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Ultrametric spaces of branches on arborescent singularities

Patrick Popescu-Pampu
(joint work with Evelia R. García Barroso and Pedro D. González Pérez)

This work started from our intention to study in which measure a theorem of Płoski from [5] can be generalized from germs of smooth complex analytic surfaces to other normal surface singularities. This theorem may be stated as follows:

Theorem. *Fix a smooth complex analytic surface singularity (S, O). For each pair of branches (that is, germs of irreducible formal curves) A, B drawn on (S, O), consider:*

$$U(A,B) := \begin{cases} \dfrac{m_O(A) \cdot m_O(B)}{A \cdot B}, & \text{if } A \neq B, \\ 0, & \text{if } A = B \end{cases}$$

where m_O denotes the multiplicity at O and $A \cdot B$ denotes the intersection number of A and B at O. Then U is an ultrametric on the set of branches on (S, O).

We discovered that, slightly reformulated purely in terms of intersection numbers, the theorem may be extended to *arborescent* singularities. We call a normal surface singularity *arborescent* if it has a simple normal crossings resolution whose dual graph is a tree – in which case all such resolutions have this property. For instance, the normal surface singularities with rational homology sphere links are precisely the arborescent singularities such that all irreducible exceptional divisors appearing in its resolutions are rational.

Our generalization of Płoski's theorem is:

Theorem. *Fix an arborescent singularity (S, O) and a branch L on it. For each pair of branches A, B on (S, O) which are distinct from L, consider:*

$$U_L(A,B) := \begin{cases} \dfrac{(L \cdot A) \cdot (L \cdot B)}{A \cdot B}, & \text{if } A \neq B, \\ 0, & \text{if } A = B \end{cases}.$$

Then U_L is an ultrametric on the set of branches on (S, O) distinct from L.

Here we work with Mumford's notion of rational-valued intersection number of Weil divisors drawn on normal surface singularities, introduced in [4].

The previous theorem generalizes Płoski's one. Indeed, each time a finite set $\mathcal{F}$ of branches is fixed on a smooth germ of surface – the simplest kind of arborescent singularity – one may choose a smooth branch L transversal to all of them, in which case the functions U and U_L coincide in restriction to $\mathcal{F}$.

Consider now a finite set $\mathcal{F}$ of branches on an arbitrary arborescent singularity (S, O). The restriction of the ultrametric U_L to $\mathcal{F}$ allows to associate canonically a rooted tree $T_L(\mathcal{F})$ to $\mathcal{F}$. Its set of leaves is $\mathcal{F}$, its set of vertices consists of the closed balls defined by the ultrametric and its root may be seen as the union of $\mathcal{F}$ with an infinitely distant supplementary point. We prove the following interpretation of this rooted tree in terms of dual graphs:

Theorem. *Fix an arborescent singularity (S, O), a branch L on it and a finite set $\mathcal{F}$ of branches distinct from L. Consider an embedded resolution π with simple normal crossings of the sum of L with the branches of $\mathcal{F}$. Denote by $D_{L,\pi}(\mathcal{F})$ the union of the geodesics joining the strict transforms of L and of the branches of $\mathcal{F}$ inside the dual graph of their total transform by π, seen as a tree rooted at the strict transform of L. Then the rooted trees $T_L(\mathcal{F})$ and $D_{L,\pi}(\mathcal{F})$ are canonically isomorphic.*

In the special case in which both (S, O) and the branch L are smooth, we recover a theorem of Favre and Jonsson [2].

The tree $D_{L,\pi}(\mathcal{F})$ has also a valuative interpretation, generalizing the one given by Favre and Jonsson [2] for the same case when both S and L are smooth.

As in [2], we work with valuations of the local ring $\mathcal{O}$ of (S, O) with values in $[0, +\infty]$ and which are allowed to take the value $+\infty$ on other elements of $\mathcal{O}$ than the function 0. Their set is naturally partially ordered : $\nu_1 \leq \nu_2$ if and only if $\nu_1(f) \leq \nu_2(f)$ for all $f \in \mathcal{O}$.

Every branch drawn on (S, O) defines a valuation – taking Mumford's intersection number of the principal divisor of each element of $\mathcal{O}$ with the branch. Every irreducible exceptional divisor on a resolution of (S, O) defines also a valuation – taking the order of vanishing of each element of $\mathcal{O}$ along this divisor.

