

On the Cohomology Rings of Holomorphically Fillable Manifolds

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Dedicated to Lê Dũng Tráng, for his 60th birthday.

ABSTRACT. An odd-dimensional differentiable manifold is called *holomorphically fillable* if it is diffeomorphic to the boundary of a compact strongly pseudoconvex complex manifold, *Stein fillable* if this last manifold may be chosen to be Stein and *Milnor fillable* if it is diffeomorphic to the abstract boundary of an isolated singularity of normal complex analytic space. We show that the homotopical dimension of a manifold-with-boundary of dimension at least 4 restricts the cohomology ring (with any coefficients) of its boundary. This gives restrictions on the cohomology rings of Stein fillable manifolds, on the dimension of the exceptional locus of any resolution of a given isolated singularity, and on the topology of smoothable singularities. We give also new proofs of structure theorems of Durfee & Hain and Bungart about the cohomology rings of Milnor fillable and holomorphically fillable manifolds respectively. The various structure theorems presented in this paper imply that in dimension at least 5, the classes of Stein fillable, Milnor fillable and holomorphically fillable manifolds are pairwise different.

1. Introduction

The foundational papers [G 85], [E 89], [E 90], [EG 91] of Eliashberg and Gromov showed that one can get information on the structure of a contact manifold N , whenever this manifold bounds an even dimensional manifold W with a holomorphic or symplectic structure compatible in some way with the contact structure on the boundary: one says that N is *filled* by W . Since then, many notions of *fillability* for contact manifolds have been introduced: *holomorphic*, *Stein*, *Milnor*, *Liouville*, *Weinstein*, *strong symplectic*, *weak symplectic*, etc (see Geiges [G 08]).

In this paper we restrict to the notions of holomorphic, Stein and Milnor fillability. An odd-dimensional, closed, orientable manifold is called *holomorphically fillable* if it is diffeomorphic to the boundary of a compact strongly pseudoconvex complex manifold. It is called *Stein fillable* if this manifold can be chosen to be Stein. It is called *Milnor fillable* if it is diffeomorphic to the abstract boundary

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(or link) of an isolated singular point of normal complex analytic space. Using a resolution of the singularity, we see that a Milnor fillable manifold is automatically holomorphically fillable. Notice that we have dropped the contact structures from these definitions, due to the fact that in this article we prove theorems which involve only the cohomology rings of each kind of fillable manifolds. Nevertheless, we began this article by speaking about them because it is the interest in Milnor fillable *contact* manifolds which led us to those theorems, before discovering that some of them were already known.

Holomorphically fillable 3-manifolds are necessarily Stein fillable, as was proved by Bogomolov & de Oliveira [BO 97]. In higher dimensions, this is no longer the case. For example, Eliashberg, Kim & Polterovich [EKP 06] have shown that the projective spaces $\mathbb{R}P^{2n-1}$, which are always holomorphically fillable, are not Stein fillable whenever $n \geq 3$.

In Section 2 we give the background on (strictly) plurisubharmonic functions, strongly pseudoconvex spaces, Stein spaces, fillable manifolds and resolutions of isolated singularities of normal complex analytic spaces needed in the rest of the paper.

In Section 3, we show that the homotopical dimension of a compact manifold-with-boundary restricts the cohomology ring with arbitrary coefficients of its boundary (Theorem 3.1). This extends the method used by Eliashberg, Kim & Polterovich to show that $\mathbb{R}P^{2n-1}$ is not Stein fillable. A Stein manifold being homotopically of dimension at most equal to its complex dimension, we get in particular constraints on the cohomology rings of Stein fillable manifolds (Corollary 3.3). Then we consider manifolds N which are total spaces of oriented circle bundles $N \xrightarrow{p} \Sigma$, and we apply Theorem 3.1 by showing that suitable hypotheses on the cohomology ring of Σ and on the Euler class of the bundle give lower bounds on the homotopical dimension of any filling of N (Proposition 3.5).

In Section 4, we give applications of the results of the previous section to isolated singularities of complex analytic spaces. First, we give a lower bound on the dimension of the exceptional locus of any resolution in terms of the cohomology ring of the boundary (Proposition 4.1). The generic fibers of a smoothing of an isolated singularity being Stein and their boundaries being diffeomorphic to the boundary of the singularity, we also get constraints on the topology of smoothable singularities (Proposition 4.5). We consider in more detail the isolated singularities obtained by contracting the zero-section of an anti-ample line bundle L on a projective manifold Σ . Suitable hypotheses on the integral cohomology ring $H^*(\Sigma; \mathbb{Z})$ and on the Chern class of L imply that the boundary of the resulting singularity (Milnor fillable, therefore holomorphically fillable) is not Stein fillable. In particular, such a germ is *non-smoothable*. As a special case we get (Corollary 4.6):

Let (X, x) be the germ of normal analytic space with isolated singularity obtained by contracting the 0-section of an anti-ample line bundle on an abelian variety Σ of complex dimension ≥ 2 , and whose first Chern class is not primitive in $H^2(\Sigma, \mathbb{Z})$. Then the boundary of (X, x) is not Stein fillable. In particular, (X, x) is not smoothable.

We answer like this partially the concluding question asked by Biran in [B 05] (see Remark 4.7).

In Section 5 we give a new proof of a theorem of Durfee & Hain [DH 88] (first announced in [D 86]), describing restrictions on the cohomology rings with rational coefficients of Milnor fillable manifolds of arbitrary dimension (Theorem 5.1). This generalizes a theorem obtained by Sullivan [S 75] in dimension 3. A crucial ingredient in our proof is a theorem of Goresky & MacPherson [GM 82], describing the kernel of the map from the homology of the boundary to the homology of the manifold, in the case of a divisorial resolution of an isolated singularity (Theorem 2.11). This theorem generalizes to any dimension the fact that the intersection form associated to a resolution of a normal surface singularity is non-degenerate. Both our proof and the one of Durfee & Hain are based on a deep purity theorem of Beilinson, Bernstein, Deligne & Gabber [BBD 82] (in our proof, this is hidden inside Goresky & MacPherson's theorem).

In Section 6 we apply the results of Section 5 in order to give a new proof of a theorem of Bungart [B 92], showing that the rational cohomology rings of holomorphically fillable manifolds are also constrained (Theorem 6.2).

We deduce examples in any odd dimension ≥ 3 of holomorphically fillable manifolds which are not Milnor fillable (Corollary 6.5). Combining them with the examples of the previous sections, we see that in all odd dimensions ≥ 5 , the classes of Stein, Milnor or holomorphically fillable manifolds are pairwise distinct.

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2. Plurisubharmonic functions and various notions of fillability

For details on the various notions and results recalled in this section, one can consult Grauert & Remmert [GR 79], Peternell [P 94], Bennequin [B 90] and Eliashberg [E 97].

Let X be a complex manifold. Denote by TX the (real) tangent bundle of the underlying real differentiable manifold and by $J : TX \rightarrow TX$ the (integrable) almost complex structure associated to the complex structure of X . The operator $d^{\mathbb{C}} := J^* \circ d$ (that is, $d^{\mathbb{C}} f := df \circ J$ for any smooth function defined on X) is real and intrinsically associated to the complex structure of X . In terms of the operators $\partial = d' = \sum_k \frac{\partial}{\partial z_k} dz_k$ and $\bar{\partial} = d'' = \sum_k \frac{\partial}{\partial \bar{z}_k} d\bar{z}_k$, one has:

$$\begin{cases} d = \partial + \bar{\partial} \\ d^{\mathbb{C}} = i(\partial - \bar{\partial}) \end{cases} .$$

Let f be a real-valued smooth function defined on X .

We associate to f the following tensors on X :

$$\begin{aligned}\alpha_f &:= -d^{\mathbb{C}}f, \\ \omega_f &:= d\alpha_f = -dd^{\mathbb{C}}f, \\ g_f(u, v) &:= \omega_f(u, Jv), \forall u, v \in TV, \\ h_f &:= g_f + i\omega_f.\end{aligned}$$

The kernel of the restriction of α_f to any regular level $X_a := f^{-1}(a)$ of f is the complex tangent bundle $TX_a \cap J(TX_a)$ of X_a . The exterior form ω_f is real of type $(1, 1)$. It is the associated *Levi form* of the function f . The associated hermitian form h_f (\mathbb{C} -antilinear in the first coordinate and \mathbb{C} -linear in the second coordinate) is also called the *Levi form* of f . If needed, we distinguish between the two versions of Levi form by specifying that we deal with *the exterior Levi form* or *the hermitian Levi form*.

