbf{SL}_2} \times \mathcal{M}^{\mathrm{s}}_{\mathbf{SL}_2}) \arrow[r] & \mathrm{Cl}(\mathcal{M}^{\mathrm{s}}_{\mathbf{SL}_2} \times \mathcal{M}^{\mathrm{s}}_{\mathbf{SL}_2})
\end{tikzcd}$$

ensures that this is equivalent to the surjectivity of the restriction morphism $\mathrm{Pic}(\mathcal{M}_{\mathbf{SL}_2} \times \mathcal{M}_{\mathbf{SL}_2}) \longrightarrow \mathrm{Pic}(\mathcal{M}^{\mathrm{s}}_{\mathbf{SL}_2} \times \mathcal{M}^{\mathrm{s}}_{\mathbf{SL}_2})$. Since $\mathrm{Pic}(\mathcal{M}_{\mathbf{SL}_2} \times \mathcal{M}_{\mathbf{SL}_2})$ is isomorphic (by the see-saw principle) to $\mathrm{Pic}(\mathcal{M}_{\mathbf{SL}_2}) \times \mathrm{Pic}(\mathcal{M}_{\mathbf{SL}_2})$, this results from the following lemma (which has been shown to me by Arnaud Beauville):

Lemma 5.2.3. — *If U and V are two smooth unirational varieties, we have an isomorphism*

$$\mathrm{Pic}(U) \times \mathrm{Pic}(V) \xrightarrow{\sim} \mathrm{Pic}(U \times V).$$

Choose a smooth compactification X (resp. Y) of U (resp. V) such that $X - U = \bigcup_i E_i$ with $E_i \in \mathrm{Div}(X)$ (resp. $Y - V = \bigcup_j F_j$ with $F_j \in \mathrm{Div}(Y)$). We have $\mathrm{Pic}(U) = \mathrm{Pic}(X)/\langle E_i \rangle$ and $\mathrm{Pic}(V) = \mathrm{Pic}(Y)/\langle F_j \rangle$. Moreover, since

X and Y are proper and unirational, the see-saw principle implies that the natural map $\mathrm{Pic}(X) \times \mathrm{Pic}(Y) \longrightarrow \mathrm{Pic}(X \times Y)$ is an isomorphism. The lemma now follows from the isomorphism $\mathrm{Pic}(U \times V) = \mathrm{Pic}(X \times Y)/\langle E_i \times Y, X \times F_j\rangle$.

Remark 5.2.4. — We know from [**BLS98**, §13] that $\mathcal{M}_{\mathbf{Spin}_r}$ is not locally factorial for $r \geqslant 7$. On the contrary, the exceptional isomorphisms imply that it is locally factorial when $r = 3, 5$ or 6. Proposition 5.2.2 states that the same holds when $r = 4$. It follows from [**LS97**, 13.5] that the quotient $\mathcal{M}^+_{\mathbf{SO}_r}$ is never locally factorial: this is the reason why pulling-back the divisors to $\mathcal{M}_{\mathbf{SL}_2} \times \mathcal{M}_{\mathbf{SL}_2}$ is relevant when studying the rank 4 case.

5.2.5. Let us consider the linear system

$$\mathfrak{d} \subset |\mathcal{L}_{\mathbf{SL}_2} \boxtimes \mathcal{L}_{\mathbf{SL}_2}|$$

spanned by the D'_κ associated to the *even* theta-characteristics. The divisors D'_κ come (via $\varphi_{\mathcal{L}} \times \varphi_{\mathcal{L}}$) from some divisors on $\mathbb{P}V \times \mathbb{P}V$ which are linearly equivalent to $\mathcal{O}_{\mathbb{P}V}(1) \boxtimes \mathcal{O}_{\mathbb{P}V}(1)$ (and invariant under the involution switching the two factors); in other words, there is a commutative diagram

$$\begin{tikzcd} \mathcal{M}_{\mathbf{SL}_2} \times \mathcal{M}_{\mathbf{SL}_2} \arrow[r, "\varphi_{\mathcal{L}} \times \varphi_{\mathcal{L}}"] \arrow[d, dashed] & \mathbb{P}V \times \mathbb{P}V \arrow[dl, dashed, "p"] \\ \mathfrak{d}^* & \end{tikzcd}$$

where the map $p\colon \mathbb{P}V \times \mathbb{P}V \dashrightarrow \mathfrak{d}^*$ may be defined in a very explicit way, which we are now going to write down.

5.2.6. Since it is finite, the quotient map π induces a morphism between divisor groups $\pi^*\colon \mathrm{Div}(\mathcal{M}^+_{\mathbf{SO}_4}) \longrightarrow \mathrm{Div}(\mathcal{M}_{\mathbf{SL}_2} \times \mathcal{M}_{\mathbf{SL}_2})$, whose image consists of divisors invariant under the (diagonal) action of J_2. By 5.2.1, the pfaffian divisor D'_κ is defined by a section s_κ of $\mathcal{L}_{\mathbf{SL}_2} \boxtimes \mathcal{L}_{\mathbf{SL}_2}$, and its invariance property implies that s_κ must be *an eigenvector for the natural action of the Mumford group* $\mathcal{G}(\mathcal{L}_{\mathbf{SL}_2})$ *on* $\mathrm{H}^0(\mathcal{M}_{\mathbf{SL}_2}, \mathcal{L}_{\mathbf{SL}_2}) \otimes \mathrm{H}^0(\mathcal{M}_{\mathbf{SL}_2}, \mathcal{L}_{\mathbf{SL}_2})$ induced by the diagonal action of $\mathcal{G}(\mathcal{L}_{\mathbf{SL}_2})$ on $\mathcal{L}_{\mathbf{SL}_2} \boxtimes \mathcal{L}_{\mathbf{SL}_2}$: recall that $\mathcal{G}(\mathcal{L}_{\mathbf{SL}_2})$, which lifts to $\mathcal{L}_{\mathbf{SL}_2}$ the action of J_2 on $\mathcal{M}_{\mathbf{SL}_2}$, naturally acts on $V = \mathrm{H}^0(\mathcal{M}_{\mathbf{SL}_2}, \mathcal{L}_{\mathbf{SL}_2})$. According to what has been recalled in B.5, the induced representation (of weight 2) on $V \otimes V$ splits as a direct sum of one-dimensional non-isomorphic representations

$$V \otimes V \simeq \bigoplus_{\kappa} k \cdot \xi_\kappa, \tag{5.2.6.1}$$

where $k \cdot \xi_\kappa$ is the eigenspace corresponding to a character $\chi_\kappa \in \mathrm{X}(\mathcal{G}(\mathcal{L}_{\mathbf{SL}_2}))$ of weight 2.

The decomposition (5.2.6.1) shows that any section s_κ defining D'_κ must be one of the $\xi_{\kappa'}$ (up to some scalars). Note that, if κ is odd, ξ_κ defines an antisymmetric function on the tensor product, so that the corresponding divisor on $\mathcal{M}_{\mathbf{SL}_2} \times \mathcal{M}_{\mathbf{SL}_2}$ contains the diagonal $\Delta_{\mathcal{M}_{\mathbf{SL}_2}}$; on the other hand, if κ is even, the support of D'_κ cannot contain $(\mathcal{O} \oplus \mathcal{O}, \mathcal{O} \oplus \mathcal{O})$ (at least in the case where C has no vanishing thetanull). Therefore D'_κ must correspond, for κ even, to a $s_{\kappa'}$ with κ' even.

The following lemma checks that the expected correspondence actually holds[(4)]:

Lemma 5.2.7. — *The divisor $D'_\kappa \in \mathfrak{d}$ is the trace on $\mathcal{M}_{\mathbf{SL}_2} \times \mathcal{M}_{\mathbf{SL}_2}$ of* $\mathrm{div}(\xi_\kappa) \in |\mathcal{O}_{\mathbb{P}V}(1) \boxtimes \mathcal{O}_{\mathbb{P}V}(1)|$. *In particular, $\mathfrak{d}$ is identified with the linear system consisting of all symmetric sections of $\mathcal{O}_{\mathbb{P}V}(1) \boxtimes \mathcal{O}_{\mathbb{P}V}(1)$.*

We prove this by restricting these divisors to $J \times J$, mapped into $\mathcal{M}_{\mathbf{SL}_2} \times \mathcal{M}_{\mathbf{SL}_2}$ via the product of the Kummer maps $j \colon L \in J \longmapsto L \oplus L^{-1}$. Note that we will abusively write here $\mathrm{div}(\xi_\kappa)$ for the divisor naturally defined by ξ_κ on $\mathbb{P}V \times \mathbb{P}V$ as well as for its trace on $J \times J$.

Let us consider the successive inverse images of $\Theta_\kappa \subset \mathcal{M}_{\mathbf{SO}_4}$ (which is the pull back via the forgetful morphism $\mathcal{M}^+_{\mathbf{SO}_4} \longrightarrow \mathcal{M}_{\mathbf{SL}_4}$ of the theta divisor $\Theta_\kappa \subset \mathcal{M}_{\mathbf{SL}_4}$) trough the commutative diagram

(5.2.7.1)
$$\begin{tikzcd}
J \times J \arrow[r, "j\times j"] \arrow[d, "(m,d)"'] & \mathcal{M}_{\mathbf{SL}_2} \times \mathcal{M}_{\mathbf{SL}_2} \arrow[d, "\pi"] & \\
J \times J \arrow[r] \arrow[d, "j\times j"'] & \mathcal{M}^+_{\mathbf{SO}_4} \arrow[r, dashed] \arrow[d] & {|4\Theta|^+} \\
\mathcal{M}_{\mathbf{SL}_2} \times \mathcal{M}_{\mathbf{SL}_2} \arrow[r, "\oplus"] & \mathcal{M}_{\mathbf{SL}_4} &
\end{tikzcd}$$

where (m,d) maps (L_1, L_2) to $(L_1 \otimes L_2, L_1 \otimes L_2^{-1})$.

The inverse image by $\oplus$ of the divisor $\Theta_\kappa \subset \mathcal{M}_{\mathbf{SL}_4}$ is $\Theta^{(2)}_\kappa \times \mathcal{M}_{\mathbf{SL}_2} + \mathcal{M}_{\mathbf{SL}_2} \times \Theta^{(2)}_\kappa$ (where $\Theta^{(2)}_\kappa$ is the theta divisor $\{E \in \mathcal{M}_{\mathbf{SL}_2} \mid h^0(C, E \otimes \kappa) \geqslant 1\} \subset \mathcal{M}_{\mathbf{SL}_2}$),

(4) Another proof would be to consider the translates $(\alpha, 1) \cdot s_\kappa$ as Beauville did in [**Bea91**] with the translates of ξ_κ; rather than recopying his argument, we prefer the following proof which uses it.

which in turn pulls back via $j\times j$ to $2(T^*_\kappa\Theta\times J)+2(J\times T^*_\kappa\Theta)$ (where $T_\kappa\colon J \longrightarrow J^{g-1}$ is the translation by κ). Note that $(T^*_\kappa\Theta \times J) + (J \times T^*_\kappa\Theta)$ is nothing but the pfaffian divisor deduced from κ and the rank 4 orthogonal bundle $p^*_{13}(\mathcal{P} \oplus \mathcal{P}^{-1}) \oplus p^*_{23}(\mathcal{P} \oplus \mathcal{P}^{-1})$ on $J \times J \times C$ (where $\mathcal{P}$ denotes a Poincaré bundle on $J \times C$).

