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présenté par

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TOPOLOGIE DE CONTACT ET SINGULARITÉS COMPLEXES

Soutenu le 9 Décembre 2008 devant le jury composé de :

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“Envisagée de ce point de vue,
la vie apparaît
comme un courant
qui va d’un germe à un germe
par l’intermédiaire d’un organisme développé.
Tout se passe comme si l’organisme lui-même
n’était qu’une excroissance, un bourgeon
que fait saillir le germe ancien
travaillant à se continuer
en un germe nouveau.”

Henri Bergson
(**L’évolution créatrice**)

Table des matières

CHAPITRE 1

Généralités

1. Travaux

1.1. Travaux présentés pour l'habilitation.

• Dans le Chapitre 2 :

- *"On the Milnor fibers of cyclic quotient singularities"* (en collaboration avec A. Némethi), disponible sur ArXiv :0805.3449. Soumis.

- *"On the cohomology rings of holomorphically fillable manifolds."* A paraître dans Contemporary Mathematics. Disponible sur ArXiv :0712.3484

- *"Milnor open books and Milnor fillable contact 3-manifolds"* (en collaboration avec C. Caubel et A. Némethi). Topology **45** (2006), 673-689.

- *"On the contact boundaries of normal surface singularities"* (en collaboration avec C. Caubel), C. R. Acad. Sci. Paris, Ser. I **339** (2004) 43-48.

• Dans le Chapitre 3 :

- *"A finiteness theorem for the dual graphs of surface singularities"* (en collaboration avec J. Seade). A paraître dans International Journal of Mathematics. Disponible sur ArXiv :0805.1842

- *"The geometry of continued fractions and the topology of surface singularities"*. In *"Singularities in Geometry and Topology 2004"*. Advanced Studies in Pure Mathematics **46**, 2007, 119-195.

• Dans le Chapitre 4 :

- *"Families of higher dimensional germs with bijective Nash map"* (en collaboration avec C. Plénat). Kodai Math. Journal.**31** (2) (2008), 199-218.

- *"A class of non-rational surface singularities for which the Nash map is bijective"* (en collaboration avec C. Plénat). Bulletin de la SMF **134** no. 3 (2006), 383-394.

1.2. Autres travaux.

• Articles de spécialité :

- *"Introduction to Jung's method of resolution of singularities"*, disponible sur ArXiv : math.CV/0703353.

- *"Iterating the hessian : a dynamical system on the moduli space of elliptic curves and dessins d'enfants"*. A paraître dans *"Noncommutativity and Singularity"*, J.P. Bourguignon, M.Kotani, Y.Maeda, N.Tose eds.

- *"On higher dimensional Hirzebruch-Jung singularities"*. Rev. Mat. Complut. **18** (2005), no.1, 209-232.

- *"The analytical invariance of the semigroup of quasi-ordinary hypersurface singularities"*, Duke Math. J. **124** (2004), no.1, 67-104.

- *"Sur le contact d'une hypersurface quasi-ordinaire avec ses hypersurfaces polaires"*, Journal of the Inst. of Math. Jussieu (2004) **3** (1), 105-138.

- *"Approximate Roots"*, dans *"Valuation Theory and its Applications."* vol. II, F.-V. Kuhlmann, S.Kuhlmann, M.Marshall editors, Fields Institute Communications **33**, AMS 2003, 285-321.

- *"Two-dimensional iterated torus knots and quasi-ordinary surface singularities"*, C. R. Acad. Sci. Paris, Ser. I **336** (2003) 651-656.

- *"On the invariance of the semigroup of a quasi-ordinary surface singularity"*, C. R. Acad. Sci. Paris, Ser. I **334** (2002) 1101-1106.

- *"On a canonical placement of knots in irreducible 3-manifolds"*, C. R. Acad. Sci. Paris, Ser. I **334** (2002) 677-682.

• Thèse :

- *"Arbres de contact des singularités quasi-ordinaires et graphes d'adjacence pour les 3-variétés réelles"*, disponible sur le site :
http ://tel.ccsd.cnrs.fr/documents/archives0/00/00/28/00/index_fr.html

• Textes de vulgarisation :

- *"Sphères et développement embryonnaire"*, dans *"La sphère sous toutes ses formes"* Dossier hors-série oct/décembre 2003 de Pour la Science, 114-117.

- *"Pour nouer, il faut courber"* (2000), développement d'un exposé fait aux journées "Sciences en fête", disponible sur le site :
http ://www.dma.ens.fr/culturemath/contenu/dossiers.html#courbure

2. Survol des travaux présentés pour l'habilitation

Tout d'abord un mot sur la langue : j'ai choisi d'écrire la description de mes travaux en anglais, car plusieurs des personnes potentiellement intéressées par ce texte ne comprennent pas le français. Comme d'autre part dans mes recherches se retrouvent deux domaines de la géométrie qui ont peu communiqué jusqu'à présent (la théorie des singularités analytiques complexes et la topologie de contact), j'ai décidé de présenter les définitions, intuitions et constructions de base des deux côtés, en espérant que cela puisse aider une personne plutôt familière avec l'un des domaines à s'orienter dans l'autre.

Je tiens ensuite à remercier Bernard Teissier, mon directeur de thèse, qui m'a offert un horizon suffisamment large pour que je ressente la richesse de l'inconnu et qui est resté un interlocuteur des plus précieux pour son écoute attentive et ses conseils. Je remercie également mes collaborateurs Clément Caubel, András Némethi, Camille Plénat et José Seade, qui ont tellement fait pour rendre ces années de recherche passionnantes.

Dès ma thèse, j'ai travaillé en théorie des singularités des variétés analytiques complexes. Mais après ma thèse j'ai changé complètement le type des structures que j'ai regardées. Ainsi, pendant la thèse je me suis principalement intéressé aux singularités quasi-ordinaires d'hypersurfaces et à des problèmes d'invariance analytique d'objets qui leur sont associés via des séries de Newton-Puiseux. Après la thèse, je me suis tourné vers les structures de contact canoniquement présentes sur les bords des singularités isolées et vers leurs remplissages symplectiques obtenus comme fibres de Milnor de leurs divers lissages. Ces réflexions m'ont amené aussi à obtenir de nouveaux résultats sur la topologie des singularités de surfaces et à étudier le problème des arcs de Nash. Ce sont ces divers aspects de ma recherche que je présenterai dans la suite.

• Le chapitre 2 a comme but de présenter le résultat de réflexions sur les liens entre la théorie des singularités des espaces analytiques complexes et les topologies de contact et symplectique. Le problème dominant est le suivant : *décrire les remplissages de Stein du bord de contact d'une singularité isolée obtenus comme fibres de Milnor*. Les principaux résultats sont :

(avec C. Caubel et A. Némethi, voir Théorème 3.3) *Le livre ouvert de Milnor associé à une fonction holomorphe à singularité isolée définie sur un germe d'espace analytique complexe à singularité isolée porte la structure de contact standard.*

(avec C. Caubel et A. Némethi, voir Théorème 3.6) *La structure de contact standard sur le bord d'une singularité de surface normale est un invariant topologique de la singularité.*

(avec A. Némethi, prouvant en particulier une conjecture de P. Lisca, voir la section 5.6) *Les fibres de Milnor associées aux composantes irréductibles de la base de déformation miniverselle d'une singularité quotient cyclique de surface sont deux à deux non-difféomorphes par des difféomorphismes étendant les identifications naturelles de leurs bords et constituent, à de tels difféomorphismes près, tous les remplissages de Stein des structures de contact standard sur les espaces lenticulaires.*

J'y explique aussi des théorèmes de restriction sur les anneaux de cohomologie des bords fortement pseudo-convexes des variétés de Stein compactes à bord, des singularités isolées et des variétés complexes compactes à bord. Certains avaient déjà été démontrés par Durfee & Hain et Bungart. Je pense que sont nouvelles les obstructions topologiques à être un bord d'une variété de Stein, à la lissification d'une singularité isolée et à l'existence de petites résolutions, le tout en dimension complexe ≥ 3 (voir la section 6).

• Le chapitre 3 contient le résultat de mes réflexions sur la topologie des singularités de surfaces analytiques complexes. Le problème dominant qui a guidé cette réflexion est le suivant : *décrire les types topologiques possibles des singularités normales de surfaces qui sont respectivement des hypersurfaces/ des intersections complètes/ Gorenstein/ numériquement Gorenstein.* Les principaux théorèmes sont :

(voir le Théorème 4.5) *La structure de plombage tracée sur le bord d'une singularité normale de surface par sa résolution minimale à croisements normaux est invariante à isotopie près par les difféomorphismes préservant l'orientation.*

(avec J. Seade, voir le Théorème 5.3) *A topologie du diviseur exceptionnel de la résolution minimale fixée, il existe un nombre fini de possibilités pour le cycle canonique d'une singularité numériquement Gorenstein réalisant cette topologie. En particulier, ceci est vrai pour les intersections complètes.*

J'y explique aussi comment une dualité élémentaire entre cônes supplémentaires par rapport à un réseau bidimensionnel permet de comprendre géométriquement les calculs faits à l'aide de fractions continues apparaissant en théorie des singularités des courbes et des surfaces.

• Le chapitre 4 contient le résultat de réflexions sur le problème des arcs de Nash. Le problème dominant est le suivant : *décrire les diviseurs essentiels d'une singularité isolée normale en dimension* ≥ 3. Je pense que la possibilité d'étendre les réflexions des précédents chapitres aux dimensions plus grandes passe par une meilleure compréhension de ces diviseurs essentiels.

Le principal théorème obtenu, permettant de donner les premiers exemples non triviaux de singularités normales non toriques ayant une application de Nash bijective, est :

(avec C. Plénat, voir Corollaire 3.13) *Soit* $(X, 0)$ *un germe normal à singularité isolée d'espace analytique complexe. Considérons une résolution divisorielle projective* π *de* $(X, 0)$. *Soit* F *une composante irréductible du lieu exceptionnel* $Exc(\pi)$ *de* π. *Supposons que pour toute autre composante irréductible* G *de* $Exc(\pi)$, *il existe un diviseur effectif entier* D *de support* $Exc(\pi)$ *dans lequel la multiplicité de* F *est strictement inférieure à celle de* G, *et tel que le fibré en droites* $\mathcal{O}(-D)$ *est ample en restriction à* $Exc(\pi)$. *Alors* F *est une composante essentielle et elle est contenue dans l'image de l'application de Nash.*

• Le chapitre 5 contient les questions principales apparues pendant les recherches précédentes, et qui continuent à nourir ma réflexion.

3. Remarques sur mon enseignement

Pendant les trois dernières années, j'ai donné un cours-TD intégré de M1 (première année de Master) intitulé *Topologie algébrique et géométrique*. J'ai démarré avec le théorème d'Euler sur les polyèdres convexes, et j'ai terminé par les axiomes d'Eilenberg-Steenrod d'une théorie homologique. Mon fil directeur a été le problème de la classification des espaces topologiques. J'ai essayé de maintenir l'équilibre entre une manipulation concrète à travers de nombreux dessins (les étudiants ont commencé par apprendre à dessiner les polyèdres réguliers, que la plupart ne connaissaient même pas de nom) et la rigueur des définitions précises. Mais celles-ci n'ont jamais été parachutées, nous y sommes arrivés graduellement, expérimentalement.

Nous avons lu plusieurs textes originaux : des extraits de l'article de Riemann sur les fonctions abéliennes [R 57], décrivant la surface qui porte son nom associée à une fonction algébrique ; l'introduction et des extraits de l'article de Poincaré [P 95], fondateur de la topologie algébrique ; l'un de ses articles sur l'enseignement des mathématiques [P 89] (qui à mon sens devrait être au programme de l'Agrégation) ; l'article d'Eilenberg et Steenrod [ES 45]. Nous avons beaucoup travaillé à partir des exemples de variétés de dimension 3 obtenus en recollant des faces de cubes ou d'octaèdres, décrits par Poincaré [P 95] : ils constituèrent par exemple la base expérimentale pour voir que la caractéristique d'Euler s'annulait en dimension 3, et qu'il fallait donc enrichir notre panoplie d'invariants, ainsi que la base expérimentale pour conjecturer une première version du théorème de dualité (ce que plusieurs étudiants firent correctement au vu des données expérimentales).

J'ai tenu à faire sentir aux étudiants la durée du développement mathématique, ainsi que l'évolution du langage et des problèmes étudiés. J'ai voulu aussi leur mettre entre les mains des articles de recherche et les faire réfléchir sur les diverses manières de faire un cours de maths. Tout cela à un moment-clé de leurs études, où ils devront choisir s'ils s'orienteront plutôt vers la recherche ou vers l'enseignement.

La plupart d'entre eux ont apprécié cela. Voici quelques-uns de leurs commentaires écrits dans le questionnaire final anonyme que j'ai distribué chaque année. Ils constituent des réponses aux questions écrites en caractères gras.

• Les renseignements historiques vous ont-ils permis de mieux comprendre les maths ?

"*Les renseignements historiques étaient intéressants, surtout lorsque ceux-ci permettaient de comprendre la démarche intellectuelle qui mène aux théorèmes, ou à l'élaboration d'une théorie.* "

"*C'est quelque chose d'extraordinaire : pour une fois, on nous dit d'où ça vient, pourquoi on a pensé à faire ça. On n'a pas cette impression que tout vient naturellement et cela nous permet d'avoir différentes approches d'un problème.*"

"*Les renseignements historiques ont beaucoup apporté, tant à l'intérêt qu'à la compréhension. Ils ont rendu ce cours plus "intuitif" et permis d'en comprendre la logique.*"

"*Peut-être pas mieux comprendre les maths, mais mieux comprendre le cheminement des découvertes. Je pense que cela m'a éclairé sur ce qu'est réellement la recherche.*"

"*Disons plutôt que ces renseignements m'ont conforté dans l'idée de l'absolue beauté des maths, abstraites et pourtant si proches de la réalité (état d'esprit perdu un peu lors des dernières années d'étude).* "

"*C'est un des aspects les plus intéressants du cours à mon sens. Ça permet de replacer les choses dans leur contexte et d'avoir une meilleure idée de comment les maths sont construites. Ça manque dans les autres cours.* "

• Avez-vous trouvée intéressante l'étude de l'article de Poincaré ? Qu'est-ce qui vous a le plus marqué le concernant ?

"*Sa capacité à faire le lien entre différents domaines des mathématiques et surtout la réflexion qu'il mène en même temps sur ses recherches m'ont marqué.*"

"*L'article de Poincaré est, bien qu'assez ancien, très actuel. Ce qui m'a le plus marqué, c'est la "vision" de Poincaré et aussi sa façon d'expliquer qui est très claire.*"

"*L'étude de l'article de Poincaré était très intéressante, il paraissait très opaque au début, puis petit à petit, on comprend certaines choses... (pas encore tout !) Les exemples m'ont marqué, car ça ressemblait à un jeu.*"

"*L'étude de l'article de Poincaré permet de se "faire la main" sur des premiers exemples et de lire dans le texte des articles de l'époque, ce qui apporte beaucoup. J'ai été surpris par le caractère intuitif de ces écrits.* "

"*Enfin de la littérature mathématique ! Ce qui m'a marqué est la façon dont il peut rendre "accessible" certaines parties de ses idées. Mais surtout : il m'a permis de me faire (enfin) une idée de ce que sont les maths et la "création". Il a décrit les développements intuitifs de l'esprit mathématique, pouvant servir dans d'autres domaines. C'est surtout son texte sur l'enseignement qui m'a le plus marqué.*"

"*Oui, c'est intéressant de voir ce que les fondateurs d'un domaine pensaient vraiment.*"

Références

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[P 89] Poincaré, H. "*La logique mathématique dans la science mathématique et dans l'enseignement*", L'Enseignement Math. **1** (1889), 157-162.

[P 95] Poincaré, H. "*Analysis Situs*", Journal de l'Ecole Polytechnique, **1** (1895), 1-121.

[R 57] Riemann, B. "*Théorie des fonctions abéliennes*" Dans "*Oeuvres mathématiques de Riemann*", trad. L. Laugel, Gauthier-Villars, 1898. Réed. J. Gabay, 1990, 89-164. Traduction de plusieurs articles parus dans J. für die reine und angew. Math. **54** (1857).

CHAPTER 2

Milnor fillable contact manifolds and their Stein fillings

1. Motivations

After my PhD, I spent the academic year 2001-2002 in the École Normale Supérieure of Lyon. There I had the opportunity to listen to some talks of Emmanuel Giroux on contact topology, and more precisely on his newly elaborated theory on the interaction between contact structures and open books on a given odd-dimensional manifold. His extremely geometrical style appealed a lot to my imagination and gave me the strong will to learn to see and manipulate contact structures.

This will was intensified by the fact that there is a natural contact manifold associated to each isolated singular point of a complex analytic space (its contact boundary). In my thesis I had studied (not necessarily isolated) singularities of complex spaces, and my main preoccupation was to make dictionaries between algebraic aspects and geometric or topologic ones. While listening to Emmanuel Giroux, I felt that learning to think in terms of contact structures would enrich the geometric panoply of structures in terms of which I could try translations of algebraic aspects.

After my recruitment as Maître de Conférences in the University Paris 7, I began discussing with Clément Caubel about contact structures and singularities, as he had also felt attracted by research at the interface between the two fields. One of the main questions we discussed was to determine whether the natural contact structure on the boundary of a normal surface singularity was or not a topological invariant of the singularity. As a result of those discussions we arrived at a strategy for proving this. We succeeded in realising all the steps of the strategy with the collaboration of András Némethi and we presented our results in the papers [**24**] and [**25**]. I explain them in section 3. Before that, section 2 is dedicated to the necessary background on contact topology.

In November 2004, I was invited by Ricardo Benedetti to the University of Pisa to talk about those results. There I met Paolo Lisca, with whom I discussed a lot about contact structures and their symplectic fillings. In particular, he told me that he had classified the minimal symplectic fillings of the contact boundaries of cyclic quotient surface singularities up to diffeomorphisms and that he conjectures that they correspond in an explicit way to the Milnor fibers of the smoothings of those singularities. I felt immediately extremely attracted by that conjecture, as it seemed to me to be the door by which I could enter to a world of better understanding of the relation between the Stein structures of the Milnor fibers and the contact structures of the boundaries of smoothable surface singularities. I began discussing this problem with András Némethi. This year we finally succeeded in proving a strong version of Lisca's conjecture (see [**126**]). I explain it in section 5. Before that, section 4 is dedicated to the needed background on deformations of singularities.

In the meantime I kept in mind a question which Etienne Ghys asked me in 2001, namely, to know if odd-dimensional tori could be diffeomorphic to boundaries of isolated singular points of complex analytic spaces. Sullivan had proved in 1975 that this was not the case for the 3-dimensional torus, but apparently it was an open question for higher dimensions. I never worked deeply on this question, but I never forgot it.

In 2007 the question interacted suddenly in my mind with a theorem of Marc de Cataldo and Luca Migliorini which I learnt from a talk of Jan Schepers. This starting point allowed me to surf through the literature till I found a theorem of Goresky and MacPherson which enabled me to prove a structure theorem about the cohomology rings of boundaries of isolated singularities, having as a corollary the fact that odd-dimensional tori are never boundaries of complex singularities of dimension at least 2. I could apply this structure theorem to obtain also a structure theorem for strongly convex boundaries of compact complex manifolds. A little later, Eduard Looijenga made me discover that those theorems had already been proved by Durfee and Hain [**39**] on one side, and Bungart [**21**] on another side. As those results seem not to be well-known and as my proof of Durfee and Hain's theorem is shorter than theirs, I wrote the survey [**140**]. There I also explained a similar structure theorem for the boundaries of compact manifolds in terms of the homotopical dimension of the entire manifold. This allows to give simple topological obstructions for a manifold to be the boundary of a Stein manifold and for a singularity to be smoothable or to have small resolutions. I explain those results in section 6.

2. Contact manifolds

I recommend as a general reference on contact geometry Geiges' recent textbook [**54**]. Here I explain only the notions and the views which allow to understand the context of my research.

2.1. Basic definitions.

In everyday language, two objects are *in contact* if they touch. If we think that both are limited by smooth surfaces, then at each point of contact their bounding surfaces are tangent. Therefore the two surfaces have not only that point in common, but also a tangent plane at that point. For this reason, the pair formed by a point in space and a plane through that point is called *a contact element*. This may be generalized to an arbitrary abstract manifold: *a contact element* of it is simply a hyperplane of the tangent space at some point.

The set of contact elements attached to a manifold is also naturally endowed with a structure of manifold: it is simply the projectified cotangent bundle $PT^*(V)$ of the initial manifold V. But it has more structure, which may be concretely felt by realizing that there are special parametrized curves inside it. Such curves represent the movement of a contact element of V, that is, of a mobile point of V and simultaneously of a tangent hyperplane attached to the mobile. The special curves are those such that at each moment, the velocity of the mobile in V is contained in the corresponding hyperplane (this is called in a visual way '*the skating condition*' by Arnold). One sees that the velocities of those curves, when thought inside $PT^*(V)$, sweep a hyperplane of the tangent space of $PT^*(V)$ at each point. Therefore the manifold of contact elements has a natural hyperplane distribution. It is called its *contact structure*.

By working in local coordinates on V, one sees that $PT^*(V)$ may be covered by charts of coordinates $(q_1, ..., q_n, p_1, ..., p_n, z)$ in which the contact structure is defined by the equation:

$$dz + \sum_{i=1}^{n} p_i dq_i = 0 \tag{2.1}$$

Using this formula and Frobenius' theorem, one shows that this hyperplane distribution is not tangent to a foliation of codimension 1 or, said technically, that it is not completely integrable. In fact it is as far as possible from being completely integrable: it is *completely non-integrable*, a notion existing only on odd-dimensional manifolds. It was precisely this condition of complete non-integrability that was taken as definition of a general contact structure:

DEFINITION 2.2. Let M be an oriented $(2n-1)$-dimensional manifold. A **contact structure** on M is a hyperplane distribution ξ in TM given by a global 1-form α such that $\alpha \wedge (d\alpha)^{\wedge(n-1)}$ vanishes nowhere. We say that the pair (M, ξ) is a **contact manifold** and that α is a **contact form**. The form α is called **positive** if $\alpha \wedge (d\alpha)^{\wedge(n-1)}$ defines the chosen orientation of M. If n is even, then the orientation defined by $\alpha \wedge (d\alpha)^{\wedge(n-1)}$ does not depend on the choice of the defining form α, hence one can speak about **positive contact structures**.

The contact form of the left-hand side of equation (2.1) is called *the standard contact form* on $\mathbb{R}^{2n+1}$ and the associated contact structure is *the standard contact structure* on $\mathbb{R}^{2n+1}$.

As explained before, one has also a so-called *standard* contact structure on any manifold of (oriented) contact elements.

2.2. Contact structures coming from complex geometry.

Another very important class of examples (which is the central one for the results presented here) comes from complex analytic geometry. Start from a connected complex manifold X of complex dimension $n \geq 2$ and from a real smooth hypersurface M of it. Denote by $J : TX \to TX$ the (integrable) almost complex structure associated to the complex structure of X, where TX denotes the tangent bundle of the underlying smooth manifold of X. Then $J(TM)$ cannot be equal to TM, because this last space is odd-dimensional. Therefore $\xi := TM \cap J(TM)$ is a J-invariant subspace of real codimension 1 of TM, that is, a hyperplane distribution with a natural complex structure $J|_\xi$. We will call it *the complex distribution* of $M \hookrightarrow X$.

For various hypersurfaces M one can get all the degrees of integrability of this distribution, from the completely integrable case till the completely non-integrable (or contact) one. A general situation when ξ is automatically contact is got when M is *strongly pseudoconvex.*

DEFINITION 2.3. Let ρ be a smooth function on X. It is called **strictly plurisubharmonic** (abbreviated **spsh**) if $-dd^c\rho > 0$, where $d^c\rho := d\rho \circ J \in T^*X$. If a *cooriented* real hypersurface of X may be defined locally in the neighborhood of any of its points as a regular level of a spsh function which grows from its negative to its positive side, then it is called **strongly pseudoconvex**.

It is important to care about the coorientation of the hypersurface: seen from one side it is pseudoconvex, from the other it is pseudoconcave. The terminology was chosen such that the positive side is the pseudoconcave one, distinguished by the fact that holomorphic curves tangent to the hypersurface are locally contained in that side.