In the sequel we will be mainly interested in a special class of real-valued functions on X :

DEFINITION 2.1. The function $f : X \rightarrow \mathbb{R}$ is called **plurisubharmonic** (abbreviated **psh**) if g_f is positive semidefinite. It is called **strictly plurisubharmonic** (abbreviated **spsh**) if g_f is positive definite, that is, if it defines a riemannian structure on the smooth manifold X .

Notice that f is spsh if and only if the associated Levi form is Kähler. (Strictly plurisubharmonic functions are the analogs on complex manifolds of (strictly) convex functions on real manifolds endowed with an affine structure (not to be confused with affine algebraic manifolds).

The notion of (s)psh function can be defined also if X is a reduced but not necessarily smooth complex space: the function $f : X \rightarrow \mathbb{R}$ is called *(s)psh* if in a neighborhood of each point of X , it is the restriction to X of a smooth (s)psh function defined on a complex manifold into which one has locally embedded X . This definition does not depend on the choice of local embedding.

Strictly plurisubharmonic functions have the following easily provable properties:

PROPOSITION 2.2. 1) *The restriction of a (s)psh function to a complex subspace is again (s)psh.*

2) *If f is (s)psh and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is (strictly) convex and smooth, then $\phi \circ f : X \rightarrow \mathbb{R}$ is (s)psh.*

3) *spsh functions form an open set among smooth functions in the C^2 topology with compact supports.*

The notions of (s)psh functions are local. The following notion is instead global: the smooth real-valued function f defined on the reduced complex analytic space X is called an *exhaustion function* if it is proper and bounded from below (which is equivalent to the fact that it attains its absolute minimum).

DEFINITION 2.3. The reduced complex analytic space X is called **strongly pseudoconvex** if it carries an exhaustion function f which is strictly plurisubharmonic outside a compact set. X is called a **Stein space** if f may be chosen to be strongly pseudoconvex all over X .

The following characterizations of Stein spaces may be obtained by combining theorems of Grauert and Narasimhan (see Grauert & Remmert [**GR 79**, page 152] or Peternell [**P 94**, sections 1–4]):

THEOREM 2.4. *Let X be a reduced paracompact complex analytic space. The following properties are equivalent:*

- 1) X is a Stein space.
- 2) X is holomorphically convex (that is, the holomorphically convex hull of any compact set is again compact) and holomorphically separable (that is, the global holomorphic functions separate the points).
- 3) X is strongly pseudoconvex and has no compact analytic subsets of positive dimension.

There are also characterizations of Stein spaces using coherent cohomology, but we won't use them in this paper. As a corollary of the previous theorem, we get:

THEOREM 2.5. *If $X \rightarrow Y$ is a finite morphism (that is, a proper morphism with finite fibers) and Y is a Stein space, then X is also Stein.*

In particular, any closed subspace of a Stein space is Stein. Still more particularly, any closed subspace of \mathbb{C}^n is Stein. In fact, in this way one does not restrict very much the class of Stein spaces. Indeed, as a result of works of Remmert, Bishop and Narasimhan, one has the following embedding theorem (see Bell & Narasimhan [BN 90, Theorem 3.1]):

THEOREM 2.6. *A Stein space X can be embedded holomorphically in some \mathbb{C}^n if and only if it has bounded local embedding dimension (that is, the dimension of the Zariski tangent spaces is a bounded function on X).*

Suppose that the reduced analytic space X is strongly pseudoconvex. Then it has a maximal compact analytic subset $K \subset X$ with a finite number of irreducible components. Consider the Remmert reduction morphism $X \xrightarrow{r} Y$ (see Peternell [P 94, Section 2]). It contracts to a point each *connected* component of K , it is an isomorphism outside K and $r_*\mathcal{O}_X = \mathcal{O}_Y$. The reduced space Y is then a Stein space, by Theorem 2.4. If X is a manifold, then Y is a normal Stein space with a finite number of isolated singularities. For this reason, in the sequel we will consider only complex spaces with isolated singularities. If f is an exhaustion function on the complex space X and $X_a := f^{-1}(a)$ is a regular level of f in whose neighborhood f is spsh, then we say that the compact sublevel $X_{\leq a} := f^{-1}((-\infty, a])$ is a *compact strongly pseudoconvex manifold*, whose boundary is X_a . If X is Stein, we say that $X_{\leq a}$ is a *compact Stein manifold*.

We will do Morse theory starting from spsh functions defined on complex manifolds X . The fundamental observation is:

PROPOSITION 2.7. *If the complex manifold X has complex dimension $n \geq 1$, then the indices of the critical points of a spsh Morse function on X are $\leq n$.*

This remark made by Thom around 1957 was the starting point of proofs through Morse theory of Lefschetz' hyperplane section theorem, by Andreotti & Frankel [AF 59] and Bott [B 59]. Milnor [M 63] noticed that an immediate consequence of those proofs is:

THEOREM 2.8. *A Stein manifold X of complex dimension n has the homotopy type of a CW-complex of dimension at most n . As a consequence, all the homology and cohomology groups with arbitrary coefficients of X vanish in degree at least $n + 1$.*

The vanishing of the previous cohomology groups of the mentioned degree with *real* coefficients was first proved by Serre [S 53]. An analogous vanishing theorem was proved for arbitrary Stein spaces by Narasimhan [N 67]. The analog of the first sentence of the theorem was then proved for affine algebraic spaces by Karchyauskas [K 79] and for arbitrary Stein spaces by Hamm [H 86]. In this paper we will need only the (co)homological version of the theorem for Stein manifolds, but (and this is very important) *with integral coefficients*. We would also like to mention that in [E 90], Eliashberg characterized the differentiable manifolds of even dimension ≥ 6 which admit a Stein structure.

In [E 97], Eliashberg explains Proposition 2.7 using symplectic geometry in the following way, which is excellent for understanding the interdependence of the objects α_f, ω_f, g_f defined before. Consider the gradient of f with respect to the riemannian metric g_f . It equals the Liouville vector field of α_f with respect to the symplectic form ω_f , therefore its flow exponentially dilates this symplectic form. This implies that the stable cells of the gradient associated to the critical points of f are isotropic (that is, ω_f vanishes in restriction to them), therefore their dimension is at most n .

In this article, the most important example of an spsh function is the squared-distance function to a point in some space \mathbb{C}^n (which was also the type of function used by Andreotti & Frankel [AF 59]). We will also consider restrictions of such functions to complex analytic subspaces of \mathbb{C}^n , which are again spsh by Proposition 2.2.

In particular, let (X, x) be an irreducible germ of reduced complex analytic space which is smooth outside x . We also say that (X, x) is an *isolated singularity*. Choose an embedding of a representative of (X, x) in some $(\mathbb{C}^n, 0)$. Denote by $\rho : (X, x) \rightarrow (\mathbb{R}_+, 0)$ the restriction of the squared-distance function to the origin. We call it a *euclidean rug function* associated to the isolated singularity (X, x) (notice that in [CNP 06] we had introduced more general euclidean rug functions; the name is inspired by Thom's article [T 59]). Its levels $\rho^{-1}(\epsilon)$ are all smooth and diffeomorphic for $\epsilon \in (0, \epsilon_0]$, where $\epsilon_0 > 0$ is sufficiently small. Their diffeomorphism type does not depend on the choice of the embedding, and is called *the (abstract) boundary* or *the (abstract) link* of (X, x) . We say that a compact representative of (X, x) of the form $\rho^{-1}([0, \epsilon_0])$ with the properties stated before is a *compact Milnor representative* of (X, x) . This notion may be extended to any reduced germ, with singularity that needs not be isolated.

DEFINITION 2.9. The odd-dimensional manifold N is called **holomorphically fillable** if it is diffeomorphic to the boundary of a compact strongly pseudoconvex space X with at most isolated singularities. N is called **Stein fillable** if X may be chosen to be a Stein *manifold*. N is called **Milnor fillable** if it is diffeomorphic to the abstract boundary of an isolated singularity.

Note that by Hironaka's theorem of resolution of singularities and the fact that the Remmert reduction of a strongly pseudoconvex space is a bimeromorphism, we would obtain equivalent definitions of holomorphic fillability by asking X to be either smooth or to be Stein with at most isolated singularities.

