
ORTHOGONAL AND SYMPLECTIC BUNDLES ON CURVES AND QUIVER REPRESENTATIONS

by

Olivier Serman

Abstract. — We show how quiver representations and their invariant theory naturally arise in the study of some moduli spaces parametrizing bundles defined on an algebraic curve, and how they lead to fine results regarding the geometry of these spaces.

Résumé (Fibrés orthogonaux et symplectiques sur les courbes et représentations de carquois)

On montre comment la théorie des représentations de carquois apparaît naturellement lors de l'étude des espaces de modules de fibrés principaux définis sur une courbe algébrique, et comment elle permet d'analyser la géométrie de ces variétés.

Introduction

Let X be a smooth projective curve defined over an algebraically closed field k of characteristic 0. It is a natural question to try to find an algebraic variety which parametrizes objects of some given kind defined on the curve X .

A first example is provided by the study of line bundles of degree 0 on X . It has been known essentially since Abel and Jacobi that there is actually an abelian variety, the Jacobian variety J_X , which parametrizes line bundles of degree 0 on X . We know a great deal about this variety, whose geometry is closely related to the geometry of X .

Weil's suggestion in [33] that vector bundles (which appear in his paper as “ \mathbf{GL}_r -divisors”) should provide a relevant non-abelian analogue of this situation opened the way to a large field of investigations, which led to the construction in the 1960's of the moduli spaces of *semi-stable* vector bundles of given rank and degree on X , achieved mainly by Mumford, Narasimhan and Seshadri. Ramanathan then extended

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this construction to prove the existence of moduli spaces for semi-stable principal G -bundles on X for any connected reductive group G .

These varieties, which will be denoted by \mathcal{M}_G in this paper, have been intensively investigated since their construction, especially for $G = \mathbf{GL}_r$. They have more recently drawn new attention for the fundamental role they appeared to play in various subjects, like Conformal Field Theory or Langland's geometric correspondence.

In these notes we consider the following question:

If $H \rightarrow G$ is a morphism between two reductive groups, what can we say about the induced morphism $\mathcal{M}_H \rightarrow \mathcal{M}_G$ between moduli spaces?

This is a frequently encountered situation. For example, choosing for H a maximal torus $T \simeq (\mathbf{G}_m)^l$ in G gives a morphism from the moduli space \mathcal{M}_T^0 of topologically trivial T -bundles (which is isomorphic to $(J_X)^l$) to the variety \mathcal{M}_G . When X is the projective line \mathbf{P}^1 , we know from [13] that any principal G -bundle on \mathbf{P}^1 comes from a principal T -bundle. If X is an elliptic curve, [17] shows that the morphism $\mathcal{M}_T^0 \rightarrow \mathcal{M}_G$ is a finite morphism from $\mathcal{M}_T^0 \simeq X^l$ onto the connected component of \mathcal{M}_G consisting of topologically trivial semi-stable G -bundles. For higher genus curves, let us just say that the morphism $\mathcal{M}_{\mathbf{G}_m}^0 = J_X \rightarrow \mathcal{M}_{\mathbf{SL}_2}$, which sends a line bundle L to the vector bundle $L \oplus L^{-1}$, gives a beautiful way to investigate the geometry of the moduli spaces of semi-stable rank 2 vector bundles on X (see [4]).

We study here the case of the classical groups $H = \mathbf{O}_r$ and \mathbf{Sp}_{2r} , naturally embedded in the general linear group. The moduli variety $\mathcal{M}_{\mathbf{O}_r}$ then parametrizes semi-stable orthogonal bundles (E, q) of rank r on X , and the morphism $\mathcal{M}_{\mathbf{O}_r} \rightarrow \mathcal{M}_{\mathbf{GL}_r}$ just forgets the quadratic form q . In the same way, $\mathcal{M}_{\mathbf{Sp}_{2r}}$ parametrizes semi-stable symplectic bundles, and $\mathcal{M}_{\mathbf{Sp}_{2r}} \rightarrow \mathcal{M}_{\mathbf{SL}_{2r}}$ forgets the symplectic form. We will also consider \mathbf{SO}_r -bundles, that is oriented orthogonal bundles. We have shown in [30] that the forgetful morphisms

$$\mathcal{M}_{\mathbf{O}_r} \rightarrow \mathcal{M}_{\mathbf{GL}_r} \quad \text{and} \quad \mathcal{M}_{\mathbf{Sp}_{2r}} \rightarrow \mathcal{M}_{\mathbf{SL}_{2r}}$$

are both closed immersions. In other words, these morphisms identify the varieties of semi-stable orthogonal or symplectic bundles with closed subschemes of the variety of all vector bundles. Note that this means that the images in $\mathcal{M}_{\mathbf{GL}_r}$ of these two forgetful morphisms are normal subschemes. The proof involves an infinitesimal study of these varieties, which naturally leads to some considerations coming from representation theory of quivers (for example, we use the fact that $\mathcal{M}_{\mathbf{GL}_r}$ is étale locally isomorphic to the variety parametrizing semi-simple representations of a given quiver). We present in Section 3 a proof of this result which simplifies a little the one given in [30].

The moduli spaces \mathcal{M}_G are in general not regular (nor even locally factorial), and a basic question is to describe their singular locus and the nature of the singularities. If X has genus $g \geq 2$, the singular locus of $\mathcal{M}_{\mathbf{SL}_r}$ has a nice description, which has

been known for long (see [21]): a semi-stable vector bundle defines a smooth point in $\mathcal{M}_{\mathbf{SL}_r}$ if and only if it is a *stable* vector bundle, except when $r = 2$ and $g = 2$ (in this very particular case, $\mathcal{M}_{\mathbf{SL}_2}$ is isomorphic to \mathbf{P}^3). For G -bundles one has to consider *regularly stable* bundles, which are stable G -bundle P whose automorphism group $\text{Aut}_G(P)$ is equal to the center $Z(G)$ of G . Such a bundle defines a smooth point in \mathcal{M}_G , and one can expect the converse to hold, barring some particular cases.

We answer this question for classical groups. We prove in Section 4 that the smooth locus of $\mathcal{M}_{\mathbf{SO}_r}$ is exactly the regularly stable locus, except when X has genus 2 and $r = 3$ or 4. For symplectic bundles we prove that the smooth locus of $\mathcal{M}_{\mathbf{Sp}_{2r}}$ is exactly the set of regularly stable symplectic bundles (for $r \geq 2$). This proof, which requires a precise description of bundles associated to points of the moduli spaces, cannot be extended to another group G without a good understanding of the nature of these bundles.

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1. The moduli spaces \mathcal{M}_G

1.1. Let X be a smooth projective irreducible curve of genus $g \geq 1$, defined over an algebraically closed field of characteristic 0.

We can associate to X its *Jacobian variety* J_X , which parametrizes line bundles of degree 0 on the curve. It is a projective variety, whose closed points correspond bijectively to isomorphism classes of degree 0 line bundles on X . Moreover, J_X has the following *moduli property*:

- if \mathcal{L} is a family of degree 0 line bundles on X parametrized by a scheme T , the classifying map φ which maps a point $t \in T$ on the point in J_X associated to the line bundle \mathcal{L}_t defines a morphism $\varphi: T \longrightarrow J_X$,
- J_X is “universal” for this property.

We should also mention here that J_X comes with a (non-unique) *Poincaré bundle* \mathcal{P} on the product $J_X \times X$. It is a line bundle on $J_X \times X$, whose restriction \mathcal{P}_a to $\{a\} \times X$ is exactly the line bundle associated to the point $a \in J_X$.

The Jacobian variety inherits many geometric properties from its moduli interpretation: let us just note here that it is an abelian variety which naturally carries a principal polarization. This extra data allows to describe sections of line bundles on J_X in terms of *theta functions*. This functional interpretation of geometric objects

defined on J_X provides a powerful tool to investigate the beautiful relations between the curve and its Jacobian.

1.2. It has thus been natural to look for some possible generalizations of this situation. To do this, we can remark that line bundles are exactly principal \mathbf{G}_m -bundles. Replacing the multiplicative group \mathbf{G}_m by any reductive group G leads to the consideration of *principal G -bundles* on X .

When G is the linear group \mathbf{GL}_r , they are vector bundles on X . Topologically, vector bundles on the curve X are classified by their rank r and degree d , and the natural question is to find an algebraic variety whose points correspond to isomorphism classes of vector bundles on X of fixed rank and degree. The idea that such varieties parametrizing vector bundles should exist and give the desired non-abelian generalization of the Jacobian variety goes back to Weil (see [33]). However, the situation cannot be as simple as it is for line bundles. Indeed, the collection $\mathcal{V}_{r,d}$ of all vector bundles of rank r and degree d on X is not *bounded*: we cannot find any family of vector bundles parametrized by a scheme T such that every vector bundle in $\mathcal{V}_{r,d}$ appears in this family. So we need to exclude some bundles in order to have a chance to get a variety enjoying a relevant moduli property.

As we have said in the introduction, the construction of these moduli spaces of vector bundles on X has been carried out in the 1960's, mainly by Mumford and by Narasimhan and Seshadri. They happened to show that one has to restrict to *semi-stable* bundles to obtain a reasonable moduli variety. This notion was introduced first by Mumford in [20] in the light of Geometric Invariant Theory.

Let us define the *slope* of a vector bundle E as the ratio $\mu(E) = \deg(E)/\text{rk}(E)$.

Definition 1.3. — A vector bundle E on X is said to be *stable* (resp. *semi-stable*) if we have, for any proper subbundle $F \subset E$, the slope inequality

$$\mu(F) < \mu(E) \quad (\text{resp. } \mu(F) \leq \mu(E)).$$

We will mainly be concerned in the following with degree 0 vector bundles. In this case, saying that a bundle is stable just means that it does not contain any subbundle of degree ≥ 0 .

Mumford's GIT allowed him to provide the set of isomorphism classes of stable bundles of given rank and degree with the structure of a quasi-projective variety.

Theorem 1.4 (Mumford). — *There exists a coarse moduli scheme $\mathcal{U}_X^{\text{st}}(r, d)$ for stable vector bundles of rank r and degree d on X . Its points correspond bijectively to isomorphism classes of stable bundles of rank r and degree d .*

This result precisely means that, if $F_{X,r,d}^{\text{st}}$ denotes the moduli functor which associates to a scheme T the set of isomorphism classes of families of stable vector bundles of rank r and degree d on X parametrized by T ,

- (i) there is a natural transformation $\varphi : F_{X,r,d}^{\text{st}} \longrightarrow \text{Hom}(-, \mathcal{U}_X^{\text{st}}(r, d))$ such that any natural transformation $F_{X,r,d}^{\text{st}} \longrightarrow \text{Hom}(-, N)$ factors through a unique morphism $\mathcal{U}_X^{\text{st}}(r, d) \longrightarrow N$,
- (ii) the set of closed points of $\mathcal{U}_X^{\text{st}}(r, d)$ is identified (via φ) to the set $F_{X,r,d}^{\text{st}}(\text{Spec } k)$ of isomorphism classes of stable vector bundles of rank r and degree d .

(Of course, the natural transformation φ associates to a family \mathcal{F} of stable bundles parametrized by T the corresponding classifying morphism $t \in T \longmapsto \mathcal{F}_t \in \mathcal{U}_X^{\text{st}}(r, d)$.)

In particular, once we agree to exclude non stable bundles, we obtain a collection of vector bundles which carries a natural algebraic structure. Hopefully, those bundles that we have to forget form a very small class inside the set of all vector bundles, at least when X has genus $g \geq 2$. Indeed, stability (as well as semi-stability) is an open condition: if \mathcal{F} is a family of vector bundles of rank r on X parametrized by T , the stable locus $T^{\text{st}} = \{t \in T | \mathcal{F}_t \text{ is stable}\}$ is open in T (see also Remark 1.8).

