

NUMERICALLY GORENSTEIN SURFACE SINGULARITIES ARE HOMEOMORPHIC TO GORENSTEIN ONES

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Abstract

Consider a normal complex analytic surface singularity. It is called Gorenstein if the canonical line bundle is holomorphically trivial in some punctured neighborhood of the singular point and is called numerically Gorenstein if this line bundle is topologically trivial. The second notion depends only on the topological type of the singularity. Laufer proved in 1977 that, given a numerically Gorenstein topological type of singularity, every analytical realization of it is Gorenstein if and only if one has either a Kleinian or a minimally elliptic topological type. The question to know if any numerically Gorenstein topology was realizable by some Gorenstein singularity was left open. We prove that this is indeed the case. Our method is to plumb holomorphically meromorphic 2-forms obtained by adequate pullbacks of the natural holomorphic symplectic forms on the total spaces of the canonical line bundles of complex curves. More generally, we show that any normal surface singularity is homeomorphic to a \mathbb{Q} -Gorenstein singularity whose index is equal to the smallest common denominator of the coefficients of the canonical cycle of the starting singularity.

1. Introduction

Let $(X, 0)$ be a normal complex analytic surface singularity. It is called *Gorenstein* if the canonical line bundle of some representative of it is holomorphically trivial outside zero, and is called *numerically Gorenstein* if the same line bundle is topologically trivial outside zero. For example, hypersurfaces and complete intersections with isolated singularities are normal and Gorenstein.

One has fundamental systems of neighborhoods of $0 \in X$ which are homeomorphic to cones over an oriented 3-manifold, called the *boundary* or the *link* of $(X, 0)$. By a theorem of Neumann [14, Theorem 2], the knowledge of the boundary up to oriented homeomorphisms is equivalent to the knowledge of the dual graph of the exceptional divisor of any good resolution, weighted by the genera and the self-intersections of its irreducible components. We say that two singularities are *homeomorphic* if and only

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if they have boundaries which are oriented homeomorphic. By Neumann's theorem, this means that they have good resolutions with isomorphic weighted dual graphs.

The *canonical cycle* of a resolution of the surface singularity is by definition the unique cycle supported by the exceptional divisor E , which has the same intersections with the irreducible components of E as a canonical divisor. Its coefficients are determined by the weighted dual graph, and therefore they are topological invariants of the singularity. The coefficient of each component of E is called its *discrepancy*. The fact that a singularity is *numerically Gorenstein* is a topological property: it is equivalent to the fact that, for some resolution, all discrepancies are integral.

Instead, the fact that a singularity is Gorenstein is *not* in general a topological property. In fact, Laufer [9, Theorem 4.3] proved that *all* analytical realizations of a given topology are Gorenstein if and only if one has either a Kleinian or a minimally elliptic topological type.

It seems that the problem *to decide if any numerically Gorenstein normal surface singularity is homeomorphic to a Gorenstein one* was left open. We learned this problem from Jonathan Wahl (see the introduction of [16]). We prove (Theorem 4.1):

Any numerically Gorenstein normal surface singularity is homeomorphic to a Gorenstein one, having, moreover, an exceptional set of the minimal good resolution analytically isomorphic to that associated to the initial singularity.

One gets automatically a characterization of the weighted graphs which are realized by Gorenstein surface singularities: *they are precisely the negative definite ones with integral discrepancies*. This fact was known before in the case of weighted homogeneous surface singularities, as a consequence of Watanabe's criterion of [21, (2.9)] for such a singularity to be Gorenstein (another proof of it is given by Wahl [19, (4.4.2)]).

Our proof starts from the fact that *the holomorphic cotangent bundle of a complex manifold is naturally endowed with a holomorphic symplectic form*. We get in this way a holomorphic symplectic form on the canonical bundle $K_C = T^*C$ of any smooth, complex, compact curve C . We pull back this form using meromorphic maps to total spaces of convenient line bundles. In order to take care of the components of E with discrepancy -1 , which are to be treated separately, we construct also special meromorphic 2-forms on the total spaces of line bundles on \mathbb{P}^1 .

The fundamental facts which make our constructions sufficient are the following:

- They lead to meromorphic 2-forms having *local normal forms without moduli* in holomorphic coordinates (namely, of the type $u^a v^b du \wedge dv$), which allows us to *plumb* them *holomorphically*.

- If the initial numerically Gorenstein singularity is not Kleinian, cusp, or simple elliptic (cases when it is automatically Gorenstein), then all the components with discrepancy -1 are rational and no two of them intersect.

By plumbing tubular neighborhoods of the 0-sections *using the identifications given by the coordinate systems of the normal forms*, we get smooth, complex analytic surfaces endowed with a meromorphic 2-form which is *symplectic* outside the compact divisor obtained as the image of the 0-sections of the line bundles. Therefore, whenever the divisor is contractible, one gets a normal Gorenstein singularity.

Teissier and Wahl suggested we extend the theorem to all normal surface singularities, the role of Gorenstein singularities being played now by the \mathbb{Q} -Gorenstein ones (i.e., those normal singularities such that some power of the canonical bundle is holomorphically trivial in a pointed neighborhood of the singular point).

We were able to prove this extension after Veys mentioned a result of his (see Theorem 2.6), implying that *if the initial normal surface singularity is not log canonical, then for the minimal good resolution, all components of E with discrepancy -1 are rational, they intersect at most two other components of E , and they are pairwise disjoint*.

In order to state the extension, we need a little more vocabulary. The *index* of a \mathbb{Q} -Gorenstein singularity is the smallest exponent r such that $K_{X \setminus 0}^r$ is holomorphically trivial and the *topological index* is defined in the same way for *all* normal singularities, by asking $K_{X \setminus 0}^r$ to be *topologically trivial*. By using the same kind of constructions as before, we prove Theorem 5.4:

Any normal surface singularity is homeomorphic to a \mathbb{Q} -Gorenstein one whose index is equal to the topological index of the initial one, having, moreover, an exceptional set of the minimal good resolution analytically isomorphic to that associated to the initial singularity.

