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A FINITENESS THEOREM FOR DUAL GRAPHS OF SURFACE SINGULARITIES

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Consider a fixed connected, finite graph Γ and equip its vertices with weights p_i which are non-negative integers. We show that there are a finite number of possibilities for the coefficients of the canonical cycle of a numerically Gorenstein surface singularity having Γ as the dual graph of the minimal resolution, the weights p_i of the vertices being the arithmetic genera of the corresponding irreducible components. As a consequence we get that if Γ is not a cycle, then there are a finite number of possibilities of self-intersection numbers which one can attach to the vertices which are of valency ≥ 2, such that one gets the dual graph of the minimal resolution of a numerically Gorenstein surface singularity. Moreover, we describe precisely the situations when there exists an infinite number of possibilities for the self-intersections of the component corresponding to some fixed vertex of Γ.

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1. Introduction

Let $(X, 0)$ be a germ of a normal complex analytic surface and $(\tilde{X}, E) \xrightarrow{\pi} (X, 0)$ a resolution of it, its exceptional divisor E not being supposed to have normal crossings. One associates to this resolution a dual graph Γ, whose vertices are weighted by the absolute values $e_i := -E_i^2 > 0$ of the self-intersections and by the arithmetic genera $p_i := p_a(E_i) \geq 0$ of the corresponding irreducible components E_i of E. We see the two collections of weights as functions with values in $\mathbb{N}$ defined on the set of vertices of Γ, and we denote them by $\underline{e}$ and $\underline{p}$ respectively.

It is known since the works of Du Val [16] and Mumford [13] that the intersection form associated to the weighted graph $(\Gamma, \underline{e})$ is negative definite. In this case we say that $(\Gamma, \underline{e})$ and $(\Gamma, \underline{e}, \underline{p})$ are also *negative definite*. Conversely, Grauert [8] showed that if a compact connected reduced divisor on a smooth complex analytic surface has a weighted dual graph which satisfies this condition, then it may be contracted to a normal singular point of an analytic surface.

An important problem, studied by several authors (see for instance Yau [18] and Laufer [11]), is to decide, amongst negative definite dual graphs $(\Gamma, \underline{e}, \underline{p})$, which ones correspond to hypersurface singularities in $\mathbb{C}^3$. This problem is rather deep and has resisted several attempts to solve it. This article arose as a new attempt to give a step in that direction. To be precise, we look at the following weaker problem:

Describe the dual graphs corresponding to numerically Gorenstein surface singularities.

This second problem was also posed by Jia, Luk & Yau [9].

Recall that the germ $(X, 0)$ is called *Gorenstein* if the canonical line bundle is holomorphically trivial on a pointed neighborhood of 0 in X. It is called *numerically Gorenstein* if the same line bundle is smoothly trivial. An isolated hypersurface singularity is a particular case of isolated complete intersection singularity, which is a particular case of a Gorenstein isolated singularity, which is a particular case of numerically Gorenstein isolated singularity. This explains in which sense the previous problem is weaker than the initial one.

This article is devoted to the study of the dual graphs of numerically Gorenstein surface singularities; for short, we call these singularities *$\mathbb{Z}$-Gorenstein*. The starting point is Durfee's observation in [6] (see also [10]), that an isolated surface singularity $(X, 0)$ is $\mathbb{Z}$-Gorenstein iff the *canonical cycle* Z_{can} of every resolution is integral. This 2-cycle, supported on the exceptional divisor, is uniquely characterized by the fact that it satisfies the adjunction formula:

$$2p_a(E_i) - 2 = E_i^2 + Z_{can} \cdot E_i \, ,$$

for each irreducible component E_i of the exceptional divisor, where $p_a(E_i)$ is the arithmetic genus of the (possibly singular) curve E_i. This system of equations shows that the coefficients of Z_{can} are determined by the dual graph of the considered resolution, decorated by the weights e_i and p_i attached to the vertices. Therefore, we can speak about *$\mathbb{Z}$-Gorenstein graphs* $(\Gamma, \underline{e}, \underline{p})$.