The announced general family of contact manifolds given by complex geometry is presented in the next proposition:

Proposition 2.4. *The complex distribution of any strongly pseudoconvex hypersurface of a complex manifold is a (naturally oriented) contact structure.*

The simplest example of this type of construction is given by $X = \mathbb{C}^n$, with $n \geq 2$ and $\rho := \sum_{j=1}^{n} |z_j|^2$. This is a proper spsh function. The complex distribution on any euclidean sphere centered at the origin is therefore a contact structure. As homotheties centered at the origin leave both the foliation of $\mathbb{C}^n \setminus 0$ by such spheres and the almost complex structure invariant, they realize contactomorphisms between all such contact spheres. Therefore, one gets a well-defined contact structure on $\mathbb{S}^{2n-1}$, called *the standard contact structure* on it.

2.3. From Darboux to Martinet.

Let us discuss now the problem of *classification* of contact structures. As in any problem of classification, one has to decide first which objects are considered equivalent. One has to separate the continuously varying characters (the so-called *moduli*) from the discrete ones. When one *varies the underlying space*, one gets like this a 'family' of objects. If one has instead a *fixed underlying space* and only the structure on it is varied, one speaks about *homotopy*. A particular case of homotopy is obtained by changing the structure using a path of isomorphisms of the underlying fixed space, in which case one speaks about *isotopy*.

Let us be more formal for the case of contact structures. A *homotopy* between two contact structures is a smooth path of contact structures connecting them. An *isotopy* between two contact structures is a homotopy of the form $(\phi_t^* \xi)_t$, where $(\phi_t)_t$ is a smooth path of self-diffeomorphisms of M. Two contact structures ξ and ξ' on M are *homotopic*, resp. *isotopic*, resp. *isomorphic* or *contactomorphic* if there is a homotopy, resp. an isotopy, resp. a diffeomorphism of M which sends ξ on ξ'. One usually tries to classify contact structures on a given manifold *up to isotopy* or *up to contactomorphism*.

Any contact structure may be seen as a hyperplane field (or distribution). One has to be careful that a homotopy between the underlying hyperplane fields of two contact structures is not necessarily a homotopy of contact structures, as the path under consideration may cross inside the space of hyperplane fields the discriminant formed by those which are not contact.

Again, a general problem of classification of structures splits into a local and into a global one. For example, complex structures and foliations have no local invariants, riemannian metrics do have (measured for instance by the Levi-Civita connexion and the associated curvature tensors).

Contact forms have no local invariants:

Theorem 2.5. (Darboux) *Any contact form may be written in suitable local coordinates as the left-hand side of (2.1).*

Globally the situation is distinct, due to the fact that there is a canonical vector field attached to any contact form: its so-called *Reeb vector field*, uniquely determined by the requirements to be in the kernel of $d\alpha$ and to have length 1 when measured by α. Then any dynamical invariants of the Reeb vector field are invariants of the contact form, which makes one feel that by deforming a form, the global structure may change drastically. In fact one can get subtle invariants from the study of Reeb vector fields. This is the subject of *contact homology*, but we won't speak about it here.

When one keeps instead of the whole contact form only the contact structure defined by a contact form, the situation is completely different. Indeed, on a closed manifold there is then an Ehresmann-type theorem:

THEOREM 2.6. *(*Gray [**69**]*) Two homotopic contact structures on a closed manifold are isotopic.*

The previous theorem shows that on closed manifolds, contact structures have no moduli, that is, that their classification up to isotopy is discrete. This is the reason why, when looking at the tangent distribution to a real hypersurface of a complex manifold, one does not keep the field of complex operators $J : \xi \to \xi$ as a supplementary structure. Indeed, then one would keep moduli, that is, the analog of Gray's theorem would not be true.

The first question which one asks in any classification problem is that of *existence* of the objects. While in all odd dimensions ≥ 5 the problem of characterization of closed manifolds which admit a contact structure is still open, in dimension 3 one has:

THEOREM 2.7. *(*Martinet*;* Lutz*) Any closed oriented 3-manifold carries a positive contact structure. Moreover, one may find a positive contact structure in any homotopy class of plane fields.*

The first affirmation was proved by Martinet [**114**] and the second one by Lutz [**113**].

2.4. The dichotomy tight/overtwisted.

From now on and till the end of the section, we will restrict to closed oriented 3-manifolds.

Martinet's theorem posed the problem to find invariants which could distinguish contact structures with homotopic underlying plane fields. The breakthrough in this problem came from Bennequin's thesis [**9**]:

THEOREM 2.8. *(*Bennequin*) On the 3-sphere $\mathbb{S}^3$ there exist two non-isotopic contact structures with homotopic underlying plane fields.*

The geometry underlying this result having led in the hands of Eliashberg to the essential dichotomy between *tight* and *overtwisted* contact structures, I want to explain it a little.

Another model than (2.1) of the standard contact structure on $\mathbb{R}^3$ is given by the following equation in cylindrical coordinates:

$$dz + r^2 d\theta = 0.$$

This equation shows that the defined structure is invariant by rotations around the z-axis and by translations parallel to it. Therefore, in order to understand the structure, it is enough to look at it along a half-line starting from the z-axis and perpendicular to it (we use in this discussion the standard euclidean metric attached to the cylindrical coordinate system (z, r, θ)). Choose for example the positive x-axis. The contact planes are tangent to it at each point. At the origin the plane is perpendicular to the z-axis, but as one moves away from the origin, the planes turn with non-vanishing speed in the same sense, tending to a vertical position at infinity, after a quarter of turn.

Let us concentrate now on what is happening inside a bi-infinite circular solid cylinder centered on the z-axis. By translational-invariance, we see that one gets an induced contact structure on the solid torus obtained by taking the quotient of the cylinder by such a (non-trivial) translation. It happens that one obtains in this way models for the tubular neighborhoods of *transverse knots* in contact 3-manifolds. More precisely, if K is a knot in the contact 3-manifold (M, ξ), everywhere transverse to ξ, then it has a fundamental system of tubular neighborhoods contactomorphic with the previous models. This allows to modify the contact structure ξ only inside such a neighborhood by working on the model. There one does the following operation: along the x-axis, insert a full twist between two given positions, and propagate the resulting field of planes in the whole solid cylinder by respecting the rotational and translational invariance. This structure descends of course again to the solid torus. The resulting modification of a contact structure in a neighborhood of a transverse knot is called *a Lutz twist*.

It is a nice exercice to see that a contact structure obtained by a Lutz twist is homotopic through plane fields to the initial contact structure. If it were homotopic through contact structures, Gray's theorem would show that both are in fact isotopic. But it is not always the case.

Consider again the model construction of a Lutz twist inside the solid cylinder in $\mathbb{R}^3$. One gets at least one concentric cylinder with smaller positive radius on the boundary of which the contact planes are perpendicular to the z-axis. Therefore, any meridional disc of such a cylinder is tangent to the contact structure all along its boundary. Such a disc exists therefore in any contact manifold obtained by a Lutz twist. Bennequin proved Theorem 2.8 by proving:

THEOREM 2.9. (Bennequin) *The standard contact structure ξ_{st} on $\mathbb{S}^3$ does not contain any embedded smooth disc tangent to ξ_{st} all along the boundary.*

Eliashberg saw that the classification of contact manifolds which contain such a disc is much simpler than for those which do not contain one. In order to deal with these differences, he introduced the following terminology:

DEFINITION 2.10. An **overtwisted disc** in a 3-dimensional contact manifold is an embedded disc which is tangent to the contact planes all along the boundary. An **overtwisted contact structure** is one which contains an overtwisted disc. A **tight contact structure** is one which is not overtwisted.

In this terminology, he proved:

THEOREM 2.11. (Eliashberg [**41**]) *Two oriented overtwisted contact structures on a closed 3-manifold are isotopic if and only if their underlying oriented plane fields are homotopic.*

In particular, two contact structures obtained by doing Lutz twists starting from the same contact structure are isotopic. Therefore the abundance of possibilities of choice of transversal knots and of twisting inside tubular neighborhoods is only apparent: this is a very surprising aspect of theorem 2.11.

Combining the theorems 2.7 and 2.11, one sees that for each homotopy class of oriented plane fields, there exists exactly one isotopy class of overtwisted oriented contact structures whose underlying plane field is in that homotopy class.

From that point on, the classification centered on that of tight contact structures. But *how to determine if a contact structure is tight*? Bennequin's proof for the standard contact structure in $\mathbb{S}^3$ passed through some very subtle knot theory and did not seem to be transplantable to other 3-manifolds. In fact the first general criterion of tightness was proved using completely different techniques, namely using Gromov's theory [**72**] of pseudoholomorphic curves in symplectic manifolds. Informally, it says that a contact manifold which can be *filled* adequately by a compact symplectic manifold is necessarily tight. In order to explain this precisely, we have to introduce first various notions of *fillings*.

2.5. Various notions of fillability.

For more details about fillings of contact 3-manifolds, see [**132**].

We have seen that contact geometry exists only in odd dimensions. But it interacts very deeply with an even-dimensional geometry, namely *symplectic geometry*.

Definition 2.12. A **symplectic form** on an even-dimensional vector space is a non-degenerate exterior form of degree 2. A **symplectic form** on an even-dimensional manifold is a *closed* non-degenerate smooth form of degree 2. A **symplectic manifold**is a manifold endowed with a symplectic form.

As a first example of the presence of symplectic structures in the contact world, note that part of Definition 2.2 may be rephrased as saying that α is a contact form if and only if $d\alpha$ is a symplectic form in restriction to $\ker\alpha$.

Let us come back to the examples of contact manifolds originating in complex geometry. Consider more particularly as ambient space a *Stein manifold*. Such manifolds are the complex analytic equivalents of affine manifolds of algebraic geometry. The following theorem gives several characterizations of them (see [**68**, page 152], [**133**, Sect. 1-4]):

Theorem 2.13. (Cartan, Serre, Grauert, Narasimhan) *Let X be a complex manifold whose connected components are of bounded dimensions. Then the following are equivalent:*

1) X admits a proper spsh function.

2) X is holomorphically convex (that is, for any infinite discrete set in X, there is a global holomorphic function which is unbounded on it) and its structural sheaf is generated by global functions.

3) X may be properly holomorphically embedded into some $\mathbb{C}^n$.

4) All the positive dimensional cohomology groups of any coherent analytic sheaf on X vanish.

Therefore the following notion is well-defined:

Definition 2.14. A complex manifold which satisfies any of the previous properties is called a **Stein manifold**.

One has an analog of Theorem 2.13 for possibly singular reduced spaces X, the condition of boundedness of the dimension being replaced by that of boundedness of the embedding dimension (which is obviously true for a subspace of some $\mathbb{C}^n$). One obtains then the notion of *Stein space*.

Characterizations 3) and 4) of Stein manifolds are the ones which emphasize their analogy with affine manifolds. Indeed, complex affine manifolds are by definition the algebraic submanifolds of some $\mathbb{C}^n$ and they may also be characterized as those algebraic manifolds such that all the positive dimensional cohomology groups of any coherent algebraic sheaf on X vanish.

As an algebraic embedding is also analytic, we see that affine manifolds are also Stein. The converse is not true even in the world of algebraic manifolds: there are examples of algebraic manifolds which are not affine but whose underlying complex analytic structure is Stein. The first example is due to Serre, and is explained in Hartshorne [**75**]. Even if we won't examine this kind of phenomenon in this survey, I mention it because it seems not to be well-known.

Look now at a Stein manifold from the point of view 1) of Theorem 2.13. That is, consider a Stein manifold X and a proper spsh function $\rho : X \to \mathbb{R}$ bounded from below. Let $M := X_{\rho=a}$ be a regular level of ρ. We call the compact *sublevel* $Y := X_{\rho\leq a}$ *a compact Stein manifold*. One should note that Y is a compact smooth manifold-with-boundary, but that it is not a compact complex manifold. By Proposition 2.4, the complex distribution on M is a contact structure. This motivates:

Definition 2.15. A contact manifold which is contactomorphic to the contact boundary of a compact Stein manifold is called **Stein fillable**, and any such compact Stein manifold is **a Stein filling** of the initial manifold.

A more general notion is obtained by asking that the bounded from below and proper function ρ be spsh only in a neighborhood of its considered regular level M. One obtains then the notion of *compact complex manifold with boundary*, and a related notion of filling:

Definition 2.16. A contact manifold which is contactomorphic to the complex distribution on a strongly convex boundary of a compact complex manifold with boundary is called **holomorphically fillable**, and any such compact complex manifold is a **holomorphic filling** of the initial manifold.

Consider again a pair (X, ρ) as before, that is, $\rho : X \to \mathbb{R}$ is a proper spsh function bounded from below. If one denotes:

$$\alpha_\rho := -d^c\rho$$
$$\omega_\rho := d\alpha_\rho$$

then:

$$\alpha_\rho|_M \text{ is a contact form defining } \xi$$
$$\omega_\rho \text{ is a symplectic form on } X$$

One may forget part of the previous structures and relations in order to arrive at concepts of symplectic geometry, which make no reference to an almost-complex structure:

DEFINITION 2.17. Let (M, ξ) be a closed oriented contact manifold, such that ξ is positive and cooriented.

A **strong symplectic filling** of (M, ξ) is a compact symplectic manifold (Y, ω) with boundary $\partial Y = M$ such that there exists a primitive α of ω *in a neighborhood of* M whose restriction to M is a defining form of ξ.

A **weak symplectic filling** of (M, ξ) is a compact symplectic manifold (Y, ω) with boundary $\partial Y = M$ such that the restriction of ω to ξ is a field of positive symplectic forms on ξ.

The systematic study of such purely symplectic notions of fillability was started in Eliashberg-Gromov [**46**].

A Stein filling of a contact manifold is obviously a strong symplectic filling and a strong symplectic filling is necessarily a weak symplectic filling. All three notions are different, as we will explain in a moment.

Let us formulate the historically first general criterion of tightness for 3-dimensional contact structures (see also [**10**]):

THEOREM 2.18. *(*Gromov [**72**]*,* Eliashberg [**42**]*) A weakly symplectically fillable contact structure is necessarily tight.*

This gives another proof and places in a broader context Bennequin's theorem 2.9. Indeed, as explained before, the standard contact structure on $\mathbb{S}^3$ is Stein filled by the standard ball in $\mathbb{C}^2$.

The results which allow to differentiate the previous notions of fillability in dimension 3 are the following:

• There exist weakly symplectically fillable contact structures which are not strongly symplectically fillable (Eliashberg [**44**] for $\mathbb{T}^3$ and Ding & Geiges [**33**] for arbitrary torus bundles over the circle).

• There exist oriented irreducible 3-manifolds which admit no positive tight contact structures (Etnyre & Honda [**50**] proved this for the Poincaré homology sphere with the orientation opposite to the one obtained as the boundary of the complex surface singularity E_8). Therefore there exist reducible 3-manifolds which admit no tight contact structure at all (one simply takes the connected sum of two copies of the Poincaré homology sphere with both its orientations).

• There exist tight contact manifolds which are not weakly symplectically fillable (Etnyre and Honda [**51**] constructed such structures on some small Seifert manifolds).

• There exist strongly symplectically fillable contact manifolds which are not Stein fillable (Ghiggini [**55**] constructed such a structure on some small Seifert manifolds).

On the other hand, any holomorphically fillable contact 3-manifold is necessarily Stein fillable (Bogomolov and de Oliveira [**13**]). This is false in higher dimensions. For example, Eliashberg, Kim & Polterovich [**47**] have shown that the projective spaces $\mathbb{RP}^{2n-1}$, which are always holomorphically fillable, are not Stein fillable whenever $n \geq 3$. We will speak more about this in Section 6 of this chapter.

2.6. Classification results.

By a theorem of Lisca & Stipsicz [**111**], the oriented Seifert fibered 3-manifolds which do not admit any positive tight contact structure are completely known: they are exactly the manifolds M_n obtained by performing a $(2n-1)$-surgery along the torus knot $T_{2,2n+1}$ in $\mathbb{S}^3$. For $n = 1$, one obtains the Poincaré homology sphere with changed standard orientation, which was the original example of this phenomenon given by Etnyre & Honda [**50**].

It is still not known if there exists a closed *irreducible* 3-manifold without any tight contact structure with either one of its orientations. By Thurston's geometrization conjecture, proved now using Perelman's work, any such manifold admits either a Seifert or a hyperbolic structure. But Gompf [**62**] proved that any Seifert manifold admits a positive tight contact structure with at least one of its orientations. Therefore a possible counterexample could exist only among hyperbolic manifolds. In fact one should search only among hyperbolic manifolds which are rational homology spheres, as on any other irreducible 3-manifold there exist weakly symplectically fillable contact structures (Eliashberg & Thurston [**48**]).

Starting with [**56**], Giroux began to develop cut-and-paste techniques in contact geometry. These were fundamental in order to prove that there are 3-manifolds which admit an infinite number of non-isotopic contact structures with homotopic underlying plane fields (Giroux [**58**]) and to start the classification of tight contact structures on particular 3-manifolds. Till now were completely classified *up to isotopy* the tight contact structures on the following classes of manifolds:

- lens spaces (Giroux [**58**], Honda [**81**]).
- torus bundles over the circle (Giroux [**58**]).
- circle bundles over surfaces (Giroux [**59**], Honda [**82**]).
- some Seifert 3-manifolds (see the references in [**111**]).

Moreover, one has the following general results:

THEOREM 2.19. *(*Colin, Giroux, Honda [**30**]*,* [**31**]*) On any closed oriented 3-manifold, there is a finite number of homotopy classes of plane fields realizable as tight contact structures.*

THEOREM 2.20. *(*Colin, Giroux, Honda, Kazez, Matić*, see* [**30**], [**31**]*) A closed, oriented 3-manifold, carries an infinite number of isotopy classes of tight contact structures if and only if it is toroidal (that is, it contains an embedded incompressible torus).*

The first of these theorems is the end result of a process of successive approximations and radical changes of technique: first Eliashberg [**43**] proved that there is a finiteness result for the Euler classes of the underlying plane fields of tight contact structures (using the *adjunction inequalities* valid for such structures), then Kronheimer & Mrowka [**96**] proved the finiteness of homotopy classes for strongly symplectically fillable contact structures using Seiberg-Witten theory and finally Colin, Giroux & Honda [**30**] proved Theorem 2.19 by putting a contact structure in normal form with respect to a triangulation.

In the enterprise of classification, it was essential to be able to determine whether a constructed contact structure is tight or not. In addition to the fillability criteria, gluing criteria were also developed. Moreover, if the lifted contact structure is tight on some cover, then the initial one is obviously also tight. But these criteria were not enough to pursue the classification. Fortunately, around 2001, Giroux introduced a new tool in the study of contact structures: open books. This is the subject of the next subsection.

2.7. Open books carrying contact structures.

For more details on the content of this subsection, see Colin [**29**] or Etnyre [**49**].

Martinet's proof in [**114**] that any closed oriented 3-manifold admits a positive contact structure started from a presentation of the 3-manifold as the result of surgery on a link in $\mathbb{S}^3$. The fact that any 3-manifold can be obtained in this way had been proved by Lickorish and Wallace.

An alternative proof was proposed by Thurston and Winkelnkemper [**160**], starting from another kind of presentation of a 3-manifold: as *an open book*.

Definition 2.21. An **open book** with **binding** N in a manifold M is a couple (N, θ), where N is a (not necessarily connected) 2–codimensional closed oriented submanifold of M with trivial normal bundle and $\theta : M \setminus N \to \mathbb{S}^1$ is a smooth fibration which in a neighborhood $N \times \mathbb{D}^2$ of N coincides with the angular coordinate. The closures of the fibers of θ are called the **pages** of the open book.

We say that the open books (N, θ) and (N', θ') in the manifolds M, respectively M', are **isomorphic** if there exists a diffeomorphism $\phi : (M, N) \to (M', N')$ which preserves the orientations and carries the fibers of θ into the fibers of θ'.

Notice that $d\theta$ induces natural co-orientations on the binding and the pages of the open book. Thus, any fixed orientation of M induces a natural orientation on N. If N itself is oriented *a priori*, then we say that the open book is *compatible with the orientations of M and N* if the two orientations of N coincide.

The previous terminology was introduced by Winkelnkemper [**171**]. This kind of structure had appeared before, in various contexts, as can be seen in the historical survey [**172**]. In particular, it appeared in Milnor's work on singularity theory, as we explain in subsection 3.3.

Giroux realised that the contact structure constructed by Thurston and Winkelnkemper has a special relation to the open book used to construct it and that this relation was in fact also fundamental in Bennequin's work [**14**]. He formulated this relation in [**60**] in all dimensions through the following definition:

Definition 2.22. A positive contact structure ξ on an oriented manifold M is **carried by an open book** (N, θ) if it admits a defining contact form α which verifies the following:
- α induces a positive contact structure on N;
- $d\alpha$ induces a positive symplectic structure on each fiber of θ.

If a contact form α satisfies these conditions, we say that it is **adapted** to (N, θ).

The construction of Thurston and Winkelnkemper shows that in dimension 3, any open book on a closed oriented manifold carries a positive contact structure (a fact no longer true in higher dimensions). In dimension 3, if such a structure exists, then it is unique:

Theorem 2.23. *(*Giroux [**60**]*, see also* [**49**]*) In dimension 3, two positive contact structures carried by the same open book are isotopic.*

Therefore, in order to describe a positive contact structure on a manifold, it is enough to describe an open book which carries it. Conversely, one has:

Theorem 2.24. *(*Giroux [**60**]*) In dimension 3, any contact structure is carried by some open book.*

Giroux and Mohsen generalized this theorem to all dimensions (see [**60**]).

Two open books which carry the same contact structure are not necessarily isotopic. But in dimension 3, Giroux [**60**] proved that they become isotopic after a process of *stabilisation*, that is, plumbing with positive Hopf bands. This allows to translate problems about isomorphism classes of contact structures into problems about open books up to stabilisation.

A word about the application of open books to the problem of determination whether a contact structure is tight or not. For the moment there is no known algorithm which allows to determine in terms of the monodromy of the open book whether the associated contact structure is tight. But using carrying open books, Ozsváth & Szábo constructed a new invariant $c(\xi)$ of contact structures, which vanishes for overtwisted structures. Therefore, its non-vanishing ensures tightness. It was shown that there are non-fillable contact structures ξ with $c(\xi) \neq 0$, which proves that it is stronger than the fillability criterion 2.18.

3. Milnor fillable contact manifolds

In this section I present the results of the papers [**24**] and [**25**], done in collaboration with Caubel and Némethi. For the needed general notions about complex analytic singularities, one may consult Chapter 3.

3.1. A prototype: Milnor's study of hypersurface singularities.

In the paper [**121**], Mumford proved that if the boundary of a normal surface singularity is simply connected, then one has in fact a regular point, and the boundary is therefore diffeomorphic to a sphere. Around 1965, using recent work of Pham, Brieskorn proved that this is false in any higher dimension: he exhibited isolated hypersurface singularities defined by equations of the form:

$$z_0^{a_0} + \cdots + z_n^{a_n} = 0$$

(nowadays such singularities are called *Brieskorn-Pham singularities*) and whose boundaries are diffeomorphic to standard spheres.

Moreover, he showed that one could obtain like this also *exotic spheres*, that is, manifolds homeomorphic to a standard sphere but not diffeomorphic to it. This was the first construction of such spheres as explicit algebraic sets, after the discovery of their existence by Milnor [**117**]. It intrigued a lot Milnor, who began to think about this discovery. This led to his book [**119**], which founded the topological theory of hypersurface singularities. See Brieskorn [**19**] and Durfee [**38**] for details about the preceding story.

I want to describe briefly the main geometric actors introduced by Milnor [**119**]. Briefly speaking, they allow to *localize* the study of monodromy of the family of levels of a holomorphic function in the neighborhood of a critical point.

Suppose that $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a holomorphic function having an isolated critical point at 0. Then Milnor considered the following objects associated to it:

(1) a sufficiently small euclidean ball B_ϵ centered at the origin.
(2) the intersection $N_\epsilon := f^{-1}(0) \cap \partial B_\epsilon$ of the critical level with the boundary of the ball.

(3) the pieces $F_{\epsilon,\lambda}$ contained in B_ϵ of nearby regular levels $f^{-1}(\lambda)$, for $\lambda \neq 0$ sufficiently small.
(4) the family $(F_{\epsilon,\lambda})_{|\lambda|=const}$ of such pieces, for a fixed absolute value of the level.
(5) the map $\theta : \partial B_\epsilon \setminus N_\epsilon \to \mathbb{S}^1$ defined by the argument of f.