Recall from [**Bea91**, A.5] that $(m,d)^*\left((T^*_\kappa\Theta \times J) + (J \times T^*_\kappa\Theta)\right)$ is a divisor defined by an explicit section $\xi_\kappa \in \mathrm{H}^0(J\times J, \mathcal{O}_J(2\vartheta)\boxtimes\mathcal{O}_J(2\vartheta))$. We have thus found that the equality $2\,(j\times j)^*D'_\kappa = 2\,\mathrm{div}(\xi_\kappa)$ holds in $\mathrm{Div}(J\times J)$.

The lemma now follows from the fact that j induces an isomorphism $j^*\colon \mathrm{H}^0(\mathcal{M}_{\mathbf{SL}_2}, \mathcal{L}_{\mathbf{SL}_2}) \longrightarrow \mathrm{H}^0(J, \mathcal{O}(2\vartheta))$: D'_κ is the restriction of the divisor on $\mathbb{P}V \times \mathbb{P}V$ given by the section $\xi_\kappa \in \mathrm{H}^0(\mathbb{P}V\times\mathbb{P}V, \mathcal{O}_{\mathbb{P}V}(1)\boxtimes\mathcal{O}_{\mathbb{P}V}(1))$; to prove the last part of the lemma, we just have to recall that the (ξ_κ)'s with κ even form a basis of $\mathbf{S}^2V$.

Remark 5.2.8. — Let us remark that the linear independency of the D'_κ directly follows from an easy description of the push-forward $\pi_*(\mathcal{L}\boxtimes\mathcal{L})$. Indeed, this coherent (and reflexive) sheaf splits, *over the regularly stable locus*, as the direct sum of the pfaffian line bundles

$$\pi_*(\mathcal{L}_{\mathbf{SL}_2}\boxtimes\mathcal{L}_{\mathbf{SL}_2})_{|\mathcal{M}^{\mathrm{r},+}_{\mathbf{SO}_4}} \simeq \bigoplus_\kappa \mathcal{P}_\kappa;$$

to see this, we just have to notice that the choice of a linearization of $\mathcal{L}\boxtimes\mathcal{L}$ over the regularly stable locus (say associated to any isomorphism $\mathcal{L}\boxtimes\mathcal{L}\simeq\pi^*\mathcal{P}_\kappa$) induces a decomposition into invertible eigenbundles $\pi_*(\mathcal{L}\boxtimes\mathcal{L}) = \bigoplus_{\eta\in X(J_2)} L_\eta$, with $L_0\simeq\mathcal{P}_\kappa$ and $L_\eta\otimes L_{\eta'}\simeq L_{\eta\eta'}\otimes\mathcal{P}_\kappa$. The J_2-action on $\mathcal{M}_{\mathbf{SO_4}}$ conjugate the L_η, and the expected fact follows from the identity $\alpha^*\mathcal{P}_\kappa\simeq\mathcal{P}_{\alpha\otimes\kappa}$ (see [**BLS98**, 5.5]). This implies that $h^0(\mathcal{M}^{\mathrm{r},+}_{\mathbf{SO}_4}, \mathcal{P}_\kappa) = 1$ for all κ, and that the pfaffian divisors $D'_\kappa \in |\mathcal{L}_{\mathbf{SL}_2}\boxtimes\mathcal{L}_{\mathbf{SL}_2}|$ are linearly independent.

Note that this is a special case of [**PR01**, 8.2], which should be compared to the decomposition $2_{J*}\mathcal{O}_J(4\vartheta)\simeq\oplus_\kappa\mathcal{O}_J(T^*_\kappa\Theta)$: if we add to the diagram of the proof of 5.2.7 the commutative square

$$\begin{tikzcd} J \arrow[r, "\Delta"] \arrow[d, "2_J"] & J\times J \arrow[d, "(m,d)"] \\ J \arrow[r, "(id,\mathcal{O})"] & J\times J \end{tikzcd}$$

we get an identification between $\ker\left(\mathrm{Pic}(\mathcal{M}^{r,+}_{\mathbf{SO}_4}) \longrightarrow \mathrm{Pic}(\mathcal{M}_{\mathbf{SL}_2}\times\mathcal{M}_{\mathbf{SL}_2})\right)$ and $\ker(2^*_J\colon \mathrm{Pic}(J)\to\mathrm{Pic}(J))$ (which is by the way the one of [**BLS98**, 5.2]).

5.2.9. We have up to now introduced a linear system $\mathfrak{d}$ defining a rational map $p\colon \mathbb{P}V \times \mathbb{P}V \dashrightarrow \mathfrak{d}^*$ such that, set-theoretically, the inverse image $\pi^{-1}(\mathrm{Bs}(\theta))$ of the base locus of θ is contained in $(\varphi_{\mathcal{L}} \times \varphi_{\mathcal{L}})^{-1}(\mathrm{Bs}(p))$. The following diagram may help to understand the situation:

(5.2.9.1)
$$\begin{tikzcd} & \mathbb{P}V \times \mathbb{P}V \arrow[d, dashed, "p"] \\ \mathcal{M}_{\mathbf{SL}_2} \times \mathcal{M}_{\mathbf{SL}_2} \arrow[ur, "\varphi_{\mathcal{L}}\times\varphi_{\mathcal{L}}"] \arrow[r, dashed] \arrow[d, "\pi"] & \mathfrak{d}^* \\ \mathcal{M}^+_{\mathbf{SO}_4} \arrow[r, dashed, "\theta"] & {|4\Theta|^+.} \end{tikzcd}$$

It is enough to check that p has no base point: but Lemma 5.2.7 says that p is the projective morphism given by the subspace of all symmetric sections in $\mathrm{H}^0\left(\mathbb{P}V \times \mathbb{P}V, \mathcal{O}(1) \boxtimes \mathcal{O}(1)\right)$. In other words, this is the composite of the Segre embedding $\mathbb{P}V \times \mathbb{P}V \longrightarrow \mathbb{P}(V \otimes V)$ and the projection $\mathbb{P}(V \otimes V) \dashrightarrow \mathfrak{d}^* \simeq \mathbb{P}(\mathbf{S}^2V)$ with center $\mathbb{P}(\mathbf{\Lambda}^2V)$. Since the tensor $\lambda \otimes \mu$ is not antisymmetric, the Segre variety in $\mathbb{P}(V \otimes V)$ does not meet $\mathbb{P}(\mathbf{\Lambda}^2V)$, and p is thus a morphism.

This implies the main result of this section:

Theorem 5.2.10. — *For any curve C, the theta map $\mathcal{M}^+_{\mathbf{SO}_4} \longrightarrow |4\Theta|^+$ has no base point.*

We therefore know that $\theta\colon \mathcal{M}^+_{\mathbf{SO}_4} \longrightarrow |4\Theta|^+$ is always finite onto its image.

We can in fact say more about p:

Proposition 5.2.11. — *The image in $\mathbb{P}(\mathbf{S}^2V)$ of the Segre variety is the quotient of $\mathbb{P}V \times \mathbb{P}V$ by the involution σ switching the two factors, and p is the quotient map. In particular, p is a finite morphism of degree 2 onto its image.*

This quotient is indeed the projective homogeneous space

$$\mathrm{Proj}\Big(\bigoplus_d \mathrm{H}^0\big(\mathbb{P}V \times \mathbb{P}V, \mathcal{O}(d) \boxtimes \mathcal{O}(d)\big)^{\sigma^*}\Big),$$

and we thus have to describe the σ^*-invariant polynomials in $k[X_i, Y_j]$ of bidegree (d, d). We know from a theorem of Noether about invariants for actions of finite groups (see for example [**Wey39**, VIII.15]) that the whole invariant algebra $k[X_i, Y_j]^{\sigma^*}$ is generated by its elements of degree at most 2: more explicitely, this algebra is generated by the elements X_i+Y_i and $X_iY_j+X_jY_i$. An immediate examination of the bigraduation shows that a σ^*-invariant polynomial of bidegree (d,d) is contained in the subalgebra generated by the functions $X_iY_j + X_jY_i$. This gives us an embedding $(\mathbb{P}V \times \mathbb{P}V)/\sigma \hookrightarrow \mathbb{P}(\mathbf{S}^2V)$, such that the quotient morphism $\mathbb{P}V \times \mathbb{P}V \longrightarrow \mathbb{P}(\mathbf{S}^2V)$ is exactly the projective morphism defined by the linear system of all symmetric polynomials of bidegree $(1,1)$ (note that it sends the closed point (λ,μ) defined by two linear forms λ and μ on V to $\lambda\otimes\mu + \mu\otimes\lambda \in (\mathbf{S}^2V)^*$). But we have just seen that p is precisely defined by this linear system, whence our assertion.

5.2.12. For later use, it may be useful to give the explicit equations for the morphism p in some natural basis of the different spaces of sections. To do this, we first choose an isomorphism between $\mathcal{G}(\mathcal{L}_{\mathbf{SL}_2})$ and the Heisenberg group $H_2 = k^* \times \mathbb{F}_2^g \times \widehat{\mathbb{F}_2^g}$. The results recalled in Appendix B give us some natural coordinates $(X_b)_{b\in\mathbb{F}_2^g}$ (resp. $(Y_b)_{b\in\mathbb{F}_2^g}$) on the first (resp. second) factor of $\mathbb{P}V \times \mathbb{P}V$, while Lemma 5.2.7 (and B.5) gives some natural coordinates $(Z_\kappa)_{\kappa\,\text{even}}$ on $\mathfrak{d}^*$. Through these choices, p becomes identified with the map $\mathbb{P}^{2^g-1} \times \mathbb{P}^{2^g-1} \dashrightarrow \mathbb{P}^{N-1}$ (where N is $2^{g-1}(2^g+1)$) given by:

$$Z_\kappa = \sum_{b\in\mathbb{F}_2^g} \gamma(b) X_b Y_{b+c} \tag{5.2.12.1}$$

where κ corresponds to (c,γ). Note that we can deduce deduce from these equations another proof of the fact that p has no base point: the vanishing of all the Z_κ is equivalent to the equations $X_bY_{b'} + X_{b'}Y_b = 0$ for all b,b' (use the easy identities $\sum_\gamma \gamma(a) = \begin{cases} 2^g \text{ if } a=0 \\ 0 \text{ otherwise}\end{cases}$, and $\sum_{\gamma,\,\gamma(c)=1} \gamma(a) = \begin{cases} 2^{g-1} \text{ if } a\in\{0,c\} \\ 0 \text{ otherwise}\end{cases}$). This new system of polynomial equations is easy to solve.