In this enlarged context, the analogue of the natural holomorphic symplectic form on K_C is a natural meromorphic 2-form of weight r on the total space of K_C^r , for all $r \in \mathbb{N}^*$. Namely, in natural local coordinates it is written $p^{1-r} (dp \wedge dq)^r$ (see Lemma 5.1).

Wahl mentioned to us (unpublished) that he can prove the previous theorem in the weighted homogeneous case by a generalization of Watanabe's arguments of [21].

We describe now briefly the content of the paper. In Section 2 we recall the fundamental notions about normal surface singularities used in the rest of the paper. In Section 3 we construct special meromorphic 2-forms on total spaces of line bundles on smooth, compact, complex curves. In Section 4 we plumb such total spaces along weighted dual graphs, in a way which leads to the proof of Theorem 4.1. Finally, in Section 5, we prove Theorem 5.4 by generalizing the steps of the proof of Theorem 4.1.

2. Background

In the sequel, the *singularities* we consider will be *germs of normal complex analytic surfaces*. If $(X, 0)$ denotes a singularity, tacitly X will also denote a Milnor representative of it.

The divisor of a meromorphic 2-form α on a smooth surface will be denoted $\text{div}(\alpha)$. Outside it, such a form is *holomorphic symplectic*, that is, nondegenerate (as a consequence, by Darboux's theorem, it may be written $du \wedge dv$ in convenient local coordinates in the neighborhood of any point, a fact we will not use in this paper).

A resolution of a singularity is said to be *good* if its exceptional divisor has normal crossings and its components are smooth. There exists a *minimal* good resolution, unique up to unique isomorphism.

Definition 2.1

Let $(X, 0)$ be a normal surface singularity. It is called *Gorenstein* if its canonical bundle is holomorphically trivial in a punctured neighborhood of 0. It is called *numerically Gorenstein* if one has instead only *topological* triviality.

In other words, a Gorenstein singularity is one which may be endowed with a 2-form which is holomorphic and nonvanishing in a punctured neighborhood of 0, that is, which is holomorphic symplectic there.

Starting from a normal surface singularity, denote by $(\tilde{X}, E) \xrightarrow{\pi} (X, 0)$ a good resolution, where E is the reduced exceptional divisor. By taking pullbacks by the morphism π , one identifies the space of holomorphic 2-forms on $X \setminus 0$ with the one of holomorphic 2-forms on $\tilde{X} \setminus E$. In particular, one may construct a holomorphic symplectic form on $X \setminus 0$ by constructing a holomorphic symplectic form on $\tilde{X} \setminus E$. It is the approach taken in this paper.

Denote by Γ the *dual (intersection) graph* of E : its vertices correspond bijectively to the irreducible components of E and between two distinct vertices i and j there are as many (unoriented) edges as the intersection number $e_{ij} := E_i \cdot E_j \geq 0$ of the corresponding components. In particular, Γ has no loops. Moreover, each vertex i of Γ is weighted by the self-intersection number $-e_i := E_i^2 < 0$ of the associated component E_i inside \tilde{X} , and by the genus g_i of E_i .

Denote by $V(\Gamma)$ the set of vertices of Γ . If $i \in V(\Gamma)$, its *valency* is the number of edges connecting it to other vertices.

By classical results of Du Val [5] and Mumford [13], the intersection form of the weighted dual graph is negative definite. Conversely, Grauert [6] proved that if the form associated to a reduced compact divisor E on a smooth surface is negative

definite, then E can be *contracted* to a normal singular point of an analytic surface, well defined up to unique analytic isomorphisms.

As the intersection form is negative definite, there exists a unique divisor *with rational coefficients* Z_{can} supported by E such that $Z_{\text{can}} \cdot E_i = K_{\tilde{X}} \cdot E_i$, for all $i \in V(\Gamma)$.

Definition 2.2

The divisor Z_{can} is called the *canonical cycle* of E (or of the resolution π). If $Z_{\text{can}} = \sum_{i \in V(\Gamma)} k_i E_i$, then the coefficient k_i is called the *discrepancy* of E_i .

The adjunction relations, applied to the curves E_i inside the surface \tilde{X} , may be written as

$$(k_i + 1)e_i = -2g_i + 2 + \sum_{\substack{j \in V(\Gamma) \\ j \neq i}} k_j e_{ij}, \quad \forall i \in V(\Gamma). \tag{1}$$

Durfee [3] proved the following characterization of numerically Gorenstein singularities (which explains the attribute *numerically*).

PROPOSITION 2.3

The normal surface singularity $(X, 0)$ is numerically Gorenstein if and only if for any (or some) good resolution, all discrepancies are integral.

One has the following generalization of the notion of normal Gorenstein singularity.

Definition 2.4

The normal surface singularity $(X, 0)$ is called \mathbb{Q} -Gorenstein if there exists a positive integer r such that the pluricanonical bundle K_X^r is holomorphically trivial in a punctured neighborhood of 0. The smallest such exponent r is then called the *index* of the singularity.

Equivalently, $(X, 0)$ is \mathbb{Q} -Gorenstein if and only if the canonical class K_X (consisting a priori of Weil divisors) is (holomorphically) \mathbb{Q} -Cartier and the index is the smallest integral factor making it Cartier. The normal Gorenstein singularities are precisely the \mathbb{Q} -Gorenstein ones of index 1.

In dimension 2, \mathbb{Q} -Gorenstein is equivalent to being a cyclic quotient of a Gorenstein singularity (the *canonical cover*), a fact which is no longer true in higher dimensions, where the canonical cover may not be Cohen-Macaulay. But in this paper we do not use this viewpoint on \mathbb{Q} -Gorenstein surface singularities.

If $(X, 0)$ is \mathbb{Q} -Gorenstein and K_X^r is holomorphically trivial on $X \setminus 0$, the pullback by the resolution morphism π of a holomorphic trivializing section of K_X^r becomes a meromorphic section of $K_{\tilde{X}}^r$ whose divisor is exactly rZ_{can} . Therefore, in this case r is a common denominator of the discrepancies. By the same method as the one used by Durfee to prove Proposition 2.3, one shows that the smallest common denominator of the discrepancies is the smallest exponent $r \in \mathbb{N}^*$ such that K_X^r is topologically trivial on $X \setminus 0$. This motivates us to introduce the following.

