Mori dream spaces and Fano varieties

Cinzia Casagrande

University of Torino

March 30, 2012

Expanded notes for a minicourse given at GAG – Géométrie Algébrique et Géométrie Complexe, CIRM, Marseille, 12-16 March 2012

These lectures have mainly two aims. The first is to explain the notion of Mori dream space, introduced by Hu and Keel, and to show the main geometrical properties, after [HK00]. In particular we describe rational contractions of a Mori dream space, and show the existence of Mori programs. The second aim is to explain the proof by Birkar, Cascini, Hacon, and McKernan [BCHM10] that Fano varieties are Mori dream spaces, using the characterization by Hu and Keel of Mori dream spaces in terms of finite generation of section algebras, and the finite generation of log-canonical algebras after [BCHM10].

I am grateful to Antonio Laface for useful conversations and explainations on Mori dream spaces and Cox rings, and to the organizers of GAG 2012 for the invitation to give this minicourse, and for the very nice conference.

Contents

1 Preliminaries

We work over the field of complex numbers $\mathbb{C}$.

1.1. Cones of divisors. Let Y be a normal projective variety. We denote by $\mathcal{N}^1(Y)$ the real vector space of Cartier divisors, with real coefficients, up to numerical equivalence. Its dimension is the **Picard number** ρ_Y. This vector space contains three important convex cones.

The **effective cone** $\mathrm{Eff}(Y)$ is the convex cone in $\mathcal{N}^1(Y)$ generated by classes of effective divisors. It is not closed in general, see [Deb01, Ex. 1.35].

The **nef cone** $\mathrm{Nef}(Y)$ is the cone of classes in $\mathcal{N}^1(Y)$ having non-negative intersection with all curves in Y. This cone is closed by definition, but in general it is not polyhedral nor rational.

Finally, we recall that a Cartier divisor D on Y is movable if its stable base locus $\mathrm{B}(D)$ has codimension ≥ 2, where

$$B(D) := \bigcap_{m \in \mathbb{Z}_{>0}} \mathrm{Bs}|mD|.$$

The **movable cone** $\mathrm{Mov}(Y)$ is the convex cone in $\mathcal{N}^1(Y)$ generated by the classes of movable divisors; again, it is not closed in general (see [Deb01, Ex. 1.35]). It is not difficult to see that if D is a Cartier divisor, then $[D] \in \mathrm{Mov}(Y)$[1] if and only if D is movable.

There are inclusions:

$$\mathrm{Nef}(Y) \subseteq \overline{\mathrm{Mov}(Y)} \subseteq \overline{\mathrm{Eff}(Y)}.$$

We will occasionally also consider the real vector space $\mathcal{N}_1(Y)$ of one-cycles in Y, with real coefficients, up to numerical equivalence. This is dual to $\mathcal{N}^1(Y)$ via the intersection pairing, and has dimension ρ_Y. Moreover $\mathrm{NE}(Y)$ is the convex cone, in $\mathcal{N}_1(Y)$, generated by classes of effective curves. Its closure is dual to the nef cone.

1.2. Pull-back of divisors under a dominant rational map. Let X be a normal and $\mathbb{Q}$-factorial projective variety and $f\colon X \dashrightarrow Y$ a dominant rational map, where Y is normal and projective. Let D be a Cartier divisor on Y. We denote by $f^*(D)$ the unique Weil divisor on X which is the pull-back of D on the open subset of X where f is regular. Notice that $f^*(D)$ is $\mathbb{Q}$-Cartier.

The behaviour of the pull-back under rational maps can be quite different than in the regular case. For instance, let $\sigma\colon \mathbb{F}_1 \to \mathbb{P}^2$ be the blow-up of a point, and $E \subset \mathbb{F}_1$ be the (-1)-curve. Then $(\sigma^{-1})^*(E) = 0$, and $\sigma^*((\sigma^{-1})^*(E)) = 0$, so that the pull-back is not functorial.

Lemma 1.3. *If $D \equiv 0$, then $f^*(D) \equiv 0$.*[2]

As a consequence, f yields a linear map $f^*\colon \mathcal{N}^1(Y) \to \mathcal{N}^1(X)$.

[1] [ ] denotes the numerical equivalence class.
[2] $\equiv$ denotes numerical equivalence.

Proof. Up to replacing D with a multiple, we can assume that $f^*(D)$ is Cartier. Consider a resolution of f:

$$\begin{tikzcd} X' \arrow[d, "\mu"'] \arrow[dr, "f'"] & \\ X \arrow[r, dashed, "f"] & Y \end{tikzcd}$$

where X' is smooth and projective, and set $E := \mu^*(f^*(D)) - (f')^*(D)$. Then E is μ-exceptional, and $E \equiv \mu^*(f^*(D))$ because $(f')^*(D) \equiv 0$. By negativity of contractions [KM98, Lemma 3.39], both E and $-E$ are effective, so that $E = 0$ and $\mu^*(f^*(D)) \equiv 0$. Using the projection formula, this yields $f^*(D) \equiv 0$. ■

2 Contracting rational maps

Let X be a normal and $\mathbb{Q}$-factorial projective variety.

A **contraction** of X is a surjective morphism with connected fibers $f\colon X \to Y$, where Y is normal and projective. We refer to [Laz04a, §2.1.A and 2.1.B] for basic properties of contractions and their relation with semiample divisors on X. The contraction f induces a push-forward of one-cycles $f_*\colon \mathcal{N}_1(X) \to \mathcal{N}_1(Y)$; we set $\mathrm{NE}(f) := \ker f_* \cap \mathrm{NE}(X)$.

Contracting rational maps are the analog of contractions among rational maps.

Definition 2.1 ([HK00], Def. 1.1)**.** Let $f\colon X \dashrightarrow Y$ be a dominant rational map, where Y is normal and projective. We say that f is **contracting**, or a **rational contraction**, if there exists a resolution of f

$$\begin{tikzcd} X' \arrow[d, "\mu"'] \arrow[dr, "f'"] & \\ X \arrow[r, dashed, "f"] & Y \end{tikzcd} \tag{$\star$}$$

where X' is smooth and projective, μ is birational, and for every μ-exceptional effective divisor E on X' we have

$$f'_*(\mathcal{O}_{X'}(E)) = \mathcal{O}_Y.$$

Some remarks on Definition 2.1:

- when f is regular, since $\mu_*(\mathcal{O}_{X'}(E)) = \mathcal{O}_X$, we recover the condition $f_*(\mathcal{O}_X) = \mathcal{O}_Y$, namely that f is a contraction.
- It is not difficult to see that the definition does not depend on the choice of the resolution $\mu\colon X' \to X$ of f.

- By taking $E = 0$, we see that f' is a contraction; in particular either f is birational, or $\dim Y < \dim X$. In this last case we say that f is **of fiber type**.

Remark 2.2. Let $f\colon X \dashrightarrow Y$ be a birational map, with Y normal and projective. Then f is contracting if and only if there are open subsets $U \subseteq X$ and $V \subseteq Y$ such that f is an isomorphism between U and V, and $\operatorname{codim}(Y \smallsetminus V) \geq 2$.

Proof. Fix a resolution of f as in $(\star)$, and let $E \subset X'$ be a prime μ-exceptional divisor. If E is f'-exceptional, then $f'_*(\mathcal{O}_{X'}(E)) = \mathcal{O}_Y$. If instead $f'(E) \subset Y$ is a divisor, set $A := Y \smallsetminus \operatorname{Sing}(Y) \smallsetminus f'(\operatorname{Exc}(f'))$. Then $\operatorname{codim}(Y \smallsetminus A) \geq 2$, hence $f'(E)_{|A}$ is a non-trivial prime Cartier divisor, and

$$f'_*(\mathcal{O}_{X'}(E))_{|A} = \mathcal{O}_A(f'(E)_{|A})) \neq \mathcal{O}_A,$$

therefore $f'_*(\mathcal{O}_{X'}(E)) \neq \mathcal{O}_Y$.

This shows that f is contracting if and only if every μ-exceptional divisor on X' is also f'-exceptional, which yields the statement. ■

Example 2.3. The inverse of a blow-up is not a contracting rational map.