The notions of holomorphic fillability and Stein fillability were introduced in the context of the study of convexity notions in symplectic geometry by Eliashberg & Gromov [EG 91]. The notion of Milnor fillability was introduced by Caubel, Némethi and myself in the paper [CNP 06]. In all these cases, one considers a

supplementary *contact* structure on the manifold N (see also Geiges [G 08]), and one has to take care of orientation issues. Since in this article we give purely cohomological obstructions for Stein, Milnor and holomorphic fillability, we do not spend time here on these issues. We mention only, for the reader who wants to get an idea of the relation between what we explained before and contact geometry, that whenever f is spsh, the restriction of the real 1-form α_f to a regular level $f^{-1}(a)$ of f is a contact form, and that the orientation defined on this level by $\alpha_f \wedge (d\alpha_f)^{\wedge(n-1)}$ coincides with its orientation as a boundary of the sublevel $f^{-1}((-\infty, a])$. Varchenko [V 80] showed that the associated contact structures on the boundaries of Milnor representatives of isolated singularities are independent of the choice of euclidean rug function. In [CNP 06], we continued the study of such *Milnor fillable contact manifolds*.

In Section 6 we will study holomorphically fillable manifolds by considering a convenient cobordism which relates them to a disjoint union of Milnor fillable manifolds. This cobordism will be constructed using the following proposition:

PROPOSITION 2.10. *Let X be a compact Stein space with isolated singularities and $f : X \rightarrow \mathbb{R}$ a spsh exhaustion function. Denote by F a finite set which contains the singular locus of X . Then there exists a spsh function $\phi : X \rightarrow \mathbb{R}$ which coincides with f outside a compact subset of the interior of X , which attains its absolute minimum exactly on F and which is a euclidean rug function in restriction to a sufficiently small neighborhood of each point of F in X .*

PROOF. The argument is similar to one given by Bungart [B 92, page 109].

There exists an analytic morphism $X \rightarrow Y$ which identifies all the points of F and is a biholomorphism outside F . Denote by $y \in Y$ the image of F by this morphism. Theorem 2.4 implies that Y is again a compact Stein space. By post-composing f with an adequate smooth convex function which interpolates between a constant function and the identity on \mathbb{R} , we get a psh function \tilde{f} on X which is constant on F and equal to f near ∂X . Therefore it descends to a psh function on Y .

As the local embedding dimension on Y is globally bounded, by Theorem 2.6 the interior $\overset{\circ}{Y}$ of Y can be embedded in some space \mathbb{C}^n . From now on, we shall look at Y as a subspace of \mathbb{C}^n . Consider then the function $\rho : Y \rightarrow \mathbb{R}$ obtained by restricting to Y the squared-distance function to y in \mathbb{C}^n .

Consider a number $a > 0$ such that $\tilde{f} = f$ wherever $\rho > a$ (that is, such that we have modified f only inside the euclidean ball with radius \sqrt{a} centered at y). Then post-compose ρ with a smooth real-valued function defined on \mathbb{R} which is the identity on the interval $]-\infty, a]$ and identically zero on $[b, +\infty[$, where $b > a$ is chosen arbitrarily. Note that we impose nothing more than its global smoothness in between. Denote by $\tilde{\rho} : Y \rightarrow \mathbb{R}$ the function obtained like this. It is a euclidean rug function in a neighborhood of y , it is spsh wherever $\tilde{f} \neq f$ and it vanishes outside a compact.

By using Proposition 2.2, we see that the function $\psi_\epsilon := \tilde{f} + \epsilon \cdot \tilde{\rho}$ is spsh all over Y whenever $\epsilon > 0$ is sufficiently small. Then its lift ϕ_ϵ to X satisfies all the conditions asked for in the conclusion of the proposition. \square

As a preliminary to the study of holomorphically fillable manifolds, in Section 5 we will prove a theorem of Durfee & Hain, showing that there exist restrictions

on the rational cohomology rings of Milnor fillable manifolds of dimension at least 3. Our proof is different from the one of Durfee & Hain. It is based on a theorem of Goresky & MacPherson. Let us explain it.

Consider an isolated singularity (X, x) of normal complex analytic space. A *resolution* of (X, x) is a proper morphism $(\tilde{X}, E) \rightarrow (X, x)$ with smooth total space \tilde{X} , realizing an isomorphism outside the singular locus x . By Hironaka's theorem, resolutions exist. A resolution is called *divisorial* if its exceptional set E is purely of codimension 1 in \tilde{X} . All the resolutions of normal surfaces are divisorial, but this is not true in higher dimensions (the simplest example of non-divisorial resolution is recalled at the beginning of Section 4).

Suppose now that X denotes a compact Milnor representative of the germ. Let $(W, E) \xrightarrow{\pi} (X, x)$ be a divisorial resolution whose exceptional divisor E has only normal crossings. Denote by N the boundary of W . As π is an isomorphism outside x , it identifies N with the boundary of X , that is with the abstract boundary of the singularity. We will need the following theorem relating the (co)homology of the boundary N of the singularity to the (co)homology of the resolution W :

THEOREM 2.11. *Let W be a compact orientable manifold of dimension $2n$, with boundary denoted by N . The following are equivalent statements and are true if W is a divisorial resolution of a Milnor representative of a normal isolated singularity of complex dimension n and N is its boundary (homology and cohomology groups are considered with rational coefficients and all the morphisms are induced by inclusions):*

- 1) *The morphisms $H_i(N) \rightarrow H_i(W)$ vanish identically for $i \in \{n, \dots, 2n - 1\}$.*
- 2) *The morphisms $H_i(N) \rightarrow H_i(W)$ are injective for $i \in \{1, \dots, n - 1\}$ and vanish identically for $i \in \{n, \dots, 2n - 1\}$.*
- 3) *The morphisms $H^i(W) \rightarrow H^i(N)$ vanish identically for $i \in \{n, \dots, 2n - 1\}$.*
- 4) *The morphisms $H^i(W) \rightarrow H^i(N)$ are surjective for $i \in \{0, \dots, n - 1\}$ and vanish identically for $i \in \{n, \dots, 2n - 1\}$.*
- 5) *The morphisms $H_i(W) \rightarrow H_i(W, N)$ are surjective for $i \in \{1, \dots, n\}$ and are injective for $i \in \{n, \dots, 2n - 1\}$.*
- 6) *The morphisms $H^i(W, N) \rightarrow H^i(W)$ are injective for $i \in \{1, \dots, n\}$ and surjective for $i \in \{n, \dots, 2n - 1\}$.*

The equivalences between 1)...6) results from a play with the long exact (co)homology sequences of the pair (W, N) , Poincaré-Lefschetz duality and the fact that $H^i(W) \rightarrow H^i(N)$ is the adjoint morphism of $H_i(N) \rightarrow H_i(W)$ when we work over \mathbb{Q} . Point 1) of the theorem was deduced by Goresky & MacPherson in [GM 82, page 124] as a consequence of a deep decomposition theorem in intersection homology theory proved by Beilinson, Bernstein, Deligne & Gabber [BBD 82, Theorem 6.2.5, page 163]. Point 6) was proved by Steenbrink [S 83, page 518] as part of a short exact sequence of mixed Hodge structures (see also [S 87, page 117]). A Hodge-theoretic proof of point 6) was given by Navarro Aznar in [N 85, page 285]. It can also be obtained as a consequence of de Cataldo & Migliorini's [CM 05, Corollary 2.1.12].

As noted in [GM 82, page 123], for any compact oriented manifold W with boundary N , the kernel of the morphism $H_\bullet(N) \rightarrow H_\bullet(W)$ between the total homologies, induced by the inclusion $N \hookrightarrow W$, is half-dimensional inside $H_\bullet(N)$.

The previous theorem describes this kernel when W is a divisorial resolution of an isolated singularity: it is exactly $\bigoplus_{i=n}^{2n-1} H_i(N)$. In fact, this was the way in which Goresky & MacPherson stated their theorem.

In the case of a germ of surface (X, x) , the previous theorem is a consequence of the fact (proved by Du Val [V 44] and Mumford [M 61]) that the intersection form of the resolution π is negative definite. More precisely, it is equivalent to the non-degeneracy of this intersection form, as can be easily seen using some diagram chasing in the cohomology long exact sequence of the pair (W, N) , Poincaré-Lefschetz duality for the manifold-with-boundary W , and the fact that W retracts by deformation on E . Therefore, Goresky & MacPherson's theorem is a generalization of the non-degeneracy of the intersection form associated to a resolution of a normal surface singularity.

3. Constraints on the cohomology of the boundary from the homotopical dimension of the total space

In this section, all (co)homology groups are considered with coefficients in an arbitrary commutative ring. If a space has the homotopical type of a CW-complex of dimension $\leq h$, we say that *it is homotopically of dimension $\leq h$* . Whenever we will be using Poincaré or Poincaré-Lefschetz duality morphisms, we will suppose that the orientable manifolds under consideration were arbitrarily oriented.

