Essential dimension

Lecture I

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The theory of *essential dimension* was born in 1997 with the publication of "On the essential dimension of a finite group", by Joe Buhler and Zinovy Reichstein. It has since attracted a lot of attention.

The basic question is: how complicated is it to write down an algebraic or geometric object in a certain class? How many independent parameters do we need?

Let us start with the very general definition, due to Merkurjev.

Let k be a field, Fields_k the category of extensions of k. Let $F\colon \mathrm{Fields}_k \to \mathrm{Sets}$ be a functor. We should think of $F(K)$ as the set of isomorphism classes of algebraic or geometric objects we care about. If ξ is an object of some $F(K)$, a *field of definition* of ξ is an intermediate field $k \subseteq L \subseteq K$ such that ξ is in the image of $F(L) \to F(K)$. We will call a pair (L, η), where $\eta \in F(L)$ is such that its image in $F(K)$ is ξ, a *compression* of ξ.

Definition (Merkurjev)**.** The *essential dimension of* ξ, denoted by $\operatorname{ed}_k \xi$, is the least transcendence degree $\operatorname{tr\,deg}_k L$ of a field of definition L of ξ.

The *essential dimension of* F, denoted by $\operatorname{ed}_k F$, is the supremum of the essential dimensions of all objects ξ of all $F(K)$.

The essential dimension $\operatorname{ed}_k \xi$ is finite, under weak hypothesis on F. But $\operatorname{ed}_k F$ could still be $+\infty$.

Here are three classes of examples.

If X is a scheme locally of finite type over k and $F(K) = \mathrm{Hom}_k(\operatorname{Spec} K, X)$, then $\mathrm{ed}_k F = \dim X$.

Let n and d be positive integers. Let $\mathbb{F}_{n,d} \colon \mathrm{Fields}_k \to \mathrm{Sets}$ be the functor associating with K/k the set $\mathbb{F}_{n,d}$ of forms of degree d in $K[x_1, \ldots, x_n]$, up to change of coordinates. Of course, every quadratic form can be diagonalized, i.e., written in the form $\sum_{i=1}^{n} a_i x_i^2$; this implies that the corresponding object of $\mathbb{F}_{n,2}(K)$ is defined on an extension $k(a_1, \ldots, a_n)$ of transcendence degree at most n. So $\mathrm{ed}_k \mathbb{F}_{n,2} \leq n$. Can one do better?

If $g \geq 0$, denote by $\mathbb{M}_g \colon \mathrm{Fields}_k \to \mathrm{Sets}$ the functor of isomorphism classes of smooth geometrically connected projective curves of genus g.

Originally essential dimension was defined for finite groups. Let G be an algebraic group over k. If K is an extension of k, denote by $\mathrm{H}^1(K, G)$ the set of isomorphism classes of G-torsors over $\operatorname{Spec} K$. A *torsor* (or principal bundle, or principal homogeneous space) is a variety P over $\operatorname{Spec} K$, with an action $P \times_{\operatorname{Spec} k} G \to P$, such that by base-changing to some extension K' of K, the scheme $P_{K'}$ becomes isomorphic to $G_{K'}$ with the action by right translation. A torsor is trivial, that is, G-equivariantly isomorphic to G_K, if and only if $P(K) \neq \emptyset$.

The essential dimension of G is the essential dimension of the functor $\mathrm{H}^1(-, G)$ of isomorphism classes of G-torsors.

Assume that $\sigma\colon (k^n)^{\otimes r} \to (k^n)^{\otimes s}$ is a tensor on an n-dimensional k-vector space k^n, and G is the group of automorphisms of k^n preserving σ. Then G-torsors over K correspond to *twisted forms* of σ, that is, n-dimensional vector spaces V over K with a tensor $\tau\colon V^{\otimes r} \to V^{\otimes s}$ that becomes isomorphic to $(k^n, \sigma) \otimes_k \overline{K}$ over the algebraic closure $\overline{K}$ of K.