Given an arbitrary finite, connected, unoriented graph Γ, whose set of vertices is denoted $V(\Gamma)$, every choice of sufficiently large positive weights $(e_i)_{i \in V(\Gamma)}$ makes it have negative definite intersection form (see [10]). More precisely, there exist weights $(e_i^0)_{i \in V(\Gamma)}$ such that one has a negative definite intersection form whenever $e_i \geq e_i^0$ for all $i \in V(\Gamma)$. By Grauert's theorem, every choice of arithmetic genus p_i for each vertex turns $(\Gamma, \underline{e}, \underline{p})$ into the dual graph of a resolution of a normal surface singularity. Furthermore (see [10]), for each fixed function $\underline{e}$ such that $(\Gamma, \underline{e})$ is negative defnite, there are infinitely many choices of a function $\underline{p}$ that make $(\Gamma, \underline{e}, \underline{p})$

$\mathbb{Z}$-Gorenstein.

In this paper we look at the converse situation, which is more delicate:

If we fix $(\Gamma, \underline{p})$*, how many choices are there for* $\underline{e}$ *such that* $(\Gamma, \underline{e}, \underline{p})$ *is a* $\mathbb{Z}$*-Gorenstein graph?*

As noticed before, the knowledge of $(\Gamma, \underline{e}, \underline{p})$ determines the coefficients of the canonical cycle of a singularity whose minimal resolution has this dual graph. We succeeded in proving the following finiteness result, which is a consequence of the main theorem of this paper (see Theorem 4.1):

Let Γ *be an arbitrary finite, connected, unoriented graph equipped with weights* $p_i \geq 0$ *assigned to each vertex* $i \in V(\Gamma)$*. Then there are at most a finite number of choices of weights corresponding to the coefficients of the canonical cycle of the minimal resolution of a* $\mathbb{Z}$*-Gorenstein surface singularity with graph* Γ *and arithmetic genera* p_i*.*

As a consequence of this result we get that if Γ is not a cycle, then there are at most a finite number of choices of weights e_i for the vertices with valence at least 2, when one varies $\underline{e}$ such that $(\Gamma, \underline{e}, \underline{p})$ is $\mathbb{Z}$-Gorenstein and minimal (that is, without vertices i such that $p_i = 0$ and $e_i = 1$). Moreover, we describe precisely in all the cases the non-finiteness appearing in the choice of the weights $(e_i)_{i \in V(\Gamma)}$ (see Proposition 5.2).

Before describing briefly the content of each section, we would like to mention that, as Jonathan Wahl told us, it is unknown if each $\mathbb{Z}$-Gorenstein graph occurs as the dual graph associated to a resolution of a Gorenstein (and not merely $\mathbb{Z}$-Gorenstein) normal surface singularity.

In the first section, we explain the necessary background about dual graphs, associated quadratic forms, anticanonical cycles and numerically Gorenstein singularities. In the second section we give examples of families of numerically Gorenstein singularities, whose study allowed us to conjecture the results proved in the following sections. In the third section we prove our main theorem, and in the last one we describe precisely the vertices i of the graph Γ, to which may be associated in an infinite number of ways a value e_i extendable to a weight $\underline{e}$ making $(\Gamma, \underline{p}, \underline{e})$ $\mathbb{Z}$-Gorenstein. This places the examples of the second section in a clearer light. We conclude with some questions.

2. Dual graphs and anticanonical cycles

Let $(X, 0)$ be a germ of normal complex analytic surface. Denote by $(\tilde{X}, E) \xrightarrow{\pi} (X, 0)$ its *minimal resolution*, where E is the reduced fibre over 0. Therefore E can be seen as a connected reduced effective divisor in $\tilde{X}$, called *the exceptional divisor* of π. Note that E has not necessarily normal crossings. In particular, its irreducible components are not necessarily smooth.