5.3. A question about theta functions

5.3.1. The diagram (5.2.9.1) naturally invites us to ask whether there should be an arrow $\mathfrak{d}^* \longrightarrow |4\Theta|^+$ making the whole diagram commutative, so that we would have a very explicit factorization of the theta map as the composite of a (rational) map defined by a linear system $\mathfrak{d}$ associated to $\mathcal{L}_{\mathbf{SL}_2} \boxtimes \mathcal{L}_{\mathbf{SL}_2}$ with a map $\mathfrak{d}^* \dashrightarrow |4\Theta|^+$ coming from a base point free system of quadrics on $\mathfrak{d}^*$. Since this diagram has been obtained by writing precisely the inverse image of the hyperplane sections $H_\kappa \subset |4\Theta|^+$, we need to know whether these hyperplane sections span $\big(\mathrm{H}^0(J^{g-1}, 4\Theta)^+\big)^*$. Unfortunately, we did not manage to prove this.

We discuss here this condition, and we will explain in the next section how to deduce from it the factorization.

Remark 5.3.2. — Geometrically, this condition means that no *even* divisor $D \in |4\Theta|^+$ contains every even theta-characteristic.

According to [**Bea06a**], an equivalent way to express the condition is to say that the linear system $|\mathcal{L}^+_{\mathbf{SO}_4}|$ is spanned by the generalized theta divisors Θ_κ given by the *even* theta-characteristic κ. Let us formulate now a quite optimistic conjecture:

Conjecture 5.3.3. — *The linear system $|\mathcal{L}^+_{\mathbf{SO}_4}|$ is spanned by the theta divisors Θ_κ associated to the even theta-characteristic κ.*

As we will see, this holds in genus 2. In higher genus, it would not be too surprising that it only holds for generic curves. In fact, the condition that C has no vanishing thetanull may turn out to be enough: we will propose another Conjecture, which is in this case a reformulation of the previous one.

5.3.4. It follows from (5.2.7.1) that it is enough[(5)] to prove the linear independence of $\big(2(T^*_\kappa\Theta \times J) + 2(J \times T^*_\kappa\Theta)\big)_{\kappa \text{ even}}$ in $|\mathcal{O}_J(2\vartheta) \boxtimes \mathcal{O}_J(2\vartheta)|$. In other words, if θ_κ denotes a non-zero section of $\mathrm{H}^0(J, T^*_\kappa\Theta)$, we have to verify that the sections $\theta_\kappa^2 \otimes \theta_\kappa^2$, κ even, are linearly independent in $\mathrm{H}^0(J \times J, 2\vartheta \boxtimes 2\vartheta)$. Note that we know that the sections $\theta_\kappa^2 \otimes \theta_{\kappa'}^2$ span $\mathrm{H}^0(J \times J, 2\vartheta \boxtimes 2\vartheta)$ (for example since their linear span is invariant for the action of the theta group $\mathcal{G}(2\vartheta \boxtimes 2\vartheta)$, whose representation in $\mathrm{H}^0(J \times J, 2\vartheta \boxtimes 2\vartheta)$ is irreducible).

If C has no vanishing theta-null, the morphism Δ^* induced by the diagonal embedding $J \hookrightarrow J \times J$ gives a surjective morphism onto the space $\mathrm{H}^0(J, 4\vartheta)^+$ of even fourth order theta functions, and the conjecture becomes:

Conjecture 5.3.5. — *If (A, ϑ) is a principally polarized Abelian variety, the fourth powers $(\theta_\kappa^4)_{\kappa \text{ even}}$ of the even theta functions of order 1 form a basis of the space $\mathrm{H}^0(A, 4\vartheta)^+$ of even fourth order theta functions.*

This means that the multiplication morphism

$$\sum_{\kappa \text{ even}} \mathrm{H}^0(A, \vartheta_\kappa)^{\otimes 4} \longrightarrow \mathrm{H}^0(A, 4\vartheta)^+$$

[(5)] This is even an equivalence when C has no vanishing theta-null.

is an isomorphism (recall that ϑ_κ is here the symmetric theta divisor associated to the theta-characteristic κ).

Remark 5.3.6. — Note that this together with [**Bea06a**] would give a natural pairing $\left(\mathrm{H}^0(J^{g-1}, 4\Theta)^+\right)^* \xrightarrow{\sim} \mathrm{H}^0(J, 4\vartheta)^+$. More precisely, this would imply that the morphism

$$J \xrightarrow{\Delta} J \times J \longrightarrow \mathcal{M}^+_{\mathbf{SO}_4} \xrightarrow{\theta} |4\Theta|^+$$

which sends L to the divisor $2\left(T_L^*\Theta + T_{L^{-1}}^*\Theta\right)$ in J^{g-1} induces an isomorphism between spaces of global sections. The Wirtinger duality explained in [**Mum74**] asserts that there is, for every theta-characteristic κ, a commutative diagram

$$\begin{tikzcd} & {|2\vartheta|^*} \arrow[dd, "\wr"] \\ J \arrow[ur, "\varphi"] \arrow[dr, "\delta_\kappa"'] & \\ & {|2\vartheta|} \end{tikzcd}$$

where δ is $L \mapsto T_L^*\vartheta_\kappa + T_{L^{-1}}^*\vartheta_\kappa$ (and φ the Kummer map). This shows that the Conjecture is implied (and equivalent when A has no zero theta-null) to the following fact: there is no quadric hypersurface in $|2\vartheta|^*$ containing the images $\varphi(\alpha)$ of the line bundles $\alpha \in J_2$ where κ takes the value $\varepsilon(\kappa)$.

5.3.7. Since the condition about Abelian varieties considered in Conjecture 5.3.5 is an open in the moduli of Abelian varieties of dimension g, finding one Abelian variety satisfying the conjecture would prove that it generically holds. Since it is obviously true in genus 1, a natural attempt is to consider a decomposable Abelian variety $A = A' \times A''$. Unfortunately, we easily see that the Conjecture fails for such a variety. Indeed, a theta-characteristic on A is given as a couple (κ', κ'') of theta-characteristics on A' and A''; it is even if and only if κ' and κ'' have the same parity $\varepsilon(\kappa') = \varepsilon(\kappa'')$. The direct sum $\sum\limits_{\kappa \text{ even}} \mathrm{H}^0(A, \vartheta_\kappa)^{\otimes 4}$ is thus equal to $\sum\limits_{\kappa',\kappa'' | \varepsilon(\kappa)=\varepsilon(\kappa'')} \mathrm{H}^0(A', \vartheta'_{\kappa'})^{\otimes 4} \otimes \mathrm{H}^0(A'', \vartheta''_{\kappa''})^{\otimes 4}$, whose image by the multiplication map must be contained in $\mathrm{H}^0(A', 4\vartheta')^+ \otimes \mathrm{H}^0(A'', 4\vartheta'')^+$: it cannot be surjective.

5.3.8. Another way to attack this question is to fix a theta structure for $(A, 2\vartheta)$, and to use the addition formula to obtain the identity

$$\theta^2_{\kappa\left[\substack{c\\ \gamma}\right]}(x) = \sum_{b\in\mathbb{F}_2^g} \gamma(b) X_b(x) X_{b+c}(0),$$

where $(X_b)_{b\in\mathbb{F}_2^g}$ is the basis of $\mathrm{H}^0(A, 2\vartheta)$ described in B.2 (recall that the notation $\kappa\left[\substack{c\\ \gamma}\right]$ is explained in B.4). This gives the decompositions

$$\theta^4_{\kappa\left[\substack{c\\ \gamma}\right]} = \sum_{b,b'} \gamma(b+b') X_{b+c}(0) X_{b'+c}(0) X_b X_{b'},$$

of all the θ^4_κ in the natural basis $X_b X_{b'}$ of $\mathrm{H}^0(J\times J, 2\vartheta \boxtimes 2\vartheta)$.

Going further requires a fine knowledge of the algebraic spans of the theta constants: a linear combination $\sum_{\kappa \text{ even}} \lambda_\kappa \theta^4_\kappa = 0$ implies the 2^{2g} relations

$$\forall b, b' \in \mathbb{F}_2^g, \quad \sum_{\kappa\left[\substack{c\\ \gamma}\right] \text{ even}} \gamma(b)\gamma(b')\psi_{b+c}(0)\psi_{b'+c}(0)\lambda_{\kappa\left[\substack{c\\ \gamma}\right]} = 0.$$

The vanishing of all the λ_κ would thus follow from the non vanishing of one of the $2^{g-1}(2^g+1)\times 2^{g-1}(2^g+1)$ minors of this system, and such a determinant is a polynomial of degree $2^g(2^g+1)$ in the theta constants.

5.3.9. Let us explain now how we checked that the conjecture holds in genus 2. First note that another way to consider the question is to study the linear dependence of the $D_\kappa = \pi^*\Theta_\kappa$ in $|\mathcal{L}^2_{\mathbf{SL}_2} \boxtimes \mathcal{L}^2_{\mathbf{SL}_2}|$. We know by the proof of Lemma 5.2.7 that they correspond to the sections s^2_κ, which admit an explicit description. In genus 2, $\mathcal{M}_{\mathbf{SL}_2}$ is isomorphic to $\mathbb{P}V \simeq \mathbb{P}^3$, and we are in fact dealing with sections $s^2_\kappa \in \mathrm{H}^0(\mathbb{P}V, \mathcal{O}(2)) \otimes \mathrm{H}^0(\mathbb{P}V, \mathcal{O}(2)) \simeq \mathbf{S}^2V \otimes \mathbf{S}^2V$. These 10 sections are the squares of the following polynomials (see 5.2.12)

$$\begin{gathered}
X_0Y_0 + X_1Y_1 + X_2Y_2 + X_3Y_3,\\
X_0Y_0 + X_1Y_1 - X_2Y_2 - X_3Y_3,\\
X_0Y_0 - X_1Y_1 + X_2Y_2 - X_3Y_3,\\
X_0Y_0 - X_1Y_1 - X_2Y_2 + X_3Y_3,\\
X_0Y_1 + X_1Y_0 + X_2Y_3 + X_3Y_2,\\
X_0Y_1 + X_1Y_0 - X_2Y_3 - X_3Y_2,\\
X_0Y_2 + X_1Y_3 + X_2Y_0 + X_3Y_1,\\
X_0Y_2 - X_1Y_3 + X_2Y_0 - X_3Y_1,\\
X_0Y_3 + X_1Y_2 + X_2Y_1 + X_3Y_0,\\
X_0Y_3 - X_1Y_2 - X_2Y_1 + X_3Y_0.
\end{gathered}$$

We have to check whether these 10 elements of $k[X_i, Y_j]_{(2,2)}$ are linearly independent. A brutal way to do this is to decompose these squares in the standard basis of the space of polynomials of bidegree $(2,2)$ and to compute the rank of the 10×100 matrix we have found. With the help of a computer, we found the expected rank.

Theorem 5.3.10. — *For a curve of genus* 2*, the divisors* Θ_κ *for* κ *even span* $|\mathcal{L}^+_{\mathbf{SO}_4}|$.