THEOREM 3.1. *Let W be a compact, connected, orientable manifold-with-boundary of dimension $m \geq 4$. Denote by N its boundary. Suppose that W is homotopically of dimension $\leq h$. Consider numbers $i_1, \dots, i_k \in \{1, \dots, m - 2 - h\}$ such that $i_1 + \dots + i_k \geq h + 1$. Then the morphism $H^{i_1}(N) \otimes \dots \otimes H^{i_k}(N) \rightarrow H^{i_1 + \dots + i_k}(N)$ induced by the cup-product in cohomology with arbitrary coefficients vanishes identically.*

PROOF. By the long exact cohomology sequence of the pair (W, N) , we have the exact sequences:

$$H^i(W) \xrightarrow{b^*} H^i(N) \rightarrow H^{i+1}(W, N), \forall i \in \{1, \dots, k\},$$

in which b^* is the morphism induced in cohomology by the inclusion $N \xrightarrow{b} W$. By Poincaré-Lefschetz duality applied to the oriented manifold-with-boundary W , we get $H^{i+1}(W, N) \cong H_{m-i_l-1}(W)$. Since $i_l \leq m - 2 - h$, we see that $m - i_l - 1 \geq h + 1$. But W was supposed to be homotopically of dimension $\leq h$, therefore $H_{m-i_l-1}(W) = 0$. We deduce that all the morphisms $H^i(W) \xrightarrow{b^*} H^i(N)$ are *surjective*. Consider then the following commutative diagram:

$$\begin{CD} H^{i_1}(W) \otimes \dots \otimes H^{i_k}(W) @>b^*>> H^{i_1}(N) \otimes \dots \otimes H^{i_k}(N) \\ @VVV @VVV \\ H^{i_1 + \dots + i_k}(W) @>b^*>> H^{i_1 + \dots + i_k}(N) \end{CD}$$

Since $i_1 + \dots + i_k \geq h + 1$, we get $H^{i_1 + \dots + i_k}(W) = 0$. Therefore the composed morphism defined by the commutative diagram vanishes identically. But the upper horizontal morphism is surjective, as a tensor product of surjective morphisms, therefore the right-hand vertical morphism does also vanish identically. \square

REMARK 3.2. One has to suppose that $h \leq m - 3$ in order to make the set of k -uples (i_1, \dots, i_k) satisfying the requested inequalities non-empty for some $k \geq 1$.

The next corollary gives restrictions on the cohomology rings of Stein fillable manifolds. In the next section we apply it to give examples of Milnor fillable manifolds which are not Stein fillable.

COROLLARY 3.3. *Let N be a Stein fillable manifold of dimension $2n - 1 \geq 5$. Consider numbers $i_1, \dots, i_k \in \{1, \dots, n - 2\}$ such that $i_1 + \dots + i_k \geq n + 1$. Then the morphism $H^{i_1}(N) \otimes \dots \otimes H^{i_k}(N) \longrightarrow H^{i_1 + \dots + i_k}(N)$ induced by the cup-product in cohomology with arbitrary coefficients vanishes identically.*

PROOF. Combine Theorem 2.8 and Theorem 3.1. \square

REMARK 3.4. 1) We have asked that N be of dimension at least 5, because the previous theorem says nothing about 3-dimensional Stein fillable manifolds (see Remark 3.2).

2) It would be interesting to find topological properties of Stein fillable manifolds which use in an essential way their *orientation*. The properties stated before being expressed purely in terms of the cohomology ring of the manifold, they are not of this type.

3) It would also be interesting to find manifolds which admit both a Stein fillable contact structure and a holomorphically fillable but not Stein fillable contact structure.

The next proposition applies Theorem 3.1 to total spaces of circle bundles.

PROPOSITION 3.5. *Let $N \xrightarrow{p} \Sigma$ be an oriented circle bundle over an orientable closed connected manifold Σ of dimension $m - 2 \geq 2$. Denote by $e \in H^2(\Sigma)$ its Euler class. Consider numbers $i_1, \dots, i_k \in \{1, \dots, m - 2 - h\}$ such that $i_1 + \dots + i_k \geq h + 1$. If the morphism $H^{i_1}(\Sigma) \otimes \dots \otimes H^{i_k}(\Sigma) \longrightarrow H^{i_1 + \dots + i_k}(\Sigma)$ induced by the cup-product is surjective, but $H^{i_1 + \dots + i_k - 2}(\Sigma) \xrightarrow{\cup e} H^{i_1 + \dots + i_k}(\Sigma)$ is not surjective, then N does not bound a manifold which is homotopically of dimension $\leq h$.*

PROOF. Consider the following part of the Gysin long exact sequence associated to the circle bundle p :

$$H^{i_1 + \dots + i_k - 2}(\Sigma) \xrightarrow{\cup e} H^{i_1 + \dots + i_k}(\Sigma) \xrightarrow{p^*} H^{i_1 + \dots + i_k}(N)$$

By hypothesis, the morphism on the left is not surjective, therefore the one on the right is not vanishing identically.

Look then at the following commutative diagram:

$$\begin{array}{ccc} H^{i_1}(\Sigma) \otimes \dots \otimes H^{i_k}(\Sigma) & \xrightarrow{p^*} & H^{i_1}(N) \otimes \dots \otimes H^{i_k}(N) \\ \downarrow & & \downarrow \\ H^{i_1 + \dots + i_k}(\Sigma) & \xrightarrow{p^*} & H^{i_1 + \dots + i_k}(N) \end{array}$$

The left-hand vertical morphism being surjective by hypothesis and the horizontal morphism on the bottom being non-zero, as seen before, we deduce that the morphism defined by the diagram is non-zero. Therefore, the right-hand vertical morphism is non-zero. Theorem 3.1 allows then to conclude. \square

4. Applications to Milnor fillable manifolds

In this section we apply Theorem 3.1 and Proposition 3.5 in order to give lower bounds on the dimensions of the exceptional sets of resolutions of isolated singularities (Proposition 4.1 and Corollary 4.2), and to construct classes of Milnor fillable but not Stein fillable manifolds in any odd dimension ≥ 5 (Proposition 4.5 and Corollary 4.6). Therefore, we keep working with arbitrary coefficient rings in cohomology.

In Section 2 we recalled the notion of divisorial resolution of an isolated singularity. We recall now the simplest example of normal singularity which admits non-divisorial resolutions. Consider the affine quadratic cone X in \mathbb{C}^4 , defined by the equation $xy = zt$. It is the cone over a smooth projective quadric $\Sigma \xrightarrow{s} \mathbb{C}\mathbb{P}^3$, which can also be seen as the image of the Segre embedding of $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$, which shows that Σ is doubly ruled. The variety X has an isolated singular point at 0 , which can be resolved by blowing it up. The exceptional set is then isomorphic to the projectified tangent cone, that is, to Σ : we have a divisorial resolution. The total space of this resolution is isomorphic to the total space of the line bundle $s^*\mathcal{O}(-1)$ over Σ . The initial affine cone X can be seen as obtained by contracting the zero section in this total space. But instead of contracting all of Σ , one can contract to a point each line in one of the rulings of the quadric, obtaining another resolution whose exceptional set is a smooth rational curve which parametrizes the lines having been contracted. As there are two rulings, one gets two resolutions with exceptional sets of dimension 1. The induced birational map between the total spaces of these resolutions is the simplest example of what algebraic geometers call a *flop* (see Kollár [K 91]).

The following proposition gives topological obstructions to the existence of resolutions with exceptional sets of small dimension.

PROPOSITION 4.1. *Let (X, x) be an isolated singularity of normal complex analytic space of complex dimension n . Denote by N its abstract boundary. If one can find numbers $i_1, \dots, i_k \in \{1, \dots, 2n - 2 - h\}$ such that $i_1 + \dots + i_k \geq h + 1$ and the morphism $H^{i_1}(N) \otimes \dots \otimes H^{i_k}(N) \rightarrow H^{i_1 + \dots + i_k}(N)$ induced by the cup-product does not vanish identically, then the exceptional set of any resolution of (X, x) has complex dimension at least $(h + 1)/2$.*

PROOF. Consider a resolution $(\tilde{X}, E) \rightarrow (X, x)$ of the germ (X, x) . One can choose as representative of \tilde{X} the preimage W of a Milnor neighborhood of x in X . Therefore, it retracts by deformation on E , which shows that it is homotopically of dimension $2d$, where d is the complex dimension of E . By our choice, W has a boundary diffeomorphic to N . Theorem 3.1 applied to our hypotheses implies that W is not homotopically of dimension $\leq h$, therefore $2d \geq h + 1$. \square

If L is a line bundle on a projective manifold Σ , denote by $N(\Sigma, L)$ the total space of the associated circle bundle. By a theorem of Grauert [G 62], if L is *anti-ample* (that is, if its dual is ample), one can contract the zero-section in the total space of L , getting like this a normal complex affine variety with an isolated singularity as the image of the zero-section. It is the simplest example of Remmert reduction of a holomorphically convex space (see Peternell [P 94, Section 2]). In this case, $N(\Sigma, L)$ is isomorphic to the boundary of the singularity. The next corollary applies Proposition 4.1 to the singularities obtained in this way.

