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Two-dimensional iterated torus knots and quasi-ordinary surface singularities

Nœuds toriques itérés bidimensionnels et singularités quasi-ordinaires des surfaces

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Abstract

We define a notion of 2-dimensional iterated torus knot, namely special embeddings of a 2-torus in the Cartesian product of a 2-torus and a 2-disc. We apply this definition to give a description of the embedded topology of the boundary of an irreducible quasi-ordinary hypersurface germ of dimension 2, in terms of the characteristic exponents of an arbitrary quasi-ordinary projection. Incidentally, we give an algorithm for computing the Jung–Hirzebruch type of its normalization. ***To cite this article: P. Popescu-Pampu, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Nous définissons une notion de nœud torique itéré bidimensionnel, à savoir des plongements particuliers d'un 2-tore dans le produit cartésien d'un 2-tore et d'un 2-disque. Nous appliquons cette définition à la description de la topologie plongée du bord d'un germe quasi-ordinaire irréductible d'hypersurface de dimension 2, en fonction des exposants caractéristiques d'une projection quasi-ordinaire arbitraire. Accessoirement, nous donnons un algorithme de calcul du type de Jung–Hirzebruch de sa normalisation. ***Pour citer cet article : P. Popescu-Pampu, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

Depuis les travaux de Brauner et Kähler, on sait décrire les bords plongés des germes irréductibles de courbes analytiques complexes planes en tant que nœuds toriques itérés. Ici nous proposons une description analogue pour les germes irréductibles quasi-ordinaires $\mathcal{S}$ de surfaces dans $\mathbf{C}^3$.

Dans des coordonnées $X_1,\ X_2,\ Y$ adéquates, la projection p_X sur le plan des coordonnées X_1, X_2, une fois restreinte à $\mathcal{S}$, est finie et non-ramifiée au-dessus de $(\mathbf{C}^*)^2$. Nous prenons comme représentants du bord plongé

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de $\mathcal{S}$, des couples $(\partial(\mathbf{D}_1^2 \times \mathbf{D}_2^2 \times \mathbf{D}_3^2),\ \mathcal{S} \cap \partial(\mathbf{D}_1^2 \times \mathbf{D}_2^2 \times \mathbf{D}_3^2))$ pour des choix adaptés de polycylindres dans les coordonnées $X_1,\ X_2,\ Y$. Notre observation essentielle est que, pour comprendre la topologie de ces couples, il suffit de comprendre celle de leur restriction au-dessus des tores bidimensionnels $\partial\mathbf{D}_1^2 \times \partial\mathbf{D}_2^2$. Comme p_X restreint à $\mathcal{S}$ est non-ramifié au-dessus de ces tores, on voit que $\mathcal{S} \cap (\partial\mathbf{D}_1^2 \times \partial\mathbf{D}_2^2 \times \mathbf{D}_3^2)$ est ce que l'on appelle un *nœud tressé* bidimensionnel (Définition 2.2) dans $\partial\mathbf{D}_1^2 \times \partial\mathbf{D}_2^2 \times \mathbf{D}_3^2$. De plus, il est isotope à un *nœud torique itéré* bidimensionnel (Définition 2.3).

En utilisant des constructions topologiques introduites dans la Section 3, nous expliquons dans la Section 7 comment reconstruire le bord plongé de $\mathcal{S}$ à partir du nœud torique itéré précédent, dans le cas où toutes les composantes des exposants caractéristiques de $\mathcal{S}$ sont non-nulles. Nous indiquons finalement les modifications à faire lorsque certaines de ces composantes sont nulles.

1. Motivations

If $(\mathcal{S}, 0)$ is a germ of reduced complex analytic space, the pairs $(\Sigma, \mathcal{S} \cap \Sigma)$ are all homeomorphic, where Σ are the boundaries of sufficiently small Euclidean balls or polycylinders defined using an arbitrary system of local parameters (Durfee's proof [3] in the algebraic case extends to analytic varieties). We call their homeomorphism type the *embedded boundary* of $\mathcal{S}$ and the homeomorphism type of $\mathcal{S} \cap \Sigma$ the *abstract boundary* of $\mathcal{S}$.