He proved that:

(1) $\epsilon > 0$ may be chosen such that the critical level $f^{-1}(0)$ intersects transversally all the spheres of radius $\leq \epsilon$ centered at the origin. One calls such a ball B_ϵ *a Milnor ball* and its boundary *a Milnor sphere* with respect to f.
(2) if B_ϵ is a Milnor ball, then the pair $(\partial B_\epsilon, N_\epsilon)$ is independent of the choices. One calls it the *embedded link* of the critical point.
(3) one may choose $\delta > 0$ such that $F_{\epsilon,\lambda}$ are diffeomorphic smooth manifolds-with-boundaries whose boundaries are diffeomorphic to N_ϵ, for each λ such that $0 < |\lambda| < \delta$.
(4) the family $(F_{\epsilon,\lambda})_{|\lambda|=const}$ is a locally trivial fibration over the circle. It is called the *Milnor fibration* associated to the considered critical point and its fibers $F_{\epsilon,\lambda}$ are called *Milnor fibers*.
(5) (N_ϵ, θ) is an open book in ∂B_ϵ, whose associated fibration is isomorphic to the Milnor fibration. One calls it the *Milnor open book*.

As explained before, the term " *open book*" was introduced later, partly because Milnor showed that this structure played such an important role in the study of critical points of holomorphic functions, that it was found that it deserves a name.

3.2. The contact boundary of an isolated singularity.

Let (X, x) be a reduced germ of complex analytic space. There exists then a preferred system of compact semi-analytic neighborhoods of x in X: those defined as sufficiently small levels of real semi-analytic non-negative functions whose zero-level has $x \in X$ as an isolated component. Following the terminology of Thom [**159**], let us call such functions *rug functions*. One can show, following the work done by Durfee [**36**] in the algebraic category, that the boundaries of those neighborhoods are all canonically homeomorphic up to isotopy (one uses a variant of Ehresmann's theorem for stratified spaces). Moreover, they are canonically oriented pseudomanifolds. We call their preferred homeomorphism class *the (abstract) boundary* of the singularity (X, x), and we denote it $\partial(X, x)$.

This is also called *the link* of (X, x) in the literature. We prefer to avoid this name, because nowadays it makes reference most of the time to embedded submanifolds of an ambient manifold (for example a collection of circles in the 3-sphere), and in our case we look at it abstractly.

When a representative of X is smooth outside the origin (one says then that (X, x) *has an isolated singularity*, which is a slight abuse of language, as X may be smooth also at 0), then by restricting the class of preferred rug function to smooth real analytic ones, one sees that one may define a preferred class of *smooth* abstract boundaries, which are canonically diffeomorphic up to smooth isotopies. Here one uses the initial version of Ehresmann's theorem.

One may restrict more the class of rug functions, by demanding them to be *spsh*. As was proved by Varchenko [**165**], one gets then a preferred class of oriented manifolds endowed with positive contact structures, which are canonically contactomorphic up to isotopies. One uses in the proof the contact version of Ehresmann's theorem, that is, Gray's theorem 2.6.

DEFINITION 3.1. The oriented contact manifold thus associated, up to contactomorphisms isotopic to the identity, to any isolated singularity (X, x) is called **the contact boundary** of (X, x), and is denoted $(\partial(X, x), \xi(X, x))$. A contact manifold isomorphic to such a contact structure on the boundary of an isolated singularity is called **Milnor fillable**.

We introduced the name 'Milnor fillable' in [**25**] in reference to the prototype [**119**] recalled in subsection 3 of the study of a germ by intersecting it with sufficiently small euclidean balls.

A Milnor fillable contact manifold is holomorphically fillable. Indeed, any resolution of the singularity gives a holomorphic filling of it. By Bogomolov & de Oliveira's theorem [**13**], in dimension 3 it is also Stein fillable. This last fact is false in higher dimensions, as I explain in section 6.

3.3. Milnor open books.

Start from an irreducible equidimensional germ (X, x) of complex analytic space *with isolated singularity*. Denote by $\mathcal{O}_{X,x}$ its local ring of germs of holomorphic functions and by $m_{X,x}$ its maximal ideal, formed by the functions which vanish at x.

A germ of holomorphic function $f \in m_{X,x}$ is said *to have an isolated singularity* at $x \in X$ if there is a representative of (X, f) such that f is regular outside x. By the general theorems of Lê & Teissier [**104**] on limits of tangent hyperplanes to a germ of complex analytic space, one sees that, given (X, x), there are always functions with isolated singularities.

In what follows, we need to restrict more than in the previous subsection the class of rug functions we are working with. Namely, we work with *euclidean rug functions*, which are by definition *spsh* functions of the form:

$$\rho := \sum_{k=1}^{N} |\phi_k|^2$$

where $\phi_1, ..., \phi_N \in m_{X,x}$. The strict plurisubharmonicity of ρ is equivalent to the fact that $(\phi_1, ..., \phi_N)$ defines an immersion of $X \setminus x$ in $\mathbb{C}^N$ ([**25**, Lemma 3.1]).

Fix an $f \in m_{X,x}$ which has an isolated singularity at x. Choose a representative of the germ (X, f) and a euclidean rug function ρ. For $\epsilon > 0$ sufficiently small, one defines:

$$\begin{array}{c} M := X_{\rho=\epsilon} \\ N(f) := M \cap f^{-1}(0) \\ \theta(f) := \arg f : M \setminus N(f) \to \mathbb{S}^1 \end{array} .$$

Following the techniques of Milnor [**119**] (for the case where $X = \mathbb{C}^n$ and ρ is the squared-distance to the origin) as extended by Hamm [**73**] to the case when X is an arbitrary germ but ρ is still the squared-distance to the origin after some embedding in $\mathbb{C}^N$, we showed in [**25**, Proposition-Definition 3.4] that:

PROPOSITION 3.2. *For $\varepsilon > 0$ sufficiently small, the pair $(N(f), \theta(f))$ is an open book in the boundary M and it is compatible with the orientations. Furthermore, its isotopy type does not depend on the choice of $\varepsilon > 0$ nor on the choice of euclidean rug function.*

We called the pair $(N(f), \theta(f))$, well-defined up to isotopy, the *Milnor open book of* f. The pair $(M, N(f))$ is called the *link* of f. Note that "the link" of f is a submanifold of an ambient manifold, as in the most common usage of the term.

Studying carefully and making intrinsic some proofs of Milnor [**119**] and Giroux [**61**], which were treating the case where $X = \mathbb{C}^n$, we proved in [**25**, Theorem 3.9]:

THEOREM 3.3. *Let (X, x) be an irreducible complex analytic germ with isolated singularity and let $f : (X, x) \to (\mathbb{C}, 0)$ be a germ of holomorphic function with isolated singularity. Then its Milnor open book carries the standard contact structure $(\partial(X, x), \xi(X, x))$.*

More precisely, we showed that for $\epsilon > 0$ sufficiently small, the open books $(N(f), \theta(f))$ carry the contact structure defined by the complex tangencies.

Answering one of my questions, Caubel extended this theorem in [**23**] to the contact boundaries of the Milnor fibers of a special class of non-isolated singularities (those defined by a map $(\mathbb{C}^{n+p}, 0) \to (\mathbb{C}^p, 0)$ "with no blowing-up").

3.4. The topological invariance of 3-dimensional contact boundaries.

For the notations used in this subsection about normal surface singularities and their resolutions, see subsection 3.1 of Chapter 3.

In 2003 Clément Caubel and myself began to discuss about the contact boundaries of singularities. We knew the work [**162**] of Ustilovsky, showing that one obtains an infinite number of pairwise non-contactomorphic contact structures on the standard differentiable spheres $\mathbb{S}^{4m+1}$ for $m \geq 1$, as contact boundaries of Brieskorn-Pham singularities . Namely, for varying $p \geq 1$, the contact boundaries of the isolated hypersurface singularities defined by the equation:

$$z_0^p + z_1^2 + \cdots + z_{2m}^2 = 0$$

are pairwise non-isomorphic. In order to distinguish them, he computed their contact homology. This led Caubel to ask question (1) of Chapter 5.

This result motivated us to ask ourselves if the same phenomenon could exist in complex dimension two. As the result of various discussions, we arrived at the following *strategy* to prove that in fact *in dimension 2 the standard contact structure is a topological invariant*:

Step 1. Show that the standard contact structure is carried by any Milnor open book.

Step 2. Show that, given the topological type of a normal surface singularity, there exists a numerically principal topological type of link which may be realised by a holomorphic function for any analytical realisation of the singularity.

Step 3. Conclude using Giroux's theorem 2.23.

It was the desire to accomplish step 1 which motivated us to prove Theorem 3.3. The desire to accomplish step 2 made us start the collaboration with András Némethi. Together with him we proved the following result of algebraic geometry (see [**25**, Theorem 4.1]):

THEOREM 3.4. *Let $\pi : (\tilde{X}, E) \to (X, x)$ be a strict normal crossings resolution of a normal surface singularity (X, x). Assume that the effective exceptional divisor D satisfies*

$$(D + E + K_{\tilde{X}}) \cdot E_i + 2 \leq 0 \quad \text{for any} \quad i \in I.$$

Then there exists a function $f \in m_{X,x}$, with an isolated singularity at x, such that $(\pi^* f)$ is a normal crossing divisor on $\tilde{X}$ with $(\pi^* \circ f)_e = D$. Moreover, for each i, the number of intersection points $n_i := (\pi^* \circ f)_s \cdot E_i$ is positive.

That is, if we impose that the pull-back of a holomorphic function *vanishes enough* on the exceptional divisor of a resolution in a way which is explicit only in terms of the topology of the resolution, then such functions *exist* and moreover may be chosen such that the divisor of their pull-back has normal crossings. This ensures that the topology of this divisor is determined by the chosen topology of the vanishing divisor. In general, if one takes an *arbitrary* element of the Lipman semigroup (see subsection 3.1 of next chapter), it is only for rational singularities that one may be sure that there exists a function whose pull-back vanishes exactly along it.

If we start from the minimal resolution with normal crossings, whose topology encodes the same information as that of the boundary of the singularity, using the work [**135**] of Pichon one extracts from the previous theorem the following proposition, which accomplishes step 2 of the strategy:

COROLLARY 3.5. *Let M be a closed connected oriented 3-manifold which is Milnor fillable. Then there exists an open book (N, θ) in M, which can be completely characterized by the topology of M, such that, for any germ (X, x) of normal complex surface with $M \simeq \partial(X, x)$, there exists a function $f \in m_{X,x}$ having an isolated singularity at 0 and whose Milnor open book $(N(f), \theta(f))$ is isomorphic to (N, θ).*

Step 3 is now automatic. One gets the announced theorem of topological invariance of the standard contact structure on the boundary of a normal surface singularity:

THEOREM 3.6. *Any Milnor fillable 3-manifold admits a unique Milnor fillable contact structure up to contactomorphism. If the manifold is a rational homology sphere, then the structure is unique up to isotopy.*

First we proved the second sentence of the previous theorem (see [**24**]). In order to conclude, I want to explain the difficulty we had to overcome in order to get the uniqueness for boundaries which were not rational homology spheres. From Theorem 3.4, we get in a way which is standard for the topological study of surface singularities, a weaker version of Corollary 3.5, in which we get a link (M, N) which may be realized by a function $f \in m_{X,x}$, for any analytical realization of the given topology. But in general (M, N) does not determine the open book with binding N up to isotopy (see [**135**], [**124**]), excepted for the case of rational homology spheres. Nevertheless, Pichon [**135**] showed that if $n_i > 0$ for all i (see the definition of n_i in the last sentence of Theorem 3.4), then the link $(\partial(X, x), N(f))$ corresponding to such a function determines the open book *up to an isomorphism*. It is false that in general this open book is determined up to an isotopy. Nevertheless, the associated contact structure could be. This motivates question (2) of Chapter 5.

Theorem 3.6 shows that singularity theory defines a *standard* contact structure on any Milnor fillable oriented 3-manifold, up to contactomorphisms. This motivates question (3) of Chapter 5.

4. Smoothable singularities and their Milnor fibers

4.1. General facts.

For more details on the material contained in this subsection, one may consult Teissier [**158**], Stevens [**156**] and Greuel, Lossen & Shustin [**71**].

One of the ways to study singularities, is to see them as *limits* of smooth spaces and to understand how the structure of the smooth spaces is captured at the limit by the singular point. The usual way in algebraic and analytic geometry to conceptualize limits is to take *families* of objects and a special member of the family. Not only the members of the family are requested to belong to the category under consideration, but also their total space, that is, the members of the family are asked to be the fibers of a morphism in the category.

The definition which this suggests is too general. For example, if one looks at the morphism of blow-up of a point in the plane and at the special fiber over that point, one would get that the projective line is a limit of points, which does not correspond to the analysed intuition. In order to get a notion more proximate to this intuition, one would like to ask at least that all the fibers of the morphism be *equidimensional*. The algebraic notion of *flatness* ensures this and in fact much more. That is why one restricts in the following way the notion of *deformation* in analytic geometry:

DEFINITION 4.1. Let (X, x) be a germ of a complex analytic space. A **deformation** of (X, x) is a germ of *flat* morphism $\psi : (Y, y) \to (S, s)$ together with an isomorphism between (X, x) and the special fiber $\psi^{-1}(s)$.

For example, when X is reduced, $f \in m_{X,x}$ is flat as a morphism $(X, x) \xrightarrow{f} (\mathbb{C}, 0)$ if and only if f does not divide zero. Such deformations over germs of smooth curves are called 1-*parameter deformations*. The simplest example is got when $X = \mathbb{C}^n$ and f has an isolated singularity at 0. Then one gets the situation considered by Milnor and recalled in subsection 3.1, in which now f is thought as a deformation of $(\{f = 0\}, 0)$.

In general, to think about a flat morphism as a deformation means to see it as a family of continuously varying fibers and to concentrate on a particular fiber. From such a family, one gets new families by rearranging the fibers, that is, by *base change*. One is particularly interested in the situations where there exist families which generate all other families by such base changes. The following definition is a reformulation of [**71**, Definition 1.8, page 234]:

DEFINITION 4.2.
1. A deformation ψ of (X, x) is **complete** if any other deformation is obtainable from it by a base-change.
2. A complete deformation ψ of (X, x) is called **versal** if for any other deformation over a base (T, t) and identification with a pull-back from ψ of the induced deformation over a subgerm $(T', t) \hookrightarrow (T, t)$, one may extend this identification over all (T, t) with a pull-back from ψ.

(3) A versal deformation is **miniversal** if the Zariski tangent space of its base (S, s) has the smallest possible dimension.

When the miniversal deformation exists, its base space is unique *up to non-unique isomorphism* (only the tangent map to the isomorphism is unique). For this reason, one does not speak about a *universal* deformation, and was coined the word "miniversal", with the variant "semi-universal".

In many references, versal deformations are defined as the complete ones in the previous definition. Then usually is stated the theorem that the base of a versal deformation is isomorphic to the product of the base of a miniversal deformation and a *smooth* germ. With this weaker definition the result is false. Indeed, starting from a complete deformation, by doing the product of its base with *any* germ (not necessarily smooth) and by taking the pull-back, we would get again a complete deformation. This shows that a complete deformation is not necessarily versal. Nevertheless, the theorem stated before is true with the previous definition of versality.

A very important property of versal deformations is that they remain versal if one moves slightly the germ on which one concentrates when thinking about the flat family as a deformation (see Pourcin [**144**] and Bingener [**11**]):

THEOREM 4.3. (**Openness of versality**) *Let $f : X \to S$ be a flat morphism of complex spaces whose singular locus is finite over S. Then the set of points $s \in S$ such that f induces a versal deformation of the germ of the fiber over s at each of its singular points is the complement of an analytic subspace of S.*

Not all germs admit versal deformations. But those we are interested in do admit:

THEOREM 4.4. (Schlessinger [**151**], Grauert [**67**]) *Let (X, x) be an isolated singularity. Then the miniversal deformation exists and is unique up to (non-unique) isomorphism.*

Let us come back to the situation we were speaking about at the beginning of the section, where a germ is seen as a limit of *smooth* spaces:

DEFINITION 4.5. A **smoothing** of a singularity is a 1-parameter deformation whose general fiber is smooth. A **smoothing component** is an irreducible component of the reduced miniversal base space over which the generic fibers are smooth.

Isolated complete intersection singularities have a miniversal deformation $(Y, y) \xrightarrow{\psi} (S, s)$ such that both Y and S are smooth, therefore irreducible (see [**158**], [**112**]). In general, the *reduced miniversal base* (S_{red}, s) may be reducible. The first example of this phenomenon was discovered by Pinkham [**136**, Chapter 8] (see also [**7**, Section 3.3]):

PROPOSITION 4.6. (Pinkham) *The germ at the origin of the cone over the rational normal curve of degree 4 in $\mathbb{P}^4$ has a reduced miniversal base space with two components, both being smoothing ones.*

Not all isolated singularities are smoothable. The most extreme case is attained with *rigid* singularities, which are not deformable at all in a non-trivial way. For example, quotient singularities of dimension ≥ 3 are rigid (Schlessinger [**152**]). For details about these phenomena, one may consult Greuel & Steenbrink [**70**]. In subsection 6.1, I give a new (as far as I know) purely topological obstruction to smoothability for singularities of dimension ≥ 3. In dimension 2 no such criterion is known in full generality. We say more about this in the next subsection.

Let us look now at the topology of the generic fibers above a smoothing component. We want to localize the study of the family in the same way as Milnor localized the study of a function on $\mathbb{C}^n$ near a singular point. This is possible (see Looijenga [**112**]):

THEOREM 4.7. *Let (X, x) be an isolated singularity. Let $(Y, y) \xrightarrow{\psi} (S, s)$ be a miniversal deformation of it. There exist (Milnor) representatives Y_{red} and S_{red} of the reduced total and base spaces of ψ such that the restriction $\psi : \partial Y_{red} \cap \psi^{-1}(S_{red}) \to S_{red}$ is a trivial C^∞-fibration. Moreover, one may choose such representatives such that over each smoothing component S_i, one gets a locally trivial C^∞-fibration $\psi : Y_{red} \cap \psi^{-1}(S_i) \to S_i$ outside a proper analytic subset.*

Hence, for each smoothing component S_i, the oriented diffeomorphism type of the oriented manifold with boundary $(\pi^{-1}(s) \cap Y_{red}, \pi^{-1}(s) \cap \partial Y_{red})$ does not depend on the choice of the generic element $s \in S_i$: it is called *the Milnor fiber* of that component. Moreover, its boundary is canonically identified with the boundary of (X, x) *up to isotopy*. In particular, the Milnor fiber of a smoothing component is diffeomorphic to a Stein filling of the contact boundary $(\partial(X, x), \xi(X, x))$. Therefore, the following problem has a meaning:

Given an isolated singularity (X, x), describe its Milnor fibers up to orientation-preserving diffeomorphisms which extend the natural identifications on the boundaries and identify them among the different Stein fillings of $(\partial(X, x), \xi(X, x))$.

In dimension two, in order to get a topological invariant of the singularity, one may look at the Milnor fibers of all normal surface singularities with a given topology. There is no a priori well-defined identification up to isotopy of all their boundaries, excepted for taut singularities (see Definition 3.24 of Chapter 3)). For them, one gets a *finite* number of Milnor fibers up to diffeomorphisms, by Theorem 4.4. This motivates questions (4) and (5) of Chapter 5.

4.2. The case of surfaces.

The notions about normal surface singularities used here are recalled in the section 3 of Chapter 3.

In dimension 2, I do not know purely topological obstructions to smoothability for *all* normal singularities. But there exist such obstructions for special *Gorenstein* (see definition 2.2 of Chapter 3) normal surface singularities (see also [**169**]), as a consequence of:

THEOREM 4.8. *(*Steenbrink [**154**]*) Let (X, x) be a Gorenstein normal surface singularity. If it is smoothable, then:*

$$\mu_- = 10 p_g(X, x) - b_1(\partial(X, x)) + (Z_K^2 + |I|). \tag{4.9}$$

In the preceding formula, μ_- denotes the negative part of the index of the intersection form on the second homology group of any Milnor fiber and $b_1(\partial(X, x))$ denotes the first

Betti number of the boundary of (X,x). It may be computed from any normal crossings resolution with exceptional divisor $E=\sum_{i\in I}E_i$ as:

$$b_1(\partial(X,x))=b_1(\Gamma)+2\sum_{i\in I}p_i,$$

where p_i denotes the genus of E_i and Γ denotes the dual graph of E. The term $Z_K^2+|I|$ may also be computed using any normal crossings resolution, and is again a topological invariant of the singularity.

The previous theorem implies that the expression in the right-hand side of (4.9) is ≥ 0, which gives non-trivial obstructions on the topology of special smoothable normal Gorenstein singularities. For example, it shows that among simple elliptic singularities (definition 3.22 of Chapter 3), the smoothable ones have minimal resolutions whose exceptional divisor is an elliptic curve with self-intersection $\in\{-9,-8,...,-1\}$. More generally, one gets like this constraints on the topology of smoothable minimally elliptic singularities, a class determined by its topology (see Theorem 3.19 of Chapter 3).

In what precedes, we have spoken only about the negative part μ_- of the index of the intersection form on the second homology of any Milnor fiber. Denoting μ_0 and μ_+ the *null*, respectively *positive* part of that index, we have the following theorem, which was proved first by Durfee [**34**] for isolated hypersurface singularities, then by Steenbrink [**154**] in this full generality (see also [**169**]):

Theorem 4.10. (Durfee, Steenbrink) *Any Milnor fiber of a normal surface singularity (X,x) satisfies:*

$$\mu_0+\mu_+=2p_g(X,x). \tag{4.11}$$

Therefore, $\mu_0+\mu_+$ is not a topological invariant of the singularity, but it is an analytical one (it does not depend on the smoothing component). In turn, μ_0 is topological (see also [**169**]):

Theorem 4.12. (Greuel & Steenbrink [**70**]) *Any Milnor fiber of a normal surface singularity (X,x) has vanishing first Betti number, which is equivalent to:*

$$\mu_0=b_1(\partial(X,x)). \tag{4.13}$$

Combining (4.11) and (4.13), one gets the following constraint from the topology on the geometric genus: $2p_g(X,x)\geq b_1(\partial(X,x))$.

As the only Gorenstein rational singularities are the Kleinian ones (see Theorem 3.16 of Chapter 3), we see that Theorem 4.8 tells us nothing about their possible non-smoothability. In fact:

Theorem 4.14. (M. Artin [**4**]) *All rational singularities are smoothable. Moreover, any component of the reduced miniversal base space is a smoothing component.*

Among the components of the reduced miniversal base space S_{red} of a rational surface singularity, Artin showed that there is a distinguished one (later called the *Artin component*) which is the image under the natural map of the miniversal base space of the total space of the minimal resolution of the singularity. Its associated Milnor fiber is diffeomorphic to a compact tubular neighborhood of the exceptional divisor of the minimal resolution of the singularity.

As shown already by the example of cyclic quotient singularities, μ_- *may depend* on the chosen smoothing component (this happens for Pinkham's example as stated in Proposition 4.6). Therefore, the condition to be *Gorenstein* is absolutely necessary in Theorem 4.8. We would like to emphasize that the phenomenon of existence of non-diffeomorphic Milnor fibers appears only in dimension ≥ 2. Indeed, for reduced curve singularities one has:

THEOREM 4.15. (Buchweitz & Greuel [**20**]) *Let (X, x) be a smoothable reduced curve singularity with $r \geq 1$ irreducible components. Then all its Milnor fibers are connected and their first Betti number μ depends only on analytical invariants of (X, x):*

$$\mu = 2\delta(X, x) - r + 1.$$

Let us come back to surfaces. In [**144**], Pinkham treated the deformation theory of germs of surfaces endowed with a $\mathbb{C}^*$-action. He looked in particular at the germs at their vertices of the affine cones over all the *rational normal curves* (that is, the images of the projective line $\mathbb{P}^1$ by all the Veronese embeddings). He showed that only for the cone over the curve of degree 4 in $\mathbb{P}^4$, the miniversal base space is disconnected.

Seen from another viewpoint, the singularity of the rational normal curve of degree $n \geq 2$ is isomorphic to the cyclic quotient singularity $(\mathcal{X}_{n,1}, 0)$ (see Definition 3.17 of Chapter 3). More generally, Christophersen [**27**] and Stevens [**155**] succeeded to determine the exact number of components of the reduced miniversal base space of $(\mathcal{X}_{p,q}, 0)$ for any $p > q > 0$. This number is defined combinatorially in a subtle way. In turn, their result motivated Lisca to make a conjecture, which was seen by Némethi and myself as the entrance to the general study of the topology of Milnor fibers of rational surface singularities. In the next section I explain Lisca's conjecture, as well as our proof of it.