The main property of contracting rational maps is that they preserve sections (in fact the definition is given precisely to have this property):

Proposition 2.4. *Let $f\colon X \dashrightarrow Y$ be a contracting rational map, and D a Cartier divisor on Y such that $f^*(D)$ is Cartier. Then $H^0(X, \mathcal{O}_X(f^*(D))) \cong H^0(Y, \mathcal{O}_Y(D))$.*

Proof. Consider a resolution of f as in $(\star)$. Then $\mu^*(f^*(D)) - (f')^*(D)$ is μ-exceptional, so there exist effective μ-exceptional divisors E_1, E_2 such that

$$\mu^*(f^*(D)) + E_1 = (f')^*(D) + E_2.$$

By the projection formula and the definition of rational contraction we have

$$\mu_* \left(\mathcal{O}_{X'}(\mu^*(f^*(D)) + E_1)\right) = \mathcal{O}_X(f^*(D)) \text{ and } f'_*(\mathcal{O}_{X'}((f')^*(D) + E_2) = \mathcal{O}_Y(D);$$

this gives the statement. ■

Corollary 2.5 ([HK00], Lemma 1.6)**.** *Let $f\colon X \dashrightarrow Y$ be a contracting rational map, and A an ample Cartier divisor on Y such that $f^*(A)$ is Cartier. Then the following holds:*

- *f is the map associated to the linear system $|mf^*(A)|$ for all $m \gg 0$;*
- *f is regular if and only if $f^*(A)$ is semiample.*

Example 2.6. Consider $\sigma^{-1}\colon \mathbb{P}^2 \dashrightarrow \mathbb{F}_1$. Both statements above are false for this (non-contracting) rational map.

Definition 2.7 ([HK00], Def. 1.8)**.** A **small $\mathbb{Q}$-factorial modification (SQM)** of X is a birational map $g\colon X \dashrightarrow \widetilde{X}$, where $\widetilde{X}$ is again normal, projective, and $\mathbb{Q}$-factorial, and g is an isomorphism in codimension 1.

Some remarks on Definition 2.7:

- both g and g^{-1} are contracting, by Remark 2.2;
- the basic example of a SQM is a flip;
- it is easy to check that g induces an isomorphism

$$g^*\colon \mathcal{N}^1(\widetilde{X}) \longrightarrow \mathcal{N}^1(X)$$

  (in particular X and $\widetilde{X}$ have the same Picard number), and that g^* preserves the effective and the movable cones:

$$g^*\left(\mathrm{Eff}(\widetilde{X})\right) = \mathrm{Eff}(X), \qquad g^*\left(\mathrm{Mov}(\widetilde{X})\right) = \mathrm{Mov}(X).$$

Remark 2.8. Let $g\colon X \dashrightarrow \widetilde{X}$ be a SQM, and $f\colon \widetilde{X} \dashrightarrow Y$ a rational contraction. Then $h := f \circ g\colon X \dashrightarrow Y$ is a rational contraction, and for every Cartier divisor D on Y such that $f^*(D)$ is Cartier, we have $h^*(D) = g^*(f^*(D))$.

3 Mori dream spaces

Definition 3.1 ([HK00], Def. 1.10)**.** Let X be a normal and $\mathbb{Q}$-factorial projective variety. We say X is a **Mori dream space (MDS)** if the following properties hold:

(1) $\mathrm{Pic}(X)$ is finitely generated (equivalently, $h^1(\mathcal{O}_X) = 0$);

(2) $\mathrm{Nef}(X)$ is generated by the classes of finitely many semiample divisors;

(3) there is a finite collection of SQMs $g_i\colon X \dashrightarrow X_i$, for $= 1, \ldots, r$, such that every X_i satisfies (2), and

$$\mathrm{Mov}(X) = \bigcup_{i=1}^{r} g_i^*\left(\mathrm{Nef}(X_i)\right).$$

Some remarks on Definition 3.1:

- (1) means that if D_1, D_2 are Cartier divisors on X such that $D_1 \equiv D_2$, then there exists $m \in \mathbb{Z}_{>0}$ such that $mD_1 \sim mD_2$.[3]

- Let X be a MDS. By (1) and (2), $\mathrm{Nef}(X)$ is a rational polyhedral cone, and every nef divisor in X is semiample. Moreover there is a bijection between the set of contractions[4] of X and the set of faces of $\mathrm{Nef}(X)$, given by the association:

$$(f\colon X \to Y) \qquad \longmapsto \qquad f^*(\mathrm{Nef}(Y)).$$

  Somehow this is the best possible behaviour with respect to contractions, but for a MDS we ask more: we want that every small elementary contraction of X has a flip, and after the flip, we want to keep the good properties of the nef cone. This is formalized in property (3).

- If X is a normal and $\mathbb{Q}$-factorial projective variety with $\rho_X = 1$, then X is a MDS if and only if $\mathrm{Pic}(X)$ is finitely generated. In particular, the only one-dimensional MDS is $\mathbb{P}^1$.

- If X is a normal and $\mathbb{Q}$-factorial projective surface satisfying (1) and (2), then X is a MDS. Indeed one has $\mathrm{Nef}(X) = \mathrm{Mov}(X)$, hence (3) is satisfied by taking just Id_X.

- If X is a MDS, then X_i is a MDS as well for every $i = 1, \dots, r$.

We will see in section 5 that every projective $\mathbb{Q}$-factorial toric variety is a MDS, and every smooth Fano variety as well.

Example 3.2. If X is a smooth projective rational surface with $-K_X$ big, then X is a MDS, see [TVAV11]. On the other hand, if we look at smooth projective K3 surfaces, some are MDS, while others are not. For a K3 surface X, being a MDS is equivalent to having finite automorphism group, see [AHL10, AL11].

We are now going to study rational contractions of a MDS.

Proposition 3.3. *Let X be a MDS and $f\colon X \dashrightarrow Y$ a rational contraction. Then there exists some $i \in \{1, \dots, r\}$ such that f factors as:*

$$\begin{tikzcd} X \arrow[r, dashed, "g_i"] \arrow[dr, dashed, "f"'] & X_i \arrow[d, "h"] \\ & Y \end{tikzcd}$$

where h is a regular contraction.

[3] $\sim$ denotes linear equivalence.

[4] We consider (regular or rational) contractions of X up to isomorphism of the target.

Proof. Let A be an ample Cartier divisor on Y such that $f^*(A)$ is Cartier. Then $f^*(A)$ is movable, so there exists some $i \in \{1, \dots, r\}$ such that $[f^*(A)] \in g_i^*(\mathrm{Nef}(X_i))$. Up to replacing A with a multiple, there exists a nef divisor M_i on X_i such that $f^*(A) \equiv g_i^*(M_i)$.

Set $h := f \circ (g_i)^{-1} \colon X_i \dashrightarrow Y$. By Remark 2.8, h is contracting, and $h^*(A) = (g_i^*)^{-1}(f^*(A)) \equiv M_i$ is nef and hence semiample. By Corollary 2.5 we conclude that h is regular. ■

Corollary 3.4. *Let X be a MDS and $f \colon X \dashrightarrow Y$ a rational contraction. Then $f^* \colon \mathcal{N}^1(Y) \to \mathcal{N}^1(X)$ is injective.*

Corollary 3.5. *Let X be a MDS. Then $g_1, \dots, g_r$ are all SQMs of X, in particular Id_X must appear among them. Up to renumbering, we will assume that $X_1 = X$ and $g_1 = Id_X$.*

Proposition 3.6 ([HK00], Prop. 1.11(3))**.** *Let X be a MDS, and let $\mathcal{M}_X$ be the set of all faces of $g_i^*(\mathrm{Nef}(X_i))$ for $i = 1, \dots, r$. Then $\mathcal{M}_X$ is a fan in $\mathcal{N}^1(X)$, with support $\mathrm{Mov}(X)$. Moreover there is a bijection between the set of rational contractions of X and $\mathcal{M}_X$, given by the association:*

$$(f \colon X \dashrightarrow Y) \qquad \longmapsto \qquad f^*(\mathrm{Nef}(Y)).$$

In particular, X has finitely many rational contractions.