COROLLARY 4.2. *Let (X, x) be an isolated normal singularity obtained by contracting the zero section in the total space of an anti-ample line bundle L over a projective manifold Σ of complex dimension $n - 1 \geq 2$. Suppose that there are numbers $h \in \{2, \dots, 2n - 3\}$, $i_1, \dots, i_k \in \{1, \dots, m - 2 - h\}$ such that $i_1 + \dots + i_k \geq h + 1$, the morphism $H^{i_1}(\Sigma) \otimes \dots \otimes H^{i_k}(\Sigma) \longrightarrow H^{i_1 + \dots + i_k}(\Sigma)$ induced by the cup-product is surjective, but $H^{i_1 + \dots + i_k - 2}(\Sigma) \xrightarrow{\cup_{c_1(L)}} H^{i_1 + \dots + i_k}(\Sigma)$ is not surjective. Then the exceptional set of any resolution of (X, x) has complex dimension at least $(h + 1)/2$.*

PROOF. This is an immediate consequence of the previous proposition and of Proposition 3.5. We use the fact that the Euler class of the circle bundle $N(\Sigma, L)$ is equal to the first Chern class of L . \square

We have seen before that the contraction of the zero-section of the line bundle $s^*\mathcal{O}(-1)$, where s is the Segre embedding of $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$, admits small resolutions. Consider instead the isolated normal singularity obtained by contracting the zero-section in the total spaces of the line bundle $s^*\mathcal{O}(-a)$, where $a \geq 2$. Corollary 4.2 applied to $L := s^*\mathcal{O}(-a)$, $n = 3$, $h = k = i_1 = i_2 = 2$, implies that this singularity does not admit small resolutions. Indeed, as one can see easily using Künneth formulae for the product manifold $\Sigma \simeq \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$, the morphism $H^2(\Sigma) \otimes H^2(\Sigma) \longrightarrow H^4(\Sigma)$ is surjective, but the morphism $H^2(\Sigma) \xrightarrow{\cup_{c_1(L)}} H^4(\Sigma)$ is not surjective if we work with integral cohomology, as its image is divisible by $a \geq 2$. This implies that the exceptional set of any resolution is of complex dimension 2.

REMARK 4.3. The previous example shows that it is essential to apply Corollary 4.2 using cohomology groups with *integral* coefficients, since the multiplication by the Chern class of the line bundle becomes surjective if one uses instead rational or real coefficients.

We pass now to the consideration of smoothings of singularities.

DEFINITION 4.4. A **smoothing** of an isolated singularity is a germ of (flat) deformation over an irreducible base whose generic fiber is smooth. If a given singularity admits smoothings, then one says that **it is smoothable**.

Consider a smoothing of an isolated normal singularity (X, x) . By working inside a Milnor representative of the total space of the smoothing and by using Ehresmann's trivialization theorem (see [V 04, page 220]), it can be shown that the diffeomorphism type of a generic fiber is well defined. It is called *the Milnor fiber* of the smoothing. Using again Ehresmann's theorem, we see that its boundary is diffeomorphic to the boundary of (X, x) . Recall from Section 2 that a Milnor representative of the total space of the smoothing is defined as a sublevel of a (automatically spsh) euclidean rug function. Theorem 2.4 implies that this Milnor representative is a Stein space. By Theorem 2.5, we see that the fibers of the smoothing are also Stein, therefore the associated Milnor fiber can be endowed with the structure of a compact Stein manifold. This Stein structure is unique only up to deformations, but this is enough in order to see that *the boundary of a smoothable normal isolated singularity is Stein fillable*.

Not all isolated singularities are smoothable. The following proposition gives smoothing obstructions from the cohomology of the boundary. They are different from the ones of Hartshorne [H 74], Rees & Thomas [RT 78], Sommese [S 79] and

Looijenga [L 86]. For details on the study of non-smoothable singularities, one can consult Greuel & Steenbrink [GS 83].

PROPOSITION 4.5. *Let (X, x) be an isolated singularity of normal complex analytic space of complex dimension n . Denote by N its abstract boundary. If one can find numbers $i_1, \dots, i_k \in \{1, \dots, n - 2\}$ such that $i_1 + \dots + i_k \geq n + 1$ and the morphism $H^{i_1}(N) \otimes \dots \otimes H^{i_k}(N) \rightarrow H^{i_1 + \dots + i_k}(N)$ induced by the cup-product does not vanish identically, then N is not Stein fillable. In particular, (X, x) is not smoothable.*

PROOF. If (X, x) were smoothable then, as explained before, its boundary would be Stein fillable. We get a contradiction from Corollary 3.3. \square

The following corollaries are examples of applications of the previous proposition. Their proofs use *integral* cohomology.

COROLLARY 4.6. *Let Σ be an abelian variety of complex dimension $n - 1 \geq 2$ and L be an anti-ample line bundle on Σ such that $c_1(L)$ is not a primitive element of the lattice $H^2(\Sigma; \mathbb{Z})$. Then the manifold $N(\Sigma, L)$ is not Stein fillable. In particular, the germ with isolated singularity obtained by contracting the 0-section of the total space of L is not smoothable.*

PROOF. Topologically, Σ is a $(2n - 2)$ -dimensional torus. Therefore, its integral cohomology ring is isomorphic to the exterior algebra $\bigwedge^{2n-2} H^1(\Sigma)$. This implies that the morphism $\otimes^{n+1} H^1(\Sigma) \rightarrow H^{n+1}(\Sigma)$ is surjective.

Since $c := c_1(L)$ is not a primitive element of $H^2(\Sigma)$, we can write $c = a \cdot k$, where $a \in \mathbb{Z}, a \geq 2$, and $k \in H^2(\Sigma)$. This implies that $\text{im}(H^{n-1}(\Sigma) \xrightarrow{\cup c} H^{n+1}(\Sigma)) \subset a \cdot H^{n+1}(\Sigma)$, which shows that the last map is not surjective.

Using Theorem 3.5, we see that all the hypotheses of Proposition 4.5 are satisfied, with $k = n + 1, i_1 = \dots = i_k = 1$. The conclusion follows. \square

REMARK 4.7. The previous corollary answers partially the following question of Biran [B 05]: “Non-fillability of circle bundles P over Σ with $\dim_{\mathbb{R}} \Sigma \geq 4$ would be a new “contact phenomenon”. An interesting example to consider seems to be $P \rightarrow \Sigma$, where Σ is an Abelian variety of complex dimension ≥ 2 ”. Our answer is partial because we have to impose an hypothesis on $c_1(L)$.

COROLLARY 4.8. *Whenever $n \geq 3$ and $a \geq 2$, the manifold $N(\mathbb{C}\mathbb{P}^{n-1}, \mathcal{O}(-a))$ is not Stein fillable. In particular, the germ with isolated singularity obtained by contracting the zero-section of the total space of $\mathcal{O}(-a)$ is not smoothable.*

PROOF. The integral cohomology ring of $\Sigma := \mathbb{C}\mathbb{P}^{n-1}$ is isomorphic to the graded algebra $\mathbb{Z}[x]/(x^n)$, where $\deg(x) = 2$. This implies that the morphism $\otimes^{n-1} H^2(\Sigma) \rightarrow H^{2n-2}(\Sigma)$ induced by cup-product is surjective.

Moreover, if $c := c_1(\mathcal{O}(-a)) = -ax$, then $\text{im}(H^{2n-4}(\Sigma) \xrightarrow{\cup c} H^{2n-2}(\Sigma)) \subset a \cdot H^{2n-2}(\Sigma)$, which shows that the last map is not surjective, as $H^{2n-2}(\Sigma)$ is a free group of rank one.

Using Theorem 3.5, we see that all the hypotheses of Proposition 4.5 are satisfied, with $k = n - 1, i_1 = \dots = i_k = 2$. The conclusion follows. \square

REMARK 4.9. 1) The germ obtained by contracting the zero-section of $\mathcal{O}(-a)$ has an alternative description as the quotient of \mathbb{C}^n by the action of the group of a -th roots of unity by coordinate-wise multiplication. Therefore the boundaries of the associated singularities are particular higher dimensional lens spaces. For $a = 2$, we obtain the real projective space $\mathbb{R}\mathbb{P}^{2n-1}$. In this way we get an alternative proof of the fact that for $n \geq 3$ this space is not Stein fillable (see Eliashberg, Kim & Polterovich [EKP 06, page 1728]).