Definition 2.5

The *topological index* of the singularity $(X, 0)$ is the smallest positive integer r such that $K_{X \setminus 0}^r$ is topologically trivial.

Recall also the following special classes of singularities:

- The *Kleinian singularities*, also known as *Du Val singularities*, *rational double points*, or *simple surface singularities* (see Durfee [4]) are, up to isomorphism, the surface singularities of the form \mathbb{C}^2/G , where G is a finite subgroup of $SU(2)$. For these singularities, the minimal resolution is good; it has $Z_K = 0$ and its dual graph Γ is one of the trees A_n, D_n, E_6, E_7, E_8 with $g_i = 0, e_i = 2, \forall i \in V(\Gamma)$.
- The *cuspidal singularities* are those normal surface singularities which have a good resolution whose dual graph Γ is homeomorphic to a circle and $g_i = 0, \forall i \in V(\Gamma)$ (see Looijenga [11, p. 16]).
- The *simple elliptic singularities* are those normal surface singularities whose minimal resolution has a smooth exceptional divisor of genus 1. As well as the cusp singularities, they are *minimally elliptic* (see Laufer [9]). As we do not need this last notion in the paper, we do not give more details about it.
- The *log canonical singularities* are those normal surface singularities whose canonical cycle satisfies the inequalities $k_i \geq -1, \forall i \in V(\Gamma)$. By a result of Sakai [17], *they are necessarily \mathbb{Q} -Gorenstein, with an index equal to their topological index*. One may find their classification in Sakai [17], Kawamata [7], Wahl [20], Alexeev [1], or Matsuki [12]. By this classification, one sees that the numerically Gorenstein log canonical singularities (therefore, the Gorenstein ones) are exactly the Kleinian, the cusp, and the simple elliptic singularities. This class may also be defined in higher dimensions. Then, one usually asks the singularity to be \mathbb{Q} -Gorenstein, in order to be able to define the discrepancies. (An exception is the recent paper [2], in which discrepancies are defined also for singularities which are not \mathbb{Q} -Gorenstein.) Sakai's result quoted before shows that this supplementary condition is not needed in dimension 2 (see also [12, Remark 4-6-3(ii)]).

In [18, 2.10], Veys proved the following theorem about the behavior of the coefficients of the canonical cycle on the dual graph of the *minimal* good resolution.

THEOREM 2.6

Let $(X, 0)$ be a normal surface singularity which is not log canonical. Denote by $V^-(\Gamma)$ the set of vertices i of the minimal good resolution such that $k_i < -1$, which is nonempty by hypothesis. Then we have the following:

- (a) The full subgraph Γ^- of Γ with vertex set $V^-(\Gamma)$ is connected.
- (b) The full subgraph of Γ with vertex set $V(\Gamma) \setminus V^-(\Gamma)$ is a disjoint union of segments whose vertices represent rational components of the exceptional divisor. Moreover, exactly one end of each segment is attached (by one edge) to Γ^- , and the discrepancies are strictly increasing along the segment, starting from that end.
- (c) All the discrepancies are (strictly) negative.

In [18, 2.11], Veys states that the previous theorem may be obtained as a corollary of results of Wahl [20]. Instead, he uses mainly results of Alexeev [1]. Before, Laufer [10, Proposition 2.1] had proved the analogue of Theorem 2.6(c) for the *minimal* (not necessarily good) resolution of any normal surface singularity which is not Kleinian.

Remark 2.7

Veys's terminology is slightly different from ours. For instance, he uses the term *log resolution* instead of *good resolution* and he works with the *log discrepancies* $a_i := k_i + 1$ instead of the discrepancies k_i . We prefer to work with the discrepancies, because in our computations they appear explicitly as exponents of coefficients of meromorphic 2-forms (see the last two sections).

As a corollary of Theorem 2.6(b), we have the following special case concerning the numerically Gorenstein singularities which are not log canonical.

COROLLARY 2.8

Start from a numerically Gorenstein singularity which is not smooth, Kleinian, a cusp singularity, or a simple elliptic one. Then the dual graph Γ of the minimal good resolution satisfies $k_i \leq -1$ for all $i \in V(\Gamma)$. Moreover, if $k_i = -1$, then E_i is a smooth rational curve, i is of valency 1, and the unique vertex j to which it is connected satisfies $k_j = -2$.

Proof

The only statement which is not a special case of the previous theorem is the last one. But it is an immediate consequence of the adjunction formula (1).

In fact, the corollary could be proved by showing first that Laufer's theorem [10, Proposition 2.1] stated before—namely, the fact that Theorem 2.6(c) holds for the minimal resolution of a normal surface singularity which is not Kleinian—is also true for the minimal good resolution of a numerically Gorenstein singularity. This is what we did before knowing Veys's results. We analyzed what happens at each step of the sequence of blowing-ups which passes from the minimal resolution to the minimal good one, and we finished by using the adjunction relations. \square

One has a generalization for all normal surface singularities which are not log canonical.

COROLLARY 2.9

Start from a normal surface singularity which is not log canonical. On the dual graph of the minimal good resolution, if i is a vertex of Γ such that $k_i = -1$, then it represents a smooth rational curve and its valency is 1 or 2. Moreover,

- *if it is of valency 1, then the unique vertex j to which it is connected satisfies $k_j = -2$;*
- *if it is of valency 2, then the two vertices j_1, j_2 to which it is connected satisfy $k_{j_1} + k_{j_2} = -2$, with $k_{j_1} \neq -1, k_{j_2} \neq -1$.*

Proof

This is again a direct consequence of Theorem 2.6 and of the adjunction formula. \square

Remark 2.10

In particular, the exceptional divisor of the minimal good resolution of non-log canonical normal surface singularities does not contain adjacent components E_i and E_j such that $k_i = k_j = -1$ (see also Remark 4.2). As Veys told us, this conjectured statement was his motivation for proving Theorem 2.6.