Proof. In order to show that $\mathcal{M}_X$ is a fan we have to show that $g_i^*(\mathrm{Nef}(X_i))$ and $g_j^*(\mathrm{Nef}(X_j))$ intersect along a common face for every $i, j = 1, \dots, r$. Let us take for simplicity $j = 1$, so that $g_1^*(\mathrm{Nef}(X_1)) = \mathrm{Nef}(X)$, and let $\xi \in \mathrm{Nef}(X) \cap g_i^*(\mathrm{Nef}(X_i))$. Up to replacing ξ with a multiple, there are semiample Cartier divisors M on X and M_i on X_i such that $\xi = [M] = [g_i^*(M_i)]$. By Proposition 3.3, we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "g_i"] \arrow[dr, "f"'] & X_i \arrow[d, "h"] \\ & Y \end{tikzcd}$$

where f is the contraction induced by $|mM|$ for $m \gg 0$, and h is the contraction induced by $|sM_i|$ for $s \gg 0$. Therefore $f^*(\mathrm{Nef}(Y))$ is a common face of $\mathrm{Nef}(X)$ and $g_i^*(\mathrm{Nef}(X_i))$, containing ξ in its relative interior. This shows that the two cones intersect along a common face. The rest of the statement follows from Proposition 3.3. ■

Example 3.7. Let X be the blow-up of $\mathbb{P}^3$ in two points p, q. Then X is toric and hence a MDS (see Corollary 5.3), with $\rho_X = 3$.

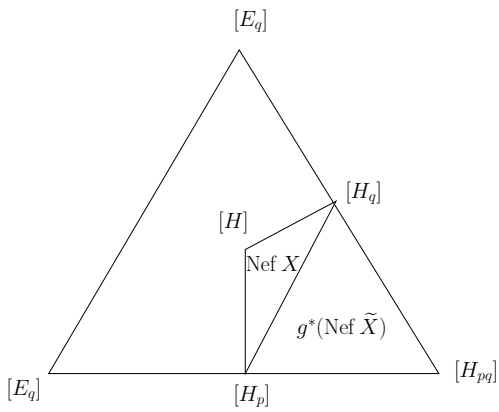
Let $l \subset X$ be the transform of the line through p and q. Then l has normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$, and is the exceptional locus of a small elementary contraction $f\colon X \to Y$. The flip of f can be described as follows.

Let $\widehat{X} \to X$ be the blow-up of l; the exceptional divisor is $\widehat{E} \cong \mathbb{P}^1 \times \mathbb{P}^1$, with normal bundle $\mathcal{O}(-1,-1)$. There exists another smooth blow-up $\widehat{X} \to \widetilde{X}$, with the same exceptional divisor $\widehat{E}$, that contracts $\widehat{E}$ in the other direction.

$$\begin{tikzcd}
 & \widehat{X} \arrow[dl] \arrow[dr] & \\
X \arrow[rr, dashed, "g"] \arrow[dr, "f"'] & & \widetilde{X} \arrow[dl] \\
 & Y &
\end{tikzcd}$$

The composition $g\colon X \dashrightarrow \widetilde{X}$ is the flip of f, and the variety $\widetilde{X}$ is smooth and contains another smooth rational curve $\widetilde{l}$ with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ (the image of $\widehat{E}$). Moreover there is a smooth morphism $\widetilde{X} \to \mathbb{P}^1$, with fiber $\mathbb{P}^1 \times \mathbb{P}^1$, such that $\widetilde{l}$ is a section (this morphism is birational to the projection $\mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ from the line through p and q). Finally, the morphism $\widetilde{X} \to \mathbb{P}^1$ factors in two different ways through a $\mathbb{P}^1$-bundle $\widetilde{X} \to \mathbb{F}_1$.

One can check that $\mathrm{Mov}(X) = \mathrm{Nef}(X) \cup g^*(\mathrm{Nef}(\widetilde{X}))$. Therefore X has exactly two SQMs: the identity and the flip g. Here is a section of the 3-dimensional cone $\mathrm{Eff}(X)$:



E_p and E_q are the exceptional divisors over p and q, and H (respectively H_p, H_q,

H_{pq}) is the transform of a general plane of $\mathbb{P}^3$ (respectively a general plane containing p, containing q, containing p and q).

The fan $\mathcal{M}_X$ contains 11 non-trivial cones. The two 3-dimensional ones are $\mathrm{Nef}(X)$ and $g^*(\mathrm{Nef}(\widetilde{X}))$, and correspond to the SQMs of X.

The five 2-dimensional cones in $\mathcal{M}_X$ correspond to rational contractions of X with a target of Picard number 2, namely: $X \to \mathrm{Bl}_p\mathbb{P}^3$, $X \to \mathrm{Bl}_q\mathbb{P}^3$, the small contraction of l, and $X \dashrightarrow \mathbb{F}_1$ twice (given by the compositions $X \dashrightarrow \widetilde{X} \to \mathbb{F}_1$).

Finally, the four 1-dimensional cones in $\mathcal{M}_X$ correspond to rational contractions of X with a target of Picard number 1, namely: $X \to \mathbb{P}^3$, $X \to \mathbb{P}^2$ twice[5], and $X \dashrightarrow \mathbb{P}^1$ (given by the composition $X \dashrightarrow \widetilde{X} \to \mathbb{P}^1$).

Let us give explicitly some properties of the bijection between $\mathcal{M}_X$ and rational contractions of X.

Remark 3.8. Let X be a MDS, $f\colon X \dashrightarrow Y$ a rational contraction, and $\sigma := f^*(\mathrm{Nef}(Y)) \in \mathcal{M}_X$. We have:

(*i*) $\dim \sigma = \rho_Y$;

(*ii*) $f \circ (g_i)^{-1}\colon X_i \dashrightarrow Y$ is regular if and only if $\sigma \subseteq g_i^*(\mathrm{Nef}(X_i))$;

(*iii*) f is birational if and only if $\sigma \not\subset \partial\,\mathrm{Eff}(X)$;

(*iv*) f is birational and $\operatorname{codim} \overline{\mathrm{Exc}(f)} \geq 2$ if and only if $\sigma \not\subset \partial\,\mathrm{Mov}(X)$.

Proof. (*i*) follows from Corollary 3.4, and (*ii*) follows from (the proof of) Proposition 3.3.

Up to composing with a SQM, we can assume that f is regular. Let A be an ample Cartier divisor on Y, and $L := f^*(A)$, so that $[L]$ is in the relative interior of σ.

Then f is birational if and only if L is big, and this is equivalent to $[L] \notin \partial\,\mathrm{Eff}(X)$, so we have (*iii*).

For (*iv*), we can assume that f is birational, so that L is semiample and big. We use the following.

Theorem 3.9 (Nakamaye, see [Laz04b], §10.3 and references therein)**.** *For every ample Cartier divisor H on X, and for every $\varepsilon \in \mathbb{Q}_{>0}$, $\varepsilon \ll 1$, the stable base locus of $L - \varepsilon H$ is* $\mathrm{Exc}(f)$.

If $\operatorname{codim} \mathrm{Exc}(f) \geq 2$, then $[L - \varepsilon H] \in \mathrm{Mov}(X)$. On the other hand $[H]$ is in the interior of the movable cone, hence $[L] = [L - \varepsilon H] + \varepsilon[H]$ is in the interior as well.

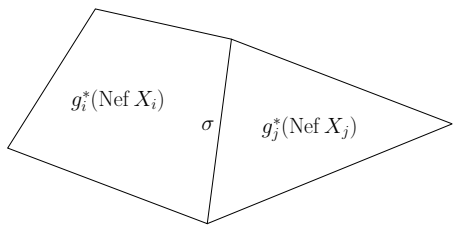
Conversely, if $\operatorname{codim} \mathrm{Exc}(f) = 1$, then every neighborhood of $[L]$ in $\mathcal{N}^1(X)$ contains points outside $\mathrm{Mov}(X)$, so that $[L] \in \partial\,\mathrm{Mov}(X)$. ■

[5] Recall that the blow-up of $\mathbb{P}^3$ in a point is a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$.