5. The Milnor fibers of cyclic quotient singularities

For the needed background on continued fractions, one may consult section 4 of Chapter 3. In all this section, we will denote simply by $[x_1, ..., x_k]$ the HJ-continued fraction $[x_1, ..., x_k]^-$, as we won't use E-continued fractions.

5.1. Lisca's conjecture.

In [**109**], Lisca announced a classification of the symplectic fillings of the standard contact structure on lens spaces up to orientation-preserving diffeomorphisms. Detailed proofs were given in [**110**]. He showed that there is a *finite* number of *minimal* fillings, all of them diffeomorphic to Stein surfaces, and he parametrized them using a special kind of sequences of integers. These sequences had appeared before in the works of Christophersen and Stevens, parametrizing the irreducible components of the reduced miniversal base of the corresponding cyclic quotient singularity. As each such component is a smoothing component, the corresponding Milnor fiber gives a Stein filling of the lens space with the standard contact structure. This motivated him to formulate the following conjecture:

CONJECTURE 5.1. (Lisca [**110**, page 768]) *The Milnor fiber of an irreducible component of the reduced miniversal base space of a cyclic quotient singularity is orientation-preserving diffeomorphic to the Stein filling of its contact boundary parametrized by the same sequence.*

This conjecture does not only claim that one gets *all* the Stein fillings from the Milnor fibers, but it claims an *explicit* correspondence between them.

Let us explain now more precisely Lisca's description of the minimal symplectic fillings of $(L(p,q), \xi_{st})$ (see Definition 3.17 of Chapter 3). First, we have to define the finite parameter space he uses, composed of certain sequences of positive integers whose associated HJ-continued fraction vanishes.

DEFINITION 5.2. A sequence $\underline{k} = (k_1, ..., k_r) \in \mathbb{N}^r$ is **admissible** if either $r = 1$ or $r \geq 2$, $\underline{k} \in (\mathbb{N} \setminus 0)^r$, $[k_1, ..., k_i] > 0$ for all $i \in \{1, ..., r-1\}$ and $[k_1, ..., k_r] \geq 0$. For $r \geq 1$, denote by:

$$K_r := \{\underline{k} = (k_1, \dots, k_r) \in \mathrm{adm}(\mathbb{N}^r) \ \mid \ [k_1, \dots, k_r] = 0\} \tag{5.3}$$

the set of **admissible sequences which represent** 0. For $\underline{k} = (k_1, \dots, k_r) \in K_r$ set $\underline{k}' := (k_r, \dots, k_1) \in K_r$.

Note that the condition of *admissibility* in the definition of K_r is really restrictive. For example, $\underline{k} = (2, 1, 1, 1, 1, 2) \notin K_6$ although $[\underline{k}] = 0$. By admissibility, if $r > 1$, then each $k_i > 0$. K_1 has only one element, namely (0).

For two coprime integers p, q with $p > q \geq 1$ and HJ-expansion $\frac{p}{p-q} = [a_1, \dots, a_r]$, set:

$$K_r(\tfrac{p}{p-q}) = K_r(\underline{a}) := \{\underline{k} \in K_r \mid \underline{k} \leq \underline{a}\} \subset K_r, \tag{5.4}$$

where $\underline{k} \leq \underline{a}$ means that $k_i \leq a_i$ for all i. Fix an element $\underline{k} \in K_r(\underline{a})$. Let $\mathcal{L}(\underline{k})$ be the framed link of Figure 5.1 with s components and decorations $k_1, \dots, k_s$ (i.e. the thick components are neglected for a moment). Let $N(\underline{k})$ be the closed, oriented 3-manifold obtained by surgery on $\mathbb{S}^3$ along the framed link $\mathcal{L}(\underline{k})$. Using the slam-dunk operation on rationally-framed links in $\mathbb{S}^3$ (see [**63**, page 163]), one sees that there exists an orientation-preserving diffeomorphism:

$$\eta : N(\underline{k}) \longrightarrow \mathbb{S}^1 \times \mathbb{S}^2. \tag{5.5}$$

DEFINITION 5.6. [**110**, page 766] Consider the diffeomorphism η from (5.5) and denote by $L(\underline{a}, \underline{k}) \subset N(\underline{k})$ the image of the thick framed link drawn in Figure 5.1. Define $W_{p,q}(\underline{k})$ to be the smooth oriented 4-manifold with boundary obtained by attaching two-handles to $\mathbb{S}^1 \times \mathbb{D}^3$ along the framed link $\eta(L(\underline{a}, \underline{k})) \subset \mathbb{S}^1 \times \mathbb{S}^2$.

The manifold $W_{p,q}(\underline{k})$ is therefore obtained by attaching index 2 handles to the 4-ball along the whole framed link described in Figure 5.1, and replacing the sublevel of a corresponding Morse function which contains the ball and the handles attached along $\mathcal{L}(\underline{k})$ with the manifold $\mathbb{S}^1 \times \mathbb{D}^3$. One may show that this construction does not depend on the choice of the orientation-preserving diffeomorphism η.

Lisca's classification theorem is:

THEOREM 5.7. *(*Lisca [**110**]*)*
(a) All the manifolds $W_{p,q}(\underline{k})$ admit Stein structures which fill $(L(p,q), \xi_{st})$, and any Stein filling (even minimal symplectic filling) of $(L(p,q), \xi_{st})$ is diffeomorphic to one of the manifolds $W_{p,q}(\underline{k})$.

(b) $W_{p,q^1}(\underline{k}^1)$ is orientation-preserving diffeomorphic to $W_{p,q^2}(\underline{k}^2)$ if and only if $(q^2, \underline{k}^2) = (q^1, \underline{k}^1)$ or $(q^2, \underline{k}^2) = ((q^1)', (\underline{k}^1)')$, where $qq' \equiv 1 \pmod{p}$ and $(k_1, ..., k_r)' := (k_r, ..., k_1)$.

Particular cases of this theorem had been proved before by Eliashberg [**42**] (for $\mathbb{S}^3$) and McDuff [**116**] (for the spaces $L(p, 1)$, for all $p \geq 2$). I find interesting to remark that $(L(p, 1), \xi_{st})$ is the contact boundary of the singularity at the vertex of the cone over the rational normal curve of degree p, which is a family of singularities especially emphasized by Pinkham [**144**]. McDuff discovered the possiblity of existence of more than one minimal symplectic fillings with the example of $L(4, 1)$, which corresponds to the singularity for which Pinkham had discovered the possibility of existence of more than one smoothing components.

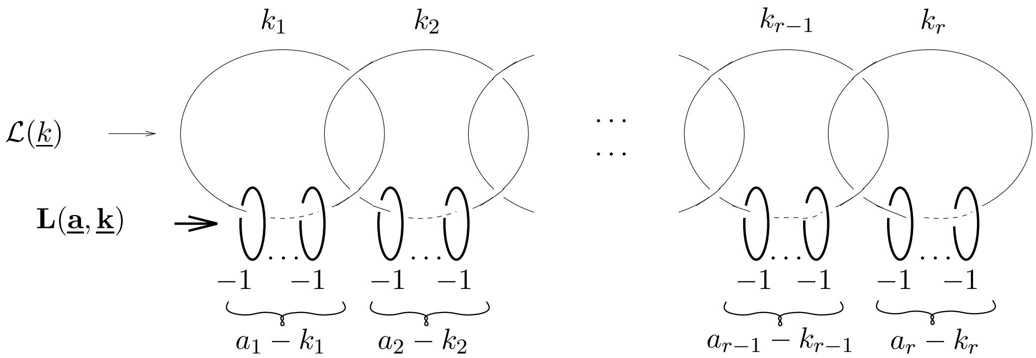

FIGURE 5.1. The framed link $L(\underline{a}, \underline{k}) \subset N(\underline{k})$

In order to prove Conjecture 5.1, we wanted to identify the sequence $\underline{k}$ associated to the Milnor fibers of $(\mathcal{X}_{p,q}, 0)$. We had to understand how Lisca reconstructed it from a given Stein filling W of $(L(p, q), \xi_{st})$. He did it by homological computations, but not working only with the pair $(W, \partial W)$. Instead, he first got a closed oriented 4-manifold by gluing W along its boundary to another *fixed* 4-manifold. Seen from the side of this second manifold, the boundary of W is orientation-preserving diffeomorphic to $L(p, p - q)$. Therefore, one has to choose a fixed filling of this lens space, which is the *dual* of $L(p, q)$. An obvious choice is $\Pi(\underline{a})$, the oriented 4-manifold obtained by plumbing along the weighted dual graph of the minimal resolution of the singularity $\mathcal{X}_{p,p-q}$, dual to $\mathcal{X}_{p,q}$ (see the paragraph which follows formulae (3.18) of Chapter 3).

Therefore, if W is a Stein filling of $(L(p, q), \xi_{st})$, denote by V the closed 4-manifold obtained by gluing W and $\Pi(\underline{a})$ via an orientation-preserving diffeomorphism $\phi : \partial W \to \partial\overline{\Pi(\underline{a})}$ of their boundaries and by $\mu : \Pi(\underline{a}) \hookrightarrow V$ the inclusion morphism. Lisca's recognition criterion is:

PROPOSITION 5.8. *(*Lisca [**110**, §7]*) Denote by $\{s_i\}_{1 \leq i \leq r}$ the classes of 2-spheres $\{S_i\}_{1 \leq i \leq r}$ in $H_2(\Pi(\underline{a}))$ (listed in the same order as $\{a_i\}_{1 \leq i \leq r}$), and also their images via the monomorphism $\mu_* : H_2(\Pi(\underline{a})) \to H_2(V)$. Then for all $i \in \{1, \dots, r\}$ one has:*

$$\#\{e \in H_2(V) \mid \ e^2 = -1, \ s_i \cdot e \neq 0, \ s_j \cdot e = 0 \text{ for all } j \neq i\} = 2(a_i - k_i) \tag{5.9}$$

for some $\underline{k} \in K_r(\underline{a})$. In this way one gets the pair $(\underline{a}, \underline{k})$, and W is orientation-preserving diffeomorphic to $W_{p,q}(\underline{k})$.

5.2. The notion of ordered lens space.

Notice that, as $\{S_i\}_{1\leq i\leq r}$ and $\{S_{r-i}\}_{1\leq i\leq r}$ cannot be distinguished, the method given in Proposition 5.8 does not differentiate $(\underline{a}, \underline{k})$ from $(\underline{a}', \underline{k}')$, hence $W_{p,q}(\underline{k})$ from $W_{p,q'}(\underline{k}')$.

For this reason, when $q = q'$, one cannot distinguish the Stein fillings $W_{p,q}(\underline{k})$ and $W_{p,q}(\underline{k}')$. Therefore, in this case Conjecture 5.1 does not claim that there is a *bijection* between the irreducible components of the reduced miniversal base space and the Stein fillings: there is an involution of the set of the components (induced by the involution $\underline{k} \to \underline{k}'$ acting on the parameter set $K_r(\underline{a})$) such that the Milnor fibers are diffeomorphic (if the conjecture is true) if and only if they correspond to an orbit of this involution.

Némethi and I wondered if a stronger conjecture could not be true: instead of looking at the Milnor fibers only *up to diffeomorphism*, we could look only at diffeomorphisms which *in restriction to the boundary are isotopic to the natural diffeomorphisms of identification* (see Theorem 4.7). In order to manage the computations with continued fractions, we wanted to see what supplementary structure on a Stein filling of $(L(p,q), \xi_{st})$ allowed to recuperate $\underline{k}$ *without any ambiguity* in the spirit of Proposition 5.8. We saw that we could get this result simply by fixing a supplementary structure on the bounding lens space.

Bonahon [**14**] proved that *each lens space contains up to isotopy a unique embedded 2-dimensional torus — a so-called splitting torus —, which bounds on each side a solid torus.* The set $\mathcal{T}$ of solid tori bounded by a splitting torus, identified modulo isotopies of the ambient space, is a set of one or two elements. It has one element exactly when the solid tori can be interchanged by an isotopy. This happens precisely when $q \in \{1, p-1\}$, cf. [**14**, page 308].

Definition 5.10. An **order** of a lens space is a total order on the set $\mathcal{T}$.

Clearly, if $q \in \{1, p-1\}$, then this supports no additional information. In all other cases the order distinguishes *the first* and *the second* of the two (non-isotopic) solid tori bounded by *any* splitting torus.

The notion of order has a similar nature as the notion of orientation ($\mathcal{T}$ is analogous to the set of connected components of the orientation bundle of a manifold), but it is independent of it.

There is an ambiguity in the notation '$L(p,q)$' for an oriented lens space: $L(p,q')$ is orientation-preserving diffeomorphic to it. But if one fixes also an order, then one may extract the pair (p,q) without any ambiguity from it. From now on, we consider that '$L(p,q)$' denotes such an oriented lens space with a preferred order.

5.3. Christophersen's and Stevens' works on deformations of cyclic quotients.

The set $K_r(\underline{a})$ of admissible sequences representing zero and restricted by $\underline{a}$ appearing in Lisca's classification was introduced before by Christophersen [**27**].

Denote by $S_{red}(p,q)$ the reduced miniversal base space of the cyclic quotient singularity $(\mathcal{X}_{p,q}, 0)$. Inspired by Arndt's work [**1**], Christophersen wrote for each $\underline{k} \in K_r(\underline{a})$ an explicit system $\mathcal{E}_{\underline{k}}$ of equations which define $\mathcal{X}_{p,q}$, and an explicit deformation $\tilde{\mathcal{E}}_{\underline{k}}$ of these equations with smooth parameter space. Based in an essential way on the work [**95**] of Kollár &

Shepherd-Barron, Stevens proved in [**155**] that one gets in this way all the irreducible components of $S_{red}(p,q)$.

THEOREM 5.11. *(*Stevens*) The reduced base space $S_{red}(p,q)$ of the miniversal deformation of $\mathcal{X}_{p,q}$ has exactly $\#K_r(\underline{a})$ irreducible components.*

Through the equations of Christophersen and Stevens one has in fact an *explicit* bijection between the set $K_r(\underline{a})$ and the irreducible components of $S_{red}(p,q)$. This precises the meaning of Lisca's conjecture.

We denote by $S^{CS}_{\underline{k}}$ the irreducible component which corresponds to $\underline{k} \in K_r(\underline{a})$.

The system $\mathcal{E}_{\underline{k}}$ is best described using toric geometry. The singularity $\mathcal{X}_{p,q}$ may also be seen as the germ at the zero-dimensional orbit of the toric variety $\mathcal{Z}_{\sigma_{p,q}} = \mathrm{Spec}\ \mathbb{C}[\check{\sigma}_{p,q} \cap M]$, where $\sigma_{p,q} \subset N_{\mathbb{R}}$ is an oriented cone in N of type $\frac{p}{q}$, and $M := \mathrm{Hom}(N, \mathbb{Z})$. The functions $(z_0, \dots, z_{r+1})$ are the characters corresponding to the minimal generating set of the semigroup $\check{\sigma}_{p,q} \cap M$. Therefore, they satisfy the relations:

$$z_{i-1}z_{i+1} - z_i^{a_i} = 0 \quad \text{for all } i \in \{1, \dots, r\}. \tag{5.12}$$

The toric surface $\mathcal{Z}_{\sigma_{p,q}}$ may be embedded inside $\mathbb{C}^{r+2}$ using the regular functions $z_0, \dots, z_{r+1}$. Christophersen and Stevens write for each $\underline{k} \in K_r(\underline{a})$ the special system $\mathcal{E}_{\underline{k}}$ of binomial equations which defines the image $\mathcal{X}_{p,q}$ of $\mathcal{Z}_{\sigma_{p,q}}$ by this embedding. See [**27**, pages 83-84], [**155**, pages 316-317] or [**7**, pages 8-11] for different presentations. Some of the equations, including (5.12), are independent of $\underline{k}$. Using the special form of the equations, one defines their deformations $\tilde{\mathcal{E}}_{\underline{k}}$, see [**27, 155**].

5.4. The steps of the proof of Lisca's conjecture.

Looking at the equations defining the systems $\tilde{\mathcal{E}}_{\underline{k}}$, Némethi and I searched the simplest possible 1-parameter deformation which defines a smoothing associated to the component $S^{CS}_{\underline{k}}$. In fact, such a 1-parameter deformation is uniquely determined by the deformed equations of (5.12) (cf. [**27**], [**155**, (2.2)]). These last are:

$$z_{i-1}z_{i+1} = z_i^{a_i} + t \cdot z_i^{k_i} \quad \text{for all } i \in \{1, \dots, r\}, \tag{5.13}$$

where $t \in \mathbb{C}$. Note that, although (5.12) did not depend on $\underline{k}$, this is not the case for their deformations (5.13). Let $\mathcal{X}^t_{\underline{k}}$ be the affine space determined by the equations $\mathcal{E}^t_{\underline{k}}$ in $\mathbb{C}^{r+2}$.

LEMMA 5.14. *The deformation $t \mapsto \mathcal{X}^t_{\underline{k}}$ has negative weight and is a smoothing belonging to the component $S^{CS}_{\underline{k}}$. In particular, $\mathcal{X}^t_{\underline{k}}$ is a smooth affine space for $t \neq 0$.*

The first statement just means that the weight of the added monomial $z_i^{k_i}$ is not larger than the weight of $z_{i-1}z_{i+1} - z_i^{a_i}$, i.e. $k_i \leq a_i$. By [**167**, (2.2)], this implies that:

$$\mathcal{X}^t_{\underline{k}} \text{ is diffeomorphic to the Milnor fiber of } S^{CS}_{\underline{k}}. \tag{5.15}$$

The special form of equations (5.13) implies that one may express from them each variable z_i rationally in terms of z_0 and z_1 (an idea we had by reading Balke's paper [**5**]):

LEMMA 5.16. *For each $i \in \{1, \dots, r+1\}$, on $\mathcal{X}^t_{\underline{k}}$ one has:*

$$z_i = z_0^{-Z(a_2, \dots, a_{i-1})} P_i \tag{5.17}$$

for some $P_i \in \mathbb{Z}[t, z_0, z_1]$. *The polynomials* P_i *satisfy the inductive relations:*

$$P_{i-1} \cdot P_{i+1} = P_i^{a_i} + tP_i^{k_i} \cdot z_0^{(a_i - k_i)\cdot Z(a_2,\ldots,a_{i-1})} \tag{5.18}$$

with $P_1 = z_1$ *and with the convention* $P_0 = 1$.

In the previous formulae, $Z(x_1, ..., x_k) := Z^{-}(x_1, ..., x_k)$ is the family of polynomials defined by the equations (4.2) of Chapter 3.

Therefore, the affine surface $\mathcal{X}^t_{\underline{k}}$ is the closure in $\mathbb{C}^{r+2}$ of the graph of the rational function $(z_0, z_1) \cdots \to (z_2, ..., z_{r+1})$, where $z_2, ..., z_{r+1}$ are given by the previous lemma. Our strategy is then to identify the surface by eliminating the indeterminacies of the previous map, seen as a rational map defined on $\mathbb{P}^2 = \mathbb{C}^2 \cup L_\infty$. Denote also by L_0 the projective line whose equation in $\mathbb{C}^2$ is $z_0 = 0$. The equations stated in the previous lemma are sufficiently manipulable to allow us to show:

THEOREM 5.19. *Consider the lines* L_∞ *and* L_0 *on* $\mathbb{P}^2$ *as above. Blow up* $r - 1 + \sum_{i=1}^{r}(a_i - k_i)$ *infinitely close points of* L_0 *in order to get the dual graph in Figure 5.2 of the configuration of the total transform of* $L_\infty \cup L_0$ *(this procedure is unique topologically, and its existence is guaranteed by the fact that* $\underline{k} \in K_r(\underline{a})$*). Denote the space obtained by this modification by* $B\mathbb{P}^2$. *Then the Milnor fiber* $\mathcal{X}^t_{\underline{k}}$ *of* $S^{CS}_{\underline{k}}$ *is diffeomorphic to* $B\mathbb{P}^2 \setminus (\cup_{j=0}^{r} V_j)$.

Moreover, let T *be a small open tubular neighbourhood of* $\cup_{j=0}^{r} V_j$, *and set* $F_{p,q}(\underline{k}) = B\mathbb{P}^2 \setminus T$. *Then* $F_{p,q}(\underline{k})$ *is a representative of the Milnor fiber of* $S^{CS}_{\underline{k}}$ *as a manifold with boundary whose boundary is* $L(p, q)$.

Furthermore, the marking $\{V_i\}_i$ *as in the Figure 5.2, defines on the boundary of* $F_{p,q}(\underline{k})$ *an order. Then this ordered boundary is* $L(p, q)$.

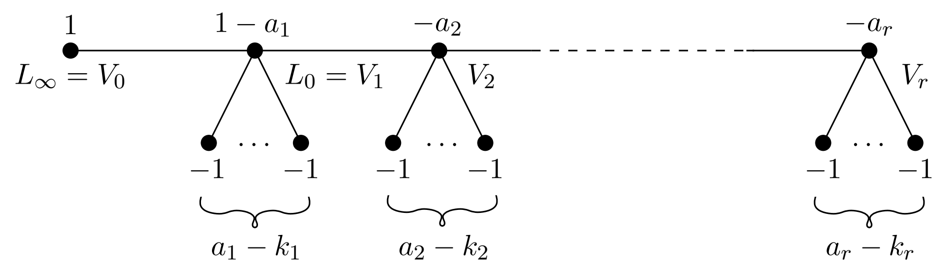

FIGURE 5.2. Illustration for the Theorem (5.19)

Using Proposition 5.8 and Theorem 5.19, we get the following Theorem, which proves a strong form of Lisca's conjecture:

THEOREM 5.20. $W_{p,q}(\underline{k})$ *is orientation-preserving diffeomorphic to* $F_{p,q}(\underline{k})$ *by a diffeomorphism which preserves the orders of the boundaries.*

5.5. Comparison with the approach of de Jong and van Straten.

On the other hand, in [**89**], de Jong and van Straten studied by an approach completely different from Christophersen and Stevens the deformation theory of cyclic quotient singularities (as a particular case of a deformation theory of so-called *sandwiched singularities*). They also parametrized the Milnor fibers of $\mathcal{X}_{p,q}$ using the elements of the set $K_r(\frac{p}{p-q})$. Therefore, one can formulate the previous conjecture for their parametrization as well.

In [**126**], we proved that this analogous conjecture is also true in the strong sense. As a consequence, we see that de Jong & van Straten parametrize in the same way by the elements of $K_r(\underline{a})$ the components of $S_{red}(p,q)$ as Christophersen & Stevens.

5.6. Conclusions.

I state briefly below the contributions brought by our paper [**126**]:

• *We introduce an additional structure associated with any (non-necessarily oriented) lens space: the 'order'. Its meaning in short is the following: geometrically it is a (total) order of the two solid tori separated by the (unique) splitting torus of the lens space; in plumbing language, it is an order of the two ends of the plumbing graph (provided that this graph has at least two vertices). Then we show that the oriented diffeomorphism type and the order of the boundary $L(p,q)$, together with the parameter $\underline{k} \in K_r(\underline{a})$ determines uniquely the filling, up to orientation-preserving diffeomorphisms fixed on the boundary.*

• *We endow in a natural way all the boundaries of the spaces involved (Lisca's fillings $W_{p,q}(\underline{k})$, Christophersen-Stevens' Milnor fibers $F_{p,q}(\underline{k})$, and de Jong-van Straten's Milnor fibers $W(\underline{a},\underline{k})$) with orders. Then we prove that all these spaces are connected by orientation-preserving diffeomorphisms which preserve the order of their boundaries: $W_{p,q}(\underline{k}) \simeq F_{p,q}(\underline{k}) \simeq W(\underline{a},\underline{k})$. This is an even stronger statement than the result expected by Lisca's conjecture since it eliminates the ambiguities present in Lisca's classification.*

• *In fact, we even provide a fourth description of the Milnor fibers: they are constructed by a minimal sequence of blow ups of the projective plane which eliminates the indeterminacies of an explicit rational function defined only in terms of $\underline{a}$ and $\underline{k}$.*

• *As a byproduct it follows that both Christophersen-Stevens and de Jong-van Straten parametrized the components of the miniversal base space in the same way (a fact not proved before, as far as we know).*

• *Moreover, we obtain that the Milnor fibers corresponding to the various irreducible components of the miniversal space of deformations of $\mathcal{X}_{p,q}$ are pairwise non-diffeomorphic by orientation-preserving diffeomorphisms whose restrictions to the boundaries preserve the order.*

Before our work, the only results identifying the Stein fillings obtainable as Milnor fibers were obtained by Ohta & Ono [**130, 131**]. In [**131**], they showed that up to diffeomorphisms, the only Stein fillings of the contact boundary of a Kleinian singularity is given by its unique Milnor fiber. In [**130**] they showed that up to diffeomorphisms, the only Stein fillings of the contact boundary of a simple elliptic singularity is given either by one of their Milnor fibers or by a compact tubular neighborhood of the exceptional divisor of the minimal resolution. We emphasize that in this case, as opposed to the situation met for rational singularities and explained after Theorem 4.14, no Milnor fiber is diffeomorphic to it.