2) As explained in the previous remark, the germs considered in the corollary are particular quotient singularities of dimension at least 3. Therefore, by a general theorem of Schlessinger [S 71], they are *rigid*, that is, they admit no non-trivial deformations at all. In particular they are non-smoothable, which gives an alternative proof of the second sentence of the corollary.

5. The rational cohomology of Milnor fillable manifolds.

In this section, all cohomology groups are considered *with rational coefficients*. This contrasts with the previous section, in which it was essential to work with integral coefficients (see Remark 4.3).

The next theorem, proved first by Durfee & Hain [DH 88], after having been announced in [D 86], states a property of the rational cohomology rings of Milnor fillable manifolds:

THEOREM 5.1. *Let N be a $(2n-1)$ -dimensional Milnor fillable manifold, where $n \geq 2$. Consider numbers $i_1, \dots, i_k \in \{1, \dots, n-1\}$ such that $i_1 + \dots + i_k \geq n$. Then the morphism $H^{i_1}(N) \otimes \dots \otimes H^{i_k}(N) \rightarrow H^{i_1 + \dots + i_k}(N)$ induced by the cup-product in cohomology with rational coefficients vanishes identically.*

PROOF. Suppose that N is diffeomorphic to the abstract boundary of a normal isolated singularity (X, x) of dimension n . Consider a divisorial resolution $(\tilde{X}, E) \rightarrow (X, x)$ of (X, x) . Choose as representative of \tilde{X} the preimage W of a Milnor representative of (X, x) .

Consider the following commutative diagram, in which the vertical morphisms are induced by the cup-product:

$$\begin{CD} H^{i_1}(W) \otimes \dots \otimes H^{i_k}(W) @>b^*>> H^{i_1}(N) \otimes \dots \otimes H^{i_k}(N) \\ @VVV @VVV \\ H^{i_1 + \dots + i_k}(W) @>b^*>> H^{i_1 + \dots + i_k}(N) \end{CD}$$

Using point 4) of Theorem 2.11, the hypotheses made on the numbers i_1, \dots, i_k imply that the upper horizontal morphism is surjective as a tensor product of surjective morphisms and that the lower horizontal one vanishes identically. The conclusion follows. \square

Note that, although the theorems 3.1 and 5.1 are formally similar, the key results used in their proofs are completely different.

In fact, Durfee & Hain stated the previous theorem for $k = 2$. The precise bounds $n - 1$ and n for i_1, \dots, i_k and $i_1 + \dots + i_k$ respectively, make the two versions equivalent. They proved their theorem using directly the theory of Beilinson, Bernstein, Deligne & Gabber, that is, without passing through the theorem of

Goresky & MacPherson. They get also restrictions on the boundaries of tubular neighborhoods of higher dimensional subspaces of algebraic varieties.

REMARK 5.2. 1) By using Poincaré duality, the previous theorem may be reformulated in the following way: *on a $(2n - 1)$ -dimensional Milnor fillable manifold, the intersection number of rational homology classes of dimension at least n is equal to zero.* Of course, we suppose that we take classes whose sum of codimensions is equal to $2n - 1$, in order to have a well-defined intersection number.

2) The previous theorem was obtained in the particular case $n = 2, k = 2, i_1 = i_2 = 1$ by Sullivan [S 75]. In this case, where (X, x) is a germ of surface, it results immediately from the fact that the intersection form of a resolution of the isolated singular point under study is non-degenerate. As explained in Section 2. Theorem 2.11 is an analog of this result in higher dimensions. From his theorem, Sullivan deduced that the boundary of an isolated singularity cannot be diffeomorphic to a 3-dimensional torus. The following generalizes this to all dimensions.

COROLLARY 5.3. *For all $n \geq 2$, the torus \mathbb{T}^{2n-1} is not Milnor-fillable.*

PROOF. As the cohomology ring of \mathbb{T}^{2n-1} is isomorphic to the exterior algebra $\bigwedge^\bullet H^1(\mathbb{T}^{2n-1})$ (a fact already used in the proof of Corollary 4.6), the morphism $\otimes^n H^1(\mathbb{T}^{2n-1}) \longrightarrow H^n(\mathbb{T}^{2n-1})$ induced by the cup-product is surjective and does not vanish identically. We apply then the previous theorem to $k = n, i_1 = \dots = i_n = 1$. □

In the next section we will see that starting from dimension 5, odd-dimensional tori are not even holomorphically fillable.

The fact that for $n \geq 2$, the torus \mathbb{T}^{2n-1} is not the boundary of an *isolated complete intersection singularity* (abbreviated *icis*) can be proved differently. Indeed, it is implied by the fact that the boundary of an icis of complex dimension n is $(n - 2)$ -connected. This was proved by Milnor [M 68] for hypersurfaces and generalized by Hamm [H 71] to arbitrary icis.

As noted by Durfee [D 86], Theorem 5.1 implies more generally that no manifold of the form $K_1 \times K_2 \times K_3$ with $\dim K_1 + \dim K_2 + \dim K_3 = 2n - 1$ and $\dim K_i \leq n - 1, \forall i \in \{1, 2, 3\}$ is Milnor fillable.

6. The rational cohomology of holomorphically fillable manifolds.

In what follows, all (co)homology groups are considered with *rational* coefficients.

PROPOSITION 6.1. *Suppose that $n \geq 3$. Let W be an oriented cobordism of dimension $2n$ from a manifold N_1 to a manifold N_2 such that (W, N_1) has the homotopy type of a relative CW-complex of dimension $\leq n$. Consider numbers $i_1, \dots, i_k \in \{1, \dots, n - 2\}$ such that $i_1 + \dots + i_k \geq n + 1$. If the morphism $H^{i_1}(N_1) \otimes \dots \otimes H^{i_k}(N_1) \longrightarrow H^{i_1 + \dots + i_k}(N_1)$ induced by the cup-product in cohomology with rational coefficients vanishes identically, then the same is true for the analogous morphism associated to N_2 .*

PROOF. For $j = 1, 2$, denote by $N_j \xrightarrow{u_j} W$ the inclusion morphism. Our hypothesis on the pair (W, N_1) implies that:

$$H^i(W, N_1) = 0, \forall i \geq n + 1.$$

Using the exact cohomology sequence of the pair (W, N_1) , we deduce that:

$$(6.1) \quad H^i(W) \xrightarrow{u_1^*} H^i(N_1) \text{ is injective, } \forall i \in \{n + 1, \dots, 2n - 1\},$$

(in fact those morphisms are bijective, but for the proof of the proposition we need only their injectivity).

By generalized Poincaré-Lefschetz duality applied to the cobordism W from N_1 to N_2 (see Hatcher [H 02, page 254]), we get $H^{i+1}(W, N_2) \cong H_{2n-i-1}(W, N_1)$, therefore:

$$H^{i+1}(W, N_1) = 0, \forall i \in \{0, \dots, n - 2\}.$$

Using the exact cohomology sequence of the pair (W, N_2) , we deduce that:

$$(6.2) \quad H^i(W) \xrightarrow{u_2^*} H^i(N_2) \text{ is surjective, } \forall i \in \{0, \dots, n - 2\}.$$

Consider then the following commutative diagram:

$$\begin{array}{ccccc} H^{i_1}(N_1) \otimes \dots \otimes H^{i_k}(N_1) & \xleftarrow{u_1^*} & H^{i_1}(W) \otimes \dots \otimes H^{i_k}(W) & \xrightarrow{u_2^*} & H^{i_1}(N_2) \otimes \dots \otimes H^{i_k}(N_2) \\ \downarrow & & \downarrow & & \downarrow \\ H^{i_1+\dots+i_k}(N_1) & \xleftarrow{u_1^*} & H^{i_1+\dots+i_k}(W) & \xrightarrow{u_2^*} & H^{i_1+\dots+i_k}(N_2) \end{array}$$

By hypothesis, the left-side vertical morphism vanishes. As the left-side lower horizontal morphism is injective by (6.1), we deduce that the middle morphism vanishes identically. As the right-side upper horizontal morphism is surjective by (6.2), we deduce that the right-side vertical morphism also vanishes. \square

As a consequence, we get the following property of the rational cohomology rings of holomorphically fillable manifolds, proved by Bungart [B 92] starting also from the results of Durfee & Hain:

THEOREM 6.2. *Suppose that $n \geq 3$. Let N be a holomorphically fillable manifold of dimension $2n - 1$. Consider numbers $i_1, \dots, i_k \in \{1, \dots, n - 2\}$ such that $i_1 + \dots + i_k \geq n + 1$. Then the morphism $H^{i_1}(N) \otimes \dots \otimes H^{i_k}(N) \rightarrow H^{i_1+\dots+i_k}(N)$ induced by the cup-product in cohomology with rational coefficients vanishes identically.*

PROOF. Suppose that N is the strictly pseudo-convex boundary of a compact holomorphic manifold Z of complex dimension $2n$. Denote by $Z \xrightarrow{r} X$ the Remmert reduction morphism of Z . Then X is a Stein space with at most isolated singularities, obtained as images of some of the connected components of the maximal compact analytic subset of Z (the other connected components contract to smooth points of X).