Denote by Γ the *dual (intersection) graph* of E: its vertices correspond bijectively to the components of E and between two distinct vertices i and j there are as many (unoriented) edges as the intersection number $e_{ij} := E_i \cdot E_j \geq 0$ of the corresponding components. In particular, Γ has no loops. Moreover, each vertex i of Γ is weighted by the number e_i, where $-e_i := E_i^2$ is the self-intersection number of the associated component E_i inside $\tilde{X}$.

Denote by $V(\Gamma)$ the set of vertices of Γ and by $\underline{e} \in \mathbb{Z}^{V(\Gamma)}$ the function which associates to each vertex its weight. To the weighted graph $(\Gamma, \underline{e})$ is associated a canonical quadratic form on the real vector space $\mathbb{R}^{V(\Gamma)}$, called *the intersection form* associated to the resolution π:

$$Q_{(\Gamma,\underline{e})}(\underline{x}) := \sum_{i \in V(\Gamma)} (-e_i x_i^2 + \sum_{\substack{j \in V(\Gamma) \\ j \neq i}} e_{ij} x_i x_j) \ = \ \sum_{i \in V(\Gamma)} x_i(-e_i x_i + \sum_{\substack{j \in V(\Gamma) \\ j \neq i}} e_{ij} x_j).$$

Often in the literature one fixes an order of the components of E, which makes $\underline{e}$ appear as a finite sequence and $(e_{ij})_{i,j}$ appear as a matrix. We did not choose to do so in order to emphasize that there is no natural order and that our considerations do not depend on any such choice.

Du Val [16] and Mumford [13] proved that the intersection form $Q_{(\Gamma,\underline{e})}$ is negative definite. In particular, $e_i > 0$ for all $i \in V(\Gamma)$. Conversely, Grauert [8] proved that if the form associated to a reduced compact effective divisor E on a smooth surface is negative definite, then E can be contracted to a normal singular point of an analytic surface.

For the following considerations on arithmetic genera, the adjunction formula and the anti-canonical cycle, we refer to Reid [15] and Barth, Hulek, Peters & Van de Ven [2].

If D is an effective divisor on $\tilde{X}$ supported on E, then it may be interpreted as a (non-necessarily reduced) compact curve, with associated structure sheaf $\mathcal{O}_D$. Its *arithmetic genus* $p_a(D)$ is by definition equal to $1-\chi(\mathcal{O}_D)$. It satisfies *the adjunction formula*:

$$p_a(D) := 1 + \frac{1}{2}(D^2 + K_{\tilde{X}} \cdot D) \tag{2.1}$$

where $K_{\tilde{X}}$ is any canonical divisor on $\tilde{X}$. This allows to extend the definition to *any* divisor supported on E, not necessarily an effective one.

Denote by p_i the arithmetic genus of the curve E_i for all $i \in V(\Gamma)$, and by g_i the arithmetic genus of its normalization, equal to its topological genus. Both genera are related by the following formula:

$$p_i = g_i + \sum_{P \in E_i} \delta_P \tag{2.2}$$

where $\delta_P \geq 0$ denotes the delta-invariant of the point P of E_i, equal to the number of ordinary double points concentrated at P. One has $\delta_P > 0$ if and only if P is singular on E_i. We deduce from (2.2) that:

$$p_i = 0 \text{ if and only if } E_i \text{ is a smooth rational curve.} \tag{2.3}$$

At this point, we have two weightings for the vertices of the graph Γ, the collection $\underline{e}$ of self-intersections and the collection $\underline{p}$ of arithmetic genera of the associated irreducible components. If E is a divisor with normal crossings and moreover all its components are smooth, then the doubly weighted graph $(\Gamma, \underline{e}, \underline{p})$ determines the embedded topology of E in $\tilde{X}$ (see Mumford [13]). In general this is not the case, because these numerical data do not determine the types of singularities of E. Nevertheless, they determine them, and consequently the embedded topology of E, up to a finite ambiguity. Indeed, there are a finite number of embedded topological types of germs of reduced plane curves (C, c) having a given value $\delta_c(C)$ (see Wall [17]).