In fact, the existence of such diffeomorphisms characterizes rational singularities, as may be seen by combining Theorem 4.10 of this chapter and Theorem 3.1 of Chapter 3. This motivates questions (6) and (7) of Chapter 5.

Note that for any non-smooth normal surface singularity, in spite of the fact that no tubular neighborhood of the exceptional divisor of the minimal resolution is Stein, one may get Stein neighborhoods after an arbitrarily C^0-small deformation of the complex structure (see Bogomolov & de Oliveira [**13**]). Therefore, such neighborhoods are orientation-preserving diffeomorphic to Stein fillings of the contact boundary.

6. The cohomology rings of holomorphically fillable manifolds

I explain in this section the results of my paper [**142**].

We have seen before various notions of fillability by holomorphic spaces: *Stein fillability, holomorphic fillability* and *Milnor fillability*. I want to explain in this section that in dimension ≥ 3, those notions are pairwise different even at the topological level, without the need to appeal to invariants of contact structures. This will be a consequence of structure theorems for the cohomology *rings* of fillable manifolds, for each notion of fillability. The structure theorems all have the formal shape:

For a fillable manifold, the product of cohomology classes with coefficients in a ring A, of small (explicit) degrees, vanishes whenever the degree of the product is (explicitly) sufficiently big.

In the previous statement, A is the ring of integers for Stein fillable manifolds and the field of rationals for the two other notions of fillability.

6.1. The case of Stein fillable manifolds.

One has the following fundamental theorem about the homotopy type of a Stein manifold (see [**118**]):

THEOREM 6.1. (Thom, Bott, Andreotti & Frankel, Milnor) *A Stein manifold has the homotopy type of a CW-complex of dimension at most equal to its complex dimension.*

I realised that this gives constraints on the cohomology *rings* of the boundaries of *compact* Stein manifolds. More generally, one gets such constraints each time one has upper bounds on the homotopical dimension of the manifold whose boundary is studied:

THEOREM 6.2. *Let W be a compact, connected, orientable manifold-with-boundary of dimension $m \geq 4$. Denote by N its boundary. Suppose that W is homotopically of dimension $\leq h$. Consider numbers $i_1, ..., i_k \in \{1, ..., m-2-h\}$ such that $i_1+\cdots+i_k \geq h+1$. Then the morphism $H^{i_1}(N) \otimes \cdots \otimes H^{i_k}(N) \longrightarrow H^{i_1+\cdots+i_k}(N)$ induced by the cup-product in cohomology with arbitrary coefficients vanishes identically.*

Using Theorem 6.1, one deduces immediately:

COROLLARY 6.3. *Let N be a Stein fillable manifold of dimension $2n-1 \geq 5$. Consider numbers $i_1, ..., i_k \in \{1, ..., n-2\}$ such that $i_1 + \cdots + i_k \geq n+1$. Then the morphism $H^{i_1}(N) \otimes \cdots \otimes H^{i_k}(N) \longrightarrow H^{i_1+\cdots+i_k}(N)$ induced by the cup-product in cohomology with arbitrary coefficients vanishes identically.*

Another consequence of Theorem 6.2 comes from the fact that a tubular neighborhood of a subvariety of a complex manifold retracts by deformation on the subvariety, which implies that its homotopical dimension is bounded above by the dimension of the subvariety. Applying this to the exceptional locus of the resolution of an isolated singularity, one gets:

PROPOSITION 6.4. *Let (X, x) be an irreducible, normal, isolated singularity of complex dimension n. Denote by N its abstract boundary. If one can find numbers $i_1, ..., i_k \in \{1, ..., 2n-2-h\}$ such that $i_1+\cdots+i_k \geq h+1$ and the morphism $H^{i_1}(N)\otimes\cdots\otimes H^{i_k}(N) \longrightarrow H^{i_1+\cdots+i_k}(N)$ induced by the cup-product does* not *vanish identically, then the exceptional set of any resolution of (X, x) has complex dimension at least $(h + 1)/2$.*

I explain now a simple illustration of this criterion. Consider the cone over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$. As explained at the end of subsection 2.2 of Chapter 3, its singularity at 0 admits small resolutions. This singularity may also be seen as the result of contraction of the zero-section in the total space of the line bundle $\mathcal{O}(-1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$, induced by the Segre embedding. Consider instead any other line bundle of the form $\mathcal{O}(-n)$, for $n \geq 2$. Then Proposition 6.4 shows that the singularity obtained by contracting its zero-section cannot admit a small resolution.

As another consequence of Theorem 6.2, one gets the following topological obstruction to smoothability (this uses the fact that a Milnor fiber of a smoothing is a Stein filling of the boundary of the considered singularity):

PROPOSITION 6.5. *Let (X, x) be an irreducible, normal, isolated, singularity of complex dimension n. Denote by N its abstract boundary. If one can find numbers $i_1, ..., i_k \in \{1, ..., n-2\}$ such that $i_1 + \cdots + i_k \geq n+1$ and the morphism $H^{i_1}(N) \otimes \cdots \otimes H^{i_k}(N) \longrightarrow H^{i_1+\cdots+i_k}(N)$ induced by the cup-product does* not *vanish identically, then N is not Stein fillable. In particular, (X, x) is not smoothable.*

As a special case, we have:

COROLLARY 6.6. *Let (X, x) be the isolated singularity obtained by contracting the* 0-*section of an anti-ample line bundle on an abelian variety Σ of complex dimension ≥ 2, and whose first Chern class is not primitive in $H^2(\Sigma, \mathbb{Z})$. Then the boundary of (X, x) is not Stein fillable. In particular, (X, x) is not smoothable.*

This answers partially the concluding question asked by Biran in [**12**].

For other obstructions to smoothability, one may consult Hartshorne [**76**] and Greuel & Steenbrink [**70**].

6.2. The case of Milnor fillable manifolds.

As a consequence of Mumford's work [**121**], boundaries of normal surface singularities are graph manifolds. But not all graph-manifolds may be obtained like this. Indeed, Sullivan [**157**] showed:

THEOREM 6.7. *(*Sullivan*) Let N be the boundary of a normal surface singularity. Then the triple cup product $H^1(N, \mathbb{Q})^{\wedge 3} \to \mathbb{Q}$ vanishes identically.*

He proved this theorem by seeing N as the boundary of a tubular neighborhood W of a resolution of the singularity under consideration, and used the fact that the associated intersection form is negative definite (see Theorem 3.1 of Chapter 3). This implies that the morphism $H^1(W) \to H^1(N)$ is surjective and that $H^2(W) \to H^2(N)$ vanishes identically, which is enough to conclude.

As an immediate consequence of Theorem 6.7, one gets:

COROLLARY 6.8. *The 3-dimensional torus is not Milnor-fillable.*

When Etienne Ghys asked me if I could generalize this result to higher dimensions, as it was normal I thought to generalize Sullivan's proof. But I knew no result analogous to the negative-definiteness of the intersection form valid in higher dimensions. I asked a few famous specialists of algebraic geometry about such an analog, without any result. Finally, in September 2007, listening to a talk by Jan Schepers, I learnt about a theorem of de Cataldo & Migliorini giving Hodge-theoretical constraints on the exceptional divisors of resolutions of isolated singularities in *any* dimension. This indication was enough to make me arrive, surfing through the literature, at the following purely topological result:

THEOREM 6.9. *(*Goresky & MacPherson [**65**]*) Let W be a divisorial resolution of a Milnor representative of a normal isolated singularity of complex dimension $n \geq 2$ and N be its boundary. Then (cohomology groups being considered with* rational *coefficients and all the morphisms being induced by inclusions), the morphisms $H^i(W) \to H^i(N)$ are* surjective *for $i \in \{0, ..., n-1\}$ and* vanish identically *for $i \in \{n, ..., 2n-1\}$.*

Goresky & MacPherson deduced their theorem as a consequence of a deep decomposition theorem in intersection homology theory proved by Beilinson, Bernstein, Deligne & Gabber [**8**, Theorem 6.2.5, page 163]. See [**142**] for other equivalent forms of the theorem and instances of the formulation of some of these equivalent forms in the literature.

As noted in [**65**, page 123], for any compact oriented manifold W with boundary N, the kernel of the morphism $H_\bullet(N) \to H_\bullet(W)$ between the total homologies, induced by the inclusion $N \hookrightarrow W$, is half-dimensional inside $H_\bullet(N)$. The previous theorem describes this kernel when W is a divisorial resolution of an isolated singularity: it is exactly $\oplus_{i=n}^{2n-1} H_i(N)$.

In the case of a germ of surface (X, x), the previous theorem is equivalent to the non-degeneracy of this intersection form, as can be easily seen using some diagram chasing in the cohomology long exact sequence of the pair (W, N), Poincaré-Lefschetz duality for the manifold-with-boundary W, and the fact that W retracts by deformation on E. Therefore, Goresky & MacPherson's theorem is a generalization of the non-degeneracy of the intersection form associated to a resolution of a normal surface singularity.

Using Theorem 6.9 and standard manipulations of singular cohomology, I got a new proof of:

THEOREM 6.10. *(*Durfee & Hain [**39**]*) Let N be a $(2n-1)$-dimensional Milnor fillable manifold, where $n \geq 2$. Consider numbers $i_1, ..., i_k \in \{1, ..., n-1\}$ such that $i_1 + \cdots + i_k \geq n$. Then the morphism $H^{i_1}(N) \otimes \cdots \otimes H^{i_k}(N) \longrightarrow H^{i_1+\cdots+i_k}(N)$ induced by the cup-product in cohomology with rational coefficients vanishes identically.*

When I proved the previous theorem, I was not aware that it had already been proved. It was Looijenga who told me this a little later in Utrecht. What is strange is that since its announcement in [**37**], it was almost never quoted; the same remark applies to Goresky & MacPherson's theorem. These results seem to have remained confined to the consciousness of specialists of mixed Hodge theory.

As a very concrete application of theorem 6.10, one gets the following generalization of corollary 6.8:

COROLLARY 6.11. *For all $n \geq 2$, the torus $\mathbb{T}^{2n-1}$ is not Milnor-fillable.*

6.3. The case of holomorphically fillable manifolds.

Let N be a holomorphically fillable manifold. Then, one may identify it with the strongly pseudoconvex boundary of a compact complex manifold. After contracting the maximal compact analytic set, one gets a Stein space with isolated singularities. One may construct on it a spsh Morse function having local minima at all the singular points and the boundary as a level set. This allows to use Theorem 6.10 near each singular point, then Poincaré-Lefschetz duality for the complement of Milnor neighborhoods of the singularities, to get:

THEOREM 6.12. *(*Bungart [**21**]*) Suppose that $n \geq 3$. Let N be a holomorphically fillable manifold of dimension $2n-1$. Consider numbers $i_1, ..., i_k \in \{1, ..., n-2\}$ such that $i_1 + \cdots + i_k \geq n+1$. Then the morphism $H^{i_1}(N) \otimes \cdots \otimes H^{i_k}(N) \longrightarrow H^{i_1+\cdots+i_k}(N)$ induced by the cup-product in cohomology with rational coefficients vanishes identically.*

When I obtained the previous theorem, a short time after Theorem 6.10, I was again not conscious that it was already known. But once I discovered the reference [**39**], a careful examination of the literature led me to discover [**21**]. Bungart's proof follows the same way as the one I found, once one knows Theorem 6.10.

As an immediate application of Theorem 6.12, one gets the following generalization of Corollary 6.11:

COROLLARY 6.13. *For all $n \geq 3$, the torus $\mathbb{T}^{2n-1}$ is not holomorphically fillable.*

By a theorem of Bourgeois [**15**] (which uses in an essential way Giroux's theory of the relation between contact structures and open books), if a closed orientable manifold M admits a contact structure, then $M \times \mathbb{T}^2$ does too. This implies that all odd-dimensional tori admit contact structures, as $\mathbb{T}^3$ does (see the next paragraph). The previous corollary shows that a contact structure on a torus of dimension at least 5 cannot be holomorphically fillable.

The 3-dimensional torus $\mathbb{T}^3$, however, is holomorphically fillable: it can be realized as a strongly pseudoconvex boundary of a tubular neighborhood of $\mathbb{S}^1 \times \mathbb{S}^1$ standardly embedded in $\mathbb{C}^2$ (see Eliashberg [**44**]). By the theorem of Sullivan quoted in the previous section and generalized in Theorem 6.10, $\mathbb{T}^3$ is not Milnor fillable. In a similar way, we get using Theorem 6.10:

PROPOSITION 6.14. *For any $n \geq 2$, the product $\mathbb{T}^n \times \mathbb{S}^{n-1}$ is holomorphically fillable but not Milnor fillable.*

Combining Corollary 6.6 and Proposition 6.14, we see that *in all odd dimensions ≥ 5, the classes of Stein, Milnor or holomorphically fillable manifolds are pairwise distinct.*

CHAPTER 3

Contributions to the topology of normal surface singularities

1. Motivations

When I began to think about Lisca's conjecture, I first read carefully his paper classifying the symplectic fillings of the standard contact structure on lens spaces. Part of my difficulty to understand that paper came from much use of Hirzebruch-Jung continued fractions. I knew rather well those continued fractions from my previous work on quasi-ordinary singularities, but here they were used in a new context. I felt the necessity to clarify this, and I started comparing the various occurrences of computations done with continued fractions in singularity theory and 3-manifold topology.

This is how I realised that all the computations I was conscious about could be explained geometrically in a unified way, using a very elementary duality of plane supplementary cones with respect to a lattice. I discovered later that this duality was already known, under equivalent more algebraic formulations, but its role in unifying computations with continued fractions seemed not to have been emphasized. As I felt that this unification brought a real economy of thought and a more global viewpoint, I decided to explain it carefully, and I wrote [**141**].

While I was writing that paper, I realised that the geometric duality I was emphasizing allowed also to strenghten an important theorem of Neumann [**127**], stating that the weighted dual graph of the minimal resolution with normal crossings of a normal surface singularity could be reconstructed from the topology of its abstract boundary. Namely, I proved that the plumbing structure corresponding to that resolution is actually reconstructible *up to isotopy*. As I saw this result as potentially useful in future work on contact boundaries, I decided to include in [**141**] also a survey about the topology of normal surface singularities (plumbing structures, graph structures, Seifert structures) and the use of continued fractions in relating them.

A very intriguing question about the topology of normal surface singularities is to characterize the topological types of isolated surface singularities in $\mathbb{C}^3$. The only known restriction on this topology I am conscious of comes from the fact that such a surface singularity is Gorenstein and smoothable. But arbitrary complete intersection singularities are also Gorenstein and smoothable, therefore one gets like this no special attribute of hypersurface singularities among them.

In 2007 I started talking with José Seade about this problem. During his stay in Paris in autumn 2007, we did a kind of brainstorming on this problem, nourished from a careful examination of various classes of examples. Suddenly emerged a very optimistic conjecture: that once the arithmetic genera of the components of the exceptional divisor of the minimal resolution are fixed, there are a finite number of possibilities for the self-intersections which could originate from a hypersurface/complete intersection/Gorenstein/numerically Gorenstein normal surface singularity (we did not know at that moment which choice could be correct). We did not believe very much in this conjecture, and as stated it is indeed false. But not very false: in fact we could prove in [**143**] a finiteness result of this kind for the *canonical cycle* of the minimal resolution for the largest possible category among the ones listed before: that of numerically Gorenstein singularities. Moreover, we described precisely the possible non-finiteness of the self-intersections. Our proof is very short but surprised us a lot: we got a contradiction from a passage to the limit in an infinite sequence of decorated graphs, a type of argument we did not see before in singularity theory.

Section 2 is dedicated to general notions about singularities *in all dimensions*: Cohen-Macaulay, Gorenstein, normal, quotient singularities and resolutions of singularities. Section 3 is dedicated to general facts about the resolutions and classifications of normal *surface* singularities. The content of those sections is also useful for the understanding of the previous chapter. Section 4 explains the duality of supplementary cones and its application to singularities. Section 5 explains the finiteness theorem on numerically Gorenstein singularities.

2. General notions on singularities in arbitrary dimensions

2.1. Special classes of singularities.

According to common usage among singularity theorists, we say that a germ (X, x) of complex analytic space is a *complex analytic singularity*, and this even if the closed point x of the germ is smooth on it. From time to time (for example, when working with points *near* x), we will work with *representatives* of the singularity.

When we speak about an *n-fold singularity*, we understand a *reduced, equidimensional* germ of dimension n. In particular, a *surface singularity* is a germ of reduced equidimensional space of complex dimension 2.

Concretely, singularities may be defined by the vanishing of a set of convergent power series. The simplest case is that of *hypersurfaces*, when one takes only one power series. More generally, one has *complete intersections*, which may be defined by the same number of power series as their codimension in the ambient smooth space. One may show that such a sequence of power series $(f_1, ..., f_k)$ satisfies the fact that at each step, if one looks at the germ defined by the first few of them, then the next one is not a zero-divisor in restriction to it: one says that the sequence is *regular*. One may show then that complete intersections are a particular case of *Cohen-Macaulay singularities*, which are maximal from the view-point of existence of regular sequences:

DEFINITION 2.1. A singularity (X, x) is called **Cohen-Macaulay** if its maximal ideal has a regular sequence with $\dim_{\mathbb{C}}(X, x)$ elements.

But complete intersections are still more particular inside the class of Cohen-Macaulay singularities, they are *Gorenstein*:

DEFINITION 2.2. A Cohen-Macaulay singularity (X, x) is **Gorenstein** if its dualising module $\omega_{X,x}$ is free (as an $\mathcal{O}_{X,x}$-module).

For details about the notions of *dualising* (or *canonical*) module and Gorenstein singularities, one may consult Eisenbud [**40**] and Reid [**148**]. The dualising module is the germ at x of the *dualising sheaf* ω_X, which is well-defined on any Cohen-Macaulay space (a space all of whose germs are Cohen-Macaulay). In restriction to the smooth locus $X \setminus \mathrm{Sing} X$, the dualising sheaf ω_X is simply the sheaf of holomorphic differential forms of maximal degree. It is more complicated to understand what it means along the singular locus. But when X is *normal*, the situation is simpler:

PROPOSITION 2.3. *Suppose that X is a* normal *complex analytic space. Then:*

$$\omega_X \simeq i_* \omega_{X \setminus \mathrm{Sing} X}$$

where $X \setminus \mathrm{Sing} X \xhookrightarrow{i} X$ denotes the inclusion morphism. In particular, if the germ (X, x) is normal, then it is Gorenstein if and only if there exists a nowhere-vanishing holomorphic form of maximal degree defined on the smooth locus of some neighborhood of x.

Recall that a *reduced* complex analytic space is called *normal* if the Riemann extension theorem (true over a smooth space) is also true over it:

DEFINITION 2.4. Let X be a reduced complex analytic space. If U is an open subspace of X, a **weakly holomorphic function** on U is a holomorphic and bounded function defined on $U \setminus Y$, where Y is a nowhere dense closed subspace of U. The space X is called **normal** if every weakly holomorphic function on U extends in a unique way to a holomorphic function on U, and this must occur for any open subset U of X.

Each reduced space X may be *normalized*: a *normalization* $\overline{X}$ of it is a normal analytic space endowed with a finite surjective morphism $\overline{X} \xrightarrow{\nu} X$ of degree 1. Such a morphism is unique up to unique isomorphism over X.

A class of singularities defined in all dimensions is that of *quotient singularities*:

DEFINITION 2.5. A **quotient singularity** is a germ analytically isomorphic to a germ of space obtained as a quotient of a smooth space by a finite group.

Quotient singularities are normal by the general construction of quotient spaces (see [**112**] or [**7**]). By a local linearization theorem, one may show that in all dimensions quotient singularities are isomorphic to germs of the form $\mathbb{C}^n/G$, where G is a finite subgroup of $GL(n, \mathbb{C})$. Say that an element of the general linear group is a *complex reflection* if it fixes pointwise a hyperplane. By a theorem of Chevalley [**26**], the quotient of $\mathbb{C}^n$ by a finite group generated by complex reflections is again isomorphic to $\mathbb{C}^n$. Now, if $G \subset GL(n, \mathbb{C})$ is an arbitrary finite group, its subgroup G_c generated by complex reflections is a normal subgroup, therefore one may construct the quotient $\mathbb{C}^n/G$ as a two-step quotient $(\mathbb{C}^n/G_c)/(G/G_c)$. One can show that the linearized action of G/G_c contains no non-trivial complex reflections: it is a so-called *small* linear group.

We see that any quotient singularity is obtainable as the germ at 0 of the quotient of $\mathbb{C}^n$ by a *small* finite linear group. Moreover, Prill [**145**] proved that the corresponding linear representation is encoded in the analytical structure of the corresponding quotient singularity.

2.2. Resolutions of singularities.

One of the ways to study singularities is to look at them as images of projections of smooth spaces. A priori one could look for such spaces in arbitrary higher dimensions, but the most studied projections are restricted in the following way:

DEFINITION 2.6. Let X be a reduced complex analytic space. A **resolution of singularities** of X is a morphism $\tilde{X} \xrightarrow{\pi} X$ such that:

- $\tilde{X}$ is smooth;
- π is proper;
- π is bimeromorphic;
- $\tilde{X} \setminus \pi^{-1}(\text{Sing } X) \xrightarrow{\pi} X \setminus \text{Sing } X$ is an isomorphism.

The **exceptional locus** $\text{Exc}(\pi)$ of π is the compact subspace $\pi^{-1}(\text{Sing } X)$ of $\tilde{X}$.

I mention that some writers do not impose the last condition in the definition of a resolution of singularities. Its presence has the advantage that the boundary of a tubular neighborhood of the singular set may be canonically identified up to an isotopy to the boundary of a tubular neighborhood of the exceptional set, which helps a lot in its topological study. I also mention that some writers call *exceptional* only the components of $\pi^{-1}(\text{Sing } X)$ whose image by π has a strictly smaller dimension.

By a fundamental theorem of Hironaka [**78**], all algebraic varieties admit resolutions of singularities, obtainable moreover by sequences of blow-ups of *smooth* centers. His proof extends readily to complex analytic *germs*, but with much more effort to complex analytic *spaces*.

When (X, x) is a germ of curve, the normalization morphism resolves the singularity. This is no longer true in higher dimensions, but normalization destroys nevertheless the singular locus in codimension 1; that is, the singular locus of a normal space is of codimension ≥ 2. If X is normal, then the exceptional locus of any resolution of singularities has everywhere dimension ≥ 1. Therefore, when $\dim X = 2$, it is a divisor of $\tilde{X}$. In general, we say that the resolution is *divisorial* if the exceptional locus is a divisor.

Starting from dimension 3, there exist singularities admitting non-divisorial resolutions, and even resolutions with exceptional sets having everywhere codimension ≥ 2 (we will call them *small resolutions*). The simplest example is given by the singularity at the origin 0 of the cone X over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$. This last surface is a smooth quadric in $\mathbb{P}^3$, therefore it is doubly ruled. Select one of the rulings, say, by the fibers of the first projection $\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{p_1} \mathbb{P}^1$. One may consider the rational map:

$$\begin{array}{lcl} X & \cdots \rightarrow & \mathbb{P}^1 \\ x & \cdots \rightarrow & p_1[x] \end{array}$$

where $[x]$ denotes the point of the quadric corresponding to the generator of the cone passing through x. This map is well-defined outside 0. Denote by $\tilde{X}$ the closure of its graph in $X \times \mathbb{P}^1$ and by $\tilde{X} \xrightarrow{\pi} X$ the natural projection on the first factor. One may show that this morphism is a resolution of singularities of X, with exceptional locus isomorphic (through the projection on the second factor) with $\mathbb{P}^1$.

3. Surface singularities

3.1. Objects associated to a resolution of surface singularity.

Let (X, x) be a germ of normal complex analytic surface. It has a well-defined (up to unique isomorphism over (X, x)) *minimal* resolution, through which any other resolution factors.