Definition 3.10. Let X be a MDS and $f\colon X \dashrightarrow Y$ a rational contraction. We say that f is **elementary** if $\rho_X - \rho_Y = 1$, equivalently if $\dim \sigma = \rho_X - 1$, where $\sigma = f^*(\mathrm{Nef}(Y))$. As in the regular case, there are three possibilities:

- f is of fiber type, and $\sigma \subset \partial\,\mathrm{Eff}(X)$;
- f is birational divisorial, and $\sigma \not\subset \partial\,\mathrm{Eff}(X)$, $\sigma \subset \partial\,\mathrm{Mov}(X)$;
- f is birational small, and $\sigma \not\subset \partial\,\mathrm{Mov}(X)$.

3.11. Existence of flips [HK00, Prop. 1.11(2)]. Let X be a MDS and $f\colon X_i \to Y$ a small elementary contraction. Then $\sigma := g_i^*(f^*(\mathrm{Nef}(Y)))$ is a $(\rho_X - 1)$-dimensional cone in $\mathcal{M}_X$, and by Remark 3.8(iv) σ is not contained in the boundary of $\mathrm{Mov}(X)$. Therefore there exists a unique $j \neq i$ such that $\sigma = g_i^*(\mathrm{Nef}(X_i)) \cap g_j^*(\mathrm{Nef}(X_j))$.



Let us consider $\varphi := g_j \circ g_i^{-1}$; we have a diagram:

$$\begin{tikzcd} X_i \arrow[rr, dashed, "\varphi"] \arrow[dr, "f"'] & & X_j \arrow[dl, "\widetilde{f}"] \\ & Y & \end{tikzcd}$$

where $\widetilde{f}$ is a small elementary contraction. This is the flip of f (see [KM98, Def. 6.5]). Indeed in $\mathcal{N}^1(X_i)$ the hyperplane $f^*(\mathcal{N}^1(Y))$ coincides with $\mathrm{NE}(f)^\perp$, and it cuts $\mathcal{N}^1(X_i)$ in two half-spaces, of classes having positive / negative intersection with $\mathrm{NE}(f)$. The positive half-space is the one containing the ample cone: therefore the two half-spaces are switched in X_i and X_j.

This shows that whenever $g_i^*(\mathrm{Nef}(X_i))$ and $g_j^*(\mathrm{Nef}(X_j))$ have a common facet,[6] the map $g_j \circ g_i^{-1}$ is a flip.

Corollary 3.12. *Let X be a MDS. Then every SQM of X factors as a finite sequence of flips, and every rational contraction of X factors as a finite sequence of flips, followed by a regular contraction.*

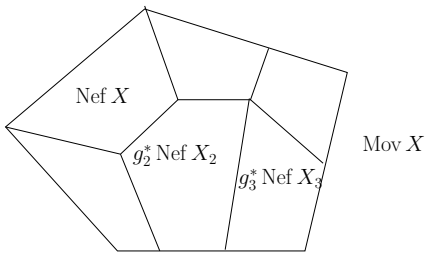
[6] A facet is a codimension 1 face.

Proof. Let $g_i\colon X \dashrightarrow X_i$ be a SQM. Let us choose a sequence of indices $j_1, \dots, j_s$ in $\{1, \dots, r\}$ such that $j_1 = 1$,[7] $j_s = i$, and

$$\dim \Big(g_{j_h}^*(\mathrm{Nef}(X_{j_h})) \cap g_{j_{h+1}}^*(\mathrm{Nef}(X_{j_{h+1}})) \Big) = \rho_X - 1 \quad \text{for every } h = 1, \dots, s-1.$$

Then $g_{j_{h+1}} \circ g_{j_h}^{-1}$ is a flip for every $h = 1, \dots, s-1$, and $g_i = (g_i \circ g_{j_{s-1}}^{-1}) \circ \cdots \circ (g_{j_3} \circ g_{j_2}^{-1}) \circ g_{j_2}$ is a factorization of g_i as a sequence of flips.

The rest of the statement follows from Proposition 3.3. ■



Both g_2 and $g_3 \circ g_2^{-1}$ are flips, and g_3 factors as $g_3 = (g_3 \circ g_2^{-1}) \circ g_2$.

4 Mori programs

In this section we show the existence of Mori programs in a MDS. Since we will need to make induction on the Picard number, the first step is the following.

Proposition 4.1. *Let X be a MDS and $f\colon X \dashrightarrow Y$ an elementary divisorial rational contraction. Then Y is a MDS.*

Proof. Up to composing with a SQM of X, we can assume that f is regular; let $E \subset X$ be the exceptional divisor. Notice that $\mathrm{NE}(f)$ is a one-dimensional face of $\mathrm{NE}(X)$, and $f^*(\mathcal{N}^1(Y)) = \mathrm{NE}(f)^\perp$ in $\mathcal{N}^1(X)$.

We show that Y is $\mathbb{Q}$-factorial. Let $D_0 \subset Y$ be a prime Weil divisor, and $\widetilde{D}_0 \subset X$ its transform. Since $E \cdot \mathrm{NE}(f) \neq 0$,[8] there exists $\lambda \in \mathbb{Q}$ such that $(\widetilde{D}_0 + \lambda E) \cdot \mathrm{NE}(f) = 0$, so that $[\widetilde{D}_0 + \lambda E] \in f^*(\mathcal{N}^1(Y))$. Therefore there exist $m \in \mathbb{Z}_{>0}$ and B a Cartier divisor on Y such that $m\lambda \in \mathbb{Z}$ and $m(\widetilde{D}_0 + \lambda E) = f^*(B)$. This implies that $\mathrm{Supp}\, B = D_0$, so that D_0 is $\mathbb{Q}$-Cartier.

[7] Recall that we have fixed $X_1 = X$ and $g_1 = \mathrm{Id}_X$.

[8] If $R \subset \mathcal{N}_1(X)$ is a half-line, and D is a divisor, we say that $D \cdot R > 0$, or $= 0$, or $\neq 0$, if the same is true for $D \cdot \gamma$, $\gamma \in R$ a non-zero element.

Since $f^*\colon \operatorname{Pic}(Y) \to \operatorname{Pic}(X)$ is injective, the Picard group of Y is finitely generated. Moreover $f^*(\operatorname{Nef}(Y))$ is a face of $\operatorname{Nef}(X)$, and it is easy to see that $\operatorname{Nef}(Y)$ is generated by the classes of finitely many semiample divisors.

We are left to give the decomposition of $\operatorname{Mov}(Y)$ as a union of the nef cones of some SQMs of Y.

Let $g_i\colon X \dashrightarrow X_i$, for $i = 1, \dots, r$, be the SQMs of X (with $g_1 = \operatorname{Id}_X$). Let us consider all elementary divisorial contractions of the varieties $X = X_1, X_2, \dots, X_r$ with exceptional divisor the transform of E:

$$f_j\colon X_{i_j} \to Y_j, \qquad \text{for } j = 1, \dots, m$$

(where $Y_1 = Y$ and $f_1 = f$).

Fix $j \in \{1, \dots, m\}$. As for Y, we see that Y_j is $\mathbb{Q}$-factorial, and that $\operatorname{Nef}(Y_j)$ is generated by the classes of finitely many semiample divisors. Moreover we have:

$$\begin{tikzcd} X \arrow[r, dashed, "g_{i_j}"] \arrow[d, "f"'] & X_{i_j} \arrow[d, "f_j"] \\ Y \arrow[r, dashed, "\varphi_j"] & Y_j \end{tikzcd}$$

where $\varphi_j := f_j \circ g_{i_j} \circ f^{-1}$ is birational, and an isomorphism in codimension 1, so that it is a SQM of Y. We show that

$$\operatorname{Mov}(Y) = \bigcup_{j=1}^{m} \varphi_j^*(\operatorname{Nef}(Y_j)),$$

which gives the statement.