1.5. Almost simultaneously, Narasimhan and Seshadri investigated the same notion of stability, but from a completely different approach inspired by Weil's seminal paper [33]. The key observation is that the Jacobian J_X of a complex curve X is a complex torus, which can be topologically identified with the space $\text{Hom}(\pi_1(X), S^1)$ of all 1-dimensional unitary representations of the fundamental group $\pi_1(X)$ of X . This transcendental correspondence between unitary characters of $\pi_1(X)$ and line bundles on X is obtained as follows: if \tilde{X} is a universal covering of X , we associate to a character $\rho : \pi_1(X) \rightarrow S^1$ the line bundle L_ρ on X defined as the quotient $\tilde{X} \times^\rho \mathbf{C}$ of the trivial bundle $\tilde{X} \times \mathbf{C}$ by the action of the fundamental group given by $(x, \lambda) \cdot \gamma = (x \cdot \gamma, \rho(\gamma)^{-1} \lambda)$ for all $\gamma \in \pi_1(X)$ (in other words, L_ρ is the $\pi_1(X)$ -invariant subbundle of the direct image of the trivial line bundle on \tilde{X}). Moreover, this bijection becomes an actual isomorphism for the complex structure induced on $\text{Hom}(\pi_1(X), S^1) \simeq H^1(X, \mathbf{R})/H^1(X, \mathbf{Z})$ by the natural isomorphism between $H^1(X, \mathbf{R})$ and $H^1(X, \mathcal{O}_X)$ given by Hodge theory.

As Weil suggested, *unitary representations of the fundamental group* of X had to play a prominent role in the study of vector bundles, if only because two unitary r -dimensional representations ρ_1 and ρ_2 of $\pi_1(X)$ give isomorphic vector bundles $E_{\rho_i} = \tilde{X} \times^{\rho_i} \mathbf{C}^r$ if and only if they are equivalent (this does not hold any longer for arbitrary linear representations, which ultimately led to the notion of *Higgs bundles*). However, there is a major difference for higher rank vector bundles: this construction does not allow to obtain every degree 0 vector bundle. The main result in [33] states that a vector bundle on X can be defined by a linear representation $\rho : \pi_1(X) \longrightarrow \mathbf{GL}_r$ if and only if it is a direct sum of indecomposable degree 0 vector bundles, so that we already miss the (non semi-stable) rank 2 vector bundles $L \oplus L^{-1}$ with $\deg L \geq 1$. And if we consider only unitary representations, we have to exclude more bundles.

The fundamental result [22], which is “already implicit in the classical paper of A. Weil”, states that, if X has genus at least 2, stable vector bundles of rank r and degree 0 correspond exactly to equivalence classes of irreducible unitary representations $\pi_1(X) \rightarrow \mathbf{U}_r$ of the fundamental group. As a consequence, Theorem 1.4 also shows that the set of equivalence classes of irreducible unitary representations of $\pi_1(X)$ has a natural complex structure, depending on that of X .

1.6. Seshadri went further and constructed a compactification of this variety by considering *unitary bundles*, i.e. bundles associated to any unitary representation. These bundles, which are also called *polystable bundles*, are exactly direct sums of stable bundles of degree 0 (more generally, we say that a semi-stable bundle of arbitrary degree is polystable if it splits as the direct sum of stable bundles). Using Mumford’s theory, he obtained a projective variety $\mathcal{U}_X(r, 0)$ which parametrizes isomorphism classes of polystable bundles of rank r and degree 0 on X , and contains $\mathcal{U}_X^{\text{st}}(r, 0)$ as an open subscheme.

However, no moduli property can be formulated in terms of polystable bundles. Indeed polystability behaves very badly in family; it is not even an open condition. There is in fact a more natural way to think about the variety $\mathcal{U}_X(r, 0)$, based on the following relation between polystable and semi-stable vector bundles. The crucial fact is that Jordan-Hölder theorem holds in the category of all semi-stable vector bundles of degree 0 on X , so that we can associate to any such bundle E the Jordan-Hölder graded object $\text{gr } E$ (sometimes called *semisimplification* of E), which is defined as the direct sum $\text{gr } E = \bigoplus F_i/F_{i-1}$ of the stable subquotients given by any Jordan-Hölder composition series $0 = F_0 \subset F_1 \subset \dots \subset F_l = E$ for E . We say that two semi-stable vector bundles are S -equivalent if the associated graded objects are isomorphic. The point is that S -equivalence classes of degree 0 vector bundles coincide with isomorphism classes of polystable bundles: the equivalence class of a vector bundle E is characterized by the isomorphism class of the corresponding graded object, which is a polystable bundle.

Seshadri proved that the classifying map $t \in T \mapsto \text{gr } \mathcal{E}_t$ associated to any family \mathcal{E} of semi-stable vector bundles of rank r and degree 0 on X parametrized by a variety T defines a morphism $T \rightarrow \mathcal{U}_X(r, 0)$, and that the variety $\mathcal{U}_X(r, 0)$ is in fact a coarse moduli space for semi-stable vector bundles of rank r and degree 0, whose closed points correspond to S -equivalence classes of vector bundles.

In arbitrary degree the corresponding result also holds:

Theorem 1.7 (Seshadri). — *There exists a projective variety $\mathcal{U}_X(r, d)$ which is a coarse moduli scheme for semi-stable bundles of rank r and degree d on X . Its closed points correspond bijectively to S -equivalence classes of semi-stable vector bundles, or, equivalently, to isomorphism classes of polystable vector bundles of rank r and degree d on X .*

It is a normal irreducible projective variety. Moreover, when X has genus $g \geq 2$, it has dimension $r^2(g-1)+1$, and contains as a dense open subscheme the moduli variety $\mathcal{U}_X^{\text{st}}(r, d)$ of stable vector bundles. Although it is commonly denoted by $\mathcal{U}_X(r, d)$, we will preferably use here the notation $\mathcal{M}_{\mathbf{GL}_r}^q$, which keeps track of the identification between rank r vector bundles and principal \mathbf{GL}_r -bundles.

Remark 1.8. — If X is a curve of genus $g \geq 2$, it is not difficult to show that, unless $g = 2$ and $r = 2$, the strictly semi-stable locus $\mathcal{U}_X(r, d) \setminus \mathcal{U}_X^{\text{st}}(r, d)$ is a closed subscheme of codimension at least 2, which means that stable bundles represent a very large part of the set of all semi-stable bundles. In the same way, semi-stable bundles form a very large class inside the collection of all vector bundles. More precisely, we can show using Harder-Narasimhan filtrations that, if \mathcal{F} is a family of vector bundles on X parametrized by a smooth scheme T such that the Kodaira-Spencer infinitesimal deformation map $T_t \rightarrow \text{Ext}^1(\mathcal{F}_t, \mathcal{F}_t)$ is everywhere surjective, then the complement $T \setminus T^{\text{ss}}$ of the semi-stable locus T^{ss} has codimension at least 2 (see [32, 4.IV]).

1.9. Building up on these ideas, Ramanathan considered in his thesis [28] the case of principal G -bundles on a curve X for any complex connected reductive group G . Topologically, principal bundles with connected structure group G on X are classified by their topological type which is a discrete invariant belonging to $H^2(X, \pi_1(G)) \simeq \pi_1(G)$. Ramanathan's aim was to construct coarse moduli schemes for G -bundles on X of a given topological type $\delta \in \pi_1(G)$. We have of course to restrict ourselves to a certain class of G -bundles.

The first step is to define semi-stability for principal G -bundles. It is done by considering reductions of structure group to parabolic subgroups of G . Here we need to recall a few definitions involving principal bundles (see [31]). If P is a G -bundle on X and F a quasi-projective variety acted upon by G , the associated fiber bundle $P(F)$ (also denoted by $P \times^G F$) is the fiber bundle defined as the quotient $(P \times F)/G$, where G acts on $P \times F$ by $(p, f) \cdot g = (p \cdot g, g^{-1} \cdot f)$. If $\rho: G \rightarrow G'$ is a group morphism, $P(G')$ is a principal G' -bundle: it is called *extension of structure group* of P from G to G' and sometimes denoted by $\rho_* P$. Conversely, if P' is a G' -bundle, we call *reduction of structure group* of P' to G a pair (P, α) consisting of a G -bundle P on X together with an isomorphism $\alpha: P(G') \xrightarrow{\sim} P'$ between the associated bundle $\rho_* P = P(G')$ and P' .

Reductions of structure group of a G' -bundle P' correspond to sections σ of the fiber bundle $P'/G = P' \times^{G'} G'/G \rightarrow X$ as follows: if $\sigma: X \rightarrow P'/G$ is such a section, the pull-back via σ of the G -bundle $P' \rightarrow P'/G$ defines a G -bundle $\sigma^* P'$ on X whose extension $\sigma^*(P') \times^G G'$ is naturally isomorphic to P' . Moreover, two sections give isomorphic G -bundles if and only if they differ by an automorphism of P' .

Definition 1.10 (Ramanathan). — A G -bundle P on X is *stable* (resp. *semi-stable*) if, for every parabolic subgroup $\Pi \subset G$, for every non trivial dominant character χ of Π , and for every Π -bundle Q defining a reduction of structure group of P to Π , the line bundle $\chi_* Q$ has degree $\deg(\chi_* Q) < 0$ (resp. ≤ 0).

This seemingly technical definition gives back for $G = \mathbf{GL}_r$ the classical definition 1.3. Moreover, in characteristic 0, a G -bundle P is semi-stable if and only if its adjoint vector bundle $\text{Ad}(P) = P \times^G \mathfrak{g}$ is. (In positive characteristic, we need to introduce strongly semi-stable bundles to get an analogous result.)

1.11. Then we need to know how S -equivalence has to be generalized. We have to recall the following facts (and refer to [28] for details). Each equivalence class defines a Levi subgroup $L \subset G$ and a stable L -bundle Q such that the associated bundle $Q(G)$ belongs to the given class. Moreover, the G -bundle $Q(G)$ is uniquely defined, up to isomorphism, by its equivalence class. This bundle is the analogue of the Jordan-Hölder graded object characterizing S -equivalence classes for vector bundles. Such bundles are called *unitary G -bundles*.

It should be noted that the theorem of Narasimhan and Seshadri remains true in this context (whence the terminology of unitary bundles): if $K \subset G$ denotes a maximal compact subgroup of a connected semisimple group G , then any morphism $\pi_1(X) \rightarrow K$ defines a unitary G -bundle on X , and we get in this way a bijection between conjugacy classes of representations $\pi_1(X) \rightarrow K$ and isomorphism classes of unitary G -bundles (see [27] for the corresponding statement for connected reductive groups).

Now we can recall the main result of [28].

Theorem 1.12 (Ramanathan). — Let G be a complex connected reductive group and $\delta \in \pi_1(G)$. There exists a coarse moduli scheme \mathcal{M}_G^δ for semi-stable principal G -bundles on X of topological type δ . It is an irreducible normal projective variety, whose points correspond bijectively to S -equivalence classes of semi-stable G -bundles.

1.13. Let us briefly recall the main lines of the construction of \mathcal{M}_G for a semisimple group G (following [6]).