3. Construction of special meromorphic 2-forms on line bundles

Our idea for proving the theorem stated in the title of the paper is to construct first *meromorphic 2-forms* on total spaces of line bundles that are *holomorphic symplectic outside the union of the 0-section and of some fibers* and then to *plumb* them in such a way that the symplectic forms glue. This section is dedicated to the first step and the next one to the second step.

When we say that L is a line bundle, L will denote the total space (and not the associated sheaf of regular sections). In particular, by a *form defined on L* , we will mean a form defined on the total space of L .

PROPOSITION 3.1

Let C be a compact, connected, and smooth complex curve of genus $g \geq 0$. Fix an integral divisor D on C and an integer $m \in \mathbb{Z}^$ which divides $2g - 2 - \deg(D)$. Then*

there exists a line bundle $L \xrightarrow{\psi} C$ of degree $m^{-1}(2g - 2 - \deg(D))$, endowed with a meromorphic 2-form α such that we have the following:

- (a) $\operatorname{div}(\alpha) = (m - 1)C + \psi^*(D)$, where C is seen here as the 0-section of L .
- (b) For each point $A \in C$, if $a \in \mathbb{Z}$ denotes the multiplicity of D at that point, one has a system of local coordinates (q, f) centered at A , seen as a point of the 0-section of L , such that q is constant on the fibers of π , that $f = 0$ defines C and, moreover,

$$\alpha = f^{m-1}q^a df \wedge dq.$$

Proof

Denote by L_D the line bundle over C whose sheaf of sections is $\mathcal{O}(D)$, and denote by s_D its canonical meromorphic section defining D . Tensoring by s_D , one gets a meromorphic map between total spaces of line bundles:

$$H_D := K_C \otimes L_{-D} \cdots \xrightarrow{\tau_D} K_C, \tag{2}$$

where $K_C \xrightarrow{\gamma} C$ denotes the canonical line bundle of C , equal to its holomorphic cotangent bundle. Denote by α_C the canonical holomorphic symplectic form on the surface K_C . If q denotes a local coordinate on the open set $U \subset C$ and p is the associated dual coordinate on $\gamma^{-1}(U) \subset K_C$ (i.e., the general 1-form on U is written pdq), one has the following expression for α_C on $\gamma^{-1}(U)$:

$$\alpha_C = dp \wedge dq. \tag{3}$$

Denote then by α_D the meromorphic 2-form on H_D obtained as the pullback of α_C by τ_D :

$$\alpha_D := \tau_D^*(\alpha_C). \tag{4}$$

Consider now any m th root $H_D^{1/m}$ of the line bundle H_D and the m th tensor power meromorphic map:

$$H_D^{1/m} \cdots \xrightarrow{\mu_{D,m}} H_D. \tag{5}$$

Such line bundles $H_D^{1/m}$ are the solutions H of the equation $m \cdot H = H_D$ in the Picard group $(\operatorname{Pic}(C), +)$ written additively. The set of solutions is nonempty by the hypothesis that $m \neq 0$ divides $\deg(H_D) = 2g - 2 - \deg(D)$ and it is a principal space (a torsor) under the subgroup of $|m|$ -torsion elements of the Jacobian $\operatorname{Pic}^0(C)$ of C . This Jacobian is a real $(2g)$ -dimensional torus, therefore there are $|m|^{2g}$ such line bundles. We may take any one of them.

Denote by $\alpha_{D,m}$ the meromorphic 2-form on $H_D^{1/m}$ obtained as the pullback of α_D by $\mu_{D,m}$:

$$\alpha_{D,m} := \mu_{D,m}^*(\alpha_D). \tag{6}$$

We claim that *the pair $(L, \alpha) := (H_D^{1/m}, \alpha_{D,m})$ satisfies both properties (a) and (b).*

The rest of the proof is devoted to an explanation of this claim.

The map $\tau_D \circ \mu_{D,m}$ (see (2) and (5)) is regular and without critical points on the complement $H_D^{1/m} \setminus (C \cup \psi_{D,m}^{-1}(|D|))$, where $H_D^{1/m} \xrightarrow{\psi_{D,m}} C$ denotes the bundle projection and $|D|$ denotes the support of D . It is therefore enough to check point (b) in order to deduce also point (a).

We will proceed in two steps, by computing first the form α_D in adapted local coordinates and then the form $\alpha_{D,m}$.

- Consider any local coordinate q on C centered at A . Choose then a local trivializing section σ_A of L_D in the neighborhood of A such that $s_D = q^a \sigma_A$. Denote by $(\sigma_A)^*$ the dual trivializing section of $L_{-D} = L_D^*$. Then $dq \otimes (\sigma_A)^*$ is a local trivializing section of H_D , with respect to which a general section is written $h dq \otimes (\sigma_A)^*$. Then (q, h) is a system of local coordinates on the total space H_D in the neighborhood of A .

We have $\tau_D(h dq \otimes (\sigma_A)^*) = h dq s_D((\sigma_A)^*) = h q^a dq$, which shows that with respect to the local coordinate systems (q, h) on H_D and (q, p) on K_C , the map τ_D is given by $(q, h) \rightarrow (q, hq^a)$. Combined with formula (3), this implies that the meromorphic 2-form α_D defined by (4) has the following expression in the coordinate system (q, h) on H_D :

$$\alpha_D = q^a dh \wedge dq. \tag{7}$$

- We do now the same kind of reasoning for the map $\mu_{D,m}$ (see (5)). Choose a local trivializing section η_A of $H_D^{1/m}$ in the neighborhood of A , such that $\mu_{D,m}(\eta_A) = \eta_A^{\otimes m} = dq \otimes (\sigma_A)^*$. If a general section of $H_D^{1/m}$ is written $l \eta_A$, then (q, l) is a local coordinate system on the total space $H_D^{1/m}$ in the neighborhood of the point A . With respect to the local coordinate systems (q, l) on $H_D^{1/m}$ and (q, h) on H_D , the map $\mu_{D,m}$ is given by $(q, l) \rightarrow (q, l^m)$. Combined with formula (7), this implies that the meromorphic 2-form $\alpha_{D,m}$ defined by (6) has the following expression in the coordinate system (q, l) on $H_D^{1/m}$:

$$\alpha_{D,m} = m q^a l^{m-1} dl \wedge dq. \tag{8}$$

By making the change of variables $f := m^{1/m} l$, where $m^{1/m}$ is an arbitrary m th root of m (which exists, by the hypothesis $m \neq 0$), we get the following from equation (8):

$$\alpha_{D,m} = f^{m-1} q^a df \wedge dq.$$

The proposition is proved. □

Remark 3.2

Proposition 3.1(b) is not a consequence of Proposition 3.1(a). Indeed, there are moduli for germs of meromorphic 2-forms whose divisor has normal crossings with two irreducible components, implying that not all germs of meromorphic 2-forms whose divisor has normal crossings may be written in the form $u^a v^b du \wedge dv$ in adapted local coordinates. One gets the simple normal forms stated in Proposition 3.1(b) as a consequence of our construction by successive pullbacks of the standard symplectic form on the canonical bundle.

The previous proposition does not allow us to get meromorphic 2-forms having a pole of order 1 along the 0-section. In the next one, we construct special meromorphic 2-forms on the total spaces of arbitrary line bundles on a smooth *rational curve*, having such pole of order 1 along the 0-section. We will see that this is enough to prove the characterization of the topological types of Gorenstein singularities.

PROPOSITION 3.3

Let C be a compact, connected, and smooth rational curve. Let $L \xrightarrow{\psi} C$ be a line bundle on it, and let A be a point on C . Then there exists a meromorphic 2-form α on L such that we have the following:

- (a) $\text{div}(\alpha) = -C - 2 \psi^*(A)$.
- (b) *One has a system of local coordinates (q, f) centered at A , seen as a point of the 0-section of L , such that q is constant on the fibers of ψ , that $f = 0$ defines C , and, moreover, that*

$$\alpha = f^{-1} q^{-2} df \wedge dq.$$

Proof

Choose a covering of C by two affine charts isomorphic to \mathbb{C} , one with coordinate q_1 centered at A and the other with coordinate q_2 , such that

$$q_2 = q_1^{-1}. \tag{9}$$

Choose then a trivialization of L over the q_1 -chart, with respect to which the fiber coordinate is denoted f_1 , and another trivialization over the q_2 -chart, with respect to

which the fiber-coordinate is denoted f_2 , such that the change of trivialization over the overlap is given by the equation

$$f_2 = q_1^n f_1. \tag{10}$$

Here $n = \text{deg}(L)$. Such trivializations exist because on \mathbb{P}^1 any line bundle of degree n is isomorphic to $\mathcal{O}(n)$. Therefore,

$$df_2 \wedge dq_2 = -q_1^{n-2} df_1 \wedge dq_1. \tag{11}$$

Using also the equations (9) and (10), this implies that the meromorphic 2-form, which on the chart with coordinates (q_2, f_2) of L is given by $\alpha := -f_2^{-1} df_2 \wedge dq_2$, becomes $\alpha = f_1^{-1} q_1^{-2} df_1 \wedge dq_1$ in the chart with coordinates (q_1, f_1) . Looking at the expressions of α in both charts, we see that the proposition is proved by taking $(q, f) := (q_1, f_1)$. □

4. The topology of Gorenstein normal surface singularities

We are ready to establish the theorem announced in the title of the paper. The idea is to *plumb* using biholomorphic maps the line bundles constructed using Propositions 3.1 and 3.3, in such a way that the associated meromorphic 2-forms *glue*. We recall that the operation of *plumbing* of tubular neighborhoods of 0-sections of line bundles over compact Riemann surfaces was introduced by Mumford in [13]. Since then, this operation was nevertheless mainly used in *the differentiable category*, that is, without necessarily respecting holomorphic structures. Instead, in our approach it is essential to do *holomorphic plumbing*. This procedure was also used by Laufer (see, e.g., [8, p. 83]).

THEOREM 4.1

Consider a numerically Gorenstein surface singularity. Then it is homeomorphic to a normal Gorenstein surface singularity. Moreover, the second singularity may be chosen such that the exceptional curve of its minimal good resolution is analytically isomorphic to that of the initial singularity.

Proof

Let $(X, 0)$ be a numerically Gorenstein singularity, and let $(\tilde{X}, E) \xrightarrow{\pi} (X, 0)$ be its minimal good resolution. If the singularity is smooth, Kleinian, a cusp singularity, or a simple elliptic one, then it is automatically Gorenstein (see Laufer [9]). Therefore, we may suppose that *we are not in one of these cases*. Consequently, we may apply Corollary 2.8.

For each vertex i of Γ , denote by $V_i \subset V(\Gamma)$ the set of indices of those components of E which have nonempty intersection with E_i . For every $j \in V_i$, denote

$E_{i,j} := E_i \cap E_j$. This set may consist of several points: $E_{i,j} = \{A_{i,j,l} \mid l \in I(i, j)\}$. Of course, $E_{i,j} = E_{j,i}$, which allows us to choose the index sets such that $A_{i,j,l} = A_{j,i,l}$ for all $l \in I(i, j) = I(j, i)$. We will look at $E_{i,j}$ also as a reduced divisor on the abstract curve E_i .

By Corollary 2.8, we have $k_i \leq -1$ for all $i \in I$. We consider two cases.

- Suppose that $k_i < -1$. Then apply Proposition 3.1 to $C := E_i$, $g := g_i$, $m := k_i + 1$, and $D := \sum_{j \in V_i} k_j E_{i,j}$. The hypothesis that m divides $2g - 2 - \deg(D)$ is a consequence of formula (1).
- Suppose that $k_i = -1$. By Corollary 2.8, we know that E_i is then a smooth rational curve and that V_i has only one element j , which satisfies, moreover, $k_j = -2$. Then apply Proposition 3.3 to $C := E_i$, $L := \mathcal{O}(-e_i)$, and $\{A\} = E_i \cap E_j$.