Denote by $(\tilde{X}, E) \xrightarrow{\pi} (X, x)$ any resolution, where E denotes the reduced fibre over 0. Therefore E can be seen as a connected reduced effective divisor in $\tilde{X}$, called *the exceptional divisor* of π. The divisor E has not necessarily normal crossings. But by blowing-up the points at which E has not a normal crossing or at which its components are singular, one obtains canonically starting from π a *strict normal crossings resolution*, that is, one with normal crossings divisor having smooth irreducible components. If one starts this process from the minimal resolution, one obtains *the canonical strict normal crossings resolution*.

Denote by Γ the *dual (intersection) graph* of E: its vertices correspond bijectively to the components of E and between two distinct vertices i and j there are as many (unoriented) edges as the intersection number $e_{ij} := E_i \cdot E_j \geq 0$ of the corresponding components. In particular, Γ has no loops. We weight each vertex i of Γ by the number e_i, where $-e_i := E_i^2$ is the self-intersection number of the associated component E_i inside the smooth surface $\tilde{X}$.

Denote by $V(\Gamma)$ the set of vertices of Γ and by $\underline{e} \in \mathbb{Z}^{V(\Gamma)}$ the function which associates to each vertex its weight. To the weighted graph $(\Gamma, \underline{e})$ is associated a canonical quadratic form on the real vector space $\mathbb{R}^{V(\Gamma)}$, called *the intersection form* associated to the resolution π:

$$Q_{(\Gamma,\underline{e})}(\underline{x}) := \sum_{i \in V(\Gamma)} (-e_i x_i^2 + \sum_{\substack{j \in V(\Gamma) \\ j \neq i}} e_{ij} x_i x_j) \; = \; \sum_{i \in V(\Gamma)} x_i(-e_i x_i + \sum_{\substack{j \in V(\Gamma) \\ j \neq i}} e_{ij} x_j).$$

One has the following characterization of weighted graphs coming from singularities (see subsection 3.3 of Chapter 4 for the notion of *contractibility* in arbitrary dimension):

THEOREM 3.1.
*1) (*Du Val [**164**], Mumford [**121**]*) The intersection form $Q_{(\Gamma,\underline{e})}$ is negative definite. In particular, $e_i > 0$ for all $i \in V(\Gamma)$.*
*2) (*Grauert [**66**]*) If the intersection form associated to a reduced compact effective divisor E on a smooth surface is negative definite, then E can be contracted to a normal singular point of an analytic surface.*

For the following considerations on arithmetic genera, the adjunction formula and the anti-canonical cycle, we refer to Reid [**148**] and Barth, Hulek, Peters & Van de Ven [**6**].

If D is an effective divisor on $\tilde{X}$ supported on E, then it may be interpreted as a (non-necessarily reduced) compact curve, with associated structure sheaf $\mathcal{O}_D$. Its *arithmetic genus* $p_a(D)$ is by definition equal to $1 - \chi(\mathcal{O}_D)$. It satisfies *the adjunction formula*:

$$p_a(D) := 1 + \frac{1}{2}(D^2 + K_{\tilde{X}} \cdot D) \tag{3.2}$$

where $K_{\tilde{X}}$ is any canonical divisor on $\tilde{X}$. The previous formula allows to extend the definition to *any* divisor supported on E, not necessarily an effective one.

For all $i \in V(\Gamma)$, denote by p_i the arithmetic genus of the curve E_i, and by g_i the arithmetic genus of its normalization, equal to its topological genus. Both genera are related by the following formula:

$$p_i = g_i + \sum_{P \in E_i} \delta(E_i, P) \tag{3.3}$$

where $\delta(E_i, P) \geq 0$ denotes the *delta-invariant* of the point P of E_i, equal to the number of ordinary double points concentrated at P, and defined more generally for arbitrary curve singularities (X, x) by the formula:

$$\delta(X, x) := \dim_{\mathbb{C}}(\mathcal{O}_{\overline{X}}/\mathcal{O}_X) \tag{3.4}$$

where $\overline{X} \xrightarrow{\nu} X$ is the normalization of X (therefore, $\overline{X}$ is a multi-germ). One has $\delta(E_i, P) > 0$ if and only if P is singular on E_i.

We deduce from (3.3) that $p_i = 0$ if and only if E_i is a smooth rational curve.

At this point, we have two weightings for the vertices of the graph Γ, the collection $\underline{e}$ of self-intersections and the collection $\underline{p}$ of arithmetic genera of the associated irreducible components. If π is a strict normal crossings resolution, then the doubly weighted graph $(\Gamma, \underline{e}, \underline{p})$ determines the embedded topology of E in $\tilde{X}$ (see Mumford [**121**]). In general this is not the case, because these numerical data do not determine the types of singularities of E. Nevertheless, they determine them, and consequently the embedded topology of E, up to a finite ambiguity. Indeed, there are a finite number of embedded topological types of germs of reduced plane curves (C, c) having a given value of $\delta(C, c)$ (see Wall [**170**, page 151]).

As the singularity (X, x) was supposed to be normal, one has $\pi_* \mathcal{O}_{\tilde{X}} = \mathcal{O}_X$, that is, a holomorphic function on $X \setminus x$ extends to a holomorphic function on X if and only if its lift to $\tilde{X}$ extends there as a holomorphic function. Therefore, one may construct holomorphic functions on X by working on some resolution.

Start from $f \in m_{X,x}$. Then, denoting by $(\pi^* f)_e$ *the exceptional part* of the principal divisor $(\pi^* f)$, that is, the part of the divisor of zeroes of $\pi^* f$ which is supported by the exceptional divisor E of π, the intersection number between $(\pi^* f)$ and each component E_i vanishes, which shows that:

$$(\pi^* f)_e \cdot E_i \leq 0, \text{ for all } i \in I.$$

In fact $-(\pi^* f)_e \cdot E_i = (\pi^* f)_s \cdot E_i$, where $(\pi^* f)_s$ denotes the *strict transform* on $\tilde{X}$ of the divisor (f).

Therefore one is led to introduce *the Lipman semigroup* $\mathcal{L}(\pi)$ of π, defined purely numerically as:

$$\mathcal{L}(\pi) := \{D \in \sum_{i \in I} \mathbb{Z} E_i \mid D \cdot E_i \leq 0, \text{ for all } i \in I\}.$$

This set is a semigroup for the addition of divisors. On it we consider the partial order relation:

$$D_1 \geq D_2 \Leftrightarrow D_1 - D_2 \text{ is effective .}$$

It is a nice exercice to show that all the elements of the Lipman semigroup are effective divisors.

We have seen that the exceptional part of any holomorphic function belongs to this semigroup. The converse is not true in general, excepted for rational singularities.

As shown by M. Artin [**3**], the Lipman semigroup has a unique non-vanishing minimal element Z_{num}, called *the fundamental cycle* or *the numerical cycle* of π. This cycle may be computed algorithmically:

PROPOSITION 3.5. *(*Laufer [**98**]*) Start from $Z_0 := 0$. If Z_j is defined and there exists $i \in I$ such that $Z_j \cdot E_i > 0$, then define $Z_{j+1} := Z_j + E_i$. Then this process stops after a finite number of steps and the last element in the sequence $Z_0, Z_1, ...$ is the fundamental cycle of π.*

We will also need to manipulate another cycle associated to the resolution π and defined, as the fundamental cycle, in a purely numerical way. As the quadratic form $Q_{(\Gamma,\underline{e})}$ is negative definite, there exists a unique divisor *with rational coefficients* Z_K supported on E such that:

$$Z_K \cdot E_i = -K_{\tilde{X}} \cdot E_i \text{ , for all } i \in V(\Gamma). \tag{3.6}$$

Indeed, by the adjunction formulae (3.2), one has the following system of equations:

$$2p_a(E_i) - 2 = E_i^2 - Z_K \cdot E_i, \text{ for all } i \in I \tag{3.7}$$

which is a square system of affine equations with unknowns the coefficients of Z_K such that a matrix of the associated homogeneous system is a matrix of the intersection form of $(\Gamma, \underline{e})$.

The sign in the previous definition is motivated by the following result:

PROPOSITION 3.8. *Assuming that the resolution is minimal, Z_K is an effective divisor.*

We call Z_K the *anti-canonical cycle* of E (or of the resolution π). The name is motivated by the fact that whenever (X, x) is Gorenstein, $-Z_K$ is a canonical divisor on $\tilde{X}$. Indeed, if the singularity (X, x) is Gorenstein, consider a non-vanishing holomorphic form defined in a pointed neighborhood of x. Therefore its lift to $\tilde{X}$ is meromorphic and its locus of zeros and poles is contained in E. This locus, considered with multiplicities, is by construction a *canonical divisor* on $\tilde{X}$. Therefore, it is exactly $-Z_K$, which shows that for Gorenstein singularities, Z_K *has integral coefficients*. This property being numeric (that is, depending only on intersection-theoretical properties) and common to all normal Gorenstein singularities, it motivates:

DEFINITION 3.9. The singularity (X, x) is called **numerically Gorenstein** if Z_K is an integral divisor. As the coefficients of Z_K depend only on the decorated graph $(\Gamma, \underline{p}, \underline{e})$, we also say that this graph is **numerically Gorenstein**.

It is unknown whether each numerically Gorenstein graph may be realised by a Gorenstein singularity.

3.2. The topology of normal surface singularities.

For details about the material presented here, one may consult [**127**] or [**141**].

Let (X, x) be a normal surface singularity. Consider its abstract boundary $\partial(X, x)$. It is a connected, compact, naturally oriented 3-manifold. The study of the topology of such 3-manifolds started with Mumford's article [**121**], in which he proved that one could recognize that a point on a normal surface was smooth from the topology of its boundary:

THEOREM 3.10. (Mumford) *If the boundary of a normal surface singularity (X, x) is simply connected, then x is a smooth point of X.*

In particular, as the boundary of a germ of surface at a smooth point is diffeomorphic to $\mathbb{S}^3$, this showed that one could not get a counterexample to Poincaré's conjecture by taking the boundary of a surface singularity.

Mumford started by describing the boundary as the result of gluing elementary 3-manifolds by an operation he called *plumbing*, the list of such 3-manifolds and the instructions for gluing being determined by the weighted dual graph $(\Gamma, \underline{p}, \underline{e})$ of any *strict normal crossings* resolution. In order to see that the dual graph determines a decomposition of the boundary, one looks at $\partial(X, x)$ as boundary of a tubular neighborhood of the exceptional divisor E of the chosen resolution.

Such a tubular neighborhood W may be constructed as the union of tubular neighborhoods W_i of the irreducible components E_i of E. Therefore one obtains a first *plumbing* procedure for the gluing of these elementary tubular neighborhoods, which are simply oriented disc bundles. More precisely, the neighborhood W_i of E_i is a disc bundle of Euler number $-e_i < 0$ over the surface E_i. Whenever two vertices of the dual graph Γ are joined by an edge, one glues trivialized restrictions to discs in their respective bases of the bundles corresponding to the two vertices by switching the fibers and the sections.

The way one gets the boundary of the 4-manifold W from the boundaries of the 4-manifolds W_i may be described by restricting the preceding procedure to the boundaries of the W_i. Namely, one starts from oriented circle bundles over E_i with Euler numbers $-e_i$, and whenever two vertices are joined by an edge of Γ, one takes out saturated and trivialized solid tori from the corresponding circle bundles and glues the trivialized tori which are created like this by switching fibers and meridians: this is the *plumbing* operation for circle bundles over surfaces.

Therefore, the boundaries of normal surface singularities are particular 3-manifolds, obtained by plumbing circle bundles over surfaces following a weighted graph. For this reason, such 3-manifolds were named *graph-manifolds*. Their theory was started by Waldhausen [**168**]. He looked at the collection of tori one gets in a graph manifold as images of the tori which were identified after taking out solid tori from the total spaces of the circle bundles. The connected components of their complement are total spaces of circle bundles. Moreover, the fibers arrive from both sides of a torus such that their intersection number is always ± 1. He considered then more general families of tori, by asking only that the complement be fiberable by circles, but forgetting the condition about intersection numbers. The class of 3-manifolds which admit such a *graph structure* is the same as before, but one has more possibilities of simplification: each time one finds two *parallel* tori, that is, 2-tori which cobound a thick torus $[0, 1] \times \mathbb{T}^2$, one can eliminate one of them, and obtain again a graph structure on the same 3-manifold. Waldhausen proved that, when the initial 3-manifold is *irreducible*, that is, indecomposable as a connected sum of two other 3-manifolds non-diffeomorphic to the 3-sphere, a minimal such collection of tori is in general a topological invariant of the 3-manifold:

THEOREM 3.11. *(*Waldhausen*) With the exception of a finite explicit list of 3-manifolds, a minimal collection of tori which correspond to a graph structure on an irreducible closed 3-manifold is unique up to isotopy.*

He described also a graph-notation for graph structures and characterized using it the graph structures corresponding to the minimal collections of tori. His work was the starting point of a calculus elaborated by Neumann [**127**] for plumbing structures. Neumann did not leave the realm of plumbing structures in order to enter the more general realm of graph structures. This allowed a non-ambiguous encoding of the structure by plumbing graphs. Neumann applied his calculus to give an algorithm which allowed to determine if a given plumbing graph describes or not a singularity boundary. Using this algorithm, he showed:

THEOREM 3.12. *(*Neumann*) The boundary of a normal surface singularity is irreducible. Its oriented topological type determines the weighted dual graph* $(\Gamma, \underline{e}, \underline{p})$ *of the minimal strict normal crossings resolution up to isomorphism.*

Therefore, one may encode the oriented topological type of the singularity boundary by this graph. Moreover, one has an algorithmic way, given an oriented graph manifold, to determine if it is diffeomorphic to a singularity boundary or not.

Before that, the first example of a graph manifold which was not a singularity boundary had been given by Sullivan [**157**] (see Corollary 6.8 of the previous chapter).

Waldhausen's structure theorem for graph manifolds was extended later by Jaco & Shalen [**87**] and Johannson [**88**] into a structure theorem for any *irreducible* 3-manifolds. Namely, any such manifold contains a well-defined and unique up to isotopy family of pairwise disjoint and non-parallel incompressible tori, minimal for the property that the components of their complement are either Seifert-fiberable or do not contain new incompressible tori (which are not boundary-parallel). Such a family is now called *a JSJ-family of tori*. These theorems were the starting point of Thurston's geometrization conjecture.

3.3. Rational and minimally elliptic surface singularities.

For details about the classification of surface singularities, I recommend Némethi's surveys [**123**], [**125**] as well as Reid [**148**] and Wall [**169**].

Since Clebsch called *genus* the measure of complexity associated by Abel and Riemann to an algebraic curve, differentiating themselves from Descartes who measured this complexity in terms of the degree, the term *genus* flourished as a measure of various complexities in algebraic geometry. This happened also in singularity theory:

DEFINITION 3.13. Let (X, x) be a normal surface singularity. Its **geometric genus** is defined as:

$$p_g(X, x) := \dim_{\mathbb{C}} R^1\pi_*\mathcal{O}_{\tilde{X}}$$

where $\pi : \tilde{X} \to X$ is any resolution of singularities. Its **arithmetic genus** is defined as:

$$p_a(X, x) := \sup_{Z \geq 0} p_a(Z)$$

where Z varies among the effective divisors supported by the exceptional divisor $\mathrm{Exc}(\pi)$.

One may show that if U is a Stein representative of (X, x) and $\tilde{U}$ is its preimage by the chosen resolution, then $p_g(X, x) = \dim_{\mathbb{C}} H^1 \pi_* \mathcal{O}_{\tilde{U}}$. It is a theorem that both definitions are independent of the chosen resolution (see [**166**, Section 1] or [**7**, Section 2.3]). One has always:

$$p_g(X, x) \geq p_a(X, x) \geq p_a(Z_{num}) \geq 0.$$

By analogy with the fact that among smooth connected compact analytic curves, those of smallest genus are called *rational*, Michael Artin [**3**] introduced the same terminology for surface singularities:

Definition 3.14. A normal surface singularity (X, x) is called **rational** if its geometric genus vanishes.

An essential property of rational singularities is that there are topological criteria of rationality:

Theorem 3.15. *(*M. Artin [**2**], [**3**]*) A normal surface singularity is rational if and only if one of the following facts happen:*

1. *One has $p_a(X, x) = 0$.*
2. *One has $p_a(Z_{num}) = 0$.*

Using Laufer's algorithm [**98**], we see that point (2) allows to determine readily from the knowledge of the weighted graph of a resolution whether a singularity is rational.

Rational surface singularities have the property that their minimal resolution coincides with their minimal normal crossings resolution. Moreover, their dual graph is a tree of rational curves ($\underline{p} = \underline{0}$), as was shown by Artin [**3**].

Another important property of them is that any element of the Lipman semigroup associated to any resolution of singularities may be realised as the exceptional part of a principal divisor. Moreover, a germ D of effective divisor in the neighborhood of E is principal if and only if $D \cdot E_i = 0$ for all $i \in I$.

Quotient singularities constitute the simplest class of rational singularities. Among quotient singularities, the most famous are those described in the following theorem:

Theorem 3.16. *Let (X, x) be a normal germ of surface. The following are equivalent:*

1. *(X, x) is analytically isomorphic to the germ $(\mathbb{C}^2/G, 0)$, where G is a finite subgroup of $SU(2, \mathbb{C})$ (one says that it is* **a Kleinian singularity***).*
2. *The canonical cycle of the minimal resolution is trivial (one says that is is* **a Du Val singularity***).*
3. *(X, x) is rational of multiplicity 2 (one says that it is* **a rational double point***).*
4. *(X, x) is rational and numerically Gorenstein.*

In fact there are many more characterizations of those singularities (see Hazewinkel et al. [**77**], Durfee [**35**], Slodowy [**153**], Cassens & Slodowy [**22**], Brieskorn [**19**] for various viewpoints on them). They appeared historically in different contexts under different aspects, some of those contexts having led to the names emphasized in the previous theorem. More precisely, Klein [**91**] was studying the theory of invariants of finite subgroups of $GL(2, \mathbb{C})$, Du Val [**163**] was studying the isolated singularities of surfaces in $\mathbb{CP}^3$ which *do not affect the conditions of adjunction*, that is, such that the holomorphic 2-forms defined outside the singular point extended to *holomorphic* 2-forms on any resolution of the singular point, Artin [**3**] showed that any singularity having the same dual graph as those of Du Val's list (even without assuming that they were of embedding dimension 3), were rational of multiplicity 2 and embedding dimension 3.

One has the following classification of Kleinian singularities:

$$\begin{array}{ll} A_n & x^{n+1}+y^2+z^2=0\ (n\geq 1) \\ D_n & x^{n-1}+xy^2+z^2=0\ (n\geq 4) \\ E_6 & x^4+y^3+z^2=0 \\ E_7 & x^3y+y^3+z^2=0 \\ E_8 & x^5+y^3+z^2=0 \end{array}$$

The dual graph of the minimal resolution is each time a tree of smooth rational curves with self-intersections -2 (that is, $\underline{p}=\underline{0}$ and $\underline{e}=\underline{2}$), the number of vertices of the graph associated to X_n being n, and the shape of the graph being the same as the one of the Coxeter diagram of the root lattice of the simple complex Lie algebra with the same name. In the sequel we will call also such weighted graphs *Kleinian graphs*.

Let us pass now to the quotient singularities which are not Kleinian. They were classified by Brieskorn [**18**] (see also Matsuki [**115**, Chapter 4.6]). The simplest ones are the quotients by *cyclic* groups:

Definition 3.17. Let p, q be coprime integers such that $p > q > 0$. The **cyclic quotient** (or **Hirzebruch-Jung**) **singularity** $(\mathcal{X}_{p,q}, 0)$ is the germ at the image of the origin in $\mathbb{C}^2$ of the quotient $\mathcal{X}_{p,q}$ of $\mathbb{C}^2$ by the action $(x,y) \to (\xi x, \xi^q y)$ of the cyclic group $\{\xi \in \mathbb{C}\ ,\ \xi^p = 1\} \simeq \mathbb{Z}/p\mathbb{Z}$. Its oriented boundary is the (oriented) **lens space** $L(p,q)$.

The alternative name 'Hirzebruch-Jung' for cyclic quotient singularities comes from the fact that those singularities appear naturally in the Hirzebruch-Jung method (originating in Jung [**90**] and Hirzebruch [**79**]) of resolution of surface singularities by preliminary embedded resolution of the discriminant curve of a projection on a smooth surface: they are the singularities of the normalization of a surface having a projection whose discriminant has normal crossings (see Laufer [**97**], Lipman [**108**] or Barth et al. [**6**] for details).

In order to describe the geometry of the minimal resolution of such singularities as well as the algebra of the minimal embedding, one needs to introduce the following Hirzebruch-Jung continued fraction expansions (see Definition 4.1):

$$(3.18) \qquad \left\{ \begin{array}{lcl} \dfrac{p}{q} & = & [b_1, ..., b_s]^- \\ \dfrac{p}{p-q} & = & [a_1, ..., a_r]^- \end{array} \right. .$$

The weighted dual graph of the minimal resolution of $(\mathcal{X}_{p,q}, 0)$ is a segment with s vertices (including its extremities), weighted by $\underline{e} = \underline{b}$ and $\underline{p} = \underline{0}$. Denote by $\Pi(\underline{b})$ a tubular neighborhood of the exceptional set. It is a plumbed 4-manifold whose plumbing graph is the weighted graph of the minimal resolution.

The embedding dimension of $(X_{p,q},0)$ is equal to $r+2$, and there exists a minimal system of generators $z_0,...,z_{r+1}$ of the maximal ideal such that $z_{i-1}z_{i+1} = z_i^{a_i}$, for all $i \in \{1,...,r\}$.

The best way to understand these facts is to see the cyclic quotient singularities as *toric singularities*. In fact, they are precisely the germs of normal toric surfaces (see Oda [**129**] or Fulton [**53**]). More precisely, $(\mathcal{X}_{p,q},0) \simeq (\mathcal{Z}(\sigma_{p,q},N),0)$, where $\mathcal{Z}(\sigma_{p,q},N)$ denotes the normal affine toric surface defined by a 2-dimensional rational cone of type $\frac{p}{q}$ (that is, isomorphic to the cone generated by $(1,0)$ and $(-q,p)$, with $N=\mathbb{Z}^2$). Then the components of the exceptional divisor of the minimal resolution correspond to the minimal generating set of the semi-group $\sigma_{p,q} \cap N$, with the exception of the generators of the edges of $\sigma_{p,q}$, while a minimal generating set of the maximal ideal is formed by the characters corresponding to the minimal generating set of the semi-group $\check{\sigma}_{p,q} \cap N^*$. This shows that the HJ-continued fractions of equation (3.18) are in some sense *dual*. I will explain in section 4 that this is one manifestation of a very simple geometric duality between supplementary cones, which makes that not only the sequences $\underline{b}$ and $\underline{a}$ are dual, but that also internal structures of them correspond dually.

Rational surface singularities are the simplest surface singularities, if one takes the arithmetic genus as a measure of complexity. The next class in terms of this complexity are therefore the singularities (X,x) with $p_a(X,x)=1$. Wagreich [**166**] started their study and called them *elliptic singularities*, by analogy with elliptic curves, whose topological genus is 1. Unlike in the case of rational singularities, this class contains germs with arbitrary high geometric genus. Laufer discovered that there exists a subclass which is also defined topologically, and which has many properties in common with rational singularities. Namely, in [**100**, Theorems 3.4 and 3.10], he proved:

THEOREM 3.19. *(*Laufer*) Let (X,x) be a normal surface singularity. Working with its minimal resolution, the following facts are equivalent:*

(1) *One has $p_a(Z_{num})=1$ and $p_a(D)<1$ for all $0<D<Z_{num}$.*
(2) *The fundamental and anticanonical cycles are equal: $Z_{num}=Z_K$.*
(3) *One has $p_a(Z_{num})=1$ and any connected proper subdivisor of E contracts to a rational singularity.*
(4) *$p_g(X,x)=1$ and (X,x) is Gorenstein.*

Laufer introduced a special name (making reference to condition (3)) for the singularities satisfying one of the previous conditions:

DEFINITION 3.20. A normal surface singularity satisfying one of the equivalent conditions stated in Theorem 3.19 is called a **minimally elliptic** singularity.

One sees from the previous theorem that one may determine from the topology of (X,x) whether it is minimally elliptic or not. Moreover, all the singularities realising that topology are necessarily Gorenstein. For rational singularities, we saw that only the Kleinian ones are Gorenstein. Kleinian singularities are moreover taut (see definition 3.24), which is not the case for all the minimally elliptic ones. Nevertheless, Laufer saw that the union of the class of Kleinian singularities and minimally elliptic singularities could be characterized in a subtle way using Gorensteinness:

THEOREM 3.21. *(*Laufer*) Let us fix a topological type of normal surface singularities. Then the singularities realising that type are generically Gorenstein if and only if the topological type corresponds either to a Kleinian singularity or to a minimally elliptic singularity.*

It is not clear a priori what means a *generic* property of the singularities with given topological type. Laufer gives the following meaning to it: a property is *generic* for a given topological type of singularities if, on the base of the miniversal space of deformations with fixed topological type, the singularities having that property form a dense open set.