We fix a faithful representation $\rho: G \rightarrow \mathbf{SL}_r$, an ample line bundle $\mathcal{O}_X(1)$ on X , and an integer M such that, for every semi-stable G -bundle P , the rank r vector bundle $P(\mathbf{SL}_r) \otimes \mathcal{O}_X(M)$ is generated by its global sections and satisfies $H^1(X, P(\mathbf{SL}_r) \otimes \mathcal{O}_X(M)) = 0$. Let us consider the functor \underline{R}_G which associates to a scheme S the set of isomorphism classes of pairs (\mathcal{P}, α) consisting of a G -bundle \mathcal{P} over $S \times X$ with semi-stable fibers together with an isomorphism $\alpha: \mathcal{O}_S^\chi \xrightarrow{\sim} p_{S*}(\mathcal{P}(\mathbf{SL}_r) \otimes p_X^*\mathcal{O}_X(M))$ (where $\chi = r(M + 1 - g)$, and p_X and p_S denote the projections from $S \times X$ onto X and S). This functor, which is introduced to relate G -bundles to vector bundles, is representable by a *smooth* scheme R_G , which

will be referred to as a *parameter scheme*. The functor $\underline{R}_{\mathbf{SL}_r}$ is indeed representable by a locally closed subscheme of the Hilbert scheme $\text{Quot}_{\mathcal{O}_X^r}^{r,rM}$. If (\mathcal{U}, u) denotes the universal pair on $R_{\mathbf{SL}_r} \times X$, we can see that \underline{R}_G is exactly the functor of global sections of \mathcal{U}/G . This functor is representable by a smooth scheme R_G , which is affine over $R_{\mathbf{SL}_r}$.

We know from Simpson's construction that the moduli scheme $\mathcal{M}_{\mathbf{SL}_r}$ is the (good) quotient $R_{\mathbf{SL}_r}/\Gamma$ of the parameter scheme $R_{\mathbf{SL}_r}$ by the natural action of $\Gamma = \mathbf{GL}_r$ (for sufficiently high M). The point is that $R_{\mathbf{SL}_r}$ is exactly the open subset of semi-stable points for the action of Γ on a closed subscheme of $\text{Quot}_{\mathcal{O}_X^r}^{r,rM}$.

The parameter scheme R_G also carries a natural action of Γ , for which the structural morphism $R_G \rightarrow R_{\mathbf{SL}_r}$ is Γ -equivariant. A good quotient R_G/Γ , if it exists, provides the desired coarse moduli space for semi-stable G -bundles. According to [28, Lemma 5.1], its existence follows from the one of $R_{\mathbf{SL}_r}/\Gamma$.

This construction can be adapted to more general cases, and in particular to \mathbf{GL}_r - and \mathbf{O}_r -bundles: we find that $\mathcal{M}_{\mathbf{GL}_r}^0$ is the good quotient $R_{\mathbf{GL}_r}^0/\Gamma$ of a smooth parameter scheme $R_{\mathbf{GL}_r}^0$ which is an open subset of the Hilbert scheme $\text{Quot}_{\mathcal{O}_X^r}^{r,rM}$, and that $\mathcal{M}_{\mathbf{O}_r}$ is the good quotient of the smooth $R_{\mathbf{GL}_r}^0$ -scheme $R_{\mathbf{O}_r}$ which represents the functor of global sections of the quotient by \mathbf{O}_r of the universal bundle parametrized by $R_{\mathbf{GL}_r}^0$.

Remark 1.14. — (i) Note that properness of $\mathcal{M}_{\mathbf{SL}_r}$ follows from the construction, since this moduli space is obtained as the good quotient of the set of semi-stable points of a projective variety. For arbitrary structure group, this construction does not ensure the properness of the moduli space (while Ramanathan's original one did), and we have to use instead semi-stable reduction theorems for principal G -bundles.
(ii) Existence of moduli spaces for principal bundles has been since then proved for higher dimensional base varieties and in arbitrary characteristic (see [12] and [29]).

2. Orthogonal and symplectic bundles

Let us now specialize the preceding discussion to the classical groups \mathbf{O}_r , \mathbf{SO}_r (with $r \geq 3$) and \mathbf{Sp}_{2r} . In these cases \mathbf{O}_r - and \mathbf{Sp}_{2r} -bundles are just orthogonal and symplectic bundles, and \mathbf{SO}_r -bundles are oriented orthogonal bundles:

Definition 2.1. — An *orthogonal bundle* is a vector bundle E endowed with a non-degenerate quadratic form $q: E \rightarrow \mathcal{O}_X$ (or, equivalently, with a symmetric isomorphism $i: E \rightarrow E^*$). An *oriented orthogonal bundle* (E, q, ω) is an orthogonal bundle (E, q) together with an orientation, which comes as a section ω of the determinant line bundle $\det E \simeq \mathcal{O}_X$ satisfying $\tilde{q}(\omega) = 1$, where \tilde{q} is the quadratic form on $\det E$ induced by q .

A *symplectic* bundle is a vector bundle E endowed with a non-degenerate symplectic form $\varphi: \Lambda^2 E \longrightarrow \mathcal{O}_X$ (or with an antisymmetric isomorphism $E \longrightarrow E^*$).

From now on, we concentrate on orthogonal bundles, and generally omit the corresponding statements for symplectic bundles.

2.2. For these bundles, semi-stability condition 1.10 translates in a very convenient way: an orthogonal bundle (E, q) is semi-stable if and only if the underlying vector bundle E is semi-stable. However, an orthogonal bundle is stable if and only if it splits as the direct orthogonal sum of some mutually non isomorphic orthogonal bundles which are stable as vector bundles (see [25]).

It follows from 1.11 (see also [26, Theorem 3.18]) that if (E, q) is a unitary orthogonal bundle then E is already a polystable vector bundle. It means that E splits as a direct sum of stable vector bundles. Let us recall two elementary facts about stable vector bundles: they are simple bundles, and there are no non zero morphism between non isomorphic stable vector bundles of the same slope. Hence, the non-degenerate quadratic structure suggests to write E as

$$(2.2.1) \quad E = \bigoplus_{i=1}^{n_1} \left(F_i^{(1)} \otimes V_i^{(1)} \right) \oplus \bigoplus_{j=1}^{n_2} \left(F_j^{(2)} \otimes V_j^{(2)} \right) \oplus \bigoplus_{k=1}^{n_3} \left((F_k^{(3)} \oplus F_k^{(3)*}) \otimes V_k^{(3)} \right),$$

where $(F_i^{(1)})$ (resp. $(F_j^{(2)})$, resp. $(F_k^{(3)})$) is a family of mutually non isomorphic orthogonal (resp. symplectic, resp. non isomorphic to their dual nor to that of $F_{k'}^{(3)}$, $k' \neq k$) stable bundles, and $(V_i^{(1)})$ (resp. $(V_j^{(2)})$, resp. $(V_k^{(3)})$) are quadratic (resp. symplectic, resp. equipped with a non-degenerate bilinear form) vector spaces, whose dimension counts the multiplicity of the corresponding stable vector bundle in E . Note that the subbundles $F_k^{(3)} \oplus F_k^{(3)*}$ have been tacitly endowed with the standard hyperbolic quadratic forms.

Remark 2.3. — We gave in [30, Remark 1.3 (ii)] another way to obtain the previous description of a unitary orthogonal bundle, which is in a sense more algebraic (since it avoids the use of the result of Narasimhan and Seshadri), and shows how the unitary bundle associated to a given orthogonal bundle can be defined in terms of isotropic filtrations of the underlying vector bundle (see also [8]).

2.4. According to Ramanathan's result, there exists a moduli space $\mathcal{M}_{\mathbf{SO}_r}$ for semi-stable oriented orthogonal vector bundles of rank r on X . It is a projective scheme, whose points correspond to unitary oriented orthogonal bundles. It has two connected components $\mathcal{M}_{\mathbf{SO}_r}^+$ and $\mathcal{M}_{\mathbf{SO}_r}^-$, which are distinguished by the second Stiefel-Whitney class $w_2 \in H^2(X, \mathbf{Z}/2\mathbf{Z}) = \{\pm 1\}$.

We also have a moduli scheme $\mathcal{M}_{\mathbf{O}_r}$ for semi-stable orthogonal bundles. It has several connected components, each of them corresponding to orthogonal bundles E of a given topological type, which is determined here by the Stiefel-Whitney classes

$w_i(E) \in H^i(X, \mathbf{Z}/2\mathbf{Z})$, $i = 1, 2$ (note that the first Stiefel-Whitney class $w_1(E)$ is nothing but the determinant $\det E$).

These moduli varieties have been investigated for hyperelliptic curves by Ramanan in [25] and Bhosle in [8]. More recently, for $g \geq 2$, Beauville studied in [5] the linear system associated to the determinant line bundle on $\mathcal{M}_{SO_r}^\pm$ (which is, for $n \neq 4$, the ample generator of the Picard group of $\mathcal{M}_{SO_r}^\pm$).

We will consider too the moduli scheme $\mathcal{M}_{Sp_{2r}}$ of semi-stable symplectic bundles on X , which is an irreducible projective variety.

2.5. Let us introduce now the forgetful morphism

$$\mathcal{M}_{O_r} \longrightarrow \mathcal{M}_{GL_r}^0$$

$$(E, q) \longmapsto E$$

which forgets the quadratic structure, as well as the other forgetful morphisms $\mathcal{M}_{SO_r} \longrightarrow \mathcal{M}_{SL_r}$ and $\mathcal{M}_{Sp_{2r}} \longrightarrow \mathcal{M}_{SL_{2r}}$. It follows from the construction of the moduli schemes that these morphisms are finite (see e.g. [3]). We give in this section a set-theoretic study of these morphisms.

Proposition 2.6. — *The forgetful morphisms $\mathcal{M}_{O_r} \longrightarrow \mathcal{M}_{GL_r}^0$ and $\mathcal{M}_{Sp_{2r}} \longrightarrow \mathcal{M}_{SL_{2r}}$ are injective.*

Proof. — It is enough to check injectivity on closed points. In view of 2.2 we have to prove that any two quadratic structures on a given polystable vector bundle E define isomorphic orthogonal bundles, or, in other words, that they differ by a linear automorphism of E . The decomposition (2.2.1) of E shows that its automorphism group is isomorphic to

$$\text{Aut}_{GL_r}(E) = \prod_{i=1}^{n_1} GL(V_i^{(1)}) \times \prod_{j=1}^{n_2} GL(V_j^{(2)}) \times \prod_{k=1}^{n_3} (GL(V_k^{(3)}) \times GL(V_k^{(3)})).$$

Since a quadratic structure on E is nothing but the data of non-degenerate quadratic (resp. symplectic, resp. bilinear) forms on the vector spaces $V_i^{(1)}$ (resp. $V_j^{(2)}$, resp. $V_k^{(3)}$), the conclusion simply follows from the basic fact that any two non-degenerate quadratic (or symplectic) forms on a vector space over an algebraically closed field are equivalent. \square

Remark 2.7. — This proposition is in fact a very particular case of a more general result proved by Grothendieck in [13] (see also [2]). Indeed, this result holds for any vector bundle on any projective variety defined over an algebraically closed field (of characteristic different from 2).

2.8. Oriented orthogonal bundles behaves differently. For orthogonal bundles of odd rank, -1 gives an orthogonal automorphism which exchanges orientation, and it follows that two \mathbf{SO}_r -structures on a vector bundle of odd rank are automatically equivalent. In even rank, this is no longer true. In fact it already fails for rank 2 bundles. Indeed, \mathbf{SO}_2 is just the multiplicative group \mathbf{G}_m , which means that $L \oplus L^{-1}$ and $L^{-1} \oplus L$ (endowed with their oriented hyperbolic form) are not isomorphic as \mathbf{SO}_2 -bundles. We can give a precise criterion for unitary orthogonal bundles to admit two non equivalent orientations.

Proposition 2.9. — *A unitary orthogonal bundle (E, q) has two antecedents via the forgetful morphism*

$$\mathcal{M}_{\mathbf{SO}_r} \longrightarrow \mathcal{M}_{\mathbf{O}_r}$$

if and only if every orthogonal bundle appearing in the decomposition (2.2.1) of its underlying vector bundle has even rank.