By combining both cases, we see that for all $i \in V(\Gamma)$, there exists a line bundle $L_i \xrightarrow{\psi_i} E_i$ of degree $-e_i$ on E_i and a meromorphic 2-form α_i on it, such that we have the following:

- $\text{div}(\alpha_i) = k_i E_i + \sum_{j \in V_i} k_j \psi_i^*(E_{i,j})$.
- For each point $A_{i,j,l} \in E_{i,j}$, one has a system of local holomorphic coordinates $(q_{i,j,l}, f_{i,j,l})$ centered at $A_{i,j,l}$ on the total space L_i , such that $q_{i,j,l}$ is constant on the fibers of ψ_i , that $f_{i,j,l} = 0$ defines the 0-section $E_i \subset L_i$ and, moreover, that

$$\alpha_i = (q_{i,j,l})^{k_j} (f_{i,j,l})^{k_i} df_{i,j,l} \wedge dq_{i,j,l}. \tag{12}$$

Consider then neighborhoods of the 0-sections $E_i \subset L_i$, and *plumb* them by using the following *holomorphic* identifications of neighborhoods of the points $A_{i,j,l}$ and $A_{i,j,l}$ contained in the previous charts, whenever i and j are adjacent vertices of Γ and $l \in I(i, j)$:

$$\begin{cases} q_{i,j,l} = \lambda_{i,j} f_{j,i,l} \\ q_{j,i,l} = \lambda_{j,i} f_{i,j,l} \end{cases} \quad \text{with } \lambda_{i,j}, \lambda_{j,i} \in \mathbb{C}^*.$$

The nonvanishing complex numbers $\lambda_{i,j}, \lambda_{j,i}$ are to be chosen such that by this change of coordinates, the form α_i given by (12) is transformed into α_j , which says precisely that *they glue on the surface obtained by the plumbing*.

These compatibility conditions translate into the following system of equations, which is nontrivial because, when we plumb, we *permute* the order of the coordinates:

$$\lambda_{i,j}^{k_j+1} = -\lambda_{j,i}^{k_i+1} \quad \text{for all pairs } (i, j) \text{ of adjacent vertices of the graph } \Gamma. \tag{13}$$

As before, our hypothesis allows us to apply Corollary 2.8, which implies that there is no pair of adjacent vertices such that $k_i = k_j = -1$ (otherwise, equation

(13) would be contradictory). In fact, this is the only obstruction to the existence of solutions of the system (13): orient each edge of Γ such that, whenever an edge goes from a vertex i to a vertex j , we have $k_j < -1$. Define then $\lambda_{j,i} := 1$ and $\lambda_{i,j}$ to be a $(k_j + 1)$ th root of -1 . We get obviously a solution of (13).

After such a choice, the surface \tilde{Y} obtained by the plumbing is smooth complex analytic and the images of the 0-sections $E_i \subset L_i$ form a divisor canonically identified to E and with the given weighted dual graph. By Grauert's theorem (see [6]), this divisor E may be contracted in \tilde{Y} , yielding a normal singularity $(Y, 0)$. By our construction, the meromorphic 2-forms α_i glue into a meromorphic 2-form $\tilde{\alpha}$ on \tilde{Y} , which is symplectic outside E . Therefore, it descends into a holomorphic symplectic form on $Y \setminus 0$, which means that $(Y, 0)$ is Gorenstein.

The second statement of the theorem is a consequence of the fact that, in our construction, one starts only from an abstract compact curve $E = \bigcup_{i \in V(\Gamma)} E_i$ with simple normal crossings, decorated with weights e_i such that the associated intersection form is negative definite and such that the coefficients k_i determined by the adjunction system (1) are integers. As a result of our construction, one gets a Gorenstein singularity having a resolution whose exceptional divisor is E , and such that $E_i^2 = -e_i$ for each irreducible component E_i .

The theorem is proved. □

Remark 4.2

We could do the plumbing construction using only the meromorphic 2-forms described in Propositions 3.1 and 3.3, because we knew that we could work with a good resolution such that for any two intersecting components E_i, E_j of the exceptional divisor, one does not have simultaneously $k_i = k_j = -1$ (see also Remark 2.10). Indeed, in Proposition 3.1 one has $m - 1 \neq -1$, and in Proposition 3.3 one has adjacent components of the divisor of the meromorphic 2-form with orders 1 and 2 (cf. Corollary 2.8).

Remark 4.3

Our construction yields singularities whose minimal good resolution has the following special property: *each irreducible component of the exceptional divisor has a neighborhood isomorphic to a neighborhood of the 0-section of its normal bundle.*

The following corollary settles a question left open by Neumann and Wahl [15].

COROLLARY 4.4

Any normal surface singularity whose boundary is an integral homology sphere is homeomorphic to a Gorenstein normal surface singularity.

Proof

The boundary is an integral homology sphere if and only if Γ is a tree with $g_i = 0$ for all $i \in V(\Gamma)$ and the determinant of the intersection form is ± 1 . Therefore, the adjunction system (1) shows that, in this case, the discrepancies are all integral. We conclude using Theorem 4.1. \square

5. Extension to \mathbb{Q} -Gorenstein singularities

If S is a smooth complex manifold of dimension $s > 0$ and $r \in \mathbb{Z}^*$, we will say that the holomorphic (meromorphic) sections of the pluricanonical line bundle K_S^r of exponent r are holomorphic (meromorphic) s -forms of weight r . If (q_1, \dots, q_s) is a system of local coordinates on S , such a form is written canonically as

$$f(q_1, \dots, q_s)(dq_1 \wedge \dots \wedge dq_s)^r,$$

where f is holomorphic (meromorphic).

In particular, consider $S = C$, a smooth holomorphic curve (a Riemann surface). Denote by $K_C^r \xrightarrow{\gamma_r} C$ the projection morphism. If q is a local coordinate on the open set $U \subset C$, then $(dq)^r$ is a local section of $K_C^r|_U$. One may take as a second coordinate on $\gamma_r^{-1}(U)$ the analogue of the dual coordinate p of the case $r = 1$ (see the beginning of the proof of Proposition 3.1), that is, the holomorphic function p_r such that the sections α of $K_C^r|_U$ (i.e., the 1-forms of weight r) are written as

$$\alpha = p_r(dq)^r.$$

The next lemma shows that all pluricanonical bundles over a curve are endowed with natural meromorphic 2-forms of the same weight as the exponent of the bundle, generalizing the existence of the natural symplectic form on the canonical bundle.