As the quadratic form $Q_{(\Gamma,\underline{e})}$ is negative definite, there exists a unique divisor *with rational coefficients* Z_K supported on E such that:

$$Z_K \cdot E_i = -\tilde{K} \cdot E_i\,, \ \text{ for all } i \in V(\Gamma). \tag{2.4}$$

We call Z_K the *anti-canonical cycle* of E (or of the resolution π). The name is motivated by the fact that whenever $(X, 0)$ is Gorenstein, $-Z_K$ is a canonical divisor on $\tilde{X}$. With the notations of the introduction, $Z_{can} = -Z_K$. The sign in the previous definition is motivated by the following well-known result:

Lemma 2.1. *Assuming (as we do) that the resolution is minimal, Z_K is an effective divisor.*

Proof. From (2.1) and (2.4) we get $Z_K \cdot E_i = -e_i - 2p_i + 2$. This number is necessarily non-positive. This is clear if $p_i \geq 1$. If instead $p_i = 0$, by (2.3) we see that E_i is a smooth rational curve. As π is supposed to be the minimal resolution of $(X, 0)$, we get $e_i \geq 2$ by Castelnuovo's criterion, which shows again that $-e_i - 2p_i + 2 \leq 0$. Therefore $Z_K \cdot E_i \leq 0$ for all $i \in V(\Gamma)$, which implies that Z_K is effective (cf. the proof of Proposition 2 in [1]). □

Denote:

$$Z_K = \sum_{i \in V(\Gamma)} z_i E_i$$

and $e_{ii} := -e_i$. The previous lemma shows that $\underline{z} \in \mathbb{Q}_{\geq 0}^{V(\Gamma)}$. By the adjunction formulae (2.1) and the relations (2.4), we get the following system of equations relating $\underline{e}, \underline{p}$ and $\underline{z}$:

$$2p_i - 2 = -(z_i - 1)e_i - \sum_{\substack{j \in V(\Gamma) \\ j \neq i}} z_j e_{ij} \tag{2.5}$$

Definition 2.1. The singularity $(X, 0)$ is called **numerically Gorenstein** or $\mathbb{Z}$-**Gorenstein** if Z_K is an integral divisor. As the coefficients $\underline{Z} = \{z_i\}$ of Z_K depend only on the decorated graph $(\Gamma, \underline{p}, \underline{e})$, we also say that this graph is $\mathbb{Z}$-**Gorenstein**.

Recall now that the *Du Val singularities*, also known as *Kleinian singularities, rational double points or simple surface singularities* (see Durfee [7]) are, up to isomorphism, the surface singularities of the form $\mathbb{C}^2/G$, where G is a finite subgroup of $SU(2)$. For these singularities the minimal resolution has $Z_K \equiv 0$, the dual graph Γ is one of the trees A_n, D_n, E_6, E_7, E_8 and $p_i = 0, e_i = 2$ for all the vertices i of Γ.

Lemma 2.2. *If $(\Gamma, \underline{p}, \underline{e})$ is not one of the Dynkin diagrams $\{A_n, D_n, E_6, E_7, E_8\}$ corresponding to the Du Val singularities and is $\mathbb{Z}$-Gorenstein, then $z_i > 0$ for all $i \in V(\Gamma)$.*

Proof. Suppose that $z_i = 0$. The previous equation implies that $2p_i - 2 < 0$, thus $p_i = 0$. Therefore (2.5) may be written:

$$-2 = -e_i - \sum_{\substack{j \in V(\Gamma) \\ j \neq i}} z_j e_{ij} \, .$$

The hypothesis that the resolution is minimal shows that $e_i \geq 2$, since $p_i = 0$. Hence $e_i = 2$ and $z_j = 0$ for all the neighbors j of i. Extending this argument step by step and using the connectedness of Γ, one gets $z_j = 0$ and $e_j = 2$ for all $j \in V(\Gamma)$. Therefore, the decorated graph must be as stated, by a classical characterization of Du Val singularities (see [2]). □