In particular, orthogonal bundles of even rank whose underlying vector bundle is stable have two distinct reductions of structure group to \mathbf{SO}_r . Such bundles always exist for curves of genus $g \geq 2$, and therefore the generic fiber of $\mathcal{M}_{\mathbf{SO}_r}^\pm \longrightarrow \mathcal{M}_{\mathbf{O}_r}$ has two elements when r is even.

Proof. — The two orientations defined on a given orthogonal bundle (E, q) give isomorphic \mathbf{SO}_r -bundles if and only if there is an orthogonal automorphism of (E, q) which exchanges the orientation. It follows from decomposition (2.2.1) that the isometry group of E is the subgroup of $\text{Aut}_{\mathbf{GL}_r}(E)$ equal to

$$(2.9.1) \quad \text{Aut}_{\mathbf{O}_r}(E) = \prod_{i=1}^{n_1} \mathbf{O}(V_i^{(1)}) \times \prod_{j=1}^{n_2} \mathbf{Sp}(V_j^{(2)}) \times \prod_{k=1}^{n_3} \mathbf{GL}(V_k^{(3)}),$$

where $\mathbf{GL}(V_k^{(3)})$ is identified with its image in $\mathbf{GL}(V_k^{(3)}) \times \mathbf{GL}(V_k^{(3)})$ by the morphism $g \mapsto (g, {}^t g^{-1})$. So E admits orthogonal automorphisms with non trivial determinant if and only if at least one of the bundles $V_i^{(1)}$ has odd rank.

Note that this argument is encoded by the exact sequence of non-abelian cohomology associated to the exact sequence $1 \rightarrow \mathbf{SO}_r \rightarrow \mathbf{O}_r \rightarrow \mu_2 \rightarrow 1$ (see [30, Remark 1.6]). \square

2.10. Before closing this section, we would like to describe precisely what happens in the case of elliptic curves. Moduli spaces of G -bundles on an elliptic curve have been described in [17]: if we denote by $\Gamma(T)$ the group of one parameter subgroups of a maximal torus $T \subset G$, the connected component of topologically trivial G -bundles is the quotient of $X \otimes_{\mathbb{Z}} \Gamma(T)$ by the operation of the Weyl group W_T . We give here a direct elementary proof of this fact for orthogonal and symplectic bundles.

Proposition 2.11. — Let X be an elliptic curve, and $l \geq 1$. The moduli space $\mathcal{M}_{\mathbf{SO}_{2l+1}}^+$ is isomorphic to \mathbf{P}^l , $\mathcal{M}_{\mathbf{SO}_{2l+1}}^-$ to \mathbf{P}^{l-1} , $\mathcal{M}_{\mathbf{SO}_{2l}}^-$ to \mathbf{P}^{l-2} and $\mathcal{M}_{\mathbf{SO}_{2l}}^+$ to the quotient of X^l by $(\mathbf{Z}/2\mathbf{Z})^{l-1} \rtimes \mathfrak{S}_l$.

Proof. — We know by Atiyah's classification that every semi-stable vector bundle of degree zero on the elliptic curve X is S -equivalent to a direct sum of line bundles. In particular, if κ_1, κ_2 and κ_3 are the three non zero line bundles of order 2, an orthogonal bundle E on X with trivial determinant splits as follows:

- $\mathcal{O}_X \oplus \bigoplus_{i=1}^l (L_i \oplus L_i^{-1})$ if $\text{rk}(E) = 2l + 1$ and $w_2(E) = 1$,
- $\kappa_1 \oplus \kappa_2 \oplus \kappa_3 \oplus \bigoplus_{i=1}^{l-1} (L_i \oplus L_i^{-1})$ if $\text{rk}(E) = 2l + 1$ and $w_2(E) = -1$,
- $\bigoplus_{i=1}^l (L_i \oplus L_i^{-1})$ if $\text{rk}(E) = 2l$ and $w_2(E) = 1$,
- $\mathcal{O}_X \oplus \kappa_1 \oplus \kappa_2 \oplus \kappa_3 \oplus \bigoplus_{i=1}^{l-2} (L_i \oplus L_i^{-1})$ if $\text{rk}(E) = 2l$ and $w_2(E) = -1$,

where the L_i are degree 0 line bundles on X . In all cases but the third one, there is at least one line bundle of order 2 which allows us to adjust the determinant of an orthogonal isomorphism: in these cases, we see that closed points of the moduli spaces are characterized by collections $\{M_1, \dots, M_k\}$ where $M_i \in \{L_i, L_i^{-1}\}$. This gives the expected isomorphisms, since $X^k / ((\mathbf{Z}/2\mathbf{Z})^k \rtimes \mathfrak{S}_k)$ is the k -th symmetric product of \mathbf{P}^1 , which is isomorphic to \mathbf{P}^k .

In the remaining case, a generic orthogonal bundle admits two inequivalent orientations, and $\mathcal{M}_{\mathbf{SO}_{2l}}^+$ is a quotient of X^l by the action of $(\mathbf{Z}/2\mathbf{Z})^{l-1} \rtimes \mathfrak{S}_l$ where $(\mathbf{Z}/2\mathbf{Z})^{l-1}$ acts on $X \times \dots \times X$ by transformations $(a_1, \dots, a_l) \mapsto (\pm a_1, \dots, \pm a_l)$ with an even number of minus signs. This finishes the proof of the proposition.

(Of course, a complete proof would consist in defining morphisms from the products of copies of X to the corresponding moduli spaces, and checking that these morphisms induce the above isomorphisms.) \square

Remark 2.12. — (i) In particular, the forgetful morphism $\mathcal{M}_{\mathbf{SO}_r}^- \rightarrow \mathcal{M}_{\mathbf{SL}_r}$ is always a closed immersion for an elliptic curve, contrary to what happens in higher genus.

(ii) When r is even, the moduli space $\mathcal{M}_{\mathbf{O}_r}^\circ$ of topologically trivial orthogonal bundles with trivial determinant is isomorphic to $\mathbf{P}^{\frac{r}{2}}$, and the forgetful morphism $\mathcal{M}_{\mathbf{SO}_r}^+ \rightarrow \mathcal{M}_{\mathbf{O}_r}^\circ$ is a 2-sheeted covering.

(iii) The same argument applies to moduli of symplectic bundles and gives isomorphisms between $\mathcal{M}_{\mathbf{Sp}_{2r}}$ and \mathbf{P}^r .

3. Differential study of the forgetful morphism and quiver representations

We have seen that the forgetful morphism $\mathcal{M}_{\mathbf{O}_r} \rightarrow \mathcal{M}_{\mathbf{GL}_r}^0$ is an injective finite morphism. It is natural to ask whether we can say more about this morphism. The answer is the main result of [30].

Theorem 3.1. — *The forgetful morphism $\mathcal{M}_{\mathbf{O}_r} \rightarrow \mathcal{M}_{\mathbf{GL}_r}^0$ is a closed immersion.*

Of course, the symplectic version of this statement also holds.

Theorem 3.2. — *The forgetful morphism $\mathcal{M}_{\mathbf{Sp}_{2r}} \rightarrow \mathcal{M}_{\mathbf{SL}_{2r}}$ is a closed immersion.*

3.3. Before going into details, let us give a few remarks about the proof. Since the forgetful morphism is injective and proper, it remains to show that it is everywhere locally a closed immersion, or, equivalently, that it is unramified (see [15, 17.2.6] and [14, 8.11.5]). Here again, it is enough to consider closed points.

To do this, we use Luna's étale slice theorem to get a good enough understanding of the local structure of the moduli spaces $\mathcal{M}_{\mathbf{O}_r}$ and $\mathcal{M}_{\mathbf{GL}_r}^0$: we obtain étale affine neighbourhoods which appear as good quotients of affine spaces by the action of some reductive groups. At this point, we have to understand the corresponding coordinate rings, which are exactly the invariant rings associated to these actions, and to check that the ring morphism induced by the forgetful morphism is surjective. In particular, it is enough to find generating sets for these invariant rings.

3.4. We begin by exhibiting étale neighbourhoods for moduli spaces of vector bundles.

Lemma 3.5 ([16, Theorem 1]). — *At a polystable vector bundle E , the moduli scheme $\mathcal{M}_{\mathbf{GL}_r}^0$ is étale locally isomorphic to a neighbourhood of the origin in the good quotient*

$$\mathrm{Ext}^1(E, E) \mathbin{\!/\mkern-5mu/\!} \mathrm{Aut}_{\mathbf{GL}_r}(E),$$

where $\mathrm{Aut}_{\mathbf{GL}_r}(E)$ acts on $\mathrm{Ext}^1(E, E)$ by functoriality.

Proof. — The construction sketched in 1.13 presents $\mathcal{M}_{\mathbf{GL}_r}^0$ as the good quotient of a smooth open subscheme $R_{\mathbf{GL}_r}^0$ of the Hilbert scheme $\mathrm{Quot}_{\mathcal{O}_X^r}^{r,r,M}$ by the natural action of the reductive group $\Gamma = \mathbf{GL}_r$. Let $q \in R_{\mathbf{GL}_r}^0$ be a point over $E \in \mathcal{M}_{\mathbf{GL}_r}^0$ whose orbit $\Gamma \cdot q$ is closed, and denote by N_q the normal space at q to this orbit. We know by Luna's étale slice theorem that there exists a locally closed subscheme $V \subset R_{\mathbf{GL}_r}^0$ containing q and invariant for the action of the isotropy group $\Gamma_q \subset \Gamma$ of q , together with a Γ_q -equivariant morphism $V \rightarrow N_q$ sending q onto 0, such that the morphisms

$$V \mathbin{\!/\mkern-5mu/\!} \Gamma_q \rightarrow \mathcal{M}_{\mathbf{GL}_r}^0 \quad \text{and} \quad V \mathbin{\!/\mkern-5mu/\!} \Gamma_q \rightarrow N_q \mathbin{\!/\mkern-5mu/\!} \Gamma_q$$

are étale. Deformation theory shows that N_q is isomorphic to the space of extensions $\mathrm{Ext}^1(E, E)$ of E by itself, while an easy argument proves that the isotropy group Γ_q is isomorphic to $\mathrm{Aut}_{\mathbf{GL}_r}(E)$. \square

3.6. Let us now carry out the same analysis for orthogonal bundles. Recall that, if $P = (E, q)$ is an orthogonal bundle, we denote by $\mathrm{Ad}(P)$ its adjoint bundle $\mathrm{Ad}(P) = P \times^{\mathbf{O}_r} \mathfrak{so}_r$. The symmetric isomorphism $\sigma: E \rightarrow E^*$ given by the quadratic structure identifies $\mathrm{Ad}(P)$ with the subbundle of $\mathcal{E}nd(E)$ consisting of germs of endomorphisms f satisfying $\sigma f + f^* \sigma = 0$ (or, equivalently, with the vector bundle $\Lambda^2 E^*$). The first cohomological space $H^1(X, \mathrm{Ad}(P))$ is thus isomorphic to the space $\mathrm{Ext}_{\mathrm{asym}}^1(E, E) \subset \mathrm{Ext}^1(E, E)$ of antisymmetric extensions of E by itself.

Lemma 3.7. — At a unitary orthogonal bundle $P = (E, q)$, the moduli scheme $\mathcal{M}_{\mathbf{O}_r}$ is étale locally isomorphic to a neighbourhood of the origin in

$$H^1(X, \mathrm{Ad}(P)) // \mathrm{Aut}_{\mathbf{O}_r}(P).$$

Moreover, the forgetful morphism coincides, through the different local isomorphisms, to the natural morphism

$$H^1(X, \mathrm{Ad}(P)) // \mathrm{Aut}_{\mathbf{O}_r}(P) \longrightarrow \mathrm{Ext}^1(E, E) // \mathrm{Aut}_{\mathbf{GL}_r}(E)$$

induced by the inclusion $H^1(X, \mathrm{Ad}(P)) \subset \mathrm{Ext}^1(E, E)$.