At least since the 1920s, with the works of Brauner and Kähler, one knows how to construct algorithmically the embedded boundary of a germ of irreducible complex analytic plane curve as an *iterated torus knot*, the initial data being the characteristic exponents of the germ (see [7] and its references). For higher-dimensional irreducible hypersurface germs, there is, in general, no analog of the characteristic exponents. Nevertheless, there is a special class of germs, the *quasi-ordinary* ones, which admit such a generalization (see Section 4).

An important work of Gau–Lipman (see [5,9]) shows that the so-called *normalized* characteristic exponents of irreducible quasi-ordinary hypersurface germs are equivalent information with the embedded boundary. However, their results are not explicit. In [9], p. 3, Lipman remarks that no description of the embedded boundary in terms of an analog of the iterated torus knots is known in higher dimensions.

Here we give such a description for the case of *irreducible quasi-ordinary surfaces*. We leave to a subsequent paper the generalization to arbitrary dimensions and to possibly reducible quasi-ordinary hypersurface germs, in terms of the Eggers–Wall tree defined in [12]. Such a generalization should allow then to describe à la Jung the embedded boundary of an arbitrary hypersurface germ using embedded plumbing operations.

By another method, Costa [2] had obtained a description of the abstract boundary of an irreducible quasi-ordinary surface germ, by describing which identifications must be done on the lens space which is the abstract boundary of the normalization.

2. Some classes of two-dimensional knots

We denote by $\mathbf{D}^k := \{x \in \mathbf{R}^k,\ \|x\| \leqslant 1\}$ closed balls, by $\mathbf{S}^k := \partial\mathbf{D}^{k+1}$ spheres and by $\mathbf{T}^k := (\mathbf{S}^1)^k$ tori, all of dimension $k \geqslant 1$. We use indices in order to distinguish various of them, with the convention $\mathbf{S}_i^1 = \partial\mathbf{D}_i^2$. If C is a closed curve in a torus $\mathbf{T}^k$, we denote by $[C]$ its image in $\pi_1(\mathbf{T}^k)$. As this group is Abelian, we do not need to specify the base-point. We use the same notation $[M]$ to denote the fundamental class in $H_d(M, \mathbf{Z})$ of an oriented d-dimensional manifold M. We denote by $\{*\}$ an arbitrary point of a given set.

The following lemma gives a normal form for sublattices of finite index of a 2-lattice endowed with a fixed basis.

Lemma 2.1. *Let $\mathcal{W}$ be a rank 2-lattice endowed with a basis (V_1, V_2). If $\overline{\mathcal{W}}$ is a sublattice of finite index, then there exists a unique triple $(R, S, T) \in \mathbf{N}^* \times \mathbf{N} \times \mathbf{N}^*$ verifying $S < R$, such that $(\overline{V}_1 := RV_1, \overline{V}_2 := SV_1 + TV_2)$ is a basis of $\overline{\mathcal{W}}$.*

We say that $(\overline{V}_1, \overline{V}_2)$ and (R, S, T) are *the basis*, respectively *the triple associated to $\overline{\mathcal{W}}$ with respect to the* (*ordered*) *basis* (V_1, V_2) of $\mathcal{W}$.

Consider a torus $\mathbf{T}_0^2 = \mathbf{S}_1^1 \times \mathbf{S}_2^1$. Let $p : \mathbf{T}_0^2 \times \mathbf{D}^2 \to \mathbf{T}_0^2$ and $p_i : \mathbf{T}_0^2 \to \mathbf{S}_i^1$, for $i \in \{1, 2\}$, be the canonical projections. Let $e : \mathbf{T}^2 \to \mathbf{T}_0^2 \times \mathbf{D}^2$ be an embedding.

Definition 2.2. We say that e is a *braided embedding* and $e(\mathbf{T}^2)$ is a *braided knot* if $e(\mathbf{T}^2)$ is transversal to the fibers of p.

In what follows *we will only consider braided embeddings* e. Then $(p \circ e)_*(\pi_1(\mathbf{T}^2))$ is a sublattice of finite index of $\pi_1(\mathbf{T}_0^2)$. We suppose that we have trivialized $\mathbf{T}^2 \simeq \mathbf{S}^1 \times \mathbf{S}^1$ such that $((p \circ e)_*[\mathbf{S}^1 \times \{*\}], (p \circ e)_*[\{*\} \times \mathbf{S}^1])$ is the basis associated to $(p \circ e)_*(\pi_1(\mathbf{T}^2))$ with respect to the basis $([\mathbf{S}_1^1 \times \{*\}], [\{*\} \times \mathbf{S}_2^1])$ of $\pi_1(\mathbf{T}_0^2)$. One should be careful with the choices of orientations.