Denote by F the (finite) set of singular points of X . Let ϕ be a function as described in the conclusion of Proposition 2.10. Suppose (without reducing the generality) that the absolute minimum of ϕ , attained by hypothesis on the finite set F , is equal to 0 (it is enough to add a constant to ϕ in order to get this). Suppose moreover that ϕ is a Morse function with only one critical point on each critical level, which can be realized by a sufficiently small smooth perturbation on a compact subset of X (we use the stability of spsh functions formulated in Proposition 2.2, point 3)). For $\epsilon > 0$ sufficiently small, the level $N := \phi^{-1}(\epsilon)$ is a disjoint union of manifolds diffeomorphic to the boundaries of the singularities of X .

Denote by $\nu > 0$ the maximum value of ϕ , attained by construction exactly on $N_2 := N$. Then $W := \rho^{-1}([\epsilon, \nu])$ is an oriented cobordism from N_1 to N_2 . As ϕ is spsh, we deduce from Proposition 2.7 that (W, N_1) has the homotopy type of a relative CW-complex of dimension $\leq n$. By Theorem 5.1, for each connected component C of N_1 , the morphism $H^{i_1}(C) \otimes \cdots \otimes H^{i_k}(C) \rightarrow H^{i_1 + \cdots + i_k}(C)$ vanishes identically, which implies that the same is true for N_1 . Therefore we can apply Proposition 6.1, and the conclusion follows. \square

A similar theorem holds for Stein fillable manifolds, with the essential difference that it is then true for cohomology groups *with arbitrary coefficient rings* (Theorem 3.3). This difference allows to detect with our methods holomorphically fillable manifolds which are not Stein fillable, as we did in Section 4.

REMARK 6.3. 1) By using Poincaré duality, the previous theorem may be reformulated in the following way: *on a $(2n - 1)$ -dimensional holomorphically fillable manifold, the intersection number of rational homology classes of dimension at least $n + 1$ is equal to zero* (compare with Remark 5.2, 1)).

2) One has to introduce the restriction $n \geq 3$ in order to have integers i_1, \dots, i_k which satisfy the conditions of the hypotheses. Therefore, as in the case of Corollary 3.3, the previous theorem says nothing about 3-dimensional manifolds.

3) With the notations of the proof of Theorem 6.2, the surjectivity of the morphism $H^0(W) \xrightarrow{u_2^*} H^0(N_2)$ (a consequence of (6.2)), shows that the boundary of a strongly pseudoconvex *connected* manifold is also connected (folklore). This result fails for strong symplectic fillings of contact manifolds, as was shown by McDuff [M 91] and Geiges [G 94].

The proof of the previous theorem shows that one can get more information on the cohomology rings of holomorphically fillable manifolds from more detailed knowledge of the topology of the isolated singularities of a Stein space which fills it. For example:

PROPOSITION 6.4. *Let N be the boundary of a compact Stein space X with isolated singularities. Fix a ring of coefficients A and numbers $i_1, \dots, i_k \in \{1, \dots, n - 2\}$ such that $i_1 + \cdots + i_k \geq n + 1$. If the morphism $H^{i_1}(M, A) \otimes \cdots \otimes H^{i_k}(M, A) \rightarrow H^{i_1 + \cdots + i_k}(M, A)$ induced by the cup-product vanishes identically for the abstract boundary M of each isolated singular point of X , then the same is true for N .*

As an immediate application of Theorem 6.2, we see that *for all $n \geq 3$, the torus \mathbb{T}^{2n-1} is not holomorphically fillable*. Indeed, one simply has to apply the previous theorem to $k = n + 1, i_1 = \cdots = i_{n+1} = 1$.

Note that the torus \mathbb{T}^{2n-1} can be realised as a Levi-flat boundary of a complex manifold: consider the product of an abelian variety and the closed unit disc in \mathbb{C} . This shows the importance of the *strong* convexity hypothesis in the definition of holomorphically fillable manifolds.

By a theorem of Bourgeois [B 02] (see also Giroux [G 02] for details on the context of research having led to it), if a closed orientable manifold M admits a contact structure, then $M \times \mathbb{T}^2$ does too. This implies that all odd-dimensional tori admit contact structures, as \mathbb{T}^3 does (see the next paragraph). The previous corollary shows that a contact structure on a torus of dimension at least 5 cannot be holomorphically fillable.

The 3-dimensional torus \mathbb{T}^3 , however, is holomorphically fillable: it can be realized as a strongly pseudoconvex boundary of a tubular neighborhood of $\mathbb{S}^1 \times \mathbb{S}^1$ standardly embedded in \mathbb{C}^2 (see Eliashberg [E 96]). By the theorem of Sullivan quoted in the previous section and generalized in Theorem 5.1, \mathbb{T}^3 is not Milnor fillable. In a similar way, we get using Theorem 5.1:

PROPOSITION 6.5. *For any $n \geq 2$, the product $\mathbb{T}^n \times \mathbb{S}^{n-1}$ is holomorphically fillable but not Milnor fillable.*

PROOF. Consider the standard embedding of $\mathbb{T}^n = \mathbb{S}_1^1 \times \cdots \times \mathbb{S}_n^1$ in \mathbb{C}^n as the product of unit circles in each factor \mathbb{C} (the indices denote different copies of \mathbb{S}^1). Since the image of this embedding is totally real, we see that it has strongly pseudoconvex regular neighborhoods (see Grauert [G 58], Eliashberg [E 97]). The boundaries of these regular neighborhoods are diffeomorphic to $\mathbb{T}^n \times \mathbb{S}^{n-1}$, which shows that this last manifold is holomorphically fillable.

Choose now points $P_i \in \mathbb{S}_i^1$, $\forall i \in \{1, \dots, n\}$ and $P \in \mathbb{S}^{n-1}$. The submanifolds $K_1 := \mathbb{S}_1^1 \times P_2 \times \cdots \times P_n \times \mathbb{S}^{n-1}$, $K_2 := \mathbb{T}^n \times P$, $K_3 := P_1 \times \mathbb{S}_2^1 \times \cdots \times \mathbb{S}_n^1 \times \mathbb{S}^{n-1}$ of $\mathbb{T}^n \times \mathbb{S}^{n-1}$ have only the point $P_1 \times \cdots \times P_n \times P$ in common, where they meet transversely. Therefore, with convenient choices of orientation, the intersection number of their homology classes is equal to 1. For $j \in \{1, 2\}$, denote by $\gamma_j \in H^{n-1}(\mathbb{T}^n \times \mathbb{S}^{n-1})$ the Poincaré dual of the homology class of K_j . We deduce that $\gamma_1 \cup \gamma_2$ does not vanish in $H^{2n-2}(\mathbb{T}^n \times \mathbb{S}^{n-1})$, which shows that the morphism $H^{n-1}(\mathbb{T}^n \times \mathbb{S}^{n-1}) \otimes H^{n-1}(\mathbb{T}^n \times \mathbb{S}^{n-1}) \rightarrow H^{2n-2}(\mathbb{T}^n \times \mathbb{S}^{n-1})$ induced by the cup-product does not vanish identically. By Theorem 5.1, we deduce that $\mathbb{T}^n \times \mathbb{S}^{n-1}$ is not Milnor fillable. \square