1. GL_n-torsors correspond to n-dimensional vector spaces, which are all defined over k. Hence $\mathrm{ed}_k \mathrm{GL}_n = 0$.
2. SL_n-torsors correspond to n-dimensional vector spaces with a volume form. Hence $\mathrm{ed}_k \mathrm{SL}_n = 0$.
3. Sp_n-torsors correspond to $2n$-dimensional vector spaces with a symplectic form. Hence $\mathrm{ed}_k \mathrm{Sp}_n = 0$.
4. O_n-torsors correspond to n-dimensional vector spaces with a non-degenerate quadratic form. Since every quadratic form can be diagonalized, we have $\mathrm{ed}_k \mathrm{O}_n \leq n$.

The algebraic groups with essential dimension 0 are called *special*. They are very special indeed.

For the rest of this lecture I will concentrate on essential dimension of affine algebraic groups. Suppose that $P \to \operatorname{Spec} K$ is a G-torsor, where G is an affine algebraic group, and $\operatorname{tr}\deg_k K < \infty$. There exists a free action of G on a variety X, such that X/G is integral, with generic point $\operatorname{Spec} K$, and the generic fiber of $X \to X/G$ is P. A compression of $P \to \operatorname{Spec} K$ corresponds to a free action of G on an integral variety Y such that there exists a dominant equivariant rational map $X \dashrightarrow Y$; the field of definition is $k(Y/G)$, and its transcendence degree is $\dim Y - \dim G$. Hence, the essential dimension of $P \to \operatorname{Spec} K$ is the least dimension of such a Y. This was the definition of essential dimension of G in the paper of Reichstein and Buhler.

In all the examples that arise in practice, $F\colon \operatorname{Fields}_k \to \operatorname{Sets}$ is naturally the restriction of a functor $F\colon \operatorname{Alg}_k \to \operatorname{Sets}$, where Alg_k is the category of k-algebras. Furthermore, F is *limit-preserving*, that is, if $\{A_i\}$ is an inductive system of k-algebras, the natural function $\varinjlim_i F(A_i) \to F(\varinjlim_i A_i)$ is bijective. This implies that $\operatorname{ed}_k \xi < \infty$ for any K/k and any $\xi \in F(K)$. From now on we will assume that F has a fixed limit-preserving extension to k-algebras.

In some cases there is a class of objects, called *versal*, or *generic*, which have the property of having the maximal possible essential dimension. A versal object is, essential, an object of some $F(K)$, from which all the other objects of any $F(L)$ can be obtained by specialization.

Let $\xi_K \in F(K)$, where K/k is finitely generated extension. We say that ξ_K is *versal* if every time we have

1. A is a finitely generated k-algebra with fraction field K,
2. an object ξ_A of $F(A)$ mapping to ξ_K in $F(K)$, and
3. an extension L of k, and an object $\eta \in F(L)$,

there exists a homomorphism of k-algebras $A \to L$ such that the induced function $F(A) \to F(L)$ carries ξ_A into η.

Clearly, if $\xi \in F(K)$ is versal, then $\operatorname{ed}_k F \leq \operatorname{tr}\deg_k K$. It is easy to check that a compression of a versal object is also versal. Hence, if ξ is versal we have $\operatorname{ed}_k F = \operatorname{ed}_k \xi$.

Not all natural functors admit a versal object; for example, it is easy to see that if F has a versal object, F can not have a positive-dimensional moduli space.

Suppose that G is an affine algebraic group over k. Choose a representation $G \to \mathrm{GL}(V)$ with a non-empty open subset $U \subseteq V$ such that the action of G on U is free. If G is finite, this is equivalent to the representation being faithful. Let K be the function field of U/G, and set $U_K \stackrel{\text{def}}{=} \operatorname{Spec} K \times_{U/G} U$. Then U_K is a G-torsor over K, and the corresponding object of $\mathrm{H}^1(K, G)$ gives a versal object for $\mathrm{H}^1(-, G)$. So, if $G \to \mathrm{GL}(V)$ is a generically free representation $G \to \mathrm{GL}(V)$, then $\operatorname{ed}_k G \leq \dim_k V - \dim G$. For example, $\operatorname{ed}_k \boldsymbol{\mu}_n^r \leq r$.

So, if G is an affine algebraic group, $\mathrm{ed}_k G$ can be interpreted as follows. Let $G \to \mathrm{GL}(V)$ be a generically free representation. Let d the minimum integer such that there exist a d-dimensional integral variety Y with a generically free action of G, and a dominant G-equivariant rational map $V \dashrightarrow Y$. Then $\mathrm{ed}_k G = d - \dim G$.