Let $N(e)$ be a closed tubular neighborhood of $e(\mathbf{T}^2)$. Its boundary $\partial N(e)$ is a 3-torus. We can choose it to be also transverse to the fibers of p. Write $\partial N(e) \simeq \mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$ in such a way that $\{*\} \times \{*\} \times \mathbf{S}^1$ is contractible in $N(e)$ and the 2-tori $\mathbf{S}^1 \times \mathbf{S}^1 \times \{*\}$ are *translations of* $e(\mathbf{T}^2)$ *in a fixed direction of* $\mathbf{D}^2$. Consider a braided embedding $j : \mathbf{T}^2 \to \mathbf{T}_0^2 \times \mathbf{D}^2$ whose image is contained in $\partial N(e)$. Then, we can express: $j_*[\mathbf{T}^2] = \alpha^1[\{*\} \times \mathbf{S}^1 \times \mathbf{S}^1] + \alpha^2[\mathbf{S}^1 \times \{*\} \times \mathbf{S}^1] + \alpha^3[\mathbf{S}^1 \times \mathbf{S}^1 \times \{*\}] \in H_2(\partial N(e), \mathbf{Z})$, $(\alpha^1, \alpha^2, \alpha^3) \in \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}^*$, $\gcd(\alpha^1, \alpha^2, \alpha^3) = 1$. Up to isotopy in $\partial N(e)$, the torus $j(\mathbf{T}^2)$ is uniquely determined by the triple $(\alpha^1, \alpha^2, \alpha^3)$.

Definition 2.3. We say that $j(\mathbf{T}^2)$ is a *cable knot of type* $(\alpha^1, \alpha^2, \alpha^3)$ *around* $e(\mathbf{T}^2)$.

Consider now $g \in \mathbf{N}^*$ and a sequence of triples $\alpha_k := (\alpha_k^1, \alpha_k^2, \alpha_k^3) \in \mathbf{Z}^2 \times \mathbf{Z}^*$, $\gcd(\alpha_k^1, \alpha_k^2, \alpha_k^3) = 1$, for $k \in \{1, \ldots, g\}$. Consider a sequence of embeddings $j_k : \mathbf{T}^2 \to \mathbf{T}_0^2 \times \mathbf{D}^2$, $(k \in \{0, \ldots, g\})$, such that $j_0(\mathbf{T}^2) = \mathbf{T}_0^2 \times \{0\}$ and $j_k(\mathbf{T}^2)$ is a cable knot of type α_k around $j_{k-1}(\mathbf{T}^2)$, for all $k \in \{1, \ldots, g\}$.

Definition 2.4. We say that $j_g(\mathbf{T}^2)$ is an *iterated torus knot of type* $(\alpha_1, \ldots, \alpha_g)$ inside $\mathbf{T}_0^2 \times \mathbf{D}^2$.

3. Constructions with mapping cylinders

The following topological constructions will be used in Section 7.

Let $p : A \to S$ be a continuous morphism between topological spaces. The *mapping cylinder of* p, denoted $C(p)$, is by definition the quotient of the disjoint union $(A \times [0, 1]) \bigsqcup S$ by the equivalence relation $(x, 1) \sim p(x)$, $\forall x \in A$. One has a canonical embedding $A \to C(p)$ obtained by passing to the quotient the map $x \in A \to (x, 0) \in A \times [0, 1]$. This construction extends to a covariant functor from the category of topological spaces over S to the category of abstract topological spaces. So, we denote by $c(\phi) : C(p_1) \to C(p_2)$ the *cylinder-morphism* associated to the commutative diagram: $\phi : (A_1 \xrightarrow{p_1} S) \to (A_2 \xrightarrow{p_2} S)$.