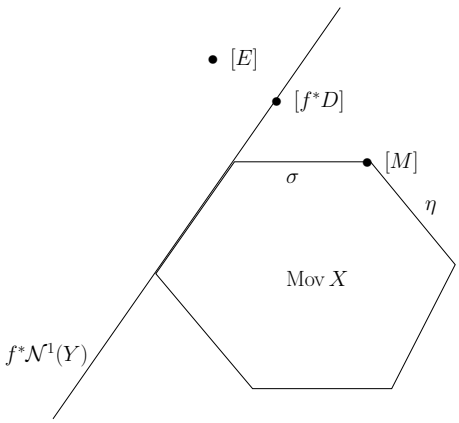
Let D be an effective, Cartier, movable divisor on Y. The divisor $f^*(D)$ does not need to be movable on X, because it could contain E in its stable base locus. However there exists $m \in \mathbb{Z}_{\geq 0}$ such that $M := f^*(D) - mE$ is movable (so that $m = 0$ if and only if $f^*(D)$ is movable). Notice that $[E] \notin \operatorname{Mov}(X)$ and $[M] \in \partial \operatorname{Mov}(X)$.

Claim. *There exists a codimension one face σ of $\operatorname{Mov}(X)$, containing $[M]$, such that if $H \subset \mathcal{N}^1(X)$ is the hyperplane spanned by σ, then $[E]$ belongs to the half-space opposite to $\operatorname{Mov}(X)$ with respect to H. (Notice that when $f^*(D)$ is movable the natural choice is $\sigma := f^*(\mathcal{N}^1(Y)) \cap \operatorname{Mov}(X)$, but in general we must choose σ different from $f^*(\mathcal{N}^1(Y)) \cap \operatorname{Mov}(X)$.)*

Proof of the claim. There are rational linear forms $\psi_1, \dots, \psi_h$ on $\mathcal{N}^1(X)$ such that $\operatorname{Mov}(X) = \{x \in \mathcal{N}^1(X) \mid \psi_i(x) \geq 0 \text{ for every } i = 1, \dots, h\}$.

Let Π be the two-dimensional linear subspace of $\mathcal{N}^1(X)$ containing $[E]$, $[f^*(D)]$ and $[M]$. Then $\operatorname{Mov}(X) \cap \Pi$ is defined by $\{(\psi_i)_{|\Pi}(x) \geq 0 \text{ for every } i = 1, \dots, h\}$.

Since $[M]$ is on the boundary of $\mathrm{Mov}(X) \cap \Pi$ in Π, and $[E] \notin \mathrm{Mov}(X) \cap \Pi$, there exists some i such that $\psi_i([M]) = 0$ and $\psi_i([E]) < 0$. Then we set $\sigma := \mathrm{Mov}(X) \cap \ker(\psi_i)$. ■



The facet σ satisfies the statement of the claim, while the facet η does not.

We carry on with the proof of Proposition 4.1. Since the fan $\mathcal{M}_X$ is supported on $\mathrm{Mov}(X)$, every face of $\mathrm{Mov}(X)$ is a union of cones of $\mathcal{M}_X$. In particular we can choose $\tau \in \mathcal{M}_X$ such that $[M] \in \tau$, $\tau \subseteq \sigma$, and $\dim \tau = \rho_X - 1$.

Then τ corresponds to an elementary rational contraction of X, which is regular after a SQM:

$$X \dashrightarrow \widetilde{X} \xrightarrow{h} Z.$$

Let $\widetilde{E} \subset \widetilde{X}$ be the transform of E. By our choice of τ, $\widetilde{E} \cdot \mathrm{NE}(h) < 0$, so that h is birational and $\mathrm{Exc}(h) \subseteq \widetilde{E}$. On the other hand $\tau \subset \partial\,\mathrm{Mov}(X)$, therefore h is divisorial by Remark 3.8(iv), and $\mathrm{Exc}(h) = \widetilde{E}$. Therefore there exists $j \in \{1, \dots, m\}$ such that $h\colon \widetilde{X} \to Z$ coincides with $f_j\colon X_{i_j} \to Y_j$.

Moreover $[M] \in \tau = g_{i_j}^*(f_j^*(\mathrm{Nef}(Y_j)))$, so there exists a nef $\mathbb{Q}$-divisor M_j on Y_j such that $M \equiv g_{i_j}^*(f_j^*(M_j))$. It is then easy to see that $D \equiv \varphi_j^*(M_j)$, therefore $[D] \in \varphi_j^*(\mathrm{Nef}(Y_j))$. ■

Let us now recall what a Mori program is. Let X be a normal and $\mathbb{Q}$-factorial projective variety, and D a divisor in X.

A contraction $f\colon X \to Y$ is **D-negative** if $D \cdot C < 0$ for every curve $C \subset X$ such that $f(C) = \{pt\}$.

A **D-negative flip** is the flip of a D-negative elementary small contraction.

A **Mori program** for D is a sequence

$$X = X_0 \overset{\varphi_0}{\dashrightarrow} X_1 \dashrightarrow \cdots \dashrightarrow X_{m-1} \overset{\varphi_{m-1}}{\dashrightarrow} X_m$$

where:

(1) each X_i is a normal and $\mathbb{Q}$-factorial projective variety, and each φ_i is birational; we set $D_{i+1} := (\varphi_i^{-1})^*(D_i)$ (and $D_0 := D$);

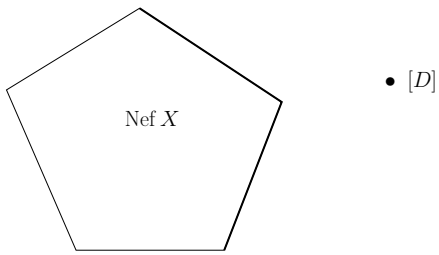
(2) φ_i is either a D_i-negative elementary divisorial contraction, or a D_i-negative flip, for every $i = 0, \ldots, m-1$;[9]

(3) either D_m is nef, or X_m has a D_m-negative elementary contraction of fiber type.

Theorem 4.2 ([HK00], Prop. 1.11(1))**.** *Let X be a MDS and D a divisor in X. Then there exists a Mori program for D. Moreover, the choice of the D_i-negative elementary contraction of X_i is arbitrary at each step, and if D_m is nef, then it is semiample.*

Proof. We proceed by induction on ρ_X. If $\rho_X = 1$, then either D is ample, or $-D$ is ample and $X \to \{pt\}$ is a D-negative elementary contraction of fiber type, or $D \equiv 0$. In this last case there exists $m \in \mathbb{Z}_{>0}$ such that mD is principal, hence D is semiample.

Let us consider the general case. If D is nef, then it is semiample. If D is not nef, let σ be a codimension 1 face of $\mathrm{Nef}(X)$ such that, if H is the hyperplane spanned by σ, then $[D]$ belongs to the half-space opposite to $\mathrm{Nef}(X)$ with respect to H. Then the elementary contraction $f\colon X \to Y$ corresponding to σ is D-negative.



Thick facets correspond to D-negative elementary contractions.

If f is of fiber type, we are done. If f is divisorial, then Y is again a MDS by Proposition 4.1, and we proceed by induction. If f is small, then by Remark 3.11 the flip $g\colon X \dashrightarrow \widetilde{X}$ of f exists, and $\widetilde{X}$ is again a MDS.

[9] One can check that D_{i+1} is the push-forward of D_i when φ_i is a divisorial contraction, and the transform of D_i when φ_i is a flip.

Therefore we only have to show that there cannot be an infinite sequence of D-negative flips. Since X has finitely many SQMs, we have to show that given a sequence of D-negative flips:

$$X = X_0 \overset{\psi_0}{\dashrightarrow} X_1 \dashrightarrow \cdots \dashrightarrow X_{s-1} \overset{\psi_{s-1}}{\dashrightarrow} X_s$$

the composition $\psi_{s-1} \circ \cdots \circ \psi_0$ cannot be an isomorphism.

For this, we use the order of D along divisors over X.

Definition 4.3. Let X be a normal and $\mathbb{Q}$-factorial projective variety. By a *divisor E over X* we mean a prime Weil divisor $E \subset Y$ where Y is normal, projective, and endowed with a birational map $\mu\colon Y \dashrightarrow X$.