Remark 5.3.11. — a) We can of course try to use this method in higher genus. We would obtain a $2^{g-1}(2^g+1) \times \left(2^{g-1}(2^g+1)\right)^2$ matrix, whose rank is expected to be maximal. The point is that the conjecture implies the independence of the sections s_κ^2 viewed in $\mathrm{H}^0\left(\mathbb{P}V \times \mathbb{P}V, \mathcal{O}(2) \otimes \mathcal{O}(2)\right)$, while the converse is true *for a curve without vanishing thetanull*: for such a curve, we know from [**Bea91**] that the map

$$(\varphi_{\mathcal{L}} \times \varphi_{\mathcal{L}})^*\colon \mathrm{H}^0\left(\mathbb{P}V \times \mathbb{P}V, \mathcal{O}(2) \boxtimes \mathcal{O}(2)\right) \longrightarrow \mathrm{H}^0\left(\mathcal{M}_{\mathbf{SL}_2} \times \mathcal{M}_{\mathbf{SL}_2}, \mathcal{L}^2_{\mathbf{SL}_2} \boxtimes \mathcal{L}^2_{\mathbf{SL}_2}\right)$$

is an isomorphism, whence our assertion.

We have been able to check in this way that the conjecture holds for curves of genus $\leqslant 5$ without vanishing thetanull.

b) It would also be interesting to begin by investigating the case of Jacobians of hyperelliptic curves.

5.4. Description of the theta map on $\mathcal{M}^+_{\mathbf{SO}_4}$ for a curve of genus 2

In this section, we first explain how to get an explicit factorization of the theta map for curves of any genus satisfying Conjecture 5.3.3. Then we show how this description allows us to give a few results on the geometry of $\mathcal{M}^+_{\mathbf{SO}_4}$ for a curve of genus 2.

5.4.1. We are considering the morphism $\mathcal{M}_{\mathbf{SL}_2} \times \mathcal{M}_{\mathbf{SL}_2} \longrightarrow |4\Theta|^+$. Since it satisfies $\pi^*\theta^*\mathcal{O}_{|4\Theta|^+}(1) \simeq \mathcal{L}^2_{\mathbf{SL}_2} \boxtimes \mathcal{L}^2_{\mathbf{SL}_2}$, this morphism is entirely determined by the corresponding map

$$(\theta \circ \pi)^*\colon\ \left(\mathrm{H}^0(J^{g-1}, 4\Theta)^+\right)^* \longrightarrow \mathrm{H}^0(\mathcal{M}_{\mathbf{SL}_2} \times \mathcal{M}_{\mathbf{SL}_2}, \mathcal{L}^2_{\mathbf{SL}_2} \boxtimes \mathcal{L}^2_{\mathbf{SL}_2});$$

we have to understand the image of a basis of $\left(\mathrm{H}^0(J^{g-1}, 4\Theta)^+\right)^*$. For a curve satisfying Conjecture 5.3.3, this space has a basis $(T_\kappa)_{\kappa \text{ even}}$ such that the linear form T_κ corresponds to the hyperplane H_κ.

We have seen earlier that $D_\kappa = \pi^*\theta^*(\mathrm{H}_\kappa)$ is defined by the square s_κ^2 of a section s_κ of $\mathcal{L}_{\mathbf{SL}_2} \boxtimes \mathcal{L}_{\mathbf{SL}_2}$ which is invariant for the involution $\tau\colon (E, E') \in$

$\mathcal{M}_{\mathbf{SL}_2}\times\mathcal{M}_{\mathbf{SL}_2}\longmapsto(E',E)$; in other words, s_κ belongs to the invariant space $\mathrm{H}^0(\mathcal{M}_{\mathbf{SL}_2}\times\mathcal{M}_{\mathbf{SL}_2},\mathcal{L}_{\mathbf{SL}_2}\boxtimes\mathcal{L}_{\mathbf{SL}_2})^{\tau^*}$, which is isomorphic to $\mathrm{H}^0(\mathfrak{d}^*,\mathcal{O}_{\mathfrak{d}^*}(1))$ (see Lemma 5.2.7). The linear map $(\theta\circ\pi)^*$ thus fits in the commutative diagram

$$\begin{tikzcd}
 & \mathrm{H}^0(\mathfrak{d}^*,\mathcal{O}_{\mathfrak{d}^*}(2)) \arrow[dl, "\mu"'] \\
\mathrm{H}^0(\mathcal{M}_{\mathbf{SL}_2}\times\mathcal{M}_{\mathbf{SL}_2},\mathcal{L}^2_{\mathbf{SL}_2}\boxtimes\mathcal{L}^2_{\mathbf{SL}_2}) & \big(\mathrm{H}^0(J^{g-1},4\Theta)^+\big)^* \arrow[l, "(\theta\circ\pi)^*"] \arrow[u, "s"']
\end{tikzcd}$$

where s is the isomorphism sending to T_κ to s_κ^2 (up to a non zero scalar), and μ is the multiplication map from $\mathrm{H}^0(\mathfrak{d}^*,\mathcal{O}_{\mathfrak{d}^*}(2))\simeq\mathbf{S}^2\mathrm{H}^0(\mathcal{M}_{\mathbf{SL}_2}\times\mathcal{M}_{\mathbf{SL}_2},\mathcal{L}_{\mathbf{SL}_2}\boxtimes\mathcal{L}_{\mathbf{SL}_2})^{\tau^*}$ to $\mathrm{H}^0(\mathcal{M}_{\mathbf{SL}_2}\times\mathcal{M}_{\mathbf{SL}_2},\mathcal{L}^2_{\mathbf{SL}_2}\boxtimes\mathcal{L}^2_{\mathbf{SL}_2})$, which is nothing but the linear map induced by the morphism $\mathcal{M}_{\mathbf{SL}_2}\times\mathcal{M}_{\mathbf{SL}_2}\longrightarrow\mathfrak{d}^*$.

This means that we have a commutative diagram

$$\begin{tikzcd}
 & \mathfrak{d}^* \arrow[d, "\sigma"] \\
\mathcal{M}_{\mathbf{SL}_2}\times\mathcal{M}_{\mathbf{SL}_2} \arrow[ur] \arrow[r, "\theta"] & {|4\Theta|^+}
\end{tikzcd}$$

where $\sigma\colon\mathfrak{d}^*\longrightarrow|4\Theta|^+$ is the projective morphism associated to the line bundle $\mathcal{O}_{\mathfrak{d}^*}(2)$ and the morphism s. More concretely, if we choose the global coordinates $(T_\kappa)_{\kappa\text{ even}}$ on $|4\Theta|^+$ (and the coordinates $(Z_\kappa)_{\kappa\text{ even}}$ on $\mathfrak{d}^*$ already used in 5.2.12), this morphism is simply given by $(Z_\kappa)\in\mathbb{P}^{N^+-1}\longmapsto(Z_\kappa^2)\in\mathbb{P}^{N^+-1}$, where $N^+=2^{g-1}(2^g+1)$ is the number of even theta-characteristics. Note that σ is the quotient morphism of $\mathfrak{d}^*$ by the action of $\{\pm1\}^{N^+}$.

5.4.2. We consider now the case of a curve of genus 2. In this case, θ_2 is an isomorphism from $\mathcal{M}_{\mathbf{SL}_2}$ onto $|2\Theta|\simeq\mathbb{P}^3$. The previous discussion shows that the theta map $\theta\colon\mathcal{M}^+_{\mathbf{SO}_4}\longrightarrow|4\Theta|^+$ fits in the following diagram (we know from

[**Pau07**] that the general theta map $\mathcal{M}_{\mathbf{SL}_4} \dashrightarrow |4\Theta|$ is generically 30-to-1)

$$\begin{tikzcd}
 & \mathbb{P}^{15} \arrow[d, dashed, "p"] \\
\mathbb{P}^3 \times \mathbb{P}^3 \arrow[ur, hook] \arrow[r, "\varphi"] \arrow[d] & \mathbb{P}^9 \arrow[d, "\sigma"] \\
\mathcal{M}^+_{\mathbf{SO}_4} \arrow[r, "\theta"] \arrow[d, "2:1"'] & \mathbb{P}^9 = |4\Theta|^+ \arrow[d, hook] \\
\mathcal{M}_{\mathbf{SL}_4} \arrow[r, dashed, "\theta", "30:1"'] & \mathbb{P}^{15} = |4\Theta|
\end{tikzcd} \tag{5.4.2.1}$$

where φ is given by the 10 polynomials previously written in 5.3.9. We know by 5.2.11 that $\mathbb{P}^3 \times \mathbb{P}^3 \longrightarrow \Sigma = \varphi(\mathbb{P}^3 \times \mathbb{P}^3)$ is exactly the quotient morphism for the involution switching the two factors; since $\deg(\varphi) \cdot \deg(\Sigma)$ is equal to the degree of the Segre variety in $\mathbb{P}^{15}$, we see that φ is a 2-sheeted cover onto a degree 10 variety in $\mathbb{P}^9$.

Since $(\sigma \circ \varphi)^* \mathcal{O}_{|4\Theta|^+}(1) \simeq \mathcal{O}_{\mathbb{P}^3}(2) \boxtimes \mathcal{O}_{\mathbb{P}^3}(2)$, the degree d of the image of $\sigma \circ \varphi$ in $\mathbb{P}^9$ satisfies the equality

$$\begin{aligned} \deg(\sigma \circ \varphi) \cdot d &= c_1(\mathcal{O}_{\mathbb{P}^3}(2) \boxtimes \mathcal{O}_{\mathbb{P}^3}(2))^6 \\ &= 2^6 \cdot \binom{6}{3}, \end{aligned}$$

and we also know a priori that $\deg(\sigma \circ \varphi)$ is divisible by 2^5 (see (5.4.2.1)). With a computer[(6)], we found points in $\mathbb{P}^9$ with exactly 32 reduced points in their fiber. This means that this degree is 32, and that the image of $\mathcal{M}^+_{\mathbf{SO}_4}$ in $\mathbb{P}^9$ has degree 40. This also shows that $\theta \colon \mathcal{M}^\circ_{\mathbf{O}_4} \longrightarrow |4\Theta|^+$ has generic degree 1, where $\mathcal{M}^\circ_{\mathbf{O}_4}$ is the connected component of $\mathcal{M}_{\mathbf{O}_4}$ containing the trivial bundle.

Theorem 5.4.3. — *For a curve of genus* 2, *the theta map* $\mathcal{M}^\circ_{\mathbf{O}_4} \longrightarrow |4\Theta|^+$ *has generic degree* 1. *Its image is a subvariety of* $\mathbb{P}^9$ *of dimension* 6 *and degree* 40.

5.4.4. We can give a precise description of the singular locus of $\mathcal{M}^+_{\mathbf{SO}_4}$. In genus 2, $\mathcal{M}_{\mathbf{SL}_2} \times \mathcal{M}_{\mathbf{SL}_2} \simeq \mathbb{P}^3 \times \mathbb{P}^3$ is smooth, and a singular point in the quotient $\mathcal{M}^+_{\mathbf{SO}_4}$ is the image of a point fixed by at least one element of J_2 (and

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(6) I should not omit to thank Samuel Boissière for the time he spent explaining to me how to do this

conversely, since the fixed locus has codimension $\geqslant 2$, every fixed point is sent to a singular point).