Suppose now we have a sequence $A \xrightarrow{e} S \times \mathbf{D}^k \xrightarrow{p} S \xrightarrow{p'} S'$ of continuous morphisms between topological spaces, where p is the projection on the first factor. We obtain the following commutative diagram, in which the vertical arrows are the canonical embeddings defined before:

$$\begin{array}{ccccc} A & \xrightarrow{e} & S \times \mathbf{D}^k & & \\ \downarrow & & \downarrow & & \\ C(p' \circ p \circ e) & \xrightarrow{c(e)} & C(p' \circ p) & \xrightarrow{d_k(p')} & C(p') \times \mathbf{D}^k \end{array} \tag{1}$$

The map $d_k(p')$ is got by retracting by deformation $\mathbf{D}^k$ to 0. More precisely, $d_k(p')$ is the quotient by the same equivalence relation as before, of the mapping $(S \times \mathbf{D}^k \times [0, 1]) \bigsqcup S' \to ((S \times [0, 1]) \bigsqcup S') \times \mathbf{D}^k$, $(x, y, t) \sqcup x' \to (x, t, (1 - t)y) \sqcup (x', 0)$.

4. Irreducible quasi-ordinary germs of surfaces

In this section we introduce only very particular quasi-ordinary objects. For the general definitions, one should consult [8,9,6,12].

Denote $X := (X_1, X_2)$ and $X^m := X_1^{m_1} X_2^{m_2}$ if $m = (m_1, m_2) \in \mathbf{Q}^2$. Let $f \in \mathbf{C}\{X\}[Y]$ be a unitary *irreducible* polynomial of degree $N \in \mathbf{N}^*$ and let $\mathcal{S} \hookrightarrow \mathbf{C}^3$ be the germ of surface defined by $\{f = 0\}$. Denote by $p_X : \mathbf{C}^3 \to \mathbf{C}^2$ the projection of the space of coordinates X, Y onto the plane of coordinates X. We will keep the same notation when restricting it to subsets of $\mathbf{C}^3$. Denote $\mathcal{D} := \{X_1 X_2 = 0\} \subset \mathbf{C}^2$ and $\mathcal{S}^* := \mathcal{S} \setminus (p_X)^{-1}(\mathcal{D})$. We will use the same notations for germs and for sufficiently small representatives of them.

If the discriminant $\Delta_Y(f)$ is non-zero on $(\mathbf{C}^*)^2$, which is equivalent to the fact that $p_X : \mathcal{S}^* \to (\mathbf{C}^*)^2$ is an unramified covering, we say that f and $\mathcal{S}$ are *quasi-ordinary*. If this is the case, by the theorem of Jung–Abhyankar (see [9]), which generalizes the theorem of Newton–Puiseux for plane branches, the set $R(f)$ of roots of f embeds canonically in the algebra $\mathbf{C}\{X^{(1/N,1/N)}\}$. If $\eta \in \mathbf{C}\{X^{(1/N,1/N)}\}$ can be written $\eta = X^m u(X)$, with $m \in \mathbf{Q}_+^2$ and $u \in \mathbf{C}\{X^{(1/N,1/N)}\}$, $u(0,0) \neq 0$, we say that m is the *dominating exponent* of η.

From now on, f *is a fixed irreducible quasi-ordinary polynomial* and $R(f)$ is seen as a subset of $\mathbf{C}\{X^{(1/N,1/N)}\}$. All the differences of roots of f have dominating exponents, which are totally ordered for the componentwise order (see [9]). We call them the *characteristic exponents* of f and we denote them by $A_1 < \cdots < A_G$, $A_k = (A_k^1, A_k^2)$, $\forall k \in \{1, \ldots, G\}$. Possibly after permuting the variables X_1, X_2, we can suppose that $(A_1^1, \ldots, A_G^1) \geqslant_{\text{lex}} (A_1^2, \ldots, A_G^2)$. This implies that $A_1^1 > 0$, a remark used in Section 7.

Let $M_0 := \mathbf{Z}^2$ be the lattice generated by the exponents of the monomials contained in $\mathbf{C}\{X\}$. Following Lipman [9], define inductively, for $k \in \{1, \ldots, G\}$, extensions of M_0 by the relations $M_k := M_{k-1} + \mathbf{Z}A_k$ and the successive indices $N_k := \text{card}(M_k/M_{k-1}) > 1$. Following González Pérez [6], define the dual lattices $W_k := \text{Hom}(M_k, \mathbf{Z})$, for $k \in \{0, \ldots, G\}$.

5. The link between the normalization and the description of the embedded boundary

In what follows, we will use some elementary facts of toric geometry. For the definitions and proofs, see Oda [10] and Fulton [4]. If $\mathcal{W}$ is a lattice and σ is a strictly convex finite rational polyhedral cone in $\mathcal{W}_{\mathbf{R}} := \mathcal{W} \otimes \mathbf{R}$, we denote by $\mathcal{Z}(\mathcal{W}, \sigma)$ the affine normal toric variety they determine.