In the sequel, we suppose that $(\Gamma, \underline{p}, \underline{e})$ is not one of the Dynkin diagrams $\{A_n, D_n, E_6, E_7, E_8\}$. By the previous lemma, $z_i \geq 1$ for all $i \in V(\Gamma)$. Let us introduce new variables, for simplicity:

$$\begin{cases} n_i := z_i - 1 \geq 0 \, , \\ v_i := \sum_{\substack{j \in V(\Gamma) \\ j \neq i}} e_{ij} \geq 0, \\ q_i := v_i + 2p_i - 2 \geq -2. \end{cases} \tag{2.6}$$

Then the adjunction formulae (2.5) become:

$$\{e_i n_i \; = \; q_i + \sum_{\substack{j \in V(\Gamma) \\ j \neq i}} e_{ij} n_j\}_{i \in V(\Gamma)}. \tag{2.7}$$

If $i \in V(\Gamma)$, v_i is the *valency* of i, that is, the number of edges connecting it to other vertices. If Γ is homeomorphic to a circle and $p_i = 0$ for all $i \in V(\Gamma)$, we say that $(\Gamma, \underline{p})$ is a *cusp-graph*. The name comes from the fact that such graphs appear as dual resolution graphs of so-called *cusp surface singularities* (see Brieskorn [3] or Looijenga [12]).

We summarize below the previous discussion:

Proposition 2.1. *Let $(X, 0)$ be a $\mathbb{Z}$-Gorenstein surface singularity which is not a Du Val singularity. Let $E = \sum E_i$ be the reduced exceptional divisor of its minimal resolution and let $Z_K = \sum z_i E_i$ be the anti-canonical cycle, characterized by the adjunction formulae (2.5). Then Z_K is an effective divisor and, with the notations (2.6), the adjunction formulae become the formulae (2.7).*

3. Examples

In this section we present some of the examples which led us to conjecture the results proved in the next two sections. In each one of them, we fix a weighted graph $(\Gamma, \underline{p})$ and we look for the weights $\underline{e}$ which make $(\Gamma, \underline{e}, \underline{p})$ correspond to a $\mathbb{Z}$-Gorenstein singularity.

i) Suppose that the graph Γ has only 1 vertex, no edges, and that we equip the vertex with some weight $p \geq 0$. Then the anti-canonical cycle is $Z_K = zE$ for some $z \in \mathbb{N}$, where E represents a (possibly singular) irreducible projective curve of arithmetic genus p. Denoting $e := -E^2$, the adjunction formulae (2.5) implies:

$$z = \frac{2p-2}{e} + 1\,.$$

Hence we have the dual graph of a $\mathbb{Z}$-Gorenstein singularity iff the weights (p, e) are chosen so that $\frac{2p-2}{e}$ is an integer. Obviously, except when $p = 1$, there are finitely many choices of such weights for fixed p.

ii) Consider a graph Γ with two vertices 1 and 2 and one edge between them, and equip the vertices with genera $p_1 = 1$ and $p_2 = 2$ respectively. The adjunction system (2.7) becomes:

$$\begin{cases} e_1 n_1 - n_2 = 1 \\ e_2 n_2 - n_1 = 3 \end{cases}.$$

An easy computation shows that there are exactly 8 solutions $(n_1, n_2; e_1, e_2)$ of the system, as follows: $(5, 4; 1, 2)$, $(3, 2; 1, 3)$, $(2, 1; 1, 5)$, $(4, 7; 2, 1)$, $(1, 1; 2, 4)$, $(2, 5; 3, 1)$, $(1, 2; 3, 2)$, $(1, 4; 5, 1)$.