Mumford's notion of intersection product allows moreover to define the value of a valuation of $\mathcal{O}$ on an arbitrary branch L. A valuation ν is called *normalized relative to* L if $\nu(L) = 1$. We prove that:

Theorem. *Fix an arborescent singularity (S, O), a branch L on it and a finite set $\mathcal{F}$ of branches distinct from L. Consider an embedded resolution with simple normal crossings of the sum of L with the branches of $\mathcal{F}$. Then the rooted tree $D_{L,\pi}(\mathcal{F})$ is isomorphic to the Hasse diagram of the poset of valuations defined by the irreducible curves represented by the vertices of $D_{L,\pi}(\mathcal{F})$, once they are normalized relative to L.*

All the theorems of Płoski and Favre-Jonsson which we generalize were proved either by working with Newton-Puiseux series or with sequences of blow-ups. We work instead only with intersection products of relatively nef rational exceptional divisors on a fixed embedded resolution of the sum of L and of the branches of $\mathcal{F}$. Our proofs are based in an essential way on the following fact:

Proposition. *Fix an arborescent singularity (S, O) and a simple normal crossings resolution of it. Denote by $(E_u)_{u\in\mathcal{V}}$ the irreducible components of its exceptional divisor and by $(E_u^*)_{u\in\mathcal{V}}$ the dual exceptional divisors, in the sense that $E_u^* \cdot E_v = \delta_{uv}$ for any $(u, v) \in \mathcal{V}^2$, where δ_{uv} is Kronecker's symbol. If E_w belongs to the segment $[E_u E_v]$ in the dual graph of the resolution, then:*

$$(E_u^* \cdot E_w^*) \cdot (E_v^* \cdot E_w^*) = (E_u^* \cdot E_v^*) \cdot (E_w^* \cdot E_w^*).$$

In turn, this proposition is based on a formula proved by Eisenbud and Neumann in [1], expressing the intersection numbers $(E_u^* \cdot E_w^*)$ in terms of determinants of subtrees of the dual tree of the resolution.

It is interesting to note that the previous proposition may be reinterpreted using spherical geometry. Consider the real vector space freely generated by the divisors $(E_u)_{u\in\mathcal{V}}$, endowed with the negative of the intersection product. It is a euclidean vector space. Look at the unit vectors $(A_u)_{u\in\mathcal{V}}$ which are positively proportional to the vectors $(E_u^*)_{u\in\mathcal{V}}$. Here is the announced interpretation: *if E_w belongs to the segment $[E_u E_v]$ in the dual graph of the resolution, then the spherical triangle with vertices A_u, A_v, A_w is right-angled at the vertex A_w.* Indeed, the equality of the previous proposition may be reformulated as the *spherical Pythagorean equality*:

$$\cos(\angle A_u A_w) \cdot \cos(\angle A_v A_w) = \cos(\angle A_u A_v),$$

which characterizes right-angled triangles in spherical geometry.

Detailed proofs of our results may be found in [3].

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Introduction to Singularities

Norbert A'Campo

This evening talk is intended especially for those Young Researchers from the Heidelberg Laureate Forum, september 18th–25th, this year, see *www.heidelberg-laureate-forum.org/event_2016/*, who were invited to attend the Conference on Singularities in Oberwolfach.

We start out with a very general approach to the term *Singularity.*

A singularity is an object in nature that does not fall in the realm of main understanding. Here *main understanding* is very vague. It could mean understanding by the basic laws in Physics, Theorems in Mathematics, ... or by the basic achievements of a theory.

In mathematical nature such an object or rather a feature could be the local behavior near a given point or global behavior of a function, a vector field, a differential form, a space, a representations of a group, etc..

Let us restrict to the case of the local behavior near a point p of differentiable functions that are defined on numerical real or complex spaces $\mathbb{R}^n$ or $\mathbb{C}^n$. So we study locally near $p \in \mathbb{R}^n$ or $p \in \mathbb{C}^n$ differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$ with derivatives of any order or holomorphic functions $f : \mathbb{C}^n \to \mathbb{C}$.

The first step is the three term expansion

$$f(p+h) = f(p) + A(h) + \mathrm{Rest}_f(p,h)$$

where $A : \mathbb{R}^n \to \mathbb{R}$ is a linear function and where the third term $\mathrm{Rest}_f(p,h)$ is relatively small compared to h, meaning

$$\lim_{h\to 0} \frac{\mathrm{Rest}_f(p,h)}{||h||} = 0$$

The linear map A is called the *differential* of f at p and is denoted by $(Df)_p$.

Examples of functions on $\mathbb{R}^n$ are the differentiable coordinate functions $x_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \cdots, n$, on $\mathbb{R}^2$ also denoted by x, y. A general function f can be expressed using a system of coordinate functions. For example $f = x^5 + y^3 : \mathbb{R}^2 \to \mathbb{R}$. Other examples are $g = x^5 + y^3 + x^2y^2$ and $k = x^4 + y^4 + x^2y^2$.

Functions with least complicated expressions are the coordinate functions. In fact, how complicated a given function f is depends heavily on the system of coordinate functions.

A natural question Q_1 is to ask which functions f can be expressed near a given point p as first coordinate function of a local coordinate system of functions.

The following main theorem gives the answer:

The answer to the question Q_1 is YES if and only if $(Df)_p \neq 0$.

Indeed, let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function, $p \in \mathbb{R}^n$ such that the differential $(Df)_p : \mathbb{R}^n \to \mathbb{R}$ at p does not vanish. The kernel $X = (Df)_p^{-1}(0)$ is a linear subspace of dimension $n-1$ in $\mathbb{R}^n$. Let Y be a 1-dimensional subspace