Let us introduce the following particular types of minimally elliptic singularities:

DEFINITION 3.22. A normal surface singularity is a **simple elliptic** singularity if it is obtained by contracting a smooth elliptic curve with negative self-intersections embedded in a smooth surface. It is a **cusp** singularity if the weighted dual graph of its minimal strict normal crossings resolution is a circle and $\underline{p} = \underline{0}$.

Simple elliptic singularities were introduced by K. Saito [**150**] as the simplest elliptic singularities in the sense of Wagreich and cusp singularities received their name from the fact that they are the singularities obtained by compactifying the *cusps* of the Hilbert modular surfaces (see Hirzebruch [**80**]). They have a common characterization with cyclic quotient singularities:

THEOREM 3.23. *(*Neumann*,* [**127**]*) If one changes the orientation of the boundary of a normal surface singularity, the resulting 3-manifold is no more orientation-preserving diffeomorphic to the boundary of an isolated surface singularity, excepted for* cyclic quotient singularities *and* cusp-singularities*.*

The previous two classes of singularities, as well as all Kleinian singularities have moreover the property that their topology determines their analytical type:

DEFINITION 3.24. A normal surface singularity or a weighted dual graph is called **taut** if its topology determines its analytical type.

In [**98**], Laufer classified all the taut weighted graphs.

4. A duality for supplementary cones with respect to a lattice

4.1. The geometric duality.

Continued fraction expansions appear naturally when one resolves germs of plane curves by sequences of plane blowing-ups, or cyclic quotient surface singularities by toric modifications.

They also appear when one passes from the natural plumbing decomposition of the abstract boundary of a normal surface singularity to its minimal JSJ decomposition. In this case it is very important to keep track of natural orientations. In general (see Theorem 3.23), if one changes the orientation of the boundary, the resulting 3-manifold is no more orientation-preserving diffeomorphic to the boundary of an isolated surface singularity. The only exceptions are *cyclic quotient singularities* and *cusp-singularities*. For both classes of singularities, one gets an involution on the set of analytical isomorphism types of the singularities in the class, by changing the orientation of the boundary. From the viewpoint of computations, Hirzebruch saw that both types of singularities have structures which can be encoded in continued fraction expansions of positive integers, and that the previous involution manifests itself in a *duality* between such expansions.

In the computations with continued fractions alluded to before, there appear in fact two kinds of continued fraction expansions:

DEFINITION 4.1. If $x_1, ..., x_n$ are variables, we consider two kinds of continued fractions associated to them:

$$[x_1, ..., x_n]^+ := x_1 + \cfrac{1}{x_2 + \cfrac{1}{\cdots + \cfrac{1}{x_n}}}$$

$$[x_1, ..., x_n]^- := x_1 - \cfrac{1}{x_2 - \cfrac{1}{\cdots - \cfrac{1}{x_n}}}$$

We call $[x_1, ..., x_n]^+$ a **Euclidean continued fraction** (abbreviated **E-continued fraction**) and $[x_1, ..., x_n]^-$ a **Hirzebruch-Jung continued fraction** (abbreviated **HJ-continued fraction**).

The first name is motivated by the fact that E-continued fractions are closely related to the Euclidean algorithm: if one applies this algorithm to a couple of positive integers (a, b) and the successive quotients are $q_1, ..., q_n$, then $a/b = [q_1, ..., q_n]^+$.

The second name is motivated by the fact that HJ-continued fractions appear naturally in the Hirzebruch-Jung method of resolution of singularities, originating in Jung [**90**] and Hirzebruch [**79**] (see also [**97**] and [**108**]).

Define two sequences $(Z^{\pm}(x_1, ..., x_n))_{n\geq 1}$ of polynomials with integer coefficients, by the initial data:

$$Z^{\pm}(\emptyset) = 1,\ Z^{\pm}(x) = x$$

and the recurrence relations:

$$Z^{\pm}(x_1, ..., x_n) = x_1 Z^{\pm}(x_2, ..., x_n) \pm Z^{\pm}(x_3, ..., x_n),\ \forall\, n \geq 2. \tag{4.2}$$

Then one proves immediately by induction on n the following equality of rational fractions:

$$[x_1, ..., x_n]^{\pm} = \frac{Z^{\pm}(x_1, ..., x_n)}{Z^{\pm}(x_2, ..., x_n)},\ \forall\, n \geq 1. \tag{4.3}$$

There is a simple formula, also attributed to Hirzebruch, which allows to pass from one type of continued fraction expansion of a number to the other one ($(2)^a$ denotes the constant sequence with a terms equal to 2):

PROPOSITION 4.4. *(*Hirzebruch*) If $(a_n)_{n\geq 1}$ is a (finite or infinite) sequence of positive integers, then:*

$$[a_1, ..., a_{2n}]^+ = [a_1 + 1, (2)^{a_2 - 1}, a_3 + 2, (2)^{a_4 - 1}, ..., (2)^{a_{2n} - 1}]^-$$

$$[a_1, ..., a_{2n+1}]^+ = [a_1 + 1, (2)^{a_2-1}, a_3 + 2, (2)^{a_4-1}, ..., (2)^{a_{2n}-1}, a_{2n+1} + 1]^-$$
$$[a_1, a_2, a_3, a_4, ...]^+ = [a_1 + 1, (2)^{a_2-1}, a_3 + 2, (2)^{a_4-1}, a_5 + 2, (2)^{a_6-1}, ...]^-.$$

If (L, σ) is a pair consisting of a 2-dimensional lattice L and a strictly convex cone σ in the associated real vector space, $P(\sigma)$ denotes the boundary of the convex hull of the set of lattice points situated inside σ and different from the origin.

Both types of expansions have geometric interpretations in terms of polygonal lines $P(\sigma)$. For Euclidean continued fractions this interpretation is attributed to Klein [**92**], while for the Hirzebruch-Jung ones it is attributed to Cohn [**28**]. Let $\lambda > 1$ and $\lambda = [l_1, l_2, ...]^+ = [n_1, n_2, ...]^-$. Then:

- (**Klein's interpretation**) *Consider a basis* (e_1, e_2) *of* L *and a half-line* h *of slope* λ *with respect to it, contained in the interior of the cone bounded by* $\mathbb{R}_+ e_1$ *and* $\mathbb{R}_+ e_2$. *Denote by* σ_1 *and* σ_2 *the cones bounded by the half-line* h *and* $\mathbb{R}_+ e_1$, *respectively* $\mathbb{R}_+ e_2$. *Then the odd-indexed numbers* $l_1, l_3, ...$ *are the integral length of the edges of the polygonal line* $P(\sigma_1)$ *and the analog for the relation between* $l_2, l_4, ...$ *and* $P(\sigma_2)$.
- (**Cohn's interpretation**) *Consider a basis* (e_1, e_2) *of* L *and a half-line* h *of slope* λ *with respect to* $(-e_1, e_2)$. *Denote by* σ *the cone bounded by* $-e_1$ *and the half-line* h. *Then, consider in order all the lattice points* $u_0, u_1, ...$ *situated on* $P(\sigma)$, *starting from* $-e_1$. *Then one has the relations* $u_{i-1} + u_{i+1} = n_i u_i$, *for all* $i \geq 1$.

It is natural to try to understand how both geometric interpretations fit together. By superimposing the corresponding drawings, I was led to consider *two supplementary cones* in a real plane, in the presence of a lattice. By *supplementary cones*, I mean two closed strictly convex cones which have a common edge and whose union is a half-plane.

From another side, I understood that the algebraic duality between continued fractions described by Hirzebruch has, as geometric counterpart, a duality between two such supplementary cones. This duality is easiest to express in the case when the edges of the cones are irrational:

Suppose that the edges of the supplementary cones σ *and* σ' *are irrational. Then the edges of each polygonal line* $P(\sigma)$ *and* $P(\sigma')$ *correspond bijectively in a natural way to the vertices of the other one: a vertex of* $P(\sigma')$ *is a primitive element of the parallel to the dual edge traced through the origin.*

Moreover, if one associates to each edge its integral length (as in Klein's interpretation) and to each vertex its weight (as in Cohn's interpretation), then:

$$(\text{weight of a vertex}) = (\text{length of the dual edge}) + 2.$$

When at least one of the edges of σ is rational, the correspondence is slightly more complicated: there is a defect of bijectivity near the intersection points of the polygonal lines with the edges of the cones (the curious reader may find it by looking carefully at Figure 4.1. It is this correspondence which gives a geometric interpretation of the formulae stated in Proposition 4.4.

The duality between supplementary cones is not new; it has already appeared from time to time under equivalent formulations (for a list, one can consult [**142**]). What I believe is new in my presentation is the conscience of its unifying role in singularity theory, as I explain in the next subsection.

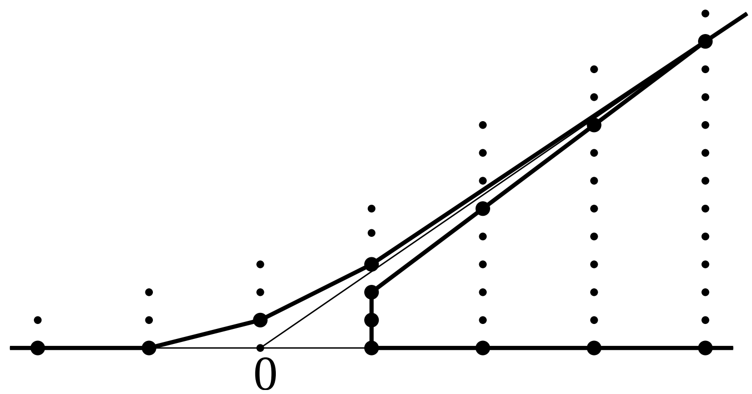

FIGURE 4.1. An example of duality between supplementary cones

4.2. Applications to curve and surface singularities.

The duality between supplementary cones gives a simple way to think about the relation between the pair (L, σ) and its dual pair $(\check{L}, \check{\sigma})$, and in particular about the relations between various invariants of toric surfaces. Indeed:

The supplementary cone of σ is canonically isomorphic over the integers with the dual cone $\check{\sigma}$, once an orientation of L is fixed.

Computations with continued fractions appear also when one passes from the canonical plumbing structure on the boundary of a normal surface singularity to its minimal graph structure. Using this, Neumann [**127**] showed that the topological type of the minimal good resolution of the germ is determined by the topological type of the link. In fact all continued fractions appearing in Neumann's work are the algebraic counterpart of pairs (L, σ) canonically determined by the topology of the boundary. Using this remark, I proved the stronger statement (see [**141**]):

THEOREM 4.5. *The plumbing structure on the boundary of a normal surface singularity associated to the minimal normal crossings resolution is determined up to isotopy by the oriented ambient manifold. In particular, it is invariant up to isotopy under the group of orientation-preserving self-diffeomorphisms of the boundary.*

In the proof of this theorem I treated separately the boundaries of Hirzebruch-Jung and cusp singularities. In both cases, I showed that the oriented boundary determines naturally a pair (L, σ) as before. If one changes the orientation of the boundary, one gets a supplementary cone. In this way, the involution defined on both sets of (taut) singularities by changing the orientation of the boundary is a manifestation of the geometric duality between supplementary cones.

Start now from a germ of plane curve $(X, 0) \hookrightarrow (\mathbb{C}^2, 0)$. One may canonically get an embedded resolution by iteratively blowing-up points on the smooth ambient surface. At the end, one gets the dual graph of the total transform of $(X, 0)$, which encodes the topology of $(\mathbb{C}^2, X, 0)$ (see [**170**]). This graph may be constructed by gluing elementary pieces corresponding to the Newton-Puiseux exponents of the irreducible components of $(X, 0)$ and to some other exponents measuring the contacts between the various components. Such elementary graphs correspond to the embedded resolutions of *monomial* *plane curves*, with equations of the form $x^a - y^b = 0$. I showed that the previous duality allowed also to understand their structure.

Summarizing the previous facts:

THEOREM 4.6. *The duality of plane supplementary cones with respect to a lattice allows to think geometrically about the following aspects of curve and surface singularities:*

(1) *The structure of the dual graph of the minimal embedded resolution of a plane monomial curve.*
(2) *The relation between the dual graph of a cyclic quotient singularity and the minimal monomial generating set of its algebra.*
(3) *The duality among cyclic quotient singularities, and the one among cusp singularities.*

For me, the moral of the story I told in [**141**] about the duality of supplementary cones is:

If one meets computations with either Euclidean or Hirzebruch-Jung continued fractions in a geometrical problem, it means that somewhere behind are present a natural 2-dimensional lattice L and a couple of lines in its associated real vector space. One has first to choose one of the two pairs of opposite cones determined by the two lines and secondly an ordering of the edges of those cones. These choices may be dictated by choices of orientations of the manifolds which led to the construction of the lattice and the cones. Therefore, in order to think geometrically about the computations with continued fractions, recognize the lattice, the lines and the orientation choices.

5. A finiteness result for the topology of numerically Gorenstein singularities

An important problem, studied by several authors (see for instance Yau [**173**] and Laufer [**102**]), is:

Describe the negative definite dual graphs $(\Gamma, \underline{e}, \underline{p})$ corresponding to hypersurface singularities in $\mathbb{C}^3$.

As far as I know, the only general obstruction on such graphs discovered so far comes from the fact that hypersurface singularities are smoothable and Gorenstein, which allows to apply Theorem 4.8 of Chapter 2. This gives a constraint on the topology in all cases where one knows that the geometric genus is determined by it (for example for minimally elliptic singularities, as shown by Theorem 3.19).

Together with José Seade, starting from the previous problem, we looked at the following weaker question (see Definition 3.9):

Describe the dual graphs corresponding to numerically Gorenstein surface singularities.

Precising Proposition 3.8, it results from the adjunction system (3.7) that, with the exception of the Kleinian singularities (when $Z_K = 0$), the anticanonical cycle $Z_K = \sum_{i \in I} z_i E_i$ of the *minimal resolution* of a numerically Gorenstein singularity has as support the whole exceptional divisor E.

In the sequel, we suppose that $(\Gamma, \underline{p}, \underline{e})$ is not one of the Kleinian graphs. Therefore, $z_i \geq 1$ for all $i \in V(\Gamma)$. Let us introduce new variables, for simplicity:

$$\left\{\begin{array}{l} n_i := z_i - 1 \geq 0\,, \\ v_i := \displaystyle\sum_{\substack{j\in V(\Gamma)\\ j\neq i}} e_{ij} \geq 0, \\ q_i := v_i + 2p_i - 2 \geq -2. \end{array}\right. \tag{5.1}$$

Then the adjunction system (3.7) becomes:

$$\{e_i n_i \;=\; q_i + \sum_{\substack{j\in V(\Gamma)\\ j\neq i}} e_{ij} n_j\}_{i\in V(\Gamma)}. \tag{5.2}$$

If $i \in V(\Gamma)$, v_i is the *valency* of i, that is, the number of edges connecting it to other vertices.

Our main theorem in the paper [**143**] says that, given a weighted graph $(\Gamma, \underline{p})$, there is a *finite* number of possible anticanonical cycles one can obtain by adding weights $\underline{e}$ such that $(\Gamma, \underline{p}, \underline{e})$ becomes numerically Gorenstein. This is an immediate consequence of:

THEOREM 5.3. *Consider a graph Γ decorated with weights $\underline{q} \in \mathbb{Z}^{V(\Gamma)}$, and the system of equations in the unknowns $(\underline{n}, \underline{e}) \in (\mathbb{N})^{V(\Gamma)} \times (\mathbb{N}^*)^{V(\Gamma)}$:*

$$\{e_i n_i = q_i + \sum_{\substack{j\in V(\Gamma)\\ j\neq i}} e_{ij} n_j\}_{i\in V(\Gamma)}. \tag{5.4}$$

Then there exist at most finitely many weights $\underline{n}$ which can be extended to solutions $(\underline{n}, \underline{e})$ of the previous system, such that the quadratic form $Q_{(\Gamma,\underline{e})}$ is negative definite.

By contrast, the possible values of $\underline{e}$ making $(\Gamma, \underline{p}, \underline{e})$ numerically Gorenstein do not necessarily form a finite set. For example, if Γ is topologically a circle and $\underline{p} = \underline{0}$ (that is, if one has a *cusp-graph*), then for any choice $\underline{e}$ such that $e_i \geq 2$ for all $i \in I$ and $e_i \geq 3$ for at least one $i \in I$, one gets a numerically Gorenstein graph. But we describe completely the obstruction to get a finite number of possibilities. Namely, if the graph $(\Gamma, \underline{p})$ is not a cusp-graph, then one can get an infinite number of values e_i only for vertices such that $p_i = 0$ and $v_i = 1$, that is, for *smooth rational leaves* of the graph.

We were very surprised by the proof we found: by contradiction, we started from an infinite sequence of weights $(\underline{e}^{(k)})_{k\in\mathbb{N}}$ making $(\Gamma, \underline{p}, \underline{e})$ numerically Gorenstein such that the associated $\underline{n}^{(k)}$ are pairwise distinct. Then, dividing each equation of the system (5.2) by the maximal $n_i^{(k)}$ and passing to the limit in a convenient subsequence, we got a non-trivial subgraph of some $(\Gamma, \underline{e}^{(k)})$ which could not be negative definite.

Such a proof is non-constructive, which led us to question (8) of Chapter 5.

CHAPTER 4

Constructions of singularities with bijective Nash map

1. Motivations

In September 2004, I participated in a conference in singularity theory in Sapporo. There I presented some of the results on contact boundaries I had obtained at that time and Camille Plénat presented her results on the Nash map. We began discussing and we saw that a suitable version of Theorem 3.4 of Chapter 2 could be combined with a criterion proved in her thesis, in order to construct new examples of surface singularities with bijective Nash map.

This gave our first paper [**139**], in which we introduced a class of surface singularities, defined only using a property of the intersection form of the minimal resolution with respect to its canonical basis, such that all the members of this family have bijective Nash map. We had obtained the first non-trivial examples of non-rational normal surface singularities with bijective Nash map, with the exception of some quasi-homogeneous singularities studied by Monique Lejeune-Jalabert.

Before that work I had never thought about the Nash map. I knew about it only from various talks I had listened to, and it was Plénat who taught me the fundamentals in this field of research. I got interested for two reasons: because it brought an application to a variant of a theorem proved for the study of the contact boundaries of singularities and because I saw it as a path towards a better understanding of the so-called *essential* components of the exceptional locus of a resolution of singularities in dimension at least three. I believe that much progress on higher-dimensional analogs of questions discussed in the previous chapters for surface singularities passes through an improvement of our understanding of such components.

After having written [**139**], we got interested in the problem of generalization of our class of examples to higher dimensions, where there were still less examples with bijective Nash map, and all the non-trivial ones had toric normalizations. The idea of this generalization came from a common remark of M. Lejeune-Jalabert and of the referee of our first paper: what was essential in our work was that a certain line bundle was ample. I knew that ampleness was a concept existing in all dimensions, therefore this was the key-concept to study. Like this we succeeded our desired generalization and we got the first non-trivial examples of higher dimensional normal non-toric singularities with bijective Nash map (see [**140**]).

2. Nash's problem on arcs

2.1. The space of arcs and Nash's map.

Let X be a reduced complex algebraic variety. Its *complex points* are simply the maps from the unique reduced irreducible complex algebraic variety of dimension zero $\mathrm{Spec}(\mathbb{C})$ (the algebro-geometric *point*) to X. Following Grothendieck's philosophy, one can consider more general *A-valued points* for any $\mathbb{C}$-algebra A, which are by definition the maps:

$$\mathrm{Spec}(A) \to X.$$

When $A = \mathbb{C}[[t]]$, $\mathrm{Spec}(A)$ is by definition the *complex abstract formal arc* and a $\mathbb{C}[[t]]$-valued point of X is a (*formal*) *arc* contained in X.

In a preprint written around 1966, published later as [**122**], Nash defined the associated *arc space* X_∞ of X, whose points represent the arcs contained in X. By looking at the Taylor expansions of the functions on X with respect to the parameter t and to their truncations at all the orders, Nash constructed this space as a projective limit of algebraic varieties of finite type over X.

If one associates to a formal arc the point of X where it is based, one gets a natural map:

$$\alpha : X_\infty \to X$$

If Y is a closed subvariety of X, denote by:

$$(X, Y)_\infty := \alpha^{-1}(Y)$$

the *space of arcs on X based at Y*.

Nash was thinking of the spaces X_∞ and $(X, Y)_\infty$ for varying $Y \subset \mathrm{Sing}(X)$ as tools for studying the structure of X in the neighborhood of its singular set. Indeed, the main object of his paper was to state a program for comparing the various resolutions of the singularities of X. Such resolutions always exist, as had recently been proven by Hironaka [**78**], but unlike in the case of surfaces, minimal ones do not necessarily exist. We quote from the introduction of [**122**] the two main problems formulated by Nash in this direction:

i) *For surfaces it seems possible that there are exactly as many families of arcs associated with a point as there are components of the image of the point in the minimal resolution of the singularities of the surface.*

ii) *In higher dimensions, the arc families associated with the singular set correspond to "essential components" which must appear in the image of the singular set in all resolutions. We do not know how complete is the representation of essential components by arc families.*

The first problem is a local one, as it deals with the structure of X only in a neighborhood of one of its (closed) points. The second one is more global, as it deals with the structure of X in the neighborhood of its entire singular set.

Following Nash's paper, the foundations for his program were worked with more detail by Lejeune-Jalabert [**106**], Nobile [**128**] and Ishii & Kollár [**86**]. They also extended the program to other categories of spaces. For example, Ishii & Kóllar [**86**] considered schemes over arbitrary fields, Lejeune-Jalabert [**106**] and Nobile [**128**] considered formal germs of varieties. Their treatment extends readily to germs of complex analytic varieties.

For such germs, the space $(X, \mathrm{Sing}(X))_\infty$ of arcs based at the singular locus of X can be canonically given the structure of a relative scheme over X, as the projective limit of relative schemes of finite type obtained by truncating arcs at each finite order.

In the sequel we will restrict to the case where (X, x) *is a germ of a complex analytic variety and* $\mathrm{Sing}(X) = \{x\}$.

The space $(X, x)_\infty$ of arcs on X based at x is a relative subscheme over X of X_∞. As it projects onto x, we see that it is in fact a true scheme (but not of finite type over $\mathbf{C}$). This implies that it makes sense to speak of the set $\mathcal{C}(X, x)_\infty$ of its irreducible components.

Denote by $\pi : \tilde{X} \to X$ a resolution of X. The *exceptional set* $\mathrm{Exc}(\pi) := \pi^{-1}(x)$ is not assumed to be of pure codimension 1, that is, the resolution is not necessarily divisorial.

To an irreducible subvariety (not necessarily a component) E of $\mathrm{Exc}(\pi)$ corresponds a divisorial valuation v_E of the fraction field $\mathrm{Frac}(\mathcal{O}_{X,x})$ of $\mathcal{O}_{X,x}$, which simply associates to any $f \in \mathrm{Frac}(\mathcal{O}_{X,x})$ the vanishing multiplicity of $\pi^*(f)$ at the generic point of E. Conversely, to each *surjective* divisorial valuation $v : \mathrm{Frac}(\mathcal{O}_{X,x}) \to \mathbb{Z} \cup \infty$ corresponds an irreducible component (even of codimension 1, that is a divisor) on some resolution of (X, x). Such divisorial valuations may be seen as the birationally invariant versions of the irreducible subvarieties over x in some resolution. Namely, if E is an irreducible subvariety of $\mathrm{Exc}(\pi)$, its *birational transform* on the total space of any other resolution is simply the center there of its associated valuation v_E.

An irreducible component of $\mathrm{Exc}(\pi)$ is called *an essential component of* π if it corresponds to an irreducible component of the exceptional set of any other resolution of X. In other words, if its birational transform is an irreducible component of the exceptional set in any resolution. An equivalence class of such essential components over all the resolutions of X is called an *essential divisor over* (X, x). If we denote by $\mathcal{E}(X, x)$ the set of essential divisors over (X, x), the essential components of the given resolution morphism π are in a canonical bijective correspondence with the elements of $\mathcal{E}(X, x)$.