iii) Consider the quotient-conical singularities of Dolgachev [4]: given any cocompact fuchsian group G of signature $\{g; \alpha_1, ..., \alpha_n\}$, we may let it act on $TH \cong H \times \mathbb{C}$ via the differential: $g\cdot(z, w) \mapsto (g(z), g_*(z)\cdot w)$, where H is the upper half plane in $\mathbb{C}$. The surface TH/G contains H/G as a divisor that can be blown down analytically. The result is a normal surface singularity $(X, 0)$ whose abstract boundary (or link) is diffeomorphic to $PSL(2, \mathbb{R})/G$ and which has a resolution with dual graph a star with a center representing a curve E_0 of genus g and weight $e_0 = 2g - 2 + n$; it has n branches of length 1, each with an end-vertex i that represents a curve of genus 0 and weight $e_i = \alpha_i$. The α_i can take any values ≥ 2. The anti-canonical cycle is $Z_K = 2E_0 + \sum_{i=1}^{n} E_i$. Thus, given such a graph, equipped with the corresponding genera, there are infinitely many choices of weights e_i for the vertices of valency 1 which make it correspond to $\mathbb{Z}$-Gorenstein singularities (cf. [5], [14]).

iv) Consider as a final example the dual graph of a cusp singularity (see [3] or [12]). This is a cycle of finite length; all vertices represent smooth rational curves $E_1, ..., E_n$ with $e_i \geq 2$ for all $i \in V(\Gamma)$ and at least one vertex i satisfies $e_i \geq 3$. The anti-canonical cycle is $Z_K = E_1 + \cdots + E_n$, regardless of the weights $\underline{e}$, which shows that such singularities are $\mathbb{Z}$-Gorenstein. We see that all the choices of weights $\underline{e}$ satisfying the previous inequality are good, for every choice of genera. Therefore there exists an infinite number of possibilities for each weight e_i.

4. The main theorem

In this section we give a structure theorem about the set of solutions of systems of the form (2.7), where we drop the conditions $q_i \geq -2$, which are necessarily satisfied (see (2.6)) if the system corresponds to potential surface singularities.

If i, j are distinct vertices of Γ, we denote by $i \leftrightarrow j$ the fact that they are adjacent, that is, connected by at least one edge.

Theorem 4.1. *Consider a graph Γ decorated with weights $\underline{q} \in \mathbb{Z}^{V(\Gamma)}$, and the system of equations in the unknowns $(\underline{n}, \underline{e}) \in (\mathbb{N})^{V(\Gamma)} \times (\mathbb{N}^*)^{V(\Gamma)}$:*

$$\{e_i n_i = q_i + \sum_{\substack{j \in V(\Gamma) \\ j \leftrightarrow i}} e_{ij} n_j\}_{i \in V(\Gamma)}. \tag{4.1}$$

Then there exist at most finitely many weights $\underline{n}$ which can be extended to solutions $(\underline{n}, \underline{e})$ of the previous system, such that the quadratic form $Q_{(\Gamma, \underline{e})}$ is negative definite.

Proof. By working on examples, we noticed that we got contradictions if we searched for solutions of the system (4.1) by traveling continuously on the graph Γ, starting from a value n_i which was too big. As we were unable to find a precise description of what "too big" meant, we had the idea to search a contradiction starting from a sequence of solutions with unbounded values of $\underline{n}$. This idea worked, as we explain now.

Suppose that there exists a sequence of solutions $((\underline{n}^{(k)}, \underline{e}^{(k)}) \in (\mathbb{N})^{V(\Gamma)} \times (\mathbb{N}^*)^{V(\Gamma)})_{k \geq 1}$ such that:

$$N^{(k)} := \max_{i \in V(\Gamma)} \{n_i^{(k)}\}_{k \to \infty} \to +\infty.$$

Selecting subsequences if necessary, we may assume that:

i) there exists $i_o \in V(\Gamma)$ such that $n_{i_o}^{(k)} = N^{(k)}$, for all $k \geq 1$;

ii) for all $i \in V(\Gamma)$, there exists $\lim\limits_{k \to \infty} \dfrac{n_i^{(k)}}{n_{i_o}^{(k)}} =: \nu_i \in [0, 1]$.

Set:

$$P := \{i \in V(\Gamma) | \nu_i > 0\};$$

then $P \neq \emptyset$ (indeed, $i_o \in P$, as $\nu_{i_o} = 1$ by i)).