References

- [AF 59] Andreotti, A., Frankel, T. *The Lefschetz theorem on hyperplane sections*. Ann. of Math. (2) **69**, no.3 (1959), 713-717.
- [BBD 82] Beilinson, A.A., Bernstein, J. N., Deligne, P. *Faisceaux pervers*. Astérisque **100**, Soc. Math. France, 1982.
- [BN 90] Bell, S.R., Narasimhan, R. *Proper Holomorphic Mappings of Complex Spaces*. Several Complex Variables VI, Encyclopaedia of Maths. Sciences vol. **69**, Barth, W., Narasimhan, R. eds., Springer Verlag 1990.
- [B 90] Bennequin, D. *Topologie symplectique, convexité holomorphe et structures de contact [d'après Y. Eliashberg, D. Mc Duff et al.]* Séminaire Bourbaki no. **725**, Astérisque 189-190, Soc. Math. France, 1990.
- [B 05] Biran, P. *Symplectic topology and algebraic families*. Proc. of the European Congress of Maths. (Stockholm 2004), European Math. Soc., 2005.
- [BO 97] Bogomolov, F.A., de Oliveira, B. *Stein Small Deformations of Strictly Pseudoconvex Surfaces*. Contemporary Mathematics **207** (1997), 25-41.
- [B 59] Bott, R. *On a theorem of Lefschetz*. Michigan Math. Journal **6** (1959), 211-216.
- [B 02] Bourgeois, F. *Odd dimensional tori are contact manifolds*. Internat. Math. Res. Notices **30** (2002), 1571-1574.
- [B 92] Bungart, L. *Vanishing cup products on pseudoconvex CR manifolds*. Contemp. Math. **137** (1992), 105-111.
- [CM 05] de Cataldo, M.A.A., Migliorini, L. *The Hodge theory of algebraic maps*. Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 5, 693-750.
- [CNP 06] Caubel, C., Némethi, A., Popescu-Pampu, P. *Milnor open books and Milnor fillable contact 3-manifolds*. Topology **45** (2006), 673-689.
- [D 86] Durfee, A.H. *Topological restrictions on the links of isolated complex singularities*. In *Complex analytic singularities*. Adv. Studies in Pure Maths. **8** (1986), 95-99.
- [DH 88] Durfee, A.H., Hain, R.M. *Mixed Hodge structures on the homotopy of links*. Math. Ann. **280** (1988), no. 1, 69-83.

- [E 89] Eliashberg, Y. *Filling by holomorphic discs and its applications*. In *Geometry of low-dimensional manifolds. 2.* (Durham, 1989), 45-67, London Math. Soc. Lecture Notes Series **151**, Cambridge Univ. Press, 1990.
- [E 90] Eliashberg, Y. *Topological characterization of Stein manifolds of dimension >2* . Internat. J. of Math. **1** (1990) no. 1, 29-46.
- [E 96] Eliashberg, Y. *Unique holomorphically fillable contact structure on the 3-torus*. Internat. Math. Res. Notices **2** (1996), 77-82.
- [E 97] Eliashberg, Y. *Symplectic geometry of plurisubharmonic functions*. (Notes by M. Abreu) In *Gauge theory and symplectic geometry*. J. Hurtubise and F. Lalonde eds., 49-67, Kluwer, 1997.
- [EG 91] Eliashberg, Y., Gromov, M. *Convex symplectic manifolds*. In *Several complex variables and complex geometry*. Part 2, Proc. Sympos. Pure Math. **52**, 135-162 (1991).
- [EKP 06] Eliashberg, L., Kim, S.S., Polterovich, L. *Geometry of contact transformations and domains: orderability versus squeezing*. Geometry & Topology **10** (2006), 1635-1747.
- [G 94] Geiges, H. *Symplectic manifolds with disconnected boundary of contact type*. Internat. Math. Res. Notices **1** (1994), 23-30.
- [G 08] Geiges, H. *An introduction to contact topology*. Cambridge Univ. Press, 2008.
- [G 02] Giroux, E. *Géométrie de contact: de la dimension trois vers les dimensions supérieures*. Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), 405-414, Higher Ed. Press, Beijing, 2002.
- [GM 82] Goresky, M., MacPherson, R. *On the topology of complex algebraic maps*. In *Algebraic Geometry. Proceedings, La Rábida, 1981*. J.M. Aroca, R. Buchweitz, M. Giusti, M. Merle eds. L.N.M. **961**, Springer Verlag, 1982.
- [G 58] Grauert, H. *On Levi's problem and the imbedding of real-analytic manifolds*. Ann. of Math. (2) **68** (1958), 460-472.
- [G 62] Grauert, H. *Über Modifikationen und exceptionnelle analytische Mengen*. Math. Ann. **146** (1962), 331-368.
- [GR 79] Grauert, H., Remmert, R. *Theory of Stein spaces*. Springer-Verlag, 1979.
- [GS 83] Greuel, G.-M., Steenbrink, J. *On the topology of smoothable singularities*. Proc. of Symp. in Pure Maths. **40** (1983), Part 1, 535-545.
- [G 85] Gromov, M. *Pseudoholomorphic curves in symplectic manifolds*. Invent. Math. **82** (1985), 307-347.
- [H 71] Hamm, H. *Lokale topologische Eigenschaften komplexer Räume*. Math. Ann. **191** (1971), 235-252.
- [H 86] Hamm, H. *Zum Homotopietyp q -vollständiger Räume*. J. Reine Angew. Math. **364** (1986), 1-9.
- [H 74] Hartshorne, R. *Topological conditions for smoothing algebraic singularities*. Topology **13** (1974), 241-253.
- [H 02] Hatcher, A. *Algebraic topology*. Cambridge Univ. Press, 2002.
- [K 79] Karchyauskas, K.K. *Homotopy properties of complex algebraic sets*. Studies in topology, Steklov Institute (Leningrad), 1979.
- [K 91] Kollár, J. *Flips, flops, minimal models, etc.* Surveys in differential geometry (Cambridge, MA, 1990), 113-199, Lehigh Univ., Bethlehem, PA, 1991.
- [L 86] Looijenga, E. *Riemann-Roch and smoothings of singularities*. Topology **25** (1986), 293-302.
- [M 91] McDuff, D. *Symplectic manifolds with contact type boundaries*. Invent. Math. **103** (1991) no.3, 651-671.
- [M 63] Milnor, J. *Morse theory*. Princeton Univ. Press, 1963.
- [M 68] Milnor, J. *Singular Points of Complex Hypersurfaces*. Princeton Univ. Press, 1968.
- [M 61] Mumford, D. *The topology of normal singularities of an algebraic surface and a criterion for simplicity*. Inst. Hautes Études Sci. Publ. Math. **9** (1961), 5-22.
- [N 67] Narasimhan, R. *On the homology groups of Stein spaces*. Invent. Math. **2** (1967), 377-385.
- [N 85] Navarro Aznar, V. *Sur la théorie de Hodge des variétés algébriques à singularités isolées*. In *Differential systems and singularities*. (Luminy, 1983), Astérisque **130** (1985), 272-307.
- [P 94] Peternell, Th. *Pseudoconvexity, the Levi problem and vanishing theorems*. Several Complex Variables VII, Encyclopaedia of Maths. Sciences vol. 69, Grauert, H., Peternell, Th., Remmert, R. eds., Springer-Verlag 1994, 221-257.
- [RT 78] Rees, E., Thomas, E. *Cobordism obstructions to deforming isolated singularities*. Math. Ann. **232** (1978), 33-54.

- [S 71] Schlessinger, M. *Rigidity of quotient singularities*. Invent. Math. **14** (1971), 17-26.
- [S 53] Serre, J.-P. *Quelques problèmes globaux relatifs aux variétés de Stein*. Colloque sur les fonctions de plusieurs variables, Bruxelles (1953), 57-68.
- [S 79] Sommese, A.J. *Non-smoothable varieties*. Comment. Math. Helv. **54** (1979), 140-146.
- [S 83] Steenbrink, J. *Mixed Hodge structures associated with isolated singularities*. Proc. of Symposia in Pure Maths. **40** (1983), Part 2, 513-536.
- [S 87] Steenbrink, J. *Mixed Hodge structures and singularities: a survey*. In *Géométrie algébrique et applications. III*. Comptes-rendus de la deuxième conférence internationale de La Rábida (1984), J-M. Aroca, T. Sanchez-Giralda, J-L. Vicente eds., Hermann, 1987.
- [S 75] Sullivan, D. *On the intersection ring of compact three manifolds*. Topology **14** (1975), 275-277.
- [T 59] Thom, R. *Les structures différentiables des boules et des sphères*. In *Colloque de géométrie différentielle globale*. Bruxelles, dec. 1958. Centre Belge de Recherches Mathématiques, 1959.
- [V 44] Du Val, P. *On absolute and non-absolute singularities of algebraic surfaces*. Revue de la Faculté des Sciences de l'Univ. d'Istanbul (A) **91** (1944), 159-215.
- [V 80] Varchenko, A.N. *Contact structures and isolated singularities*. Mosc. Univ. Math. Bull. **35** no.2 (1980), 18-22.
- [V 04] Voisin, C. *Hodge theory and complex algebraic geometry I*. Cambridge Univ. Press, 2004.

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