LEMMA 5.1

Let C be a smooth holomorphic curve, and let $r \in \mathbb{Z}^*$. Then the holomorphic 2-forms of weight r on the total space K_C^r , which are expressed in local coordinates as

$$p_r^{1-r} (dp_r \wedge dq)^r, \tag{14}$$

glue into a well-defined meromorphic 2-form $\alpha_{C,r}$ of weight r on the total space K_C^r .

Proof

Let \tilde{q} be a second local coordinate on C , defined on an open set overlapping the domain of definition of q . Denote by \tilde{p}_r the associated complementary coordinate on the total space K_C^r . Then $\tilde{p}_r(d\tilde{q})^r = p_r(dq)^r$, which implies.

$$\tilde{p}_r = p_r \cdot \left(\frac{d\tilde{q}}{dq}\right)^{-r}. \tag{15}$$

As a consequence, $d\tilde{p}_r \wedge d\tilde{q} = (d\tilde{q}/dq)^{-r+1} dp_r \wedge dq$, which implies, by taking r th powers on both sides and by applying again relation (15), the following desired equality:

$$\tilde{p}_r^{1-r} (d\tilde{p}_r \wedge d\tilde{q})^r = p_r^{1-r} (dp_r \wedge dq)^r.$$

□

The following proposition generalizes Proposition 3.1. It specializes to it for $r = 1$.

PROPOSITION 5.2

Let C be a compact, connected, and smooth complex curve of genus $g \geq 0$. Fix an integral divisor D on C , an integer $r \in \mathbb{N}^*$, and an integer $m \in \mathbb{Z}^*$ which divides $r(2g - 2) - \text{deg}(D)$. Then there exists a line bundle $L \xrightarrow{\psi} C$ of degree $m^{-1}(r(2g - 2) - \text{deg}(D))$, endowed with a meromorphic 2-form α of weight r , such that we have the following:

- (a) $\text{div}(\alpha) = (m - r)C + \psi^*(D)$, where C is seen here as the 0-section of L .
- (b) For each point $A \in C$, if $a \in \mathbb{Z}$ denotes the multiplicity of D at that point, one has a system of local coordinates (q, f) centered at A , seen as a point of the 0-section of L , such that q is constant on the fibers of ψ , that $f = 0$ defines C , and, moreover, that

$$\alpha = f^{m-r} q^a (df \wedge dq)^r.$$

Proof

We follow the steps of the proof of Proposition 3.1. Consider the natural meromorphic map got by tensoring with the meromorphic section s_D of L_D :

$$H_{D,r} := K_C^r \otimes L_{-D} \cdots \xrightarrow{\tau_{D,r}} K_C^r.$$

Then $(dq)^r \otimes \sigma_A^*$ is a trivializing section of $K_C^r \otimes L_{-D}$. If $h_r(dq)^r \otimes \sigma_A^*$ denotes the general section of $H_{D,r}$, then (q, h_r) is a local coordinate system on $H_{D,r}$ in the neighborhood of A . With respect to the coordinate systems (q, h_r) and (q, p_r) , the map $\tau_{D,r}$ is given by $(q, h_r) \rightarrow (q, h_r q^a)$. Recalling that $\alpha_{C,r}$ was defined by Lemma 5.1, this implies that

$$\begin{aligned} (\tau_{D,r})^* \alpha_{C,r} &= (\tau_{D,r})^* (p_r^{1-r} (dp_r \wedge dq)^r) = (h_r q^a)^{1-r} (d(h_r q^a) \wedge dq)^r \\ &= h_r^{1-r} q^a (dh_r \wedge dq)^r. \end{aligned}$$

Consider now an arbitrary m th root $H_{D,r}^{1/m}$ of the line bundle $H_{D,r}$, which exists by the hypothesis that $m \neq 0$ divides $\text{deg}(H_{D,r}) = r(2g - 2) - \text{deg}(D)$. In adapted local

coordinates, chosen as in the proof of Proposition 3.1, the m th tensor power map

$$H_{D,r}^{1/m} \cdots \xrightarrow{\mu_{D,r,m}} H_{D,r}$$

is given by $(q, l) \rightarrow (q, l^m)$. Therefore,

$$\alpha_{D,r,m} := (\mu_{D,r,m})^*(\tau_{D,r})^*\alpha_{C,r} = l^{m(1-r)}q^a (ml^{m-1} dl \wedge dq)^r = m^r l^{m-r} q^a (dl \wedge dq)^r.$$

By making the change of variables $f := m^{r/m}l$, where $m^{1/m}$ is an arbitrary m th root of m , we get

$$\alpha_{D,r,m} = f^{m-r}q^a (df \wedge dq)^r.$$

This shows that the pair $(L, \alpha) := (H_{D,r}^{1/m}, \alpha_{D,r,m})$ satisfies both properties (a) and (b). □

In the same way as Proposition 3.1 did not allow us to get meromorphic 2-forms with poles of order 1 along the 0-section, the previous one does not allow us to get meromorphic 2-forms of weight r with poles of order r along the 0-section. We get such poles using the following proposition, which generalizes Proposition 3.3. It specializes to it for $r = 1, m_1 = -2$, and $m_2 = 0$.