Let Γ_P be the subgraph of Γ spanned by P (that is, the subgraph of Γ whose set of vertices is P and whose edges are all the edges of Γ which connect two elements of P). It may not be connected. Denote by $\Gamma_{(P, i_o)}$ the connected component of Γ_P which contains the vertex i_o.

For all $i \in V(\Gamma_{(P, i_o)})$, one has by (4.1):

$$e_i^{(k)} \frac{n_i^{(k)}}{n_{i_o}^{(k)}} = \frac{q_i}{n_{i_o}^{(k)}} + \sum_{\substack{j \in V(\Gamma) \\ j \leftrightarrow i}} e_{ij} \frac{n_j^{(k)}}{n_{i_o}^{(k)}}.$$

By assumption ii) and the construction of Γ_P, the following limits exist:

$$\lim_{k\to\infty} \frac{n_i^{(k)}}{n_{i_o}^{(k)}} = \nu_i > 0\,,$$

for all $i \in V(\Gamma_{(P,i_o)})$, and:

$$\lim_{k\to\infty} \frac{n_j^{(k)}}{n_{i_o}^{(k)}} = 0\,,$$

for all the vertices $j \in V(\Gamma) \setminus V(\Gamma_{(P,i_o)})$ which are connected to the vertex i. Moreover, as $\lim\limits_{k\to\infty} N^{(k)} = \infty$ and $N^{(k)} = n_{i_o}^{(k)}$ for all $k \geq 1$ by assumption i), we see that $\lim\limits_{k\to\infty} \dfrac{q_i}{n_{i_o}^{(k)}} = 0$.

Thus one gets that $\lim\limits_{k\to\infty} e_i^{(k)}$ exists and:

$$\epsilon_i := \lim_{k\to\infty} e_i^{(k)} = \frac{1}{\nu_i} \sum_{\substack{j\in V(\Gamma_{(P,i_o)})\\ j\leftrightarrow i}} e_{ij}\nu_j < +\infty. \qquad (4.2)$$

As all the numbers $e_i^{(k)}$ are integers, this shows that there exists k_o such that for all $k \geq k_o$ and for all $i \in V(\Gamma_{P,i_o})$ one has: $e_i^{(k)} = \epsilon_i$. Therefore, by equation (4.2):

$$e_i^{(k_o)}\,\nu_i = \sum_{\substack{j\in V(\Gamma_{(P,i_o)})\\ j\leftrightarrow i}} e_{ij}\nu_j\,,\ \text{ for all } i \in V(\Gamma_{(P,i_o)}). \qquad (4.3)$$

Define $\underline{\mu} \in \mathbb{Z}^{V(\Gamma)}$ by $\mu_i := \nu_i$ for all $i \in V(\Gamma_{P,i_o})$ and $\mu_i := 0$ otherwise. Therefore $\mu_{i_o} = 1$, which shows that $\underline{\mu} \neq 0$. The equalities (4.3) imply:

$$Q_{(\Gamma,\underline{e}^{(k_o)})}(\underline{\mu}) = 0$$

which contradicts the fact that $Q_{(\Gamma,\underline{e}^{k_0})}$ is negative definite. □

As an immediate consequence of Theorem 4.1 we get:

Corollary 4.1. *If a vertex $i \in V(\Gamma)$ satisfies that $n_i \neq 0$ for every solution of (4.1), then e_i takes only a finite number of values. Therefore, the system (4.1) has a finite number of solutions $(\underline{n}, \underline{e}) \in (\mathbb{N}^*)^{V(\Gamma)} \times (\mathbb{N}^*)^{V(\Gamma)}$. Thus, if all vertices satisfy that $n_i \neq 0$ for every solution of (4.1), then there are a finite number of possible weights (self-intersections) for the vertices making the graph $\mathbb{Z}$-Gorenstein.*

Proof. If $n_i \neq 0$ for all $i \in V(\Gamma)$, we get from the equations (4.1) that:

$$e_i = \frac{1}{n_i}(q_i + \sum_{\substack{j\in V(\Gamma)\\ j\leftrightarrow i}} e_{ij}n_j).$$

As by Theorem 4.1 there are only finitely many possibilities for $\underline{n}$, the conclusion follows. □

5. The possible non-finiteness of self-intersections

In this section we impose again the restrictions $q_i \geq -2$, satisfied when the system (4.1) corresponds to potential normal surface singularities (see the relations (2.6)). We want to describe to what extent not only the weights $\underline{n}$ can take a finite number of values (which is ensured by Theorem 4.1), but also $\underline{e}$.