Proof. — It follows from 1.13 that $\mathcal{M}_{\mathbf{O}_r}$ is the quotient of a smooth $R_{\mathbf{GL}_r}^0$ -scheme $R_{\mathbf{O}_r}$ by the group Γ . Hence Luna's theorem applies as well as in the case of vector bundles: if q' is a point of $R_{\mathbf{O}_r}$ with closed orbit lying over P , $N_{q'}$ the normal space at q' to this orbit, and $\Gamma_{q'}$ the isotropy group of q' , we can find a slice V' through q' together with a $\Gamma_{q'}$ -equivariant morphism $V' \rightarrow N_{q'}$ giving étale morphisms

$$V' // \Gamma_{q'} \longrightarrow \mathcal{M}_{\mathbf{O}_r} \quad \text{and} \quad V' // \Gamma_{q'} \longrightarrow N_{q'} // \Gamma_{q'}.$$

Deformation theory implies that the normal space $N_{q'}$ is isomorphic to $H^1(X, \mathrm{Ad}(P)) = \mathrm{Ext}_{\mathrm{asym}}^1(E, E)$, and we can check that the isotropy group $\Gamma_{q'}$ is isomorphic to $\mathrm{Aut}_{\mathbf{O}_r}(E)$ (we abusively write $\mathrm{Aut}_{\mathbf{O}_r}(E)$ instead of $\mathrm{Aut}_{\mathbf{O}_r}(P)$).

The second part follows from the fact that the forgetful morphism is the quotient by Γ of the structural morphism $R_{\mathbf{O}_r} \rightarrow R_{\mathbf{GL}_r}^0$. We may then choose compatible

slices V and V' in order to obtain the following commutative diagram

$$\begin{array}{ccccc}
& & V' \mathbin{\!/\mkern-5mu/\!} \Gamma_{q'} & & \\
& \swarrow & \downarrow & \searrow & \\
\mathrm{Ext}_{\mathrm{asym}}^1(E, E) \mathbin{\!/\mkern-5mu/\!} \mathrm{Aut}_{\mathbf{O}_r}(E) & & V \mathbin{\!/\mkern-5mu/\!} \Gamma_q & & \mathcal{M}_{\mathbf{O}_r} \\
\downarrow & & \downarrow & & \downarrow \\
& \swarrow & \searrow & & \\
& \mathrm{Ext}^1(E, E) \mathbin{\!/\mkern-5mu/\!} \mathrm{Aut}_{\mathbf{GL}_r}(E) & & & \mathcal{M}_{\mathbf{GL}_r}^0,
\end{array}$$

which gives the expected identification. \square

3.8. We have thus translated the infinitesimal study of the forgetful morphism to a question regarding the morphism

$$\mathrm{Ext}_{\mathrm{asym}}^1(E, E) \mathbin{\!/\mkern-5mu/\!} \mathrm{Aut}_{\mathbf{O}_r}(E) \longrightarrow \mathrm{Ext}^1(E, E) \mathbin{\!/\mkern-5mu/\!} \mathrm{Aut}_{\mathbf{GL}_r}(E).$$

Theorem 3.1 is proved if we show that, for every unitary orthogonal bundle (E, q) , this morphism is unramified at the origin. Now, if we denote by $k[X]$ the coordinate ring of an affine scheme X , this morphism corresponds to the restriction morphism

$$k[\mathrm{Ext}^1(E, E)]^{\mathrm{Aut}_{\mathbf{GL}_r}(E)} \longrightarrow k[\mathrm{Ext}_{\mathrm{asym}}^1(E, E)]^{\mathrm{Aut}_{\mathbf{O}_r}(E)}$$

between invariant algebras, and it is enough to check that it is a surjective morphism.

Remark 3.9. — On the open locus of $\mathcal{M}_{\mathbf{O}_r}$ consisting of orthogonal bundles with stable underlying vector bundle, Theorem 3.1 is automatic. Indeed the isotropy groups act trivially, and there is nothing left to prove.

3.10. We now make Lemmas 3.5 and 3.7 more explicit. The polystable vector bundle E can be written as

$$E = \bigoplus_{i=1}^n F_i \otimes V_i$$

where F_1, \dots, F_n are mutually non isomorphic stable vector bundles, and V_1, \dots, V_n vector spaces. The space of extensions $\mathrm{Ext}^1(E, E)$ decomposes as

$$\mathrm{Ext}^1(E, E) = \bigoplus_{i,j} \mathrm{Ext}^1(F_i, F_j) \otimes \mathrm{Hom}(V_i, V_j)$$

and the isotropy group is isomorphic to $\mathrm{Aut}_{\mathbf{GL}_r}(E) = \prod_i \mathbf{GL}(V_i)$. Denote by d_{ij} the dimension of $\mathrm{Ext}^1(F_i, F_j)$, which is equal to $\mathrm{rk}(F_i) \mathrm{rk}(F_j)(g-1)$ for $i \neq j$, and to $\mathrm{rk}(F_i)^2(g-1)+1$ for $i = j$. Thus, if we pick bases for the extension spaces $\mathrm{Ext}^1(F_i, F_j)$, we may view an extension $\omega \in \mathrm{Ext}^1(E, E)$ as a collection $(f_{ij}^k)_{1 \leq i, j \leq n, k=1, \dots, d_{ij}}$ of morphisms between the vector spaces V_1, \dots, V_n . An element $g = (g_1, \dots, g_n) \in \mathrm{Aut}_{\mathbf{GL}_r}(E)$ acts on (f_{ij}^k) by conjugation:

$$g \cdot (f_{ij}^k) = (g_j f_{ij}^k g_i^{-1}).$$

We recognize here the setting of quiver representations (see [9]). Indeed, let us consider the quiver \mathcal{Q}_E whose set of vertices is defined by

$$(\mathcal{Q}_E)_0 = \{s_1, \dots, s_n\},$$

these vertices being connected by d_{ij} arrows from s_i to s_j , and define the dimension vector $\alpha \in \mathbf{N}^{(\mathcal{Q}_E)_0}$ by $\alpha_i = \dim V_i$. The preceding discussion shows that $\mathrm{Ext}^1(E, E)$ is exactly the representation space $R(\mathcal{Q}_E, \alpha)$ of the quiver \mathcal{Q}_E for the dimension vector α , and that the action of $\mathrm{Aut}_{\mathbf{GL}_r}(E)$ on $\mathrm{Ext}^1(E, E)$ is nothing but the usual action of the group $\mathbf{GL}(\alpha) = \prod_i \mathbf{GL}_{\alpha_i}$ on $R(\mathcal{Q}_E, \alpha)$:

Lemma 3.11. — *The $\mathrm{Aut}_{\mathbf{GL}_r}(E)$ -module $\mathrm{Ext}^1(E, E)$ is isomorphic to the $\prod \mathbf{GL}_{\alpha_i}$ -module $R(\mathcal{Q}_E, \alpha)$. In particular, it only depends (up to $\mathrm{Aut}_{\mathbf{GL}_r}(E)$ -isomorphism) on the ranks and multiplicities of the stable subbundles F_1, \dots, F_n of E .*

3.12. Suppose now that (E, q) is a unitary orthogonal bundle. Following (2.2.1), we can write E as the direct sum

$$E = \bigoplus_{i=1}^{n_1} \left(F_i^{(1)} \otimes V_i^{(1)} \right) \oplus \bigoplus_{j=1}^{n_2} \left(F_j^{(2)} \otimes V_j^{(2)} \right) \oplus \bigoplus_{k=1}^{n_3} \left((F_k^{(3)} \oplus F_k^{(3)*}) \otimes V_k^{(3)} \right).$$

Let us put $E_i^{(1)} = F_i^{(1)} \otimes V_i^{(1)}$, $E_j^{(2)} = F_j^{(2)} \otimes V_j^{(2)}$ and $E_k^{(3)} = (F_k^{(3)} \oplus F_k^{(3)*}) \otimes V_k^{(3)}$. They all have an orthogonal structure $\sigma_l^{(a)} : E_l^{(a)} \xrightarrow{\sim} E_l^{(a)*}$ induced by that of E .

The space $\mathrm{Ext}^1(E, E)$ splits into the direct sum of all extension spaces $\mathrm{Ext}^1(E_i^{(k)}, E_j^{(l)})$. An element $\omega = \sum \omega_{i,j}^{(k,l)} \in \mathrm{Ext}^1(E, E) \simeq \bigoplus \mathrm{Ext}^1(E_i^{(k)}, E_j^{(l)})$ belongs to $\mathrm{Ext}_{\text{asym}}^1(E, E)$ if and only if $\omega_{i,i}^{(k,k)} \in \mathrm{Ext}_{\text{asym}}^1(E_i^{(k)}, E_i^{(k)}) \subset \mathrm{Ext}^1(E_i^{(k)}, E_i^{(k)})$ for all i and k , and $\sigma_j^{(l)} \omega_{i,j}^{(k,l)} + \omega_{j,i}^{(l,k)*} \sigma_i^{(k)} = 0$ for all $(i, k) \neq (j, l)$. So, identifying $\mathrm{Ext}^1(E_i^{(k)}, E_j^{(l)})$ with its image in $\mathrm{Ext}^1(E_i^{(k)}, E_j^{(l)}) \oplus \mathrm{Ext}^1(E_j^{(l)}, E_i^{(k)})$ by the application $\omega_{i,j}^{(k,l)} \mapsto \omega_{i,j}^{(k,l)} - \sigma_i^{(k)-1} \omega_{i,j}^{(k,l)*} \sigma_j^{(l)}$, it appears that $\mathrm{Ext}_{\text{asym}}^1(E, E)$ is equal to the subspace of $\mathrm{Ext}^1(E, E)$ defined as

$$\bigoplus_k \left(\bigoplus_i \mathrm{Ext}_{\text{asym}}^1(E_i^{(k)}, E_i^{(k)}) \oplus \bigoplus_{i < j} \mathrm{Ext}^1(E_i^{(k)}, E_j^{(k)}) \right) \oplus \bigoplus_{k < l} \bigoplus_{i,j} \mathrm{Ext}^1(E_i^{(k)}, E_j^{(l)}).$$

Moreover, we can check that the diagonal summands involved in this decomposition are respectively isomorphic to:

$$\mathrm{Ext}_{\text{asym}}^1(E_i^{(1)}, E_i^{(1)}) = \left(H^1(X, S^2 F_i^{(1)*}) \otimes \mathfrak{so}(V_i^{(1)}) \right) \oplus \left(H^1(X, \Lambda^2 F_i^{(1)*}) \otimes S^2 V_i^{(1)*} \right),$$

$$\mathrm{Ext}_{\text{asym}}^1(E_j^{(2)}, E_j^{(2)}) = \left(H^1(X, \Lambda^2 F_j^{(2)*}) \otimes \mathfrak{sp}(V_j^{(2)}) \right) \oplus \left(H^1(X, S^2 F_j^{(2)*}) \otimes \Lambda^2 V_j^{(2)*} \right),$$

$$\begin{aligned} \mathrm{Ext}_{\mathrm{asym}}^1(E_k^{(3)}, E_k^{(3)}) &= \left(\mathrm{Ext}^1(F_k^{(3)}, F_k^{(3)}) \otimes \mathfrak{gl}(V_k^{(3)}) \right) \oplus \\ &\quad \left(H^1(X, S^2 F_k^{(3)*}) \otimes \Lambda^2 V_k^{(3)*} \right) \oplus \left(H^1(X, \Lambda^2 F_k^{(3)*}) \otimes S^2 V_k^{(3)*} \right) \oplus \\ &\quad \left(H^1(X, S^2 F_k^{(3)}) \otimes \Lambda^2 V_k^{(3)} \right) \oplus \left(H^1(X, \Lambda^2 F_k^{(3)}) \otimes S^2 V_k^{(3)} \right), \end{aligned}$$

where $\mathrm{Ext}^1(F_k^{(3)}, F_k^{(3)})$ has been identified with its image in $\mathrm{Ext}^1(F_k^{(3)}, F_k^{(3)}) \oplus \mathrm{Ext}^1(F_k^{(3)*}, F_k^{(3)*})$ by the map $\omega \mapsto \omega - \omega^*$.