Then for every $\mathbb{Q}$-divisor D on X we set

$$\operatorname{ord}_E D := \operatorname{ord}_E \mu^*(D) \in \mathbb{Q},$$

where $\operatorname{ord}_E \mu^*(D)$ is just the coefficient of E in the $\mathbb{Q}$-divisor $\mu^*(D)$.

Lemma 4.4. *Let X be a normal and $\mathbb{Q}$-factorial projective variety, D a divisor in X, $g\colon X \dashrightarrow \widetilde{X}$ a D-negative flip, and $\widetilde{D}$ the transform of D in $\widetilde{X}$.*

- *For every divisor E over X, we have* $\operatorname{ord}_E D \geq \operatorname{ord}_E \widetilde{D}$.[10]
- *There exists a divisor E_1 over X such that* $\operatorname{ord}_{E_1} D > \operatorname{ord}_{E_1} \widetilde{D}$.

Lemma 4.4 allows to conclude the proof of Theorem 4.2. Indeed we can find a divisor E_1 over X such that, if D_s is the transform of D in X_s, we have $\operatorname{ord}_{E_1} D > \operatorname{ord}_{E_1} D_s$, hence $\psi_{s-1} \circ \cdots \circ \psi_0$ cannot be an isomorphism. ■

Proof of Lemma 4.4. Up to replacing D with a multiple, we can assume that both D and $\widetilde{D}$ are Cartier. Let us consider a resolution of g, with Z normal and projective:

$$\begin{tikzcd}
 & Z \arrow[dl, "h"'] \arrow[dr, "\widetilde{h}"] & \\
X \arrow[rr, dashed, "g"] \arrow[dr, "f"'] & & \widetilde{X} \arrow[dl, "\widetilde{f}"] \\
 & Y &
\end{tikzcd}$$

Notice that if E is any divisor over X, with $E \subset Y$ and $\mu\colon Y \dashrightarrow X$ birational, we can choose Z such that $\mu^{-1} \circ h\colon Z \dashrightarrow Y$ is regular. Then if $E_Z \subset Z$ is the

[10] Notice that E can be seen as a divisor over $\widetilde{X}$ via $g \circ \mu\colon Y \dashrightarrow \widetilde{X}$.

transform of E, we have $\operatorname{ord}_E D = \operatorname{ord}_{E_Z} D$ and $\operatorname{ord}_E \widetilde{D} = \operatorname{ord}_{E_Z} \widetilde{D}$. Therefore it is enough to show that $\operatorname{ord}_B D \geq \operatorname{ord}_B \widetilde{D}$ for every prime divisor $B \subset Z$; this is clear if B is not exceptional.

Let D_Z be the transform of D (and of $\widetilde{D}$) in Z, and let $E_1, \dots, E_r \subset Z$ be the prime exceptional divisors. We have:

$$h^*(D) = D_Z + \sum_{i=1}^{r} (\operatorname{ord}_{E_i} D)\, E_i \quad \text{and} \quad \widetilde{h}^*(\widetilde{D}) = D_Z + \sum_{i=1}^{r} \left(\operatorname{ord}_{E_i} \widetilde{D}\right) E_i.$$

Set

$$H := \widetilde{h}^*(\widetilde{D}) - h^*(D) = \sum_{i=1}^{r} \left(\operatorname{ord}_{E_i} \widetilde{D} - \operatorname{ord}_{E_i} D\right) E_i.$$

We show that H is ψ-nef, where $\psi := f \circ h = \widetilde{f} \circ \widetilde{h}$.

Let $C \subset Z$ be an irreducible curve such that $\psi(C) = \{pt\}$. If $h(C)$ is a point, then $h^*(D) \cdot C = 0$. If instead $h(C)$ is a curve, then $f(h(C)) = \{pt\}$, hence $D \cdot h(C) < 0$ and $h^*(D) \cdot C < 0$. In any case $h^*(D) \cdot C \leq 0$. Similarly one shows that $\widetilde{h}^*(\widetilde{D}) \cdot C \geq 0$, hence $H \cdot C \geq 0$.

By negativity of contractions [KM98, Lemma 3.39], $-H$ must be effective; this implies that $\operatorname{ord}_{E_i} D \geq \operatorname{ord}_{E_i} \widetilde{D}$ for every $i = 1, \dots, r$.

Assume now that $\operatorname{ord}_{E_1} D = \operatorname{ord}_{E_1} \widetilde{D}$, and let $C \subset E_1$ be an irreducible curve such that $\psi(C) = \{pt\}$ and C is not contained in $E_2, \dots, E_r$. Then $E_i \cdot C \geq 0$ for $i = 2, \dots, r$, which yields $H \cdot C = 0$. Similarly as before, this implies that $h(C) = \{pt\}$ and $\widetilde{h}(C) = \{pt\}$, so that the morphism $Z \to X \times \widetilde{X}$ given by $(h, \widetilde{h})$ is not finite.

On the other hand let $\Gamma \subset X \times \widetilde{X}$ be the closure of the graph of g, and let $Z_0 \to \Gamma$ be the normalization. Since the morphism $Z_0 \to X \times \widetilde{X}$ is finite, we see that if $E_1 \subset Z_0$ is a prime exceptional divisor, we have $\operatorname{ord}_{E_1} D > \operatorname{ord}_{E_1} \widetilde{D}$. ■

Notice that when $g\colon X \dashrightarrow \widetilde{X}$ is a K-negative flip, Lemma 4.4 says precisely that discrepancies increase after the flip, see [KM98, Lemma 3.38].

Remark 4.5. Let X be a MDS and D a divisor on X. It is not difficult to check that a Mori program for D ends with a fiber type contraction if and only if $[D] \notin \operatorname{Eff}(X)$.

4.6. The effective cone. We are now going to use the existence of Mori programs to show that the effective cone of a MDS is rational polyhedral, in particular it is closed. We use the following remark on non-movable prime divisors in X.

Remark 4.7. Let X be a MDS and $D \subset X$ a prime divisor which is not movable. Then there exists a SQM $g_i\colon X \dashrightarrow X_i$ such that the transform $D_i \subset X_i$ of D is the exceptional divisor of an elementary divisorial contraction.

Proof. We run a Mori program for D. Since D is effective, the program must end with D becoming nef (see Remark 4.5). On the other hand D is not movable, hence D cannot be nef on any SQM of X. This means that in the Mori program some divisorial contraction must occur, and we get:

$$X \overset{g_i}{\dashrightarrow} X_i \xrightarrow{f} Y,$$

where g_i is a sequence of D-negative flips, and f is a D_i-negative elementary divisorial contraction ($D_i \subset X_i$ the transform of D). Since $D_i \cdot \mathrm{NE}(f) < 0$, we see that $D_i = \mathrm{Exc}(f)$. Let us also notice that since $f_*(D_i) = 0$, f is actually the last step of the Mori program. ■

Corollary 4.8. *Let X be a MDS and let $D_1, \dots, D_s \subset X$ be the exceptional divisors of all elementary divisorial rational contractions of X. Then*

$$\mathrm{Eff}(X) = \mathrm{Mov}(X) + \mathbb{R}_{\geq 0}[D_1] + \cdots + \mathbb{R}_{\geq 0}[D_s],$$

in particular $\mathrm{Eff}(X)$ *is a rational polyhedral cone in* $\mathcal{N}^1(X)$.

In fact it is not difficult to show that $\mathbb{R}_{\geq 0}[D_i]$ is a one-dimensional face of $\mathrm{Eff}(X)$ for every $i = 1, \dots, s$, see [Cas12, §2.18].

It is possible to show that the fan $\mathcal{M}_X$, supported on the movable cone, extends to a bigger fan $\mathcal{M}'_X$, supported on the effective cone. The maximal dimensional cones in this fan are called *Mori chambers*; they give a decomposition of $\mathrm{Eff}(X)$ as a union of rational polyhedral cones, and have a precise geometric meaning. We refer the reader to [HK00] and to [Oka11] for the definition and some nice properties of Mori chambers.