In view of Corollary 4.1, let us concentrate our attention on the vertices i_0 and on the solutions $(\underline{n}, \underline{e})$ of (4.1) such that $n_{i_0} = 0$.

Proposition 5.1. *If there exists a vertex i_0 such that $n_{i_0} = 0$ for some solution $(\underline{n}, \underline{e})$ of the system (4.1), then $v_{i_0} \leq 2$. Moreover:*

a) *If $v_{i_o} = 2$, then one has a cusp graph and $n_i = 0$ for all $i \in V(\Gamma)$. Therefore $Z_K = \sum E_i$, $e_j \geq 2$ for all vertices j and $e_j \geq 3$ for at least one vertex (in order to ensure that the graph is negative definite).*

b) *If $v_{i_o} = 1$, then $p_{i_0} = 0$ and the unique neighbor i_1 of i_0 satisfies $n_{i_1} = 1$. Therefore, E_i is a smooth rational curve, $z_{i_0} = 1$ and $z_{i_1} = 2$.*

c) *If $v_{i_0} = 0$ (therefore Γ has only one vertex and no edges), then $p_{i_0} = 1$ for every choice of weight e_{i_0}, we get a $\mathbb{Z}$-Gorenstein graph and $Z_K = E_{i_0}$.*

Proof. From (4.1) and (2.6), we get:

$$0 = v_{i_0} + 2p_{i_0} - 2 + \sum_{\substack{j \in V(\Gamma) \\ j \leftrightarrow i}} e_{i_0 j} n_j. \tag{5.1}$$

We conclude using the fact that all the parameters appearing in the equation are non-negative integers. □

Conversely, as shown by the examples iv) of Section 3, for all cusp-graph Γ one has an infinite number of possibilities for the weight e_i, and this for each vertex i. More precisely, the set of possibilities for $\underline{e}$ is $(\mathbb{N}^* \setminus \{1\})^{V(\Gamma)} \setminus (\{2\})^{V(\Gamma)}$. The case b) of the previous lemma is realized for example in the family iii) of Section 3. The case c) corresponds, for instance, to the Brieskorn singularities defined by the equation $z_1^a + z_2^b + z_3^c = 0$ with $1/a + 1/b + 1/c = 1$; these have a minimal resolution which is a line bundle over a non-singular elliptic curve.

The following is an immediate consequence of Proposition 5.1.

Proposition 5.2. *Suppose that $(\Gamma, \underline{p})$ is not a cusp graph. We fix a vertex i of Γ and we look at the possible values of e_i, when one varies $(\underline{n}, \underline{e})$ among the solutions of (4.1). Then:*

- *if $v_i > 1$, the number e_i takes only a finite number of values;*
- *if $v_i = 1$, the number e_i takes an infinite number of values if and only if there exists a solution with $n_i = 0$. In this case, one obtains new solutions by varying only e_i in $\mathbb{N}^* \setminus \{1\}$ and fixing all the other values of $(\underline{n}, \underline{e})$.*

As a conclusion, we ask some questions that arise naturally from our work:

- Given $(\Gamma, \underline{p})$, is the set of weights $\underline{e}$ making $(\Gamma, \underline{p}, \underline{e})$ minimal and $\mathbb{Z}$-Gorenstein always non-empty ?

• Given $(\Gamma, \underline{p})$, can one give explicit upper bounds on the values of the functions $\underline{n}$ which can be extended to a solution of (4.1) ?

• Given $(\Gamma, \underline{p})$, can one give an algorithm to compute the values of the functions $\underline{n}$ which can be extended to a solution of (4.1) ?

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