Let us describe the components of $\operatorname{Sing}\mathcal{M}^+_{\mathbf{SO}_4}$ corresponding to an element $\eta \in J_2$. The fixed locus in $\mathcal{M}_{\mathbf{SL}_2} \times \mathcal{M}_{\mathbf{SL}_2}$ is the image of the product $\mathrm{Nm}_\eta^{-1}(\eta) \times \mathrm{Nm}_\eta^{-1}(\eta)$ by the morphism $(L, L') \longmapsto (\pi_{\eta *}L, \pi_{\eta *}L')$ (this is a union of products $\mathbb{P}^1 \times \mathbb{P}^1$ of two copies of the Kummer curve of the Prym P_η). Its image in $\mathcal{M}^+_{\mathbf{SO}_4}$ is given by the following easy lemma:

Lemma 5.4.5. — *Let L and L' be two line bundles on the curve C_η associated to a 2-torsion point $\eta \in J_2$. Then there is an exact sequence*

$$0 \to \pi_{\eta *}(L^{-1} \otimes L') \to \mathcal{H}om(\pi_{\eta *}L, \pi_{\eta *}L') \to \pi_{\eta *}(\sigma_\eta^* L^{-1} \otimes L') \to 0$$

where σ_η is the involution of the 2-sheeted cover $C_\eta \longrightarrow C$.

This is an immediate adaptation of the proof of [**Bea91**, Lemme 1.3]: we deduce from the equality $\det \pi_\eta^* \pi_{\eta *} L = L \otimes \sigma_\eta^* L$ an exact sequence $0 \to \sigma_\eta^* L \to \pi_\eta^* \pi_{\eta *} L \to L \to 0$. We obtain the Lemma by applying $\mathcal{H}om(\,\cdot\,, L')$ and $\pi_{\eta *}$.

In other words, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Nm}_\eta^{-1}(\eta) \times \mathrm{Nm}_\eta^{-1}(\eta) \arrow[r, "(\pi_{\eta *}, \pi_{\eta *})"] \arrow[d, "(m,d)"'] & \mathcal{M}_{\mathbf{SL}_2} \times \mathcal{M}_{\mathbf{SL}_2} \arrow[d] \\
\mathrm{Nm}_\eta^{-1}(\mathcal{O}) \times \mathrm{Nm}_\eta^{-1}(\mathcal{O}) \arrow[r] & \mathcal{M}^+_{\mathbf{SO}_4}
\end{tikzcd}$$

where (m, d) maps (L, L') to $(L \otimes L', L^{-1} \otimes L')$, and the bottom arrow sends (L, L') to $\pi_{\eta *}L \oplus \pi_{\eta *}L'$.

Remark 5.4.6. — We can also give a concrete description of the image of this singular locus in $|4\Theta|^+$. Consider the element $\eta \in J_2$ which acts on each factor of $\mathbb{P}^3 \times \mathbb{P}^3$ by the matrix $\mathrm{diag}(1, 1, -1, -1)$ (through the thetastructure, η corresponds to $(0, \alpha)$ where α is a non trivial character of $\mathbb{F}_2^2$). Its fixed locus in $\mathbb{P}^3 \times \mathbb{P}^3$ is the disjoint union of 4 products $\mathbb{P}^1 \times \mathbb{P}^1$: if we denote by $V_{i,j}$ the closed subscheme of $\mathbb{P}^3$ defined by the equations $X_i = X_j = 0$ (or $Y_i = Y_j = 0$), this fixed locus is the union $V_{01} \times V_{01} \cup V_{01} \times V_{23} \cup V_{23} \times V_{01} \cup V_{23} \times V_{23}$. These 4 components behave differently: we see (on the equations) that $V_{01} \times V_{01}$ is mapped onto a $\mathbb{P}^2 \subset |4\Theta|^+$ (and the restriction of φ is the the composition of the morphism from $\mathbb{P}^1 \times \mathbb{P}^1$ to its symmetric product variety $\mathbb{P}^2$ followed by the quotient morphism $\sigma\colon (U_i) \in \mathbb{P}^2 \longmapsto (U_i^2) \in \mathbb{P}^2$), while $V_{01} \times V_{23}$ is mapped in a $\mathbb{P}^3$ (and φ restricts to the morphism $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3 \longrightarrow \mathbb{P}^3$, where the second map is again the restriction of σ).

5.5. Base locus of $\theta\colon \mathcal{M}^-_{\mathbf{SO}_4} \dashrightarrow |4\Theta|^-$ in genus 2

5.5.1. The exceptional isomorphism $\mathbf{Spin}_4 \simeq \mathbf{SL}_2 \times \mathbf{SL}_2$ gives a quotient

$$\begin{array}{ccc} \mathcal{M}^1_{\mathbf{SL}_2} \times \mathcal{M}^1_{\mathbf{SL}_2} & & (E_1, E_2) \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow \\ \mathcal{M}^-_{\mathbf{SO}_4} & & (\mathcal{H}om(E_1, E_2), \det) \end{array}$$

where $\mathcal{M}^1_{\mathbf{SL}_2}$ is isomorphic to the variety $\mathcal{SU}_C(2,1)$ of stable bundles of rank 2 and determinant $\xi \in J^1$. Since the pull-back $\pi^*\mathcal{L}^-_{\mathbf{SO}_4}$ is equal to $\mathcal{L}_{\mathcal{M}^1_{\mathbf{SL}_2}} \boxtimes \mathcal{L}_{\mathcal{M}^1_{\mathbf{SL}_2}}$, we cannot try to adapt the method used for studying the base locus in $\mathcal{M}^+_{\mathbf{SO}_4}$ to simplify the description. However, we will treat the genus 2 case by showing that the theta map $\mathcal{M}^1_{\mathbf{SL}_2} \times \mathcal{M}^1_{\mathbf{SL}_2} \dashrightarrow |4\Theta|^-$ is given by some explicit sections of $\mathrm{H}^0(\mathcal{M}^1_{\mathbf{SL}_2} \times \mathcal{M}^1_{\mathbf{SL}_2}, \mathcal{L} \boxtimes \mathcal{L})$.

5.5.2. We now consider the case of a genus 2 curve. We first recall that we a priori know 20 base points for $\theta\colon \mathcal{M}^-_{\mathbf{SO}_4} \dashrightarrow |4\Theta|^-$: Raynaud has constructed in [**Ray82**] some rank 4 bundles on a curve of genus 2 without theta divisor, which are now called "Raynaud bundles". The following considerations, which are due to A. Beauville (and have appeared in [**Hit**]), show that this construction gives 16 distinct bundles, 10 of them being orthogonal bundles(7) (it follows from Theorem 5.2.10 that their Stiefel-Whitney class must be non trivial).

Recall that these bundles are obtained as restrictions via some suitable embeddings $C \hookrightarrow J$ of the Fourier-Mukai transform of the inverse of the ample line bundle $\mathcal{O}_J(2\vartheta)$. One way to get some more explicit data about these bundles is to consider first their pull-back via 2_J: let E be the Fourier-Mukai transform $\mathcal{F}(\mathcal{O}(2\vartheta)^{-1})$, which is a rank 4 vector bundle on J such that $2_J^*E \simeq V^* \otimes \mathcal{O}(2\vartheta)$ (which means that E is the quotient by J_2 of the bundle $V^* \otimes \mathcal{O}(2\vartheta)$) and $\det E = \mathcal{O}(2\vartheta)$. The tensor product $E \otimes E$ is thus isomorphic to the quotient of $(V^* \otimes \mathcal{O}(2\vartheta)) \otimes (V^* \otimes \mathcal{O}(2\vartheta))$ by J_2. The natural decomposition of the representation of the Mumford group on $V^* \otimes V^*$ into one-dimensional eigenspaces recalled in 5.2.6.1 induces a J_2-isomorphism between

(7)Since these bundles must be stable, this will give 20 $\mathbf{SO}_4$-bundles...

$V^*\otimes V^*\otimes\mathcal{O}(4\vartheta)$ and $2_J^*\left(\oplus_\kappa\mathcal{O}(T_\kappa^*\Theta)\right)$, and the tensor product $E\otimes E$ thus splits as the direct sum $\oplus_\kappa\mathcal{O}(T_\kappa^*\Theta)$. This gives, for each theta characteristic κ, a non degenerate bilinear form on E with values in $\mathcal{O}(T_\kappa^*\Theta)$, which is symmetric or skew-symmetric according to the parity of κ: in other words, there are two isomorphisms

$$\mathbf{S}^2E\simeq\bigoplus_{\kappa\text{ even}}\mathcal{O}(T_\kappa^*\Theta)\quad\text{and}\quad\mathbf{\Lambda}^2E\simeq\bigoplus_{\kappa\text{ odd}}\mathcal{O}(T_\kappa^*\Theta).$$

Restrict now this bundle to the curve C via a symmetric embedding $\iota_\kappa\colon C\hookrightarrow J$ given by any theta characteristic κ. We get in this way a rank 4 vector bundle $F_\kappa=\iota_\kappa^*E$ on C with determinant K_C^2, which depends on the choice of the theta characteristic κ: since the restriction of $T_{\kappa'}^*\Theta$ to C is linearly equivalent to $\kappa'\otimes\kappa$, we see that the vector bundle F_κ satisfies

$$F_\kappa\otimes F_\kappa\simeq\kappa\otimes\big(\bigoplus_{\kappa'}\kappa'\big).$$

It is therefore natural to consider the vector bundle $F_\kappa\otimes\kappa^{-1}$, and more generally the vector bundles $G_{\kappa,\lambda}=F_\kappa\otimes\kappa^{-1}\otimes\lambda$ for all $\lambda\in J_4$: the isomorphism

$$G_{\kappa,\lambda}\otimes G_{\kappa,\lambda}\simeq\bigoplus_{\kappa'}\big(\kappa'\otimes\kappa^{-1}\otimes\lambda^2\big)$$

shows that $G_{\kappa,\lambda}$ carries a natural bilinear pairing, which is symmetric (resp. skew-symmetric) when $\kappa\otimes\lambda^2$ is even (resp. odd), and also that $G_{\kappa,\lambda}\simeq G_{\kappa,\lambda'}$ if and only if $\lambda^2\simeq\lambda'^2$.

This gives 16 distinct "Raynaud bundles" in $\mathcal{M}_{\mathbf{SL}_4}$, which are all self-dual: 10 are orthogonal bundles, and 6 are symplectic.

5.5.3. We use here again the explicit description of $\mathcal{M}^1_{\mathbf{SL}_2}$ that we have recalled in 4.5.4: if ξ is a fixed line bundle on C of degree 5, we consider $V_W=\sum\xi_w$ the direct sum of the fibers of ξ at the six Weierstrass points. There is a then a J_2-morphism $\varphi\colon\mathcal{SU}(2,\xi)\longrightarrow\mathbb{P}V_W$, where J_2 acts on $\mathbb{P}V_W$ as follows: elements of J_2 correspond to partitions $W=S\cup T$ of the set W of Weierstrass points (with $|S|$ even), and the involution of $\mathbb{P}(V_W)$ given by an element of J_2 associated to the partition $W=S\cup T$ is the one which fixes exactly $\mathbb{P}(\sum_{w\in S}\xi_w)$ and $\mathbb{P}(\sum_{w\in T}\xi_w)$.