The isometry group

$$\mathrm{Aut}_{\mathbf{O}_r}(E) = \prod_{i=1}^{n_1} \mathbf{O}(V_i^{(1)}) \times \prod_{j=1}^{n_2} \mathbf{Sp}(V_j^{(2)}) \times \prod_{k=1}^{n_3} \mathbf{GL}(V_k^{(3)}),$$

(see (2.9.1)) naturally acts on $\mathrm{Ext}_{\mathrm{asym}}^1(E, E)$ by conjugation.

This laborious description of the $\mathrm{Aut}_{\mathbf{O}_r}(E)$ -module $\mathrm{Ext}_{\mathrm{asym}}^1(E, E)$ has the following consequence:

Lemma 3.13. — *The morphism*

$$\mathrm{Ext}_{\mathrm{asym}}^1(E, E) // \mathrm{Aut}_{\mathbf{O}_r}(E) \longrightarrow \mathrm{Ext}^1(E, E) // \mathrm{Aut}_{\mathbf{GL}_r}(E)$$

only depends, up to isomorphisms, on the ranks and multiplicities of the stable bundles $F_l^{(a)}$ appearing in the decomposition (2.2.1) of the orthogonal bundle E .

3.14. Case of the trivial bundle. — In order to clarify a bit this description before proving the main result of this section (as well as to give an idea of this proof), it seems useful to consider the case of the trivial orthogonal bundle $E = \mathcal{O}_X \otimes k^r$. The space of extensions $\mathrm{Ext}^1(E, E)$ is then identified with the space $\mathrm{Mat}_r(k)^g$ of g -tuples of $r \times r$ matrices, and $\mathrm{Ext}_{\mathrm{asym}}^1(E, E)$ with the subspace $\mathrm{Mat}_r^{\mathrm{asym}}(k)^g$ of g -tuples of antisymmetric matrices. The isotropy groups $\mathrm{Aut}_{\mathbf{GL}_r}(E) = \mathbf{GL}_r$ and $\mathrm{Aut}_{\mathbf{O}_r}(E) = \mathbf{O}_r$ act diagonally by conjugation.

As we have seen in 3.8, the forgetful morphism is unramified at the trivial bundle if the restriction morphism

$$k[\mathrm{Mat}_r(k)^g]^{\mathbf{GL}_r} \longrightarrow k[\mathrm{Mat}_r^{\mathrm{asym}}(k)^g]^{\mathbf{O}_r}$$

is surjective. These invariant algebras have been described in [23]. The algebra $k[\mathrm{Mat}_r(k)^g]^{\mathbf{GL}_r}$ is generated by traces of products $(M_1, \dots, M_g) \mapsto \mathrm{tr}(M_{i_1} \cdots M_{i_l})$ (with $l \leq r^2$), while $k[\mathrm{Mat}_r(k)^g]^{\mathbf{O}_r}$ is generated by functions $(M_1, \dots, M_g) \mapsto \mathrm{tr}(A_{i_1} \cdots A_{i_l})$, where $A_{i_k} \in \{M_{i_k}, {}^t M_{i_k}\}$. The restriction of such a function to the subspace $\mathrm{Mat}_r^{\mathrm{asym}}(k)^g$ is clearly the restriction of a \mathbf{GL}_r -invariant function on $\mathrm{Mat}_r(k)^g$. Since the restriction map $k[\mathrm{Mat}_r(k)^g]^{\mathbf{O}_r} \rightarrow k[\mathrm{Mat}_r^{\mathrm{asym}}(k)^g]^{\mathbf{O}_r}$ is surjective, it proves that the forgetful morphism is indeed unramified at the trivial bundle.

Proof of Theorem 3.1. — We have to prove that the forgetful morphism is unramified. The decomposition (2.2.1) of a unitary orthogonal bundle allows us to define the

slice-type stratification of $\mathcal{M}_{\mathbf{O}_r}$. The locally closed strata consist of all unitary orthogonal bundles E having a given isometry group $\text{Aut}_{\mathbf{O}_r}(E)$. Lemma 3.13, together with Lemma 3.7, shows that the sheaf of relative differential

$$\Omega^1_{\mathcal{M}_{\mathbf{O}_r}/\mathcal{M}_{\mathbf{GL}_r}^0}$$

has constant rank on each stratum. It is thus enough to show that this sheaf vanishes on the closed ones.

Since orthogonal summands $F_k^{(3)} \oplus F_k^{(3)*}$ (with $F_k^{(3)} \not\simeq F_k^{(3)*}$) or $F_j^{(2)} \otimes V_j^{(2)}$ (where $F_j^{(2)}$ is a symplectic bundle) specialize to the trivial orthogonal bundle, a closed stratum must consist of unitary orthogonal bundles which split as

$$E = \bigoplus_{i=1}^{n_1} F_i^{(1)} \otimes V_i^{(1)}$$

where $F_1^{(1)}, \dots, F_{n_1}^{(1)}$ are mutually non isomorphic orthogonal bundles whose underlying vector bundles are stable, and $V_1^{(1)}, \dots, V_{n_1}^{(1)}$ some quadratic spaces.

Let now $E = \bigoplus_i F_i \otimes V_i$ be such an orthogonal bundle. We claim that the restriction morphism

$$k[\text{Ext}^1(E, E)]^{\text{Aut}_{\mathbf{GL}_r}(E)} \longrightarrow k[\text{Ext}_{\text{asym}}^1(E, E)]^{\text{Aut}_{\mathbf{O}_r}(E)}$$

is surjective. This means that the forgetful morphism is unramified on the closed strata, which finishes the proof of the Theorem.

Let \mathcal{Q}_E be the quiver defined in 3.10. According to Lemma 3.11, $k[\text{Ext}^1(E, E)]^{\text{Aut}_{\mathbf{GL}_r}(E)}$ is isomorphic to the coordinate ring $k[R(\mathcal{Q}_E, \alpha)]^{\mathbf{GL}(\alpha)}$ of the quotient variety $R(\mathcal{Q}_E, \alpha)/\mathbf{GL}(\alpha)$ which parametrizes isomorphism classes of semisimple representations of \mathcal{Q}_E with dimension α . This invariant algebra has been described in [18] (in the characteristic 0 case). In particular, it is generated by traces along oriented cycles in \mathcal{Q}_E (of length $\leq (\sum \alpha_i)^2$), that is by functions

$$(f_a)_a \longmapsto \text{tr}(f_{a_l} \cdots f_{a_1})$$

where $a_1 \cdots a_l$ is an oriented cycle in the quiver \mathcal{Q}_E .

On the other hand, since the inclusion $\text{Ext}_{\text{asym}}^1(E, E) \longrightarrow \text{Ext}^1(E, E)$ is equivariant for the action of the isometry group

$$\text{Aut}_{\mathbf{O}_r}(E) = \prod_i \mathbf{O}(V_i),$$

the restriction morphism $k[\text{Ext}^1(E, E)]^{\text{Aut}_{\mathbf{O}_r}(E)} \longrightarrow k[\text{Ext}_{\text{asym}}^1(E, E)]^{\text{Aut}_{\mathbf{O}_r}(E)}$ is surjective. The next proposition provides us with a set of generators for

$$k[\text{Ext}^1(E, E)]^{\text{Aut}_{\mathbf{O}_r}(E)} \simeq k[R(\mathcal{Q}_E, \alpha)]^{\prod \mathbf{O}_{\alpha_i}}.$$

If $\widetilde{\mathcal{Q}_E}$ is the quiver deduced from \mathcal{Q}_E by adding one new arrow $a^*: v' \rightarrow v$ for any arrow $a: v \rightarrow v'$, it tells us that $k[R(\mathcal{Q}_E, \alpha)]^{\prod \mathbf{O}_{\alpha_i}}$ is generated by functions

$$(f_a)_a \longmapsto \text{tr}(f_{\tilde{a}_l} \cdots f_{\tilde{a}_1})$$

where $\tilde{a}_1 \cdots \tilde{a}_l$ is an oriented cycle in the quiver $\widetilde{\mathcal{Q}_E}$, and $f_{\tilde{a}_i}$ is equal to f_{a_i} or its adjoint according to whether \tilde{a}_i is a_i or a_i^* .

The space $\text{Ext}_{\text{asym}}^1(E, E) \subset \text{Ext}^1(E, E)$ has been described in 3.12. It identifies with a subspace of $R(\mathcal{Q}_E, \alpha)$ made of representations having the following property: if $f_a: V_v \rightarrow V_{v'}$ is the linear morphism associated to an arrow $a: v \rightarrow v'$, then its adjoint morphism $f_a^*: V_{v'}^* \rightarrow V_v^*$ is, up to the sign, the linear morphism associated to an arrow $a: v' \rightarrow v$. It implies that the restrictions to $\text{Ext}_{\text{asym}}^1(E, E)$ of the preceding functions are also restrictions of $\text{Aut}_{\mathbf{GL}_r}(E)$ -invariant functions on $\text{Ext}^1(E, E)$, whence our claim. \square

Let us now state and prove the result about $\prod \mathbf{O}_{\alpha_i}$ -invariant functions used in the proof. Let Q by a quiver with n vertices, and $\alpha \in \mathbf{N}^n$ a dimension vector. Consider the group $\mathbf{O}(\alpha) = \prod \mathbf{O}_{\alpha_i}$. As a subgroup of $\mathbf{GL}(\alpha)$, it acts by conjugation on the representation space $R(Q, \alpha)$. Let \widetilde{Q} be the quiver deduced from Q by adding one new arrow $a^*: v' \rightarrow v$ for any arrow $a: v \rightarrow v'$.

Proposition 3.15. — *The algebra $k[R(Q, \alpha)]^{\mathbf{O}(\alpha)}$ of polynomial invariants for the action of $\mathbf{O}(\alpha)$ on the representation space $R(Q, \alpha)$ is generated by traces along oriented cycles in the associated quiver \widetilde{Q} . These are functions*

$$(f_a)_a \mapsto \text{tr}(f_{\tilde{a}_l} \cdots f_{\tilde{a}_1})$$

where $\tilde{a}_1 \cdots \tilde{a}_l$ is an oriented cycle in the quiver \widetilde{Q} , and $f_{\tilde{a}_i}$ is equal to f_{a_i} or its adjoint according to whether \tilde{a}_i is a_i or a_i^* .

It may be rephrased as follows. First note that any representation of Q can be extended to a representation of \widetilde{Q} by associating to a new arrow a^* the adjoint of the linear map corresponding to a . This defines a natural map $R(Q, \alpha) \rightarrow R(\widetilde{Q}, \alpha)$, and the proposition just means that the restriction morphism $k[R(\widetilde{Q}, \alpha)]^{\mathbf{GL}(\alpha)} \rightarrow k[R(Q, \alpha)]^{\mathbf{O}(\alpha)}$ is onto.

This result is a special case of [30, Theorem 2.3.3], and follows (exactly as (*loc. cit.*)) from an adaptation of the proof given in [18] to describe the invariant ring $k[R(Q, \alpha)]^{\mathbf{GL}(\alpha)}$. This special case is technically much easier. Indeed, we had to consider in [30] algebras with antimorphisms of order 4, while antiinvolution are enough here. We present here a quite detailed proof, but warmly refer the reader to the original exposition [18].