Notice that if $H \subseteq G$ and $G \to \mathrm{GL}(V)$ is a generically free representation, then its restriction to H is again generically free; hence $\mathrm{ed}_k H \leq \ \mathrm{ed}_k + \dim G - \dim H$. In particular, if H has finite index in G we have $\mathrm{ed}_k H \leq \ \mathrm{ed}_k G$.

How does one prove *lower bounds on* $\operatorname{ed}_k G$? Here are the main methods that have been used.

1. Fixed points methods, which I will not discuss.

2. Methods from birational geometry.

3. Cohomological invariants.

4. The p-local methods of Lecture 2.

Assume that $F\colon \text{Fields}_k \to \text{Sets}$ is a functor, and k' is an extension of k. We can defined $F_{k'}\colon \text{Fields}_{k'} \to \text{Sets}$ as the composite of F with the obvious functor $\text{Fields}_{k'} \to \text{Fields}_k$. It is a very easy exercise to show that $\operatorname{ed}_k F \geq \operatorname{ed}_{k'} F_{k'}$. For example, if G is an algebraic group over k, then $\operatorname{ed}_k G \geq \operatorname{ed}_{k'} G_{k'}$. So to prove lower bounds we may assume that k is algebraically closed.

Suppose that G is finite and $k = \mathbb{C}$. A compression of a representation is unirational; hence, if $\operatorname{ed}_{\mathbb{C}} G = d$, there exists a unirational variety of dimension d on which G acts generically freely. If $d = 1$ this means that G is a subgroup of PGL_2.

Theorem (Buhler and Reichstein)**.** *A non trivial finite group has essential dimension 1 if and only if it is either cyclic, or dihedral of order* $2n$*, with* n *odd.*

If $d = 2$ then G is a subgroup of the Cremona group of birational automorphisms of $\mathbb{P}^2$. Alex Duncan has used this to classify finite groups of essential dimension 2. Beauville used results of Prokhorov, who classified rationally connected threefolds with an action of a simple group, to show that the only simple groups of essential dimension 3 are A_6 and (possibly) $\mathrm{PSL}_2(\mathbb{F}_{11})$.

Cohomological invariants. Let $F\colon \mathrm{Fields}_k \to \mathrm{Sets}$ be a functor. A *cohomological invariant* of degree r for F is a natural transformation $\alpha\colon F \to \mathrm{H}^r(-,\Lambda)$, where Λ is a finite abelian group and $\mathrm{H}^r(K,\Lambda)$ is the r^{th} Galois cohomology group.

Remark (Serre)**.** Suppose $k = \overline{k}$. If F has a non-zero cohomological invariant of degree r, then $\operatorname{ed}_k F \geq r$.

This follows from the fact that $\mathrm{H}^r(K,\Lambda) = 0$ for $r > \operatorname{tr\,deg}_k K$.

For example, let us show that $\operatorname{ed}_k \boldsymbol{\mu}_n^r = r$, if $\operatorname{char} k$ does not divide n (this is not necessary) and $n > 1$. We have seen that $\operatorname{ed}_k \boldsymbol{\mu}_n^r \leq r$; to show that $\operatorname{ed}_k \boldsymbol{\mu}_n^r \geq r$ we may extend the base field to $\overline{k}$, and assume that k is algebraically closed. Then $\boldsymbol{\mu}_n$ is a finite cyclic group, which we identify with $\mathbb{Z}/n\mathbb{Z}$ by choosing a generator.

We have $\mathrm{H}^1(K, \mathbb{Z}/n\mathbb{Z}) = \mathrm{H}^1(K, \boldsymbol{\mu}_n) = K^*/K^{*n}$; we can define a cohomological invariant $\mathrm{H}^1(K, \boldsymbol{\mu}_n^r) = (K^*/K^{*n})^r \to \mathrm{H}^r(K, \mathbb{Z}/n\mathbb{Z})$ by sending $(\alpha_1, \ldots, \alpha_r)$ into the product $\alpha_1 \ldots \alpha_r$. One can check that this is not zero, for example when $K = k(x_1, \ldots, x_r)$ and setting $\alpha_i = [x_i]$. This shows that $\operatorname{ed}_k \boldsymbol{\mu}_n^r = r$.