Let $\mathcal{K}$ be an element of $\mathcal{C}(X, x)_\infty$, that is, an irreducible component of the space of arcs based at x, which is what Nash called *a family of arcs*. For each arc represented by a point of $\mathcal{K}$, one can consider the intersection point with $\mathrm{Exc}(\pi)$ of its strict transform on $\tilde{X}$. For an arc generic with respect to the Zariski topology of $\mathcal{K}$, this intersection point is situated on a unique irreducible component of $\mathrm{Exc}(\pi)$; moreover, this component is essential (Nash [**122**]). In this manner one defines a map:

$$\mathcal{N}_{X,x} : \mathcal{C}(X, x)_\infty \to \mathcal{E}(X, x)$$

which is called *the Nash map* associated to (X, x). Nash proved that the map $\mathcal{N}_{X,x}$ is always *injective* (which shows in particular that $\mathcal{C}(X, x)_\infty$ is a finite set). In our context, one can reformulate question ii) above:

When is the map $\mathcal{N}_{X,x}$ *bijective?*

This question is also known as *the Nash problem on arcs*. More generally, one may consider germs which are not necessarily normal, with isolated singularity, or irreducible. Then one may differentiate a *local Nash map* defined as before from a *global* one $\mathcal{N}_{X,\mathrm{Sing}(X)}$, where x is replaced by all $\mathrm{Sing}(X)$, and ask the same question as before.

2.2. The results known before our papers.

The Nash map is of course bijective for trivial reasons when there exists a resolution whose exceptional set is irreducible. When I spoke about *non-trivial examples* in the introduction of this chapter, I was alluding to this fact.

The bijectivity of $\mathcal{N}_{X,x}$ was proved before the paper [**139**] done in collaboration with Plénat for the following special classes of *normal surface singularities*:

• for the germs of type $(A_n)_{n\geq 1}$ by Nash himself in [**122**];

• for normal minimal singularities by Reguera [**146**]; different proofs were given by Plénat [**137**] and by Fernández-Sánchez [**52**];

• for sandwiched singularities it was sketched by Reguera [**147**], using her common work [**107**] with Lejeune-Jalabert on the wedge problem.

• for the germs of type $(D_n)_{n\geq 4}$ by Plénat [**138**];

• for the germs with a good $\mathbb{C}^*$-action such that the curve $\operatorname{Proj} X$ is not rational, it follows immediately by combining results of Lejeune-Jalabert [**105**] and Reguera [**147**].

With the exception of the last class, all the other ones consist only in *rational* singularities and can be defined purely topologically.

Our paper inspired Morales' work [**120**].

In *arbitrary dimensions*, the bijectivity of $\mathcal{N}_{X,x}$ or of the analogous map $\mathcal{N}_{X,\mathrm{Sing}(X)}$ was proved before the second paper [**140**] done in collaboration with Plénat for the following classes of germs with not necessarily normal or isolated singularities:

• for the germs which have resolutions with irreducible exceptional set, for trivial reasons;

• for germs of normal toric varieties by Ishii and Kollár in [**86**]; in this case, one can distinguish two types of Nash problems, as was done by Ishii [**85**]; Ishii [**84**] solved the Nash problem also for all (not necessarily normal) toric varieties;

• for various classes of not necessarily irreducible germs whose normalizations are disjoint unions of normal toric germs by Ishii [**84**], [**85**], Petrov [**134**] and González Pérez [**64**].

No surface or 3-fold is known for which the Nash map is not bijective. But Ishii and Kollár proved in [**86**] that it is not always bijective for algebraic varieties of dimension at least 4. Indeed, they gave a counterexample in dimension 4, which can be immediately transformed into a counterexample (with non-isolated singularity) in any larger dimension.

As one sees, only very particular classes of singularities are known to have a bijective Nash map. Therefore it is interesting to find new examples, in order to get more experimental material. This search motivated us in both papers [**139**] and [**140**].

3. A numerical criterion of bijectivity

3.1. Plénat's criterion.

The only class of normal singularities for which the essential divisors are completely known is that of germs of normal toric varieties. Indeed, Bouvier [**16**] determined combinatorially the essential divisors of all normal toric germs. Her work was based on preliminary results of Bouvier & González-Sprinberg [**17**]. Ishii [**84**] characterized the essential divisors also in the case of not necessarily normal toric varieties.

The following criteria ensure that a 1-codimensional component of the exceptional locus of a given resolution is essential (see Ishii & Kollár [**86**, Examples 2.4, 2.5, 2.6]):

PROPOSITION 3.1. *Let $\pi : \tilde{X} \to X$ be a resolution of X. Let F be an irreducible component of* $\mathrm{Exc}(\pi)$*, which is of codimension 1 in $\tilde{X}$.*

*1) (*Nash [**122**]*) If F is not birationally ruled, then F is essential.*

2) If (X, x) is a canonical singularity and F is crepant, then F is essential.

Moreover, in both cases the birational transform of F on any other resolution has again codimension 1.

The notion of *canonical singularity* generalizes to arbitrary dimensions the notion of Du Val surface singularity (see Theorem 3.16 of Chapter 3). We don't give the exact definition here, as we won't need it in the sequel.

As follows from Nash's paper, if F is now a component of $\mathrm{Exc}(\pi)$ of arbitrary codimension, its membership to the image of the Nash map ensures that it is essential. This theoretical criterion needs in its turn criteria to check the said membership.

For each irreducible component F of $\mathrm{Exc}(\pi)$, consider the smooth arcs on $\tilde{X}$ whose closed points are on $F \setminus \cup_{G \neq F} G$, where G varies among the irreducible components of $\mathrm{Exc}(\pi)$, and which intersect F transversally (that is, such that their tangent line and the tangent space to F at their intersection point are direct summands). Consider the set of their images in $(X, x)_\infty$ and denote the closure of this set by $V(F)$.

Nash [**122**] proved that $V(F)$ is an *irreducible* subvariety of $(X, x)_\infty$ and that $(X, x)_\infty$ is the union of these subvarieties, when F varies among all the irreducible components of $\mathrm{Exc}(\pi)$. Therefore, one finds among them the irreducible *components* of $(X, x)_\infty$. Such components are characterized among $(V(F))_F$ by the fact that they are maximal for the partial order defined by inclusion, and they are precisely those components of $\mathrm{Exc}(\pi)$ which are contained in the image of the Nash map. Therefore:

PROPOSITION 3.2. *If a component F of* $\mathrm{Exc}(\pi)$ *satisfies that $V(F) \nsubseteq V(G)$ for any other component G of* $\mathrm{Exc}(\pi)$*, then it is contained in the image of the Nash map.*

One needs therefore a criterion to check non-inclusions of the type $V(F) \nsubseteq V(G)$. Such a criterion was proven by Plénat in [**137**, 2.2], as a generalization of Reguera [**146**, Theorem 1.10], who considered only the class of rational surface singularities. It appears also implicitely in the toric case in Ishii [**83**, Proposition 4.8]. It is an essential ingredient of all the criteria explained here.

PROPOSITION 3.3. *(*Plénat [**137**]*) Let v_1 and v_2 be exceptional divisors over (X, x). If there exists a function $f \in m_{X,x}$ such that $v_1(f) < v_2(f)$, then $V(v_1) \nsubseteq V(v_2)$.*

In order to apply concretely this criterion, one needs to be able to prove the existence of holomorphic functions with controlled vanishing on the irreducible components of some resolution. In particular, in view of Proposition 3.2, one gets the following criterion of bijectiveness of the Nash map:

PROPOSITION 3.4. *(*Plénat [**137**]*) Let (X, x) be a normal isolated singularity. Suppose that there exists a divisorial resolution $\pi : (\tilde{X}, \mathrm{Exc}(\pi)) \to (X, x)$ such that for each pair of distinct components F and G of* $\mathrm{Exc}(\pi)$*, there exists $f \in m_{X,x}$ with $v_F(f) < v_G(f)$. Then $\mathcal{N}_{X,x}$ is bijective.*

3.2. The strategy.

I explain now the *strategy* of construction of examples of singularities with bijective Nash maps using the previous criterion, followed first for surfaces [**139**] and then for higher dimensional singularities [**140**].

Suppose that one starts from a divisorial resolution $\pi : (\tilde{X}, \mathrm{Exc}(\pi)) \to (X, x)$. The pull-back to $\tilde{X}$ of an element of $m_{X,x}$ is a holomorphic function defined in a neighborhood of $\mathrm{Exc}(\pi)$ and vanishing along the exceptional divisor. Conversely, such a function is the pull-back of a function of $m_{X,x}$, by the hypothesis of normality of (X, x). Therefore, it is enough to construct functions on $\tilde{X}$ which vanish in a controlled way along $\mathrm{Exc}(\pi)$.

If D is the part supported by the exceptional divisor of the divisor of the function g defined in a neighborhood of $\mathrm{Exc}(\pi)$, then g is an element of $H^0(\mathcal{O}_{\tilde{X}}(-D))$. Conversely, an element of this space of global sections vanishes *at least as much as* D along $\mathrm{Exc}(\pi)$, but not necessarily *exactly* to the order D. It will do so if we know that the invertible sheaf $\mathcal{O}_{\tilde{X}}(-D)$ of the germs of functions vanishing at least as much as D is *generated by its global sections*.

Therefore, what is sufficient to know, in order to apply the criterion 3.4, is that for each pair of distinct components F and G of the exceptional divisor, there is an effective divisor D supported by $\mathrm{Exc}(\pi)$ in which the multiplicity of F is *strictly less* than the multiplicity of G, and such that $\mathcal{O}_{\tilde{X}}(-D)$ is generated by its global sections.

Notice then that in fact it is enough to know that such divisors D exist with the property that $\mathcal{O}_{\tilde{X}}(-D)$ is *relatively ample* with respect to π. Indeed, then some multiple of it is very ample with respect to π, which ensures that it is generated by its global sections. Moreover, the inequalities of multiplicities of its components are preseved ! The advantage of ampleness with respect to global generation is that for it there is a *numerical* criterion, that is, a way to check it only *by computing intersection numbers*.

For singularities of dimension at least 3, very few classes are known with explicit divisorial resolutions. And when such resolutions are known, the information is not available in a form adapted to check ampleness of line bundles. Fortunately, one has a characterization proved by Grauert [**66**] of the divisors which come from resolutions of normal isolated singularities. For surfaces it says simply that the intersection form of the divisor is negative definite. In higher dimensions it has again to do with negativity properties of the normal bundle of the divisor, which are expressed in terms of *ampleness*.

In the next subsection I explain the required notions about ampleness, Kleiman's criterion of ampleness, and Grauert's criterion for a divisor to be the exceptional divisor of some resolution of singularities. Then I explain the precise realisation of the previous strategy.

3.3. Ampleness and contractibility criteria.

Consider a hyperplane in a projective space. One can move it all around, making it avoid any point and, after constraining it to pass through a point, making it have any given tangent direction at that point. Therefore, if one considers a subvariety of the projective space, its hyperplane sections will have the same properties: intuitively, one feels that it is possible to do *very ample* movements with them.

If one extracts the subvariety from its ambient projective space, one may remember the embedding by remembering the *linear system* of its hyperplane sections. This is a sublinear system of the *complete* linear system of all sections of the associated line bundle. But if one may do very ample movements inside a given linear system, they are a fortiori ampler in a larger linear system. This motivates:

Definition 3.5. A line bundle on a reduced variety is **very ample** if it is isomorphic to the pull-back by an embedding into a projective space of the associated hyperplane line-bundle $\mathcal{O}(1)$.

An analogous definition may be given for a line bundle on the total space of a morphism and *relative to this morphism*, corresponding to the possibility of embedding the morphism in a relative projective space over its base space.

The previous notion is very clear geometrically, but it is difficult to check whether a given line bundle is very ample or not. The more general notion of *ample* line bundle is better behaved from this view-point:

Definition 3.6. A line bundle on a reduced compact complex analytic variety is **ample** if some positive power of it is very ample.

If the line bundle associated to an effective Cartier divisor on a variety is ample, this does not necessarily imply that one can move the divisor in a linear system at all. It only says that some multiple of the divisor becomes very ample. More generally, and this may be taken as an alternative definition of ampleness, the line bundle L is ample if and only if for any coherent sheaf $\mathcal{F}$, there exists a positive power L^n of L such that $L^n \otimes \mathcal{F}$ is generated by its global sections.

The numerical criterion of ampleness we use says, in intuitive terms, that a line bundle is ample if and only if its degree on any curve and on any *limit* of curves is *positive*. In order to give a precise formulation, we need to give some definitions which allow to speak about limits of curves at the homological level.

Let Y be a complete algebraic variety. Let $Z_1(Y)_{\mathbb{R}}$ be the $\mathbb{R}$-vector space of real one-cycles on X, consisting of all finite $\mathbb{R}$-linear combinations of irreducible algebraic curves on Y. Two elements γ_1 and γ_2 of $Z_1(Y)_{\mathbb{R}}$ are *numerically equivalent* if one has the equality of intersection numbers:

$$D \cdot \gamma_1 \;=\; D \cdot \gamma_2$$

for every $D \in \mathrm{Div}(Y) \otimes_{\mathbb{Z}} \mathbb{R}$, where $\mathrm{Div}(Y)$ denotes the group of Cartier divisors on Y. The corresponding vector space of numerical equivalence classes of one-cycles is written $N_1(Y)_{\mathbb{R}}$.

Definition 3.7. Let Y be a complete algebraic variety. The **cone of curves**

$$NE(Y) \subset N_1(Y)_{\mathbb{R}}$$

is the cone $\mathbb{R}_+$−spanned by the classes of all effective one-cycles on Y. Its closure $\overline{NE}(Y) \subset N_1(Y)_{\mathbb{R}}$ is the **closed cone of curves** or **Kleiman-Mori cone** of Y.

One has in this language:

Theorem 3.8. **(Kleiman's criterion of ampleness)** *Let Y be a projective variety. A Cartier divisor D on Y is ample if and only if $D \cdot z > 0$ for all non zero $z \in \overline{NE}(Y)$.*

For details, we refer to Debarre [**32**] and Lazarsfeld [**103**].

In our case we need to check ampleness on an exceptional divisor, which is not necessarily irreducible. But ampleness on a reducible variety can be tested on its irreducible components (see for example Lazarsfeld [**103**, proposition 1.2.16]):

PROPOSITION 3.9. *Let Y be a projective variety and L a line bundle on Y. Then L is ample on Y if and only if the restriction of L to each irreducible component of Y is ample.*

We pass now to the characterization of the exceptional divisors of the resolutions of singularities. More generally, we look at the compact subspaces which may be contracted analytically to a point:

DEFINITION 3.10. Let Y be a reduced complex space and $E \subset Y$ a compact nowhere discrete and nowhere dense analytic set. E is called **exceptional** (in Y) if there is a complex space Z and a proper surjective holomorphic map $\phi : Y \to Z$ such that:

(1) $\phi(E)$ is a finite set;

(2) $\phi : Y \setminus E \to Z \setminus \phi(E)$ is biholomorphic;

(3) $\phi_*(\mathcal{O}_Y) = \mathcal{O}_Z$.

Then one says that ϕ **contracts** E (in Y).

One can show that a map ϕ which contracts E in Y is *unique* in the following sense: if $\phi_k : Y \to Z_k$, $k \in \{1, 2\}$ both contract E in Y, then there exists a unique analytic isomorphism $u : Z_1 \to Z_2$ such that $\phi_2 = u \circ \phi_1$.

One should note that in the minimal model theory of algebraic varieties, one considers more general contractions, which are not necessarily birational maps. We won't deal with such generalizations here.

Grauert proved a fundamental criterion of contractibility (Grauert [**66**], see also Peternell [**133**, theorem 2.12]). A particular case of it is sufficient for our purposes:

THEOREM 3.11. **(Grauert's criterion of contractibility)** *Let Y be a complex manifold and let E be a reduced projective (not necessarily smooth or irreducible) hypersurface in Y. Suppose that there exists an effective divisor A whose support is E, such that the restriction $\mathcal{O}_E(-A)$ of the line bundle $\mathcal{O}_Y(-A)$ to E is ample. Then the analytic hypersurface E is exceptional in Y.*

If Y is a surface, the converse of the theorem is also true. In this case, the hypothesis about the existence of A is equivalent to the fact that the intersection form of E is negative definite. For surfaces, the hypothesis of Grauert's criterion of contractibility is usually expressed in this last manner.

The converse of Theorem 3.11 is not true in a naive form if $\dim_{\mathbb{C}} Y \geq 3$, as shown by examples of Laufer [**101**] (see also Peternell [**133**, Example 2.14]). Nevertheless, there exists a converse if one replaces the search of an ample line bundle by that of a coherent sheaf $\mathcal{I}$ such that $\mathrm{supp}(\mathcal{O}_Y/\mathcal{I}) = E$ and $\mathcal{I}/\mathcal{I}^2$ is positive (see Peternell [**133**, Theorem 2.15]).

3.4. From ampleness to the bijectivity of the Nash map.

Now that the notion of ampleness is explained, we can state precisely the results obtained using the strategy presented in subsection 3.2. We suppose again that $\pi : (\tilde{X}, \mathrm{Exc}(\pi)) \to (X, x)$ is a *divisorial* resolution of the normal isolated singularity (X, x).

Let:

$$L(\pi) := \bigoplus_F \mathbb{Z}F$$

be the lattice freely generated by the irreducible components F of $\mathrm{Exc}(\pi)$, that is, the lattice of divisors on $\tilde{X}$ supported by $\mathrm{Exc}(\pi)$. Inside the associated real vector space $L_{\mathbb{R}}(\pi)$, consider the closed regular cone:

$$\sigma(\pi) := \bigoplus_F \mathbb{R}_+ F$$

of the *effective* $\mathbb{R}$-divisors on $\tilde{X}$ supported by $\mathrm{Exc}(\pi)$.

For each pair of distinct irreducible components F, G, consider the closed convex subcone $\sigma_{F,G}(\pi)$ of $\sigma(\pi)$ defined by:

$$\sigma_{F,G}(\pi) := \{\sum_{F'} a(F')F' \in \sigma(\pi) \mid a(F) \leq a(G)\}$$

the sum being done over the irreducible components F' of $\mathrm{Exc}(\pi)$.

THEOREM 3.12. *Fix an irreducible component F of $\mathrm{Exc}(\pi)$. Suppose that for each other component G, the cone $\sigma_{F,G}(\pi)$ contains in its interior an integral divisor D such that $\mathcal{O}_{\tilde{X}}(-D)$ is generated by its global sections. Then $V(F)$ is in the image of the Nash map $\mathcal{N}_{X,x}$. In particular, F is an essential component of π.*

As a corollary, we get:

COROLLARY 3.13. *Fix an irreducible component F of $\mathrm{Exc}(\pi)$. Suppose that for each other component G, the cone $\sigma_{F,G}(\pi)$ contains an integral divisor D such that $\mathcal{O}_{\tilde{X}}(-D)$ is ample when restricted to each component of $\mathrm{Exc}(\pi)$. Then $V(F)$ is in the image of $\mathcal{N}_{X,x}$ and F is an essential component of π.*

As another corollary, we get the announced computable criterion of bijectivity of the Nash map:

COROLLARY 3.14. *Suppose that for each pair of distinct irreducible components F, G of $\mathrm{Exc}(\pi)$, the cone $\sigma_{F,G}(\pi)$ contains an integral divisor D such that $\mathcal{O}_{\tilde{X}}(-D)$ is ample when restricted to each component of $Exc(\pi)$. Then the components of $Exc(\pi)$ are precisely the essential components over x and the Nash map $\mathcal{N}_{X,x}$ is bijective.*

In our first paper [**139**], restricted to the case of surfaces, we had obtained this theorem through another route, through a version of Theorem 3.4 of Chapter 2. We did not state it in terms of ampleness. As explained in the introduction to this chapter, it was our understanding of the fact that the notion of ampleness was the fundamental one in its proof, that made us arrive at this formulation, valid in all dimensions. The techniques of the paper [**139**] were in the meantime extended by Morales [**120**].

In the next section I explain how we applied this last theorem to construct non-trivial examples of normal 3-dimensional non-toric isolated singularities with bijective Nash map. The same techniques should work in higher dimensions (see Question (9) in Chapter 5).

4. Applications

Corollary 3.14 gives a method to construct examples of singularities (X, x) for which the Nash map $\mathcal{N}_{X,x}$ is bijective. Namely, one starts from a divisorial resolution of a germ such that the components of the exceptional locus have closed cones of curves of finite type. The condition on an effective divisor supported by the exceptional set to have an ample opposite in restriction to the exceptional set translates then into a finite system of linear inequalities. If this system has solutions inside all the cones considered in the corollary then, using the corollary, one has an example with bijective Nash map.

One could try to start from germs defined explicitly by equations and to use one of the available algorithms of resolution. Nevertheless, those algorithms do not allow to compute the closed cone of curves of a component of the exceptional set.

For this reason we decided to work differently. The strategy we followed was to start from a finite collection (F) of smooth projective varieties, with cones of curves which are closed and of finite type. Then choose line bundles over the varieties F with ample duals and glue analytically the total spaces of those line bundles along neighborhoods of suitable hypersurfaces of the F. Of course, the first thing to adjust in order to do such a gluing, is to make a pairing of the chosen hypersurfaces and to fix isomorphisms between the elements in each pair.

If the gluing succeeds, one gets a smooth analytic variety X which contains a divisor F obtained topologically by identifying the chosen pairs of hypersurfaces of the varieties F. The choices should be done in order to make F *exceptional* in X, in the sense of Definition 3.10. This should be checkable using Grauert's criterion of contractibility. Then try to construct the divisors D verifying the conditions of Corollary 3.14. The hypothesis on the finiteness of the cones of curves ensures, as explained before, that this search amounts to the resolution of a *finite* system of inequalities.

Concretely, we gave the simplest possible illustration of the previous strategy. Namely:

• we worked in the smallest interesting dimension, that is, in dimension 3;

• we started with the smallest possible number of divisors which would give a non-trivial example, that is, 2;

• we worked with the simplest possible surfaces which would necessarily give non-toric singularities, that is, with irrational ruled surfaces;

• we looked at their simplest possible embeddings in an ambient smooth 3-fold, that is, in the total spaces of line bundles;

• we chose those line bundles in the simplest way for the gluing of their total spaces to be done by an algebraic plumbing.

The details are explained in [**140**], where we show also that the obtained singularities are normal and *non-toric*. As a consequence, they provide new examples of germs with bijective Nash map for non-trivial reasons.

Suppose now that one has proved for some surface singularity that the Nash map is bijective. We would like then to answer question (10) of Chapter 5.

CHAPTER 5

Perspectives

In this final short chapter, I list the main questions and problems which occurred while doing the research presented before. They keep nourishing my reflection.

(1) (C. Caubel) Let (X, x) be a normal isolated singularity of complex analytic space of dimension at least 3. If its contact boundary is contactomorphic to the standard contact sphere, is X necessarily smooth at x?

(2) (C. Caubel, A. Némethi and myself) Is the standard contact structure on the boundary of a normal surface singularity invariant (up to isotopy) by any orientation-preserving diffeomorphism? This is known to be true when the boundary is a rational homology sphere, see [**24**].

(3) Characterize the standard contact structure among the different Stein fillable contact structures on the oriented boundary of a normal surface singularity.

(4) Do contact rational homology 3-spheres have only a finite number of Stein fillings, up to orientation-preserving diffeomorphisms ? If this is false in such a generality, does it become true for Milnor fillable ones ?

(5) Fix a topological type of normal surface singularity. Is there only a finite number of Milnor fibers (up to orientation-preserving diffeomorphisms) when one varies the analytical structure which realizes the fixed topological type?

(6) Determine the class of taut rational surface singularities for which the Milnor fibers are, up to orientation-preserving diffeomorphisms which induce on the boundary the canonical identification, all the Stein fillings of the associated contact boundary.

(7) As an extension of the previous problem, determine the class of topological types of rational surface singularities for which the Milnor fibers of the various analytical realisations are, up to orientation-preserving diffeomorphisms, all the Stein fillings of the associated contact boundary.

(8) (J. Seade and myself) Given a connected graph weighted with arithmetic genera, is there always a numerically Gorenstein surface singularity whose minimal resolution realises it?

(9) Construct examples of normal isolated singularities of arbitrary dimension with bijective Nash map, such that the number of essential components is arbitrarily big and that all the essential components are birationally ruled. This should be possible using the techniques of Chapter 4.

(10) Consider a normal surface singularity with bijective Nash map. Is it possible to reconstruct the dual graph of the minimal resolution from the structure of the space of arcs based at the singular point ?

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Index