We now investigate the theta map $\mathcal{M}^-_{\mathbf{SO}_4}\dashrightarrow|4\Theta|^-$ by completing the following diagram

$$\begin{tikzcd}
\mathcal{SU}(2,\xi) \times \mathcal{SU}(2,\xi) \arrow[r, hook, "\varphi\times\varphi"] \arrow[d, "\pi"] & \mathbb{P}V_W \times \mathbb{P}V_W \arrow[d, dotted, "p"] \\
\mathcal{M}^-_{\mathbf{SO}_4} \arrow[r, dashed, "\theta"] & |4\Theta|^-.
\end{tikzcd}$$

Since $\pi^*\theta^*\mathcal{O}_{|4\Theta|^-}(1)$ is equal to $\mathcal{L}_{\mathcal{M}^1_{\mathbf{SL}_2}} \boxtimes \mathcal{L}_{\mathcal{M}^1_{\mathbf{SL}_2}}$, it is enough to understand the morphism p^* which fits in the diagram

$$\begin{tikzcd}
\mathrm{H}^0(\mathcal{SU}(2,\xi),\mathcal{L}) \otimes \mathrm{H}^0(\mathcal{SU}(2,\xi),\mathcal{L}) & V_W \otimes V_W \arrow[l, "\sim", "(\varphi\times\varphi)^*"'] \\
\mathrm{H}^0(\mathcal{M}^-_{\mathbf{SO}_4}, \mathcal{L}^-_{\mathbf{SO}_4}) \arrow[u, hook, "\pi^*"] & \big(\mathrm{H}^0(J^1, 4\Theta)^-\big)^*. \arrow[l, "\sim", "\theta^*"'] \arrow[u, dotted, "p^*"']
\end{tikzcd}$$

We see first that the image of π^* is exactly the space of J_2-invariants[(8)] in $\mathrm{H}^0(\mathcal{SU}(2,\xi),\mathcal{L}) \otimes \mathrm{H}^0(\mathcal{SU}(2,\xi),\mathcal{L}) \simeq V_W \otimes V_W$, which is the vector space spanned by the elements $\xi_w \otimes \xi_w$. The dotted arrow is thus, in some global coordinates on each $\mathbb{P}V_w$ labelled by the Weierstrass points $w \in W$, the map $\mathbb{P}^5 \times \mathbb{P}^5 \dashrightarrow \mathbb{P}^5$ given by $\big((X_w),(Y_w)\big) \mapsto X_w Y_w$.

Now, since the closed subscheme $\mathcal{SU}(2,\xi)$ in $\mathbb{P}V_W$ is exactly defined as the intersection of the two quadrics $\sum X_w^2 = 0$ and $\sum \lambda_w X_w^2 = 0$, we see that the base locus of $\mathcal{SU}(2,\xi) \times \mathcal{SU}(2,\xi) \dashrightarrow |4\Theta|^-$ is, set-theoretically, defined by $X_{w_i} = X_{w_j} = X_{w_k} = 0 = Y_{w_{i'}} = Y_{w_{j'}} = Y_{w_{k'}}$ for each partition $W = \{w_i, w_j, w_k\} \cup \{w_{i'}, w_{j'}, w_{k'}\}$. Any of these partitions gives 16 points, lying in the same J_2-orbit. We have thus explicitely located the 20 Raynaud bundles[(9)], and there are no other orthogonal bundle without theta divisor.

Finally, we can identify the scheme structure on the base locus of θ: indeed, since

$$c_1(\mathcal{L}^-_{\mathbf{SL}_2} \boxtimes \mathcal{L}^-_{\mathbf{SL}_2})^6 = \binom{6}{3} p_1^*\big(c_1(\mathcal{L}_{\mathbf{SL}_2})^3\big) p_2^*\big(c_1(\mathcal{L}_{\mathbf{SL}_2})^3\big) = 20 \cdot 16,$$

---

(8) The fact that J_2 acts on the tensor product $\mathrm{H}^0(\mathcal{SU}(2,\xi),\mathcal{L}) \otimes \mathrm{H}^0(\mathcal{SU}(2,\xi),\mathcal{L})$ is a consequence of the linearization of the action of J_2 on $\mathcal{SU}(2,\xi) \hookrightarrow \mathbb{P}V_W$ described in 4.5.4. We have seen there that this action of J_2 lifts to an action of an extension of J_2 by μ_2 on $\mathrm{H}^0(\mathcal{SU}(2,\xi),\mathcal{L}) \simeq V_W$, whose diagonal action on the second tensor power factors through J_2.

(9) Note that the oriented orthogonal bundle corresponding to a 3 points subset $I \subset W$ and the oriented orthogonal bundle corresponding to its complement $W \setminus I$ are isomorphic orthogonal bundles, but carry opposite orientations.

we see that the base locus must be reduced. We have thus proved the following theorem, which is contained, as well as its symplectic counterpart [**Hit**], in [**Pau07**].

Theorem 5.5.4. — *When C is a curve of genus 2, the base locus $\mathcal{B}$ of the theta map $\mathcal{M}^-_{\mathbf{SO}_4} \dashrightarrow |4\Theta|^-$ is the zero-dimensional reduced scheme consisting of the 20 Raynaud orthogonal bundles.*

Since $\dim \mathcal{M}^-_{\mathbf{SO}_4} > \dim |4\Theta|^-$, we automatically deduce the following corollary:

Corollary 5.5.5. — *The theta map $\mathcal{M}^-_{\mathbf{SO}_4} \setminus \mathcal{B} \longrightarrow |4\Theta|^-$ is a surjective morphism.*

Remark 5.5.6. — It is now natural to try to give some informations about its fibers.

It is easy at a point P defined by 5 coordinates hyperplanes, corresponding to any set $W' = W \setminus \{w_0\}$ of 5 Weierstrass points. Since at most three coordinates can simultaneously vanish at a point of $\mathcal{SU}(2,\xi)$, we can check that the fiber $(\theta \circ \pi)^{-1}(P) \subset \mathcal{SU}(2,\xi) \times \mathcal{SU}(2,\xi)$ is the union of 80 elliptic curves (whose j-invariant is available); its image in $\mathcal{M}^-_{\mathbf{SO}_4}$ is a connected nodal curve, consisting of 20 rational curves. A detailed analysis shows that these 20 curves meet in 30 points, with 2 curves a point and 3 points a curve: this fiber is thus a stable curve of genus 11.

APPENDIX A

ORTHOGONAL AND SYMPLECTIC BUNDLES OVER AN ELLIPTIC CURVE

Moduli spaces of G-bundles over an elliptic curve C have been described in [**Las98**][(1)] as a quotient of $C \otimes X(T)$ (where $X(T)$ is the character group of a maximal torus $T \subset G$) by the action of the Weyl group W_T (at least the component containing the trivial bundle). In the case $G = \mathbf{SO}_r$, this implies that $\mathcal{M}^+_{\mathbf{SO}_r}$ is isomorphic to $\mathbb{P}^{\frac{r-1}{2}}$ when r is odd, and to the quotient of $C^{\frac{r}{2}}$ by the Weyl group $(\mathbb{Z}/2\mathbb{Z})^{\frac{r}{2}-1} \rtimes \mathfrak{S}_{\frac{r}{2}}$ when r is even. We give here an elementary direct proof of this result: this is somehow instructive, as it shows how orientation matters.

Proposition A.1. — *Let C be an elliptic curve, and $n \geqslant 1$. The moduli space $\mathcal{M}^+_{\mathbf{SO}_{2l+1}}$ is isomorphic to $\mathbb{P}^l$, $\mathcal{M}^-_{\mathbf{SO}_{2l+1}}$ to $\mathbb{P}^{l-1}$, $\mathcal{M}^-_{\mathbf{SO}_{2l}}$ to $\mathbb{P}^{l-2}$ and $\mathcal{M}^+_{\mathbf{SO}_{2l}}$ to the quotient of C^l by $(\mathbb{Z}/2\mathbb{Z})^{l-1} \rtimes \mathfrak{S}_l$.*

Every semi-stable bundle of degree zero on the elliptic curve C is S-equivalent to a direct sum of invertible bundles. In particular, if $(\mathcal{N}_i)_{i=1,\dots,3}$ are the three non zero line bundles of order two, an orthogonal bundle E on C with trivial determinant splits as follows:

- $\mathcal{O}_C \oplus \bigoplus_{i=1}^{l} \left(L_i \oplus L_i^{-1}\right)$ if $\mathrm{rk}(E) = 2l+1$ and $w_2(E) = 1$,
- $\mathcal{N}_1 \oplus \mathcal{N}_2 \oplus \mathcal{N}_3 \oplus \bigoplus_{i=1}^{l-1} \left(L_i \oplus L_i^{-1}\right)$ if $\mathrm{rk}(E) = 2l+1$ and $w_2(E) = -1$,
- $\bigoplus_{i=1}^{l} \left(L_i \oplus L_i^{-1}\right)$ if $\mathrm{rk}(E) = 2l$ and $w_2(E) = 1$,

[(1)] Their existence follows from [**Ram96**], even if Ramanathan states the result only for higher genus.

– $\mathcal{O}_C \oplus \mathcal{N}_1 \oplus \mathcal{N}_2 \oplus \mathcal{N}_3 \oplus \bigoplus_{i=1}^{l-2} \left(L_i \oplus L_i^{-1}\right)$ if $\mathrm{rk}(E) = 2l$ and $w_2(E) = -1$,

where the L_i are line bundles on C. In all cases but the third one, the order two line bundles allow us to adjust the determinant of an orthogonal isomorphism: in other words, any orthogonal bundle admits exactly one orientation up to (orthogonal) isomorphism, and the orthogonal bundle is characterized by the collection $\{M_1, \dots, M_k\}$ where $M_i \in \{L_i, L_i^{-1}\}$. This gives the expected isomorphism, since $C^k/\big((\mathbb{Z}/2\mathbb{Z})^k \rtimes \mathfrak{S}_k$ is the k-th symmetric product of $\mathbb{P}^1$, which is isomorphic to $\mathbb{P}^k$.

In the remaining case, a generic orthogonal bundle admits two unequivalent orientations, so that $\mathcal{M}^+_{\mathbf{SO}_{2l}}$ is a quotient of C^l by the action of $(\mathbb{Z}/2\mathbb{Z})^{l-1} \rtimes \mathfrak{S}_l$ where $(\mathbb{Z}/2\mathbb{Z})^{l-1}$ acts on $C \times \cdots \times C$ by $(a_1, \dots, a_l) \longmapsto (\pm a_1, \dots, \pm a_l)$ with an even number of minus signs, which finishes the proof of the proposition.

Of course, a complete proof would consist in defining morphisms from the quotients of C^k to the corresponding moduli space, and checking that these morphisms are isomorphisms. Note that, already in genus 1, it is more comfortable to consider moduli for $\mathbf{O}_r$-bundles: when r is even, we simply find $\mathcal{M}^+_{\mathbf{O}_r} = \mathbb{P}^{\frac{r}{2}}$.

Remark A.2. — In this case we can replace the description of the connected components of $\mathcal{M}_{\mathbf{SO}_r}$ in terms of the second Stiefel-Whitney class by the following argument: Mumford's invariance mod 2 theorem shows that the parity of the numbers of copies of any order 2 line bundle contained in a fiber of a family of orthogonal bundles on $T \times C$ does not depend on $t \in T$ (and, if $\mathcal{N}_i$ appears an odd number of times, so do the two other nonzero bundles of order 2, because $\det E = \mathcal{O}_C$).