We first need a lemma about algebras with trace and antiinvolution. Recall that a k -algebra with trace is a k -algebra R together with a linear map $\text{tr}: R \rightarrow R$ satisfying the identities $\text{tr}(a)b = b \text{tr}(a)$, $\text{tr}(ab) = \text{tr}(ba)$ and $\text{tr}(\text{tr}(a)b) = \text{tr}(a)\text{tr}(b)$ for all $a, b \in R$. A k -algebra with trace and antiinvolution is an algebra with trace R endowed with an antiinvolution $\iota: R \rightarrow R$ which commutes with the trace. The algebra $\text{Mat}_N(B)$ of $N \times N$ matrices with coefficients in a commutative ring B will be equipped with its usual trace together with the adjunction map $\tau: M \in \text{Mat}_N(B) \mapsto {}^t M$.

If R is a k -algebra with trace and antiinvolution ι , we can consider the functor $\tilde{X}_{R,N}$ (from commutative k -algebras to sets) of N -dimensional trace preserving representations of R commuting with the antiinvolutions:

$$\tilde{X}_{R,N}(B) = \{f \in \text{Hom}_k(R, \text{Mat}_N(B)) \mid f \circ \text{tr} = \text{tr} \circ f, f \circ \iota = \tau \circ f\}.$$

We claim that this functor is representable by a commutative ring \tilde{A} . Indeed, we know from [10, 2.2] that the functor $X_{R,N}$ of trace preserving representations of R is representable by a ring A . If $j: R \rightarrow \text{Mat}_N(A)$ is the corresponding universal morphism, there is a unique involution t of A such that $\text{Mat}_N(t)j = \tau j \iota$, and the quotient \tilde{A} of A by this involution represents $\tilde{X}_{R,N}$. We still denote by $\tilde{X}_{R,N}$ the affine scheme $\text{Spec } \tilde{A}$.

In particular, we have a universal morphism $\tilde{j}: R \rightarrow \text{Mat}_N(\tilde{A})$. The conjugation action of \mathbf{O}_N on $\text{Mat}_N(\tilde{A})$ induces a right action on \tilde{A} : indeed, every g in \mathbf{O}_N defines an automorphism \bar{g} of \tilde{A} such that $\text{Mat}(\bar{g})\tilde{j} = C(g)\tilde{j}$, where we denote by $C(g)$ the conjugation by g . We consider the action of \mathbf{O}_N on $\text{Mat}_N(\tilde{A})$ defined by $g \cdot M = C(g)\text{Mat}(\bar{g})^{-1}(M)$ for $g \in \mathbf{O}_N$ and $M \in \text{Mat}_N(\tilde{A})$. The universal morphism \tilde{j} then maps R into the algebra $\text{Mat}_N(\tilde{A})^{\mathbf{O}_N}$ of \mathbf{O}_N -equivariant morphisms from $\tilde{X}_{R,N} = \text{Spec } \tilde{A}$ to $\text{Mat}_N(k)$ (see [24] or [10, 1.2]).

The main result of [24] can be easily adapted to this situation (see also [7, §12]):

Lemma 3.16. — *Let R be a k -algebra with trace and antiinvolution. Then the universal morphism \tilde{j} is a surjective morphism $R \rightarrow \text{Mat}_N(\tilde{A})^{\mathbf{O}_N}$.*

Proof. — Following [24] we begin by proving this when R is a free algebra with trace and antiinvolution built on the generators $\{x_s\}_{s \in \Sigma}$. In this case one can check that $\text{Mat}_N(\tilde{A})^{\mathbf{O}_N}$ is the algebra of all \mathbf{O}_N -equivariant polynomial maps from $\text{Mat}_N(k)^\Sigma$ to $\text{Mat}_N(k)$, and our assertion immediately follows from the description of this algebra given in [23, 7.2] (which comes as a direct consequence of the result recalled in 3.14 about generators for the invariant algebra $k[\text{Mat}_N(k)^\Sigma]^{\mathbf{O}_N}$).

In the general case, we write R as the quotient of a free algebra with trace and antiinvolution T by an ideal I . If \tilde{A}_T is the universal ring associated to T , we know that the two-sided ideal in $\text{Mat}_N(\tilde{A}_T)$ generated by the image of I must be equal to $\text{Mat}_N(J)$ for some ideal J in \tilde{A}_T . The universal ring for R is then the quotient \tilde{A}_T/J . The conclusion follows from the linear reductivity of \mathbf{O}_N , which ensures that $\text{Mat}_N(\tilde{A}_T)^{\mathbf{O}_N} \rightarrow \text{Mat}_N(\tilde{A}_T/J)^{\mathbf{O}_N}$ is onto. Note that this last argument makes essential use of the characteristic 0 assumption. \square

Let us go back to the quiver Q . The associated quiver \tilde{Q} carries a natural involution σ that fixes vertices and exchanges arrows a and a^* . Let R (resp. \tilde{R}) be the algebra obtained from the path algebra of the opposite quiver Q^{op} (resp. \tilde{Q}^{op}) by adding traces. The involution $\sigma: \tilde{Q} \rightarrow \tilde{Q}$ induces an antiinvolution ι of \tilde{R} such that representations of \tilde{R} commuting with τ and ι correspond bijectively to representations

of R . In other words, σ gives an involution of the space $R(\tilde{Q}, \alpha)$ such that $R(Q, \alpha)$ is isomorphic to the subspace of $R(\tilde{Q}, \alpha)$ consisting of all representations which preserve the preceding involutions. The proof of 3.15 relies on a precise description of this space as a subspace of $\tilde{X}_{\tilde{R}, N}(k)$, where $N = \sum \alpha_i$.

Proof of Proposition 3.15. — We follow closely [18, §3]. Consider the subalgebra $\tilde{S}_n \subset \tilde{R}$ generated by the orthogonal idempotents e_1, \dots, e_n corresponding to the different vertices of \tilde{Q} . The antiinvolution ι is trivial on this subalgebra, and the scheme $\tilde{X}_{\tilde{S}_n, N}$ is the disjoint union $\tilde{X}_{\tilde{S}_n, N} = \bigcup_{\delta \in \mathbf{N}^n} \tilde{X}_\delta$, where $\delta \in \mathbf{N}^n$ ranges over the set of all dimension vectors such that $\sum \delta_i = N$, of the homogeneous varieties

$$\tilde{X}_\delta = \mathbf{O}_N / \prod_i \mathbf{O}_{\delta_i}.$$

This induces a decomposition $\tilde{X}_{\tilde{R}, N} = \bigcup \varpi^{-1} \tilde{X}_\delta$ (where $\varpi: \tilde{X}_{\tilde{R}, N} \longrightarrow \tilde{X}_{\tilde{S}_n, N}$ is the morphism induced by the inclusion $\tilde{S}_n \subset \tilde{R}$), or, equivalently, a decomposition $\tilde{A} = \prod \tilde{A}_\delta$ of the coordinate ring \tilde{A} of $\tilde{X}_{\tilde{R}, N}$.

We focus on the component $\tilde{X}_{\tilde{R}, \alpha} = \varpi^{-1} \tilde{X}_\alpha$ corresponding to the dimension vector α . Let us write the identity matrix id_N as the sum $\sum u_i$ of orthogonal idempotents u_1, \dots, u_n associated to the decomposition $k^N = \bigoplus k^{\alpha_i}$, and let p be the point in \tilde{X}_α defined by the representation of \tilde{S}_n sending e_i to u_i . The fiber $\varpi^{-1}(p)$ (which represents the subfunctor of $\tilde{X}_{\tilde{R}, N}$ consisting of representations sending e_i to u_i) naturally carries an action of the centralizer in \mathbf{O}_N of the idempotents u_i , which is isomorphic to $\mathbf{O}(\alpha) = \prod \mathbf{O}_{\alpha_i}$. Moreover, this fiber can be identified with the subspace of $R(\tilde{Q}, \alpha)$ consisting of involutions preserving representations of \tilde{Q} , which is itself isomorphic to $R(Q, \alpha)$.

Since ϖ is \mathbf{O}_N -equivariant and \tilde{X}_α is homogeneous, the invariant ring $\text{Mat}_N(k[R(Q, \alpha)])^{\mathbf{O}(\alpha)}$ of \mathbf{O}_α -equivariant maps from $\varpi^{-1}(p)$ to $\text{Mat}_N(k)$ is exactly the ring $\text{Mat}_N(\tilde{A}_\alpha)^{\mathbf{O}_N}$ of \mathbf{O}_N -equivariant maps from $\tilde{X}_{\tilde{R}, \alpha}$ to $\text{Mat}_N(k)$. But, since \mathbf{O}_N acts separately on each factor of $\text{Mat}_N(\tilde{A}) = \prod \text{Mat}_N(\tilde{A}_\delta)$, it follows from Lemma 3.16 that \tilde{j} gives a surjective morphism from \tilde{R} onto $\text{Mat}_N(\tilde{A}_\alpha)^{\mathbf{O}_N} \simeq \text{Mat}_N(k[R(Q, \alpha)])^{\mathbf{O}(\alpha)}$. The expected description of $k[R(Q, \alpha)]^{\mathbf{O}(\alpha)}$ follows by taking traces. \square

Remark 3.17. — (i) We have treated in [30] the more general following problem: let Q stand for a quiver with $n = n_1 + n_2 + n_3 + 2n_4$ vertices

$$r_1, \dots, r_{n_1}, s_1, \dots, s_{n_2}, t_1, \dots, t_{n_3}, u_1, u_1^*, \dots, u_{n_4}, u_{n_4}^*,$$

and $\alpha \in \mathbf{N}^n \simeq \mathbf{N}^{n_1} \times \mathbf{N}^{n_2} \times \mathbf{N}^{n_3} \times (\mathbf{N} \times \mathbf{N})^{n_4}$ be an admissible dimension vector (by which we mean a vector such that α_{t_k} is even and $\alpha_{u_l} = \alpha_{u_l^*}$). We define $\Gamma(\alpha)$ to be the group

$$\Gamma(\alpha) = \prod_{i=1}^{n_1} \mathbf{GL}_{\alpha_{r_i}} \times \prod_{j=1}^{n_2} \mathbf{O}_{\alpha_{s_j}} \times \prod_{k=1}^{n_3} \mathbf{Sp}_{\alpha_{t_k}} \times \prod_{l=1}^{n_4} \mathbf{GL}_{\alpha_{u_l}},$$

which is actually thought of here as a subgroup of $\mathbf{GL}(\alpha)$ via the inclusions $P \in \mathbf{GL}_{\alpha_{u_l}} \hookrightarrow (P, {}^t P^{-1}) \in \mathbf{GL}_{\alpha_{u_l}} \times \mathbf{GL}_{\alpha_{u_l^*}}$ for $l = 1, \dots, n_4$. We give a generating set for the invariant algebra $k[R(Q, \alpha)]^{\Gamma(\alpha)}$.

Let \tilde{Q} be the quiver deduced from Q as follows. We add n_1 new vertices $r_1^*, \dots, r_{n_1}^*$, and consider the involution σ on the set of vertices \tilde{Q}_0 which fixes the s_j and t_k and exchanges u_l with u_l^* , and r_i with r_i^* . We now add a new arrow $a^* : \sigma(v') \rightarrow \sigma(v)$ for any arrow $a : v \rightarrow v'$. Note that in this new quiver any two vertices r_i and $r_{i'}^*$ are never connected by a single arrow.