We consider now $\mathbf{C}^2$ as the toric surface $\mathcal{Z}(W_0, \sigma_0)$, where σ_0 is the canonical regular cone in $(W_0)_{\mathbf{R}}$. One has a canonical identification $W_0 = \pi_1((\mathbf{C}^*)^2)$. Then the topology of the covering $p_X : \mathcal{S}^* \to (\mathbf{C}^*)^2$ is determined by the finite index sublattice $(p_X)_*(\pi_1(\mathcal{S}^*))$ of W_0. Using the isomorphism between arithmetical and topological Galois groups, in [12] we proved:

Proposition 5.1. *One has the equality*: $(p_X)_*(\pi_1(\mathcal{S}^*)) = W_G$.

Let $\gamma_G : \mathcal{Z}(W_G, \sigma_0) \to \mathcal{Z}(W_0, \sigma_0)$ be the toric map obtained by passing from the lattice W_0 to W_G. In [12] we deduced from Proposition 5.1 the following description of a normalization morphism, which was also obtained algebraically by González Pérez [6]:

Proposition 5.2. *There is a normalization morphism* $\nu : (\mathcal{Z}(W_G, \sigma_0), 0) \to (\mathcal{S}, 0)$ *such that* $\gamma_G = p_X \circ \nu$.

For $k \in \{1, \ldots, G\}$, define $\xi_k := X^{A_1} + \cdots + X^{A_k} \in \mathbf{C}\{X^{(1/N,1/N)}\}$, let $f_k \in \mathbf{C}\{X\}[Y]$ be the minimal polynomial of ξ_k (which is again quasi-ordinary) and denote $\mathcal{S}_k := \{f_k = 0\} \subset \mathbf{C}^3$. Denote by $\nu_k : (\mathcal{Z}(W_k, \sigma_0), 0) \to (\mathcal{S}_k, 0)$ the normalization morphisms given by Proposition 5.2.

Consider $(r_1, r_2, r_3) \in (\mathbf{R}^*)^3$. We look at the intersection of $\mathcal{S}$ with the boundary of the polycylinder $\mathbf{D}_1^2(r_1) \times \mathbf{D}_2^2(r_2) \times \mathbf{D}_3^2(r_3)$. By choosing $r_1 \ll r_3$, $r_2 \ll r_3$ sufficiently small (which we will suppose from now on, without specifying the radii anymore), we get representatives of the embedded boundary of $\mathcal{S}$ such that

$\mathcal{S} \cap (\mathbf{D}_1^2 \times \mathbf{D}_2^2 \times \mathbf{S}_3^1) = \emptyset$. Using the equisingularity results of Ban [1] or Oh [11], *the embedded boundary of $\mathcal{S}$ is homeomorphic to the embedded boundary of $\mathcal{S}_G$.*

Our main observation, detailed in Section 7 is that, *in order to describe these embedded boundaries, it is enough to understand their intersection with the preimage by p_X of the 2-torus $\mathbf{S}_1^1 \times \mathbf{S}_2^1$*. Looking at the effect of adding one monomial X^{A_k} after the other, we see that the intersections $\mathcal{T}_k := \mathcal{S}_k \cap (\mathbf{S}_1^1 \times \mathbf{S}_2^1 \times \mathbf{D}_3^2)$ can be described as iterated 2-torus knots with k steps of iteration. The coverings $p_X : \mathcal{S}_k^* \to (\mathbf{C}^*)^2$ retract by deformation on $p_X : \mathcal{T}_k \to \mathbf{S}_1^1 \times \mathbf{S}_2^1$. So, the sublattices $(p_X)_*(\pi_1(\mathcal{T}_k))$ of $\pi_1((\mathbf{C}^*)^2) = W_0$, which allow to describe the types of the iterated torus knots (see Section 2), coincide with $(p_X)_*(\pi_1(\mathcal{S}_k^*)) = W_k$ which, by Theorem 5.2, allow to describe the normalization morphism ν_k. In the following section, we give an algorithm allowing to compute the sublattices W_k of W_0.