The same argument applies to $\mathcal{M}_{\mathbf{Sp}_{2r}}$, which is therefore isomorphic to $\mathbb{P}^r$.

We could also write the forgetful morphism in this special case, and see that it behaves in the same way (actually our proof does not use the hypothesis $g \geqslant 2$; however, there are not many other analogies: for example, $\mathcal{M}_{\mathbf{SO}_r}$ is not equidimensional, as well as $\mathcal{M}_{\mathbf{GL}_r}$).

APPENDIX B

JACOBIAN VARIETIES AND HEISENBERG REPRESENTATIONS

For the convenience of the reader, we repeat here the classical material discussed in [**Bea91**, Appendice].

B.1. We have chosen to keep the notation Θ for the canonical theta divisor $\{L \in J^{g-1} | h^0(C, L) \neq 0\}$ in J^{g-1}, and Θ_L for the generalized theta divisor with support $\{E \in \mathcal{M}_{\mathbf{SL}_r} | h^0(C, E \otimes L) \neq 0\}$. If L is a point in J^{g-1}, and $T_L \colon J \longrightarrow J^{g-1}$ the multiplication morphism, the pull back of Θ by this multiplication map will therefore be denoted by $T_L^*\Theta$. To avoid introducing too many capital Θ, we will write ϑ for a symmetric theta divisor on J (or on a principally polarized Abelian variety A), and, following the now widespread habit, we will abusively denote by 2ϑ the linear equivalence class of the double of any symmetric theta divisor on J (or A).

B.2. If A is a principally polarized Abelian variety of dimension g, we will denote by A_2 the two-torsion subgroup of A, which is exactly the subgroup of A whose action can be lifted to $\mathcal{O}(2\vartheta)$. This group thus fits in an exact sequence

$$1 \to k^* \to \mathcal{G}\left(\mathcal{O}(2\vartheta)\right) \to A_2 \to 0$$

where $\mathcal{G}\left(\mathcal{O}(2\vartheta)\right)$ is the *theta group* (or *Mumford group*) consisting of pairs (α, φ) with $\alpha \in A_2$ and $\varphi \colon T_\alpha^*\mathcal{O}(2\vartheta) \xrightarrow{\sim} \mathcal{O}(2\vartheta)$. The natural representation of $\mathcal{G}\left(\mathcal{O}(2\vartheta)\right)$ in $\mathrm{H}^0\left(A, \mathcal{O}(2\vartheta)\right)$ is its unique irreducible representation where k^* acts by homotheties.

A level 2 structure is an isomorphism between the preceding theta group and the level 2 Heisenberg group $H_2 = k^* \times \mathbb{F}_2^g \times \widehat{\mathbb{F}_2^g}$ (whose law is defined by $(t, a, \alpha) \cdot (s, b, \beta) = (ts\beta(a), a + b, \alpha\beta)$. The group H_2 thus acts on $\mathrm{H}^0\left(A, \mathcal{O}(2\vartheta)\right)$, and this vector space admits a unique (up to a scalar) basis $(X_b)_{b\in\mathbb{F}_2^g}$ such that

$$(t,a,\alpha)\cdot X_b = t\alpha(a+b)X_{a+b} \qquad \text{for all } (t,a,\alpha)\in H_2, b\in\mathbb{F}_2^g.$$

Note that we can easily describe the eigenspaces in $\mathrm{H}^0(A,\mathcal{O}(2\vartheta))$ of any $\eta\in J_2$. Let us first observe that, for any $\eta,\eta'\in J_2$, we have

$$\tilde\eta\tilde\eta' = \langle\eta,\eta'\rangle\tilde\eta'\tilde\eta$$

for any $\tilde\eta,\tilde\eta'\in H_2$ lying over η,η' (where $\langle\cdot,\cdot\rangle$ denotes the Weil pairing). The two subspaces $\left(\mathrm{H}^0(A,\mathcal{O}(2\vartheta))\right)_\eta^\pm$ fixed by $\eta\neq\mathcal{O}$ are thus conjugated under the action of J_2; in particular, they both have dimension 2^{g-1}.

Remark B.3. — For computational purpose, it is useful to consider the *finite* Heisenberg group $\widetilde{H_2}$, which is the kernel of the homomorphism $h\in H_2\longmapsto h^4$ (see [**Bea03**]). This is an extension of A_2 by μ_4, and its representation in $\mathrm{H}^0(A,\mathcal{O}(2\vartheta))$ is its unique irreducible representation on which μ_4 acts by homotheties: everything works as well as with H_2.

B.4. *Theta characteristics* of A are quadratic forms $\kappa\colon A_2\longrightarrow\{\pm1\}$ associated to the Weil pairing $\langle\cdot,\cdot\rangle\colon A_2\times A_2\longrightarrow\{\pm1\}$. The group A_2 acts (simply transitively) on the set $S(A)$ of theta characteristics by the rule

$$(\eta\cdot\kappa) = (\alpha\longmapsto\langle\eta,\alpha\rangle\kappa(\alpha)).$$

The *parity* of κ is defined by the value $\varepsilon(\kappa)$ which κ takes $2^{g-1}(2^g+1)$ times. The parity of $\eta\cdot\kappa$ is $\varepsilon(\eta\cdot\kappa)=\kappa(\eta)\varepsilon(\kappa)$, and there are $2^{g-1}(2^g+1)$ even and $2^{g-1}(2^g-1)$ odd theta-characteristics. Once a theta structure has been chosen, every element $(c,\gamma)\in\mathbb{F}_2^g\times\widehat{\mathbb{F}_2^g}$ defines (through the isomorphism between the theta group H_2) a theta characteristic $\kappa\left[\begin{smallmatrix}c\\\gamma\end{smallmatrix}\right]$ by the formula

$$\kappa\begin{bmatrix}c\\\gamma\end{bmatrix}(a,\alpha)=\gamma(a)\alpha(a+c)$$

(recall that the theta structure identifies the Weil pairing with $\langle(a,\alpha),(b,\beta)\rangle=\alpha(b)\beta(a)$). The parity of this theta characteristic is given by the formula $\varepsilon(\kappa\left[\begin{smallmatrix}c\\\gamma\end{smallmatrix}\right])=\gamma(c)$.

B.5. There is a one-to-one correspondence between characters of H_2 of level 2 (which occur in the study of the representation of H_2 in $\mathrm{H}^0(A,\mathcal{O}(2\vartheta))\otimes\mathrm{H}^0(A,\mathcal{O}(2\vartheta))$) and theta-characteristic, obtained by sending $\kappa=\kappa\left[\begin{smallmatrix}c\\\gamma\end{smallmatrix}\right]$ to the character

$$\chi_{\kappa\left[\begin{smallmatrix}c\\\gamma\end{smallmatrix}\right]}\colon(t,a,\alpha)\in H_2\longmapsto t^2\gamma(a)\alpha(c).$$

For every theta-characteristic $\kappa\left[\begin{smallmatrix}c\\\gamma\end{smallmatrix}\right]$, the element $Q_{\kappa\left[\begin{smallmatrix}c\\\gamma\end{smallmatrix}\right]} \in \mathrm{H}^0\left(A, \mathcal{O}(2\vartheta)\right) \otimes \mathrm{H}^0\left(A, \mathcal{O}(2\vartheta)\right)$ defined by

$$Q_{\kappa\left[\begin{smallmatrix}c\\\gamma\end{smallmatrix}\right]} = \sum_{b\ in\mathbb{F}_2^g} \gamma(b) X_b X_{b+c}$$

is an eigenvector under the H_2-action corresponding to the character $\chi_{\kappa\left[\begin{smallmatrix}c\\\gamma\end{smallmatrix}\right]}$. It is symmetric or anti-symmetric (with respect to the involution $X \otimes X' \mapsto X' \otimes X$) according to the parity of $\chi_{\kappa\left[\begin{smallmatrix}c\\\gamma\end{smallmatrix}\right]}$.

These observations provide a complete description of the representation of H_2 in $\mathrm{H}^0\left(A, \mathcal{O}(2\vartheta)\right) \otimes \mathrm{H}^0\left(A, \mathcal{O}(2\vartheta)\right)$: this representation splits as the direct sum of the (mutually non-isomorphic) one-dimensional representations of H_2 of level 2. More precisely, when κ ranges over the set of even (resp. odd) theta-characteristic, the elements $Q_{\kappa\left[\begin{smallmatrix}c\\\gamma\end{smallmatrix}\right]}$ form a basis of $\mathbf{S}^2\mathrm{H}^0\left(A, \mathcal{O}(2\vartheta)\right)$ (resp. $\mathbf{\Lambda}^2\mathrm{H}^0\left(A, \mathcal{O}(2\vartheta)\right)$).

Let us finally mention that these results can also be obtained in a more intrinsic fashion, for which we refer to [**Bea91**, A.5]: we can associate to any theta-characteristic κ a symmetric theta divisor ϑ_κ. If θ_κ is a non-zero section of $\mathcal{O}(\vartheta_\kappa)$ (well defined up to a scalar), then $(x, y) \mapsto \theta_\kappa(x+y)\theta_\kappa(x-y)$ defines a section of $\mathcal{O}(2\vartheta) \boxtimes \mathcal{O}(2\vartheta)$, which is in fact collinear to Q_κ.

Remark B.6. — Of course, for complex Abelian varieties, all of that may be rephrased in terms of classical theta functions (see, for example, [**Bea91**, A.7] – although some 2's are missing).

B.7. It is now natural to investigate the multiplication morphism

$$\mu\colon \mathrm{H}^0\left(A, \mathcal{O}(2\vartheta)\right) \otimes \mathrm{H}^0\left(A, \mathcal{O}(2\vartheta)\right) \longrightarrow \mathrm{H}^0\left(A, \mathcal{O}(4\vartheta)\right)$$

by studying the image of the basis given by the sections Q_κ (note that, for κ odd, the antisymmetry of Q_κ automatically implies the nullity of $\mu(Q_\kappa)$).

Its image must be contained in the space $\mathrm{H}^0\left(A, \mathcal{O}(4\vartheta)\right)^+$ of sections of $\mathcal{O}(4\vartheta)$ which are invariant for the involution induced by the unique isomorphism $(-1)^*\mathcal{O}(2\vartheta) \xrightarrow{\sim} \mathcal{O}(2\vartheta)$. This space admits as a basis the family of sections $\theta_\kappa(2\cdot)$ (for κ even).

The addition formula implies that $\mu(Q_\kappa)$ is collinear to $\theta_\kappa(0)\theta_\kappa(2\cdot)$. In particular, this tells us that μ gives an isomorphism $\mathbf{S}^2\mathrm{H}^0\left(A, \mathcal{O}(2\vartheta)\right) \xrightarrow{\sim} \mathrm{H}^0\left(A, \mathcal{O}(4\vartheta)\right)^+$ if and only if no thetanull $\theta_\kappa(0)$ (for κ even) vanishes.

B.8. We now specialize the preceding discussion to the case of the Jacobian variety J of the smooth curve C.