Theorem. — *The invariant algebra $k[R(Q, \alpha)]^{\Gamma(\alpha)}$ is generated by traces along cycles in the associated quiver \tilde{Q} , that is by functions*

$$(f_a)_a \mapsto \text{tr}(f_{\tilde{a}_1} \cdots f_{\tilde{a}_l})$$

where $\tilde{a}_1 \cdots \tilde{a}_l$ is an oriented cycle in the quiver \tilde{Q} , and $f_{\tilde{a}_i}$ is equal to f_{a_i} or its adjoint according to whether \tilde{a}_i is a_i or a_i^* .

It directly implies (together with 3.12) that $\text{Ext}_{\text{asym}}^1(E, E) // \text{Aut}_{\mathbf{O}_r}(E) \longrightarrow \text{Ext}^1(E, E) // \text{Aut}_{\mathbf{GL}_r}(E)$ is a closed immersion for every unitary orthogonal bundle E .

(ii) The proof given in [18] and its adaptation in Proposition 3.15 are only valid in characteristic 0. However, these results remain true in arbitrary characteristic. Characteristic free proofs can be found in [11] for the $\mathbf{GL}(\alpha)$ -action on $R(Q, \alpha)$ and in [19] for the $\Gamma(\alpha)$ -action. They rely on the notions of good filtrations and good pairs.

3.18. We close this section with a few comments about the forgetful morphism $\mathcal{M}_{\mathbf{SO}_r} \longrightarrow \mathcal{M}_{\mathbf{SL}_r}$ for genus ≥ 2 curves (see Remark 2.12 for elliptic curves). Of course its study reduces to that of $\mathcal{M}_{\mathbf{SO}_r} \longrightarrow \mathcal{M}_{\mathbf{O}_r}$. Lemma 3.7 and its variant for \mathbf{SO}_r -bundles easily show that, when r is odd, this morphism is an isomorphism onto its image. When r is even, it is a 2-sheeted covering of its image. Indeed, an orthogonal bundle (E, q) has two antecedents if and only if $\text{Aut}_{\mathbf{SO}_r}(E) = \text{Aut}_{\mathbf{O}_r}(E)$. Then, at a point (E, q, ω) defined by any of its two reductions to \mathbf{SO}_r , the forgetful morphism is a local isomorphism (in the étale topology).

Proposition 3.19. — *Let X be a curve of genus ≥ 2 . Then the forgetful morphism $\mathcal{M}_{\mathbf{SO}_r} \longrightarrow \mathcal{M}_{\mathbf{SL}_r}$ is a closed immersion when r is odd, and a 2-sheeted covering of its image when r is even.*

Remark 3.20. — When r is even, the infinitesimal behaviour of $\mathcal{M}_{\mathbf{SO}_r} \longrightarrow \mathcal{M}_{\mathbf{O}_r}$ remains difficult to describe explicitly. In the case of the trivial bundle $E = \mathcal{O} \otimes V$ with V a quadratic vector space of even dimension, we have to understand the action of $\text{Aut}_{\mathbf{SO}_r}(E) \simeq \mathbf{SO}_r$ on $\text{Ext}_{\text{asym}}^1(E, E) \simeq H^1(X, \mathcal{O}_X) \otimes \mathfrak{so}(V)$. The computation has been carried out in [1], and provides a set of generators for the

$k[\mathrm{Ext}_{\mathrm{asym}}^1(E, E)]^{\mathrm{Aut}_{\mathbf{O}_r}(P)}$ -algebra $k[\mathrm{Ext}_{\mathrm{asym}}^1(E, E)]^{\mathrm{Aut}_{\mathbf{SO}_r}(P)}$ in terms of *polarized pfaffians*.

4. Singular locus of $\mathcal{M}_{\mathbf{SO}_r}$ and $\mathcal{M}_{\mathbf{Sp}_{2r}}$

In this section, X is a curve of genus $g \geq 2$.

4.1. Narasimhan and Ramanan have described in [21] the singular locus of the moduli space $\mathcal{M}_{\mathbf{GL}_r}^d$ of vector bundles of rank r and degree d : they have shown that it is exactly the closed subset of strictly semi-stable vector bundles, except when X is a curve of genus 2, $r = 2$ and d even (in which case $\mathcal{M}_{\mathbf{GL}_2}^d$ is a smooth variety). Note that the fact that stable vector bundles E define non singular points in $\mathcal{M}_{\mathbf{GL}_r}^d$ is trivial. Indeed, we know that $\mathcal{M}_{\mathbf{GL}_r}^d$ is étale locally isomorphic to $\mathrm{Ext}^1(E, E) // \mathrm{Aut}_{\mathbf{GL}_r}(E)$, which is smooth when E is stable for the obvious reason that the isotropy group $\mathrm{Aut}_{\mathbf{GL}_r}(E) = \mathbf{G}_m$ then acts trivially on $\mathrm{Ext}^1(E, E)$.

4.2. For arbitrary reductive algebraic groups G , the relevant notion is that of *regularly stable bundle*: a regularly stable G -bundle is a stable G -bundle E such that $\mathrm{Aut}_G(E) = Z(G)$. The same argument shows that the smooth locus of \mathcal{M}_G contains the open subset of regularly stable bundles.

We check here that, when $G = \mathbf{SO}_r$ and \mathbf{O}_r , this inclusion is in fact an equality, except in two special cases (which are not surprising in view of the particular case occurring in [21]). Note that regularly stable oriented orthogonal bundles are stable orthogonal bundles whose underlying vector bundle is either stable or, when r is even, the direct sum of two different stable bundles of odd rank, while regularly stable orthogonal bundles are just orthogonal bundles with stable underlying vector bundle.

Theorem 4.3. — *The smooth locus of $\mathcal{M}_{\mathbf{SO}_r}$ (resp. $\mathcal{M}_{\mathbf{O}_r}$) is precisely the open set consisting of regularly stable \mathbf{SO}_r -bundles (resp. \mathbf{O}_r -bundles), except when $g = 2$ and $r = 3$ or 4.*

Proof. — The proof relies on the precise description of the closed points of $\mathcal{M}_{\mathbf{SO}_r}$. Let U be the set of points in $\mathcal{M}_{\mathbf{SO}_r}$ which correspond to those oriented orthogonal bundles (E, q, ω) which are either regularly stable, or an orthogonal sum $E = E_1 \oplus E_2$ of two different stable vector bundles, or an hyperbolic sum $E = F \oplus F^*$ where F is a stable vector bundle with $F \not\simeq F^*$. It is an open dense subset of $\mathcal{M}_{\mathbf{SO}_r}$. Moreover, if we denote by $\mathcal{M}_{\mathbf{SO}_r}^{\mathrm{rs}}$ the locus of regularly stable bundles, we see that $U \setminus \mathcal{M}_{\mathbf{SO}_r}^{\mathrm{rs}}$ is open and dense in $\mathcal{M}_{\mathbf{SO}_r} \setminus \mathcal{M}_{\mathbf{SO}_r}^{\mathrm{rs}}$ for $r \geq 4$ (when $r = 3$ we need to enlarge U by adding the bundles $\mathcal{O}_X \oplus L \oplus L^{-1}$ for degree 0 line bundles L with $L^2 \not\simeq \mathcal{O}_X$).

It is thus enough to check that the singular locus of U is exactly $U \setminus \mathcal{M}_{\mathbf{SO}_r}^{\text{rs}}$. The proof of Lemma 3.7 shows that $\mathcal{M}_{\mathbf{SO}_r}$ is étale locally isomorphic at a point defined by a unitary bundle P to the good quotient $H^1(X, \text{Ad}(P)) // \text{Aut}_{\mathbf{SO}_r}(P)$.

At a point defined by an oriented orthogonal bundle $E = E_1 \oplus E_2$ with E_1 and E_2 two different stable vector bundles (of even rank if r is even), $\mathcal{M}_{\mathbf{SO}_r}$ is étale locally isomorphic to a neighbourhood of the origin in the quotient of

$$H^1(X, \Lambda^2 E_1) \oplus H^1(X, \Lambda^2 E_2) \oplus \text{Ext}^1(E_1, E_2)$$

by the action of $\text{Aut}_{\mathbf{SO}_r}(E)/Z(\mathbf{SO}_r) \simeq \mu_2$ (where $-1 \in \mu_2$ acts by $(1, 1, -1)$). Chevalley's theorem implies that this quotient cannot be smooth, since $\text{Ext}^1(E_1, E_2)$ must have dimension at least 2.

At a point $E = F \oplus F^*$ with F a stable vector bundle non isomorphic to its dual, $\mathcal{M}_{\mathbf{SO}_r}$ is étale locally isomorphic to the quotient of

$$H^1(X, \Lambda^2 F^*) \oplus H^1(X, \Lambda^2 F) \oplus \text{Ext}^1(F, F)$$

by the action of $\text{Aut}_{\mathbf{SO}_r}(E) \simeq \text{Aut}(F) \simeq \mathbf{G}_m$, where $\lambda \in \mathbf{G}_m$ acts by $(\lambda^{-2}, \lambda^2, 1)$. We easily see that its multiplicity at the origin is equal to $\binom{2(d-1)}{d-1}$ where d is the dimension of $H^1(X, \Lambda^2 F^*)$ (see [30, 3.3.4 (ii)]). This multiplicity cannot be equal to 1, unless $g = 2$ and $r = 4$.

This concludes the proof of the Theorem for $r \geq 4$. In rank 3 we have also to consider the points $E = \mathcal{O}_X \oplus L \oplus L^{-1}$ where L is a line bundle of degree 0 whose square L^2 is not trivial. The automorphism $\lambda \in \text{Aut}_{\mathbf{SO}_3}(E) \simeq \text{Aut}(L) = \mathbf{G}_m$ then acts on

$$\text{Ext}^1(L, L) \oplus \text{Ext}^1(\mathcal{O}_X, L) \oplus \text{Ext}^1(\mathcal{O}_X, L^{-1})$$

by $(1, \lambda, \lambda^{-1})$. We easily see that the quotient is smooth if and only if $\text{Ext}^1(\mathcal{O}_X, L)$ has dimension 1, which happens exactly when X has genus 2. \square

Remark 4.4. — If $g = 2$ and $r = 3$, the same techniques can be used to describe the singular locus of $\mathcal{M}_{\mathbf{SO}_3}$. Since the sum of two copies of the adjoint representation of \mathbf{SO}_3 is coregular, the trivial bundle $\mathcal{O}_X \oplus \mathcal{O}_X \oplus \mathcal{O}_X$ defines a smooth point in $\mathcal{M}_{\mathbf{SO}_3}$ (even if this particular point is often called “the worst point”). So the singular locus is exactly the set of orthogonal bundles of the form $\eta \oplus F$ where $\eta \not\simeq \mathcal{O}_X$ is a line bundle of order 2 and F a rank 2 orthogonal bundle with $\det(F) = \eta$.

If $r = 4$, the singular locus of $\mathcal{M}_{\mathbf{SO}_4}$ is exactly the set of orthogonal bundles which are orthogonal sums $F_1 \oplus F_2$ of two rank 2 orthogonal bundles with $\det F_1 \simeq \det F_2 \simeq \eta \not\simeq \mathcal{O}_X$.

We can of course prove in the same way the following result for moduli spaces $\mathcal{M}_{\mathbf{Sp}_{2r}}$ of semi-stable symplectic bundles on X of rank $2r \geq 4$:

Theorem 4.5. — *The smooth locus of $\mathcal{M}_{\mathbf{Sp}_{2r}}$ is precisely the open set consisting of regularly stable symplectic bundles.*

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OLIVIER SERMAN, Laboratoire Paul Painlevé, UMR 8524 du CNRS, Université des Sciences et Technologies de Lille, Cité scientifique, F-59 655 Villeneuve-d'Ascq Cedex, France
E-mail : Olivier.Serman@math.univ-lille1.fr