5 Section algebras

Let X be a normal and $\mathbb{Q}$-factorial projective variety, and let $D_1, \dots, D_r$ be Cartier divisors on X. We define:

$$R(X; D_1, \dots, D_r) := \bigoplus_{m_1, \dots, m_r \in \mathbb{Z}_{\geq 0}} H^0(X, \mathcal{O}_X(\sum_{i=1}^r m_i D_i)),$$

$$\overline{R}(X; D_1, \dots, D_r) := \bigoplus_{m_1, \dots, m_r \in \mathbb{Z}} H^0(X, \mathcal{O}_X(\sum_{i=1}^r m_i D_i)).$$

These complex vector spaces can be naturally endowed with a ring structure, using multiplication of sections. In this way we get two $\mathbb{Z}^r$-graded $\mathbb{C}$-algebras, and

$R(X; D_1, \ldots, D_r)$ is a subalgebra of $\overline{R}(X; D_1, \ldots, D_r)$. Finite generation of this type of algebras has relevant implications for the geometry of X, see for instance [Deb01, §7.1] and [Laz04a, Def. 2.1.19]. For Mori dream spaces we have especially good properties.

Theorem 5.1 ([HK00], Prop. 2.9 and [Bäk11], Th. 1.2)**.** *Let X be a MDS and let $D_1, \ldots, D_r$ be Cartier divisors on X. Then $R(X; D_1, \ldots, D_r)$ and $\overline{R}(X; D_1, \ldots, D_r)$ are finitely generated $\mathbb{C}$-algebras.*

This result has been shown by Hu and Keel in the case where the classes $[D_1], \ldots, [D_r]$ are linearly independent in $\mathcal{N}^1(X)$; the proof relies on the decomposition of $\mathrm{Eff}(X)$ as a union of Mori chambers. Then Bäker has shown that finite generation in general can be deduced from this special case.

The following result gives a converse to Theorem 5.1.

Theorem 5.2 ([HK00], Prop. 2.9)**.** *Let X be a normal and $\mathbb{Q}$-factorial projective variety with finitely generated Picard group. Suppose that there exist Cartier divisors $D_1, \ldots, D_r$ on X such that:*

(1) $[D_1], \ldots, [D_r]$ are a basis of $\mathcal{N}^1(X)$;

(2) $\overline{R}(X; D_1, \ldots, D_r)$ is finitely generated.

Then X is a MDS.

Let us point out that Theorem 5.2 is in practice the only tool to show that a variety is a MDS. Its proof relies on methods of Geometric Invariant Theory, we refer the reader to [HK00] for more details. Using Cox's descrition of the homogeneous coordinate ring of a toric variety [Cox95], Theorem 5.2 easily implies the following.

Corollary 5.3 ([HK00], Cor. 2.4)**.** *Let X be a $\mathbb{Q}$-factorial projective toric variety. Then X is a MDS.*

Remark 5.4 (Cox rings)**.** Let X be a normal and $\mathbb{Q}$-factorial projective variety with Picard number ρ, and with finitely generated *and free* Picard group. Let us choose $D_1, \ldots, D_\rho$ Cartier divisors on X whose classes give a basis of $\mathrm{Pic}(X)$. The **Cox ring** of X is defined as

$$\mathrm{Cox}(X) := \overline{R}(X; D_1, \ldots, D_\rho),$$

see [HK00, Def. 2.6]. One can check that different choices of divisors yield isomorphic algebras, so that the definition is well-posed, and we can restate Theorems 5.1 and 5.2 by saying that X is a MDS if and only if $\mathrm{Cox}(X)$ is finitely generated as a complex algebra.

When $\mathrm{Pic}(X)$ is finitely generated, but not free, we may choose $D_1, \dots, D_\rho$ such that they yield a basis of $\mathrm{Pic}(X)$ modulo torsion. However, the (isomorphism class of the) section algebra $\overline{R}(X; D_1, \dots, D_\rho)$ will depend on the choice of the divisors $D_1, \dots, D_\rho$, so that we do not get an algebra canonically associated to X.

Following [ADHL10, §4.2], one can still define a Cox ring for X, as follows. Suppose that

$$\mathrm{Pic}(X) \cong \mathbb{Z}^\rho \oplus \mathbb{Z}/(a_1) \oplus \cdots \oplus \mathbb{Z}/(a_s).$$

We choose Cartier divisors $D_1, \dots, D_\rho, E_1, \dots, E_s$ such that the classes of the D_i's generate the torsion free part, and the class of E_j generates $\mathbb{Z}/(a_j)$. Then we consider:

$$A := \bigoplus_{m_i \in \mathbb{Z}, n_j \in \{0, \dots, a_j - 1\}} H^0(X, \mathcal{O}_X(\sum_{i,j}(m_i D_i + n_j E_j))).$$

It is possible to define a ring structure on A, in such a way that a different choice of divisors will yield an isomorphic $\mathbb{C}$-algebra. Notice that A is graded by $\mathrm{Pic}(X)$, instead of $\mathbb{Z}^\rho$. It is still true that X is a MDS if and only if A is finitely generated.

We refer the reader to [LV09] and [ADHL10] for the study of MDS from the point of view of Cox rings.

We are going to conclude with the following result.

Theorem 5.5 ([BCHM10], Cor. 1.3.2)**.** *Let X be a smooth Fano variety. Then X is a Mori dream space.*

Let us notice that [BCHM10, Cor. 1.3.2] is more general: indeed one can allow X to have log-terminal $\mathbb{Q}$-factorial singularities, and moreover X does not really need to be Fano, but "log-Fano" is enough. This is a much more general condition, we refer the reader to [McK10] and [PS09] for more details on log-Fano varieties. Here we will only consider the case where X is smooth and Fano.

The proof of Theorem 5.5 is achieved by using Theorem 5.2 and finite generation of log-canonical algebras:

Theorem 5.6 ([BCHM10], Cor. 1.1.2)**.** *Let Y be a smooth projective variety and $\Delta = \sum_i a_i D_i$ an effective $\mathbb{Q}$-divisor on Y such that $a_i < 1$ for all i, and* $\mathrm{Supp}\,\Delta$ *is a SNC divisor. Then the algebra*

$$R(Y, K_Y + \Delta) := \bigoplus_{m \in \mathbb{Z}_{\geq 0}} H^0\left(Y, \mathcal{O}_Y(\lfloor m(K_Y + \Delta) \rfloor)\right)$$

is finitely generated.[11]

[11] Here $\lfloor\ \rfloor$ denotes the integral part of a $\mathbb{Q}$-divisor, obtained by taking the integral parts of the coefficients.

For the proof of Theorem 5.5 we follow [BCHM10, Proof of Cor. 1.1.9, "aliter"]. We first introduce, in the following remark, a standard trick to reduce a section algebra of several divisors on a variety to the section algebra of a unique divisor on another variety. We refer the reader to [Laz04a, §2.3.B] for this construction.

Remark 5.7. Let X be a smooth projective variety with finitely generated and free Picard group. Let $D_1, \dots, D_r$ be divisors on X and set

$$E := \mathcal{O}_X(D_1) \oplus \cdots \oplus \mathcal{O}_X(D_r) \quad \text{and} \quad Y := \mathbb{P}_X(E).$$

Then Y is smooth and projective, and $\pi\colon Y \to X$ is a $\mathbb{P}^{r-1}$-bundle.

Let $\mathcal{O}_Y(1) \in \operatorname{Pic}(Y)$ be the tautological line bundle. For every $m \in \mathbb{Z}_{\geq 0}$ we have

$$\pi_* \mathcal{O}_Y(m) \cong S^m E = \bigoplus_{a_1+\cdots+a_r=m, a_i \geq 0} \mathcal{O}_X(\sum_{i=1}^r a_i D_i),$$

and hence:

$$R(Y, \mathcal{O}_Y(1)) \cong R(X; D_1, \dots, D_r).$$

For every $i = 1, \dots, r$ set

$$F_i := \mathcal{O}_X(D_1) \oplus \cdots \oplus \mathcal{O}_X(D_{i-1}) \oplus \mathcal{O}_X(D_{i+1}) \oplus \cdots \oplus \mathcal{O}_X(D_r) \quad \text{and} \quad T_i := \mathbb{P}_X(F_i).$$

The surjection $E \to F_i$ yields an embedding $T_i \hookrightarrow Y$, and T_i is a smooth divisor in Y. Since in every fiber of π the divisors $T_1, \dots, T_r$ are hyperplanes in general position, the divisor $T := T_1 + \cdots + T_r$ is SNC.