One can also use cohomological invariants to show that $\operatorname{ed}_k \mathrm{O}_n = n$. Let q be a non-degenerate quadratic form in n variables, written in diagonal form $a_1 x_1^2 + \cdots + a_n x_n^2$. The symmetric functions of the classes $[a_i] \in K^*/K^{*2} = \mathrm{H}^1(K, \mathbb{F}_2)$ are invariants of q, the Stiefel–Whitney classes of q. The n^{th} symmetric functions is non-zero for the generic form; hence $\operatorname{ed}_k \mathrm{O}_n \geq n$.

The quantity $\operatorname{ed}_{\mathbb{C}} \mathrm{S}_n$ is an interesting number, linked with classical questions in the theory of equations. The idea of essential dimension came from Klein's work. It is easy to see that $\operatorname{ed}_{\mathbb{C}} \mathrm{S}_2 = \operatorname{ed}_{\mathbb{C}} \mathrm{S}_3 = 1$, $\operatorname{ed}_{\mathbb{C}} \mathrm{S}_4 = 2$. Since S_n contains a copy of $\boldsymbol{\mu}_2^{\lfloor n/2 \rfloor}$, we have $\operatorname{ed}_{\mathbb{C}} \mathrm{S}_n \geq \operatorname{ed}_{\mathbb{C}} \boldsymbol{\mu}_2^{\lfloor n/2 \rfloor} = \lfloor n/2 \rfloor$. On the other hand, consider the (versal) action of S_n on $\mathbb{A}^n$ by permuting coordinates; we have a rational S_n-equivariant map $\mathbb{A}^n \subseteq (\mathbb{P}^1)^n \dashrightarrow (\mathbb{P}^1)^n /\!/ \mathrm{PGL}_2$. If $n \geq 5$, then the action of S_n on $(\mathbb{P}^1)^n /\!/ \mathrm{PGL}_2$ is faithful. So

$$\lfloor n/2 \rfloor \leq \operatorname{ed}_{\mathbb{C}} \mathrm{S}_n \leq n - 3$$

for $n \geq 5$. Hence $\operatorname{ed}_{\mathbb{C}} S_5 = 2$ and $\operatorname{ed}_{\mathbb{C}} \mathrm{S}_6 = 3$. This is in the original paper of Buhler and Reichstein. A. Duncan showed that $\operatorname{ed}_{\mathbb{C}} \mathrm{S}_7 = 4$, using Prokhorov's results in birational geometry. Nothing else is known.

PGL_n is the automorphism group of the matrix algebra M_n. Therefore PGL_n-torsors over K correspond to twisted forms of the multiplication tensor $\mathrm{M}_n^{\otimes 2} \to \mathrm{M}_n$, that is, to K-algebras A that become isomorphic to M_n over $\overline{K}$. These are the *central simple algebras* of degree n.

Main open problem: what is $\mathrm{ed}_k \mathrm{PGL}_n$?

Assume that $n = p$ is a prime, and k contains all p^{th} roots of 1. Then a central simple algebra of degree p over an extension K of k is either a matrix algebra, or a non-commutative division algebra over K. A very important class of central simple algebras of degree p is that of the *cyclic* ones. A cyclic algebra of degree p over K has a presentation of the type $x^p = a$, $y^p = b$ and $yx = \omega xy$, where $a, b \in K^*$ and ω is a primitive p^{th} root of 1. A cyclic algebra is defined over $k(a, b)$, hence it has essential dimension at most 2.

One of the main open questions in the theory of division algebras is whether a division algebra of prime degree p is always cyclic. This is known for $p = 2$ (easy) and for $p = 3$ (a theorem of A.A. Albert). Hence the essential dimension of PGL_2 and PGL_3 is 2. It is widely conjectured that this is false for $p \geq 5$; a method for proving this would be to show that the essential dimension of PGL_p is larger than 2.

Theorem (Lorenz, Reichstein, Rowen and Saltman)**.** *If p is a prime and $p \geq 5$, then*

$$2 \leq \operatorname{ed}_k \mathrm{PGL}_p \leq \frac{(p-1)(p-2)}{2}.$$

This should give some idea of the extent of our ignorance.