In this situation, the notion of theta-characteristic of J (hopefully) coincides with the usual definition of theta-characteristic of C as square roots of the canonical bundle K_C: a line bundle κ on C satisfying $\kappa^2 \simeq K_C$ indeed induces a theta-characteristic $\eta \in J_2 \longmapsto (-1)^{h^0(C,\eta\otimes\kappa)+h^0(C,\kappa)}$, whose parity is the one of $h^0(C,\kappa)$. We will thus identify the theta-characteristics of J with the ones of C.

The divisor ϑ_κ on J associated to κ is just the pull-back $T_\kappa^*\Theta$ of the canonical theta divisor $\Theta \subset J^{g-1}$. The vanishing of the section $\theta_\kappa \in \mathrm{H}^0(J,\vartheta_\kappa)$ at the origin thus simply means that κ belongs to the support of Θ. In particular, the morphism $\mathbf{S}^2\mathrm{H}^0\left(J,\mathcal{O}(2\vartheta)\right) \xrightarrow{\sim} \mathrm{H}^0\left(J,\mathcal{O}(4\vartheta)\right)^+$ is an isomorphism if and only if $h^0(C,\kappa) = 0$ for all even theta-characteristic κ.

APPENDIX C

NONABELIAN COHOMOLOGY

We recall here from [**Gir71**, III.3] the few basic facts about nonabelian cohomology that we have mentionned in this thesis.

Let X be any variety over k, and G any algebraic group. Recall that the first cohomology space $\mathrm{H}^1_{\text{ét}}(X, G)$ (which can be defined à la Čech) parametrizes isomorphism classes of G-bundles over X: this is a naturally a *pointed set*, the distinguished element corresponding to the trivial G-bundle on X.

We present here the long exact sequences of nonabelian cohomology which can be associated to a subgroup $H \subset G$.

C.1. In the general case, there exists an exact sequence of pointed sets

$$* \to \mathrm{H}^0(X, H) \to \mathrm{H}^0(X, G) \to \mathrm{H}^0(X, G/H) \xrightarrow{\delta} \mathrm{H}^1_{\text{ét}}(X, H) \to \mathrm{H}^1_{\text{ét}}(X, G)$$

where the connecting map δ is defined as follows: an element σ of $\mathrm{H}^0(X, G/H)$ may be viewed as a section of the trivial G-bundle on X mod H, and $\delta(\sigma)$ is the H-bundle associated to this section by Lemma 1.1.3. The last assertion of this lemma admits the following translation: $\mathrm{H}^0(X, G)$ acts on $\mathrm{H}^0(X, G/H)$, and two sections in $\mathrm{H}^0(X, G/H)$ have the same image by δ if and only if they are conjugate by this action.

This exact sequence only gives the preimage of the pointed element of $\mathrm{H}^1_{\text{ét}}(X, H)$. Still, we can get some informations about the preimage of any H-bundle P on X by considering the long exact sequence deduced from the inclusion of *twisted group sheaves* ${}^P H \hookrightarrow {}^P G$ obtained by twisting H and G by P for the action of H by *inner* automorphisms. These twisted group sheaves are nothing other than the sheaves of automorphisms $\mathcal{A}ut_H(P)$ and $\mathcal{A}ut_G(P(G))$. Moreover, we can define a natural isomorphism

$\theta_P\colon \mathrm{H}^1_{\text{ét}}(X,H) \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}(X,{}^PH)$ as follows[(1)]: using the opposite group structures on H and ${}^PH$ allows us to consider P as an ${}^PH$-bundle P^o endowed with a left action of H. If Q is an H-bundle, $\theta_P(Q)$ is the associated ${}^PH$-bundle $Q \times^H P^o$ (in particular, θ_P sends the class of P to the trivial ${}^PH$-bundle). This isomorphism together with the exact sequence

$$* \to \mathrm{H}^0(X,{}^PH) \to \mathrm{H}^0(X,{}^PG) \to \mathrm{H}^0(X,{}^PG/{}^PH) \xrightarrow{\delta} \mathrm{H}^1_{\text{ét}}(X,{}^PH) \to \mathrm{H}^1_{\text{ét}}(X,{}^PG)$$

allows us to describe the preimage of any H-bundle P.

However, this construction is not well suited for concrete computations: in Proposition 2.1.2.5, we have rather used Lemma 1.1.3 to ensure that a semistable vector bundle on a smooth curve C has at most one pre-image by the morphism $\mathrm{H}^1_{\text{ét}}(C,\mathbf{O}_r) \longrightarrow \mathrm{H}^1_{\text{ét}}(C,\mathbf{GL}_r)$. The previous exact sequence gives a way to recover this result and to consider how it can be extended to the case of any vector bundle. If P is an orthogonal bundle on C, the first step is to understand the quotient sheaf $\mathcal{A}ut_{\mathbf{GL}_r}(P(\mathbf{GL}_r))/\mathcal{A}ut_{\mathbf{O}_r}(P)$ and its global sections, which is not so easy.

C.2. When H is a normal subgroup of G, the cohomology set $\mathrm{H}^1_{\text{ét}}(X,G/H)$ makes sense, and the extended sequence of pointed sets

$$* \to \mathrm{H}^0(X,H) \to \mathrm{H}^0(X,G) \to \mathrm{H}^0(X,G/H) \xrightarrow{\delta} \mathrm{H}^1_{\text{ét}}(X,H) \to \mathrm{H}^1_{\text{ét}}(X,G) \to \mathrm{H}^1_{\text{ét}}(X,G/H)$$

is exact.

For any H-bundle P, the sequence of twisted group sheaves $1 \to {}^PH \to {}^PG \to G/H \to 1$ is exact: the quotient sheaf is always constant. The nice point here is that the group $\mathrm{H}^0(X,G/H)$ naturally acts on $\mathrm{H}^1_{\text{ét}}(X,H)$: if $\pi\colon G \longrightarrow G/H$ denotes the quotient map, a section $c \in \mathrm{H}^0(X,G/H)$ sends a H-bundle P to the H-bundle $P \times^H \pi^{-1}(c)$ (see [**Gir71**, Remarque III.3.3.2]).

Proposition C.3. — [**Gir71**, Proposition III.3.3.3] *Two H-bundles have the same associated G-bundle if and only if they are conjugate under the action of* $\mathrm{H}^0(X,G/H)$.

The stabilizer in $\mathrm{H}^0(X,G/H)$ *of an H-bundle P is the image of the morphism* $\mathrm{H}^0(X,{}^PG) \longrightarrow \mathrm{H}^0(X,G/H)$ *obtained by twisting by P the quotient map* $G \longrightarrow G/H$.

[(1)]This discussion requires to consider $\mathcal{G}$-bundles for some group sheaves $\mathcal{G}$.

This reduces the description of the fiber of $\mathrm{H}^1_{\text{ét}}(X,H) \longrightarrow \mathrm{H}^1_{\text{ét}}(X,G)$ containing P to the study of the morphism $\mathrm{Aut}_H(P) \longrightarrow \mathrm{Aut}_G(P(G))$ deduced by twisting the inclusion $H \hookrightarrow G$ by P (and taking global sections).

Remark C.4. — In this situation, we also have extra informations about exactness at $\mathrm{H}^1_{\text{ét}}(X,G)$: if Q is a G-bundle, the set of all G-bundles Q' having the same image $Q/H \in \mathrm{H}^1_{\text{ét}}(X,G/H)$ is identified via the isomorphism $\mathrm{H}^1_{\text{ét}}(X,G) \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}(X,{}^Q G)$ to the image of the morphism $\mathrm{H}^1_{\text{ét}}(X,{}^Q H) \longrightarrow \mathrm{H}^1_{\text{ét}}(X,{}^Q G)$ (see [**Gir71**, Corollaire III.3.3.5]).

If we apply this to the exact sequence $1 \to \mathbf{SO}_r \to \mathbf{O}_r \to \{\pm 1\} \to 1$ on a curve C, this only shows that $\mathrm{H}^1_{\text{ét}}(C,\mathbf{O}_r)$ is the disjoint union of the sets of isomorphism classes of $\mathbf{O}_r$-bundles with fixed determinant $\eta \in J_2$, and that any of these components is the isomorphic image of the set of H'-bundles, where H' is the kernel of the morphism $\mathcal{A}ut_{\mathbf{O}_r}(\mathcal{O} \oplus \cdots \oplus \mathcal{O} \oplus \eta) \longrightarrow \mathbb{Z}/2\mathbb{Z}$.

C.5. When H is an Abelian group, we can go further: the second cohomology space $\mathrm{H}^2_{\text{ét}}(X,H)$ exist, and we can easily define à la Čech a connecting map $\mathrm{H}^1_{\text{ét}}(X,G/H) \to \mathrm{H}^2_{\text{ét}}(X,H)$. The extended long sequence need not be exact, but it is if we assume that H is *central* in G.

In this case, we can twist the different groups by any (G/H)-bundle R: we obtain an exact sequence $1 \to H \to {}^R G \to {}^R(G/H) \to 1$. The natural isomorphism $\mathrm{H}^1_{\text{ét}}(X,G/H) \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}\big(X,{}^R(G/H)\big)$ thus identifies the set of G/H-bundles mapped to the image of R in $\mathrm{H}^2_{\text{ét}}(X,H)$ with the image of the boundary map $\mathrm{H}^1_{\text{ét}}(X,{}^R G) \to \mathrm{H}^1_{\text{ét}}\big(X,{}^R(G/H)\big)$. This way to compute the fibers of $\mathrm{H}^1_{\text{ét}}(X,G/H) \to \mathrm{H}^2_{\text{ét}}(X,H)$ makes use of the set of ${}^R G$-bundles.

Remark C.6. — When working on G-bundles over schemes, this situation appears each time we consider the exact sequence $1 \to \pi_1(G) \to \widetilde{G} \to G \to 1$. Note that over smooth curves the previous description may be considerably improved: see [**BLS98**, §2].

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Espaces de modules de fibrés orthogonaux sur une courbe algébrique

On étudie dans cette thèse les espaces de modules de fibrés orthogonaux sur une courbe algébrique lisse.

On montre dans un premier temps que le morphisme d'oubli associant à un fibré orthogonal le fibré vectoriel sous-jacent est une immersion fermée : ce résultat repose sur un calcul d'invariants sur les espaces de représentations de certains carquois.

On présente ensuite, pour les fibrés orthogonaux de rang 3 et 4, des résultats plus concrets sur la géométrie de ces espaces, en accordant une attention particulière à l'application thêta.

Mots clés : schémas de modules, fibrés principaux, fibrés orthogonaux, fonctions thêta généralisées, représentations de carquois.

Moduli schemes of orthogonal bundles over an algebraic curve

We study in this thesis the moduli schemes of orthogonal bundles over an algebraic smooth curve.

We first show that the forgetful morphism from the moduli space of orthogonal bundles to the moduli space of all vector bundles is a closed immersion: this relies on an explicit description of a set of generators for the invariants on the representation spaces of some quivers.

We then give, for orthogonal bundles of rank 3 and 4, some more concrete results about the geometry of these varieties, with a special attention towards the theta map.

Key words: moduli schemes, principal bundles, orthogonal bundles, generalized theta functions, quiver representations.

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