Lemma 5.8. *With the notations above, we have $K_Y - \pi^*(K_X) \sim -T$.*

Proof. If $r = 1$, then $T = \emptyset$, π is an isomorphism, and there is nothing to prove.

Let $r = 2$. If $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$, then $Y \cong X \times \mathbb{P}^1$ and $T_i = \psi^*(p_i)$, where $\psi\colon Y \to \mathbb{P}^1$ is the projection and $p_1, p_2 \in \mathbb{P}^1$ are points. Then $K_Y - \pi^*(K_X) = \psi^*(K_{\mathbb{P}^1}) \sim \psi^*(-p_1 - p_2) = -T$.

Assume now that $r = 2$ and $\mathcal{O}_X(D_1) \ncong \mathcal{O}_X(D_2)$. Using that $\operatorname{Pic}(Y) = \mathbb{Z}\mathcal{O}_Y(1) \oplus \pi^*(\operatorname{Pic}(X))$, one can show that if $L \in \operatorname{Pic}(Y)$ is such that $L_{|T_i} \cong \mathcal{O}_{T_i}$ for $i = 1, 2$, then $L \cong \mathcal{O}_Y$. On the other hand $T_1 \cap T_2 = \emptyset$, T_i is a section of π, and hence

$$\mathcal{O}_Y(K_Y - \pi^*(K_X) + T_1 + T_2)_{|T_1} \cong \mathcal{O}_Y(K_Y + T_1)_{|T_1} \otimes \mathcal{O}_{T_1}(-K_{T_1}) \cong \mathcal{O}_{T_1}$$

by adjunction, and similarly for T_2; this yields the statement.

For $r \geq 3$, one shows the statement by induction on r, using the fact that the restriction $\operatorname{Pic}(Y) \to \operatorname{Pic}(T_1)$ is an isomorphism. By induction $K_{T_1} - \pi^*_{|T_1} K_X \sim -T_{2|T_1} - \cdots - T_{r|T_1}$, and the adjunction formula yields the statement. ∎

Proof of Theorem 5.5. We notice first of all that $\mathrm{Pic}(X)$ is finitely generated and free.

Let $D_1, \dots, D_r$ be divisors on X; we claim that $R(X; D_1, \dots, D_r)$ is finitely generated.

Let us first observe that this implies the statement. Indeed we can choose $D_1, \dots, D_r$ such that $[D_1], \dots, [D_r]$ are a basis of $\mathcal{N}^1(X)$, and such that the convex cone generated by $[D_1], \dots, [D_r]$ in $\mathcal{N}^1(X)$ contains $\mathrm{Eff}(X)$. This implies that $\overline{R}(X; D_1, \dots, D_r) = R(X; D_1, \dots, D_r)$, so that X is a MDS by Theorem 5.2.

To show finite generation of $R(X; D_1, \dots, D_r)$, we apply the construction explained in Remark 5.7. We need to show that $R(Y, \mathcal{O}_Y(1))$ is finitely generated.

By Lemma 5.8 we have $K_Y - \pi^*(K_X) \sim -T$. On the other hand

$$\mathcal{O}_Y(K_Y - \pi^*(K_X)) = \pi^*(\det E) \otimes \mathcal{O}_Y(-r) = \mathcal{O}_Y(\pi^*(D)) \otimes \mathcal{O}_Y(-r),$$

where $D := D_1 + \cdots + D_r$, so we get

$$\mathcal{O}_Y(r) = \mathcal{O}_Y(T + \pi^*(D)).$$

Since $R(Y, \mathcal{O}_Y(1))$ is finitely generated if and only if $R(Y, \mathcal{O}_Y(r))$ is (see for instance [Deb01, 7.5]), we are reduced to show that $R(Y, T + \pi^*(D))$ is finitely generated.

Now we look for an effective $\mathbb{Q}$-divisor Δ on Y such that all coefficients of Δ are < 1, $\mathrm{Supp}(\Delta)$ is SNC, and there exists $m_0 \in \mathbb{Z}_{>0}$ such that

$$m_0(K_Y + \Delta) \sim T + \pi^*(D)$$

(and $m_0(K_Y + \Delta)$ is an integral divisor). Then $R(Y, K_Y + \Delta)$ is finitely generated by Theorem 5.6, and this easily implies that $R(Y, T + \pi^*(D))$ is finitely generated.

Thus let $m \in \mathbb{Z}_{>0}$. We write:

$$\begin{aligned}\Delta \sim_{\mathbb{Q}} - K_Y + \frac{1}{m}T + \frac{1}{m}\pi^*(D) &\sim \pi^*(-K_X) + T + \frac{1}{m}T + \frac{1}{m}\pi^*(D) \\ &= \frac{m-1}{m}\,T + \frac{1}{m}\big(m\pi^*(-K_X) + 2T + \pi^*(D)\big).\end{aligned}$$

Since $-K_X$ is ample and $2T + \pi^*(D)$ is π-ample, there exists $m_0 \in \mathbb{Z}$, $m_0 \gg 0$, such that $m_0\pi^*(-K_X) + 2T + \pi^*(D)$ is very ample. Let A be a general element of the linear system $|m_0\pi^*(-K_X) + 2T + \pi^*(D)|$. Then we set $\Delta := (m_0 - 1)/m_0 T + 1/m_0 A$, and we are done. ■

References

[ADHL10] Ivan Arzhantsev, Ulrich Derenthal, Jürgen Hausen, and Antonio Laface, *Cox rings*, preprint arxiv:1003.4229v2, 2010.

[AHL10] Michela Artebani, Jürgen Hausen, and Antonio Laface, *On Cox rings of K3 surfaces*, Compositio Mathematica **146** (2010), 964–998.

[AL11] Michela Artebani and Antonio Laface, *Cox rings of surfaces and the anticanonical Iitaka dimension*, Advances in Mathematics **226** (2011), 5252–5267.

[Bäk11] Hendrik Bäker, *Good quotients of Mori dream spaces*, Proceedings of the American Mathematical Society **139** (2011), 3135–3139.

[BCHM10] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan, *Existence of minimal models for varieties of log general type*, Journal of the American Mathematical Society **23** (2010), 405–468.

[Cas12] Cinzia Casagrande, *On the birational geometry of Fano 4-folds*, Mathematische Annalen, published online, 2012.

[Cox95] David Cox, *The homogeneous coordinate ring of a toric variety*, Journal of Algebraic Geometry **4** (1995), 17–50.

[Deb01] Olivier Debarre, *Higher-dimensional algebraic geometry*, Universitext, Springer-Verlag, 2001.

[HK00] Yi Hu and Sean Keel, *Mori dream spaces and GIT*, Michigan Mathematical Journal **48** (2000), 331–348.

[KM98] János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, 1998.

[Laz04a] Robert Lazarsfeld, *Positivity in algebraic geometry I*, Springer-Verlag, 2004.

[Laz04b] ———, *Positivity in algebraic geometry II*, Springer-Verlag, 2004.

[LV09] Antonio Laface and Mauricio Velasco, *A survey on Cox rings*, Geometriae Dedicata **139** (2009), 269–287.

[McK10] James McKernan, *Mori dream spaces*, Japanese Journal of Mathematics **5** (2010), 127–151.

[Oka11] Shinnosuke Okawa, *On images of Mori dream spaces*, preprint arxiv:1104.1326, 2011.

[PS09] Yuri G. Prokhorov and Vyacheslav V. Shokurov, *Towards the second main theorem on complements*, Journal of Algebraic Geometry **18** (2009), 151–199.

[TVAV11] Damiano Testa, Anthony Várilly-Alvarado, and Mauricio Velasco, *Big rational surfaces*, Mathematische Annalen **351** (2011), 95–107.