PROPOSITION 5.3

Let C be a compact, connected, and smooth rational curve. Let $L \xrightarrow{\psi} C$ be a line bundle on it, and let A_1, A_2 be two distinct points on C . Fix an integer $r \in \mathbb{N}^*$ and two other integers $m_1, m_2 \in \mathbb{Z} \setminus \{-r\}$ such that $m_1 + m_2 = -2r$. Then there exists a meromorphic 2-form α of weight r on L such that we have the following:

- (a) $\text{div}(\alpha) = -r C + m_1 \psi^*(A_1) + m_2 \psi^*(A_2)$.
- (b) For each $l \in \{1, 2\}$, one has a system of local coordinates (u_l, v_l) centered at A_l , seen as a point of the 0-section of L , such that u_l is constant on the fibers of ψ , that $v_l = 0$ defines C and, moreover, that

$$\alpha = v_l^{-r} u_l^{m_l} (dv_l \wedge du_l)^r.$$

Proof

We follow the notation introduced in the proof of Proposition 3.3—that is, we work inside the charts $(q_1, f_1), (q_2, f_2)$, chosen moreover such that A_l is defined by $q_l = f_l = 0$. Consider the following meromorphic 2-form of weight r on L :

$$\alpha := f_2^{-r} q_2^{m_2} (df_2 \wedge dq_2)^r.$$

The equations (9), (10), and (11) imply immediately that, written in the other chart,

$$\alpha = (-1)^r f_1^{-r} q_1^{m_1} (df_1 \wedge dq_1)^r.$$

We almost get the desired theorem, with the exception of the fact that in this second chart, one has the extra factor $(-1)^r$. We have to care about it, in order to get the same shape for *all* the local normal forms, adapted to the plumbing operations to be done in the sequel (in the proof of Theorem 5.4).

- *If r is even*, there is nothing more to be done.
- *If r is odd*, then we do the change of variable $q_1 = v \cdot u_1$, where v is an arbitrary $(m_1 + r)$ th root of $(-1)^r$, which exists, by the hypothesis $m_1 \neq -r$. We get $\alpha = f_1^{-r} u_1^{m_1} (df_1 \wedge du_1)^r$. By taking $u_2 := q_2$ and $v_l := f_l$ for $l \in \{1, 2\}$, we see that α is expressed as in point (b) in both charts with coordinates (u_l, v_l) . As these charts cover L , the proposition is proved. □

Now we are ready to prove a generalization of Theorem 4.1 to all normal surface singularities. It specializes to it for $r = 1$.

THEOREM 5.4

Consider a normal surface singularity, with topological index $r \geq 1$. Then it is homeomorphic to a \mathbb{Q} -Gorenstein singularity with index r . In particular, being \mathbb{Q} -Gorenstein imposes no topological restrictions on normal surface singularities. Moreover, the second singularity may be chosen such that the exceptional curve of its minimal good resolution is analytically isomorphic to that of the initial singularity.

Proof

The proof follows the same steps as the one of Theorem 4.1 and it uses the same notation. For each vertex $i \in \Gamma$, we consider a convenient line bundle endowed with a meromorphic 2-form of weight r , constructed either by Proposition 5.2, or by Proposition 5.3. We consider three cases, according to the values of the discrepancy k_i and of the valency k_i . When $k_i = -1$, we know by Corollary 2.9 that the valency v_i is either 1 or 2 and that E_i is a rational curve.

- *Suppose that $k_i \neq -1$* . Then apply Proposition 5.2 to $C := E_i$, $g := g_i$, $m := r(k_i + 1)$, and $D := \sum_{j \in V_i} r \cdot k_j E_{i,j}$. The hypothesis that m divides $r(2g - 2) - \text{deg}(D)$ is a consequence of formula (1).
- *Suppose that $k_i = -1$ and $v_i = 1$* . Denote by j the unique element of V_i . Then apply Proposition 5.3 to $C := E_i$, $L := \mathcal{O}(-e_i)$, $\{A_1\} = E_i \cap E_j$, $m_1 = -2r$, and $m_2 = 0$.
- *Suppose that $k_i = -1$ and $v_i = 2$* . Corollary 2.9 tells us that, if $j_1 \neq j_2$ are the two elements of V_i , then $k_{j_1} + k_{j_2} = -2$ and $k_{j_1} \neq -1$, $k_{j_2} \neq -1$. Then apply Proposition 5.3 to $C := E_i$, $L := \mathcal{O}(-e_i)$, $\{A_l\} = E_i \cap E_{j_l}$, and $m_l = r \cdot k_{j_l}$, for each $l \in \{1, 2\}$.

By combining the three cases, we see that for all $i \in V(\Gamma)$, there exists a line bundle $L_i \xrightarrow{\psi_i} E_i$ of degree $-e_i$ on E_i and a meromorphic 2-form α_i of weight r on it, such that we have the following:

- $\operatorname{div}(\alpha_i) = r \cdot k_i E_i + \sum_{j \in V_i} r \cdot k_j \psi_i^*(E_{i,j})$.
- For each point $A_{i,j,l} \in E_{i,j}$, one has a system of local holomorphic coordinates $(q_{i,j,l}, f_{i,j,l})$ centered at $A_{i,j,l}$ on the total space L_i , such that $q_{i,j,l}$ is constant on the fibers of ψ_i , that $f_{i,j,l} = 0$ defines the 0-section $E_i \subset L_i$, and, moreover, that

$$\alpha_i = (f_{i,j,l})^{r \cdot k_i} (q_{i,j,l})^{r \cdot k_j} (df_{i,j,l} \wedge dq_{i,j,l})^r. \tag{16}$$

Then, we plumb neighborhoods of the 0-sections of these bundles by identifying neighborhoods of the points $A_{i,j,l}$ and $A_{j,i,l}$, in such a way that the forms α_i and α_j glue. Again, this is possible because we have the simple normal forms (16), and because no two adjacent vertices i and j of Γ satisfy $k_i = k_j = -1$, a fact ascertained by the Corollary 2.9 of Veys’s Theorem 2.6 (the analogue of equation (13) is now the equation $\lambda_{i,j}^{r(k_j+1)} = (-1)^r \lambda_{j,i}^{r(k_i+1)}$). After contracting the union of the 0-sections of the bundles L_i , one gets a singularity $(X, 0)$ such that the 2-form of weight r obtained by gluing the forms α_i gives a holomorphic trivialization of $K_{X \setminus 0}^r$. The theorem is proved. □

The analogue of Remark 4.3 holds obviously also in this more general context.

To conclude, we mention that the problems of characterizing the topological types of isolated complete intersections, hypersurfaces, and smoothable isolated surface singularities are still open.

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