6. The algorithm of normalization

Let $\mathcal{W}$ be a lattice of rank 2 and $\sigma \subset \mathcal{W}_{\mathbf{R}}$ a strictly convex finite rational cone of dimension 2. If $T_1, T_2 \in \mathcal{W}$ are the primitive generators of the edges of σ (unique up to permutation), then there exists a unique basis (E, F) of $\mathcal{W}$ such that: $T_1 = E,\ T_2 = aF - bE$ with $0 \leqslant b < a,\ \gcd(a, b) = 1$. Denote $\mathcal{Z}(a, b) := \mathcal{Z}(\mathcal{W}, \sigma)$. Denote by 0 the closed orbit of $\mathcal{Z}(a, b)$. The analytic germ $(\mathcal{Z}(a, b), 0)$ is called *the Jung–Hirzebruch singularity of type $\mathcal{A}_{a,b}$*. If $\sigma \subset \mathcal{W},\ \sigma' \subset \mathcal{W}'$ are cones which determine the pairs of integers $(a, b),\ (a', b')$ as explained before, then ($\mathcal{Z}(a, b)$ and $\mathcal{Z}(a', b')$ are isomorphic as toric surfaces) $\Leftrightarrow$ ($\mathcal{A}_{a,b}$ and $\mathcal{A}_{a',b'}$ are isomorphic as analytic singularities) $\Leftrightarrow$ ($a = a'$ and ($b = b'$ or $bb' \equiv 1 \bmod a$)). For the proofs, see [10,4].

Let (U_1, U_2) be the canonical basis of M_0, with $X_i = \chi^{U_i}$, and let $(V_1, V_2) \in W_0^2$ be its dual basis. Denote by (V_1^k, V_2^k) and (R_k, S_k, T_k) the basis and respectively the triple associated to W_k with respect to the basis (V_1, V_2) of W_0, as in Lemma 2.1. In the following proposition, proved in [12], we compute recursively the triples (R_k, S_k, T_k) and the indices N_k for $k \in \{1, \dots, G\}$, in terms of the characteristic exponents $A_1, \dots, A_G$. We deduce the Jung–Hirzebruch type of the normalization of $\mathcal{S}$ (and so the type of the lens space which is its abstract boundary), bringing more precision to the results of Costa [2].

Proposition 6.1. *Define $(R_0, S_0, T_0) := (1, 0, 1)$. If $k \in \{1, \dots, G\}$ and $(R_{k-1}, S_{k-1}, T_{k-1})$ is known, define the numbers $P_k^i, Q_k^i, Q_k, D_k, J_k^i, K_k^1 \in \mathbf{N},\ i \in \{1, 2\}, \gcd(P_k^1, Q_k^1) = 1, \gcd(P_k^2, Q_k^2) = 1$, as follows: $\frac{P_k^1}{Q_k^1} := R_{k-1} A_k^1$, $\frac{P_k^2}{Q_k^2} := S_{k-1} A_k^1 + T_{k-1} A_k^2,\ \ Q_k := \operatorname{lcm}(Q_k^1, Q_k^2),\ \ D_k := \gcd(Q_k^1, Q_k^2),\ \ J_k^i := \frac{Q_k^i}{D_k}$, for $i \in \{1, 2\},\ \ K_k^1 \in \{0, \dots, Q_k^1 - 1\}$ $(K_k^1 = 0,$ if $Q_k^1 = 1)$ and $(K_k^1 = -J_k^1 P_k^2 (P_k^1)^{-1}$ in $\mathbf{Z}/Q_k^1\mathbf{Z}$, if $Q_k^1 \neq 1)$. Then:*

$$\begin{cases} R_k = Q_k^1 R_{k-1}, \\ S_k \in \{0, \dots, R_k - 1\}, \quad S_k \equiv (K_k^1 R_{k-1} + J_k^2 S_{k-1}) \bmod R_k, \\ T_k = J_k^2 T_{k-1}. \end{cases}$$

Moreover $N_k = Q_k$. If we denote: $T'_G := \gcd(R_G, S_G),\ \ R'_G := R_G / T'_G,\ \ S'_G := S_G / T'_G$, then the normalization of $\mathcal{S}$ has a Jung–Hirzebruch singularity of type $\mathcal{A}_{R'_G, S'_G}$.

7. The embedded boundary

In order to compute the type of the iterated 2-torus knot $\mathcal{T}_G$, one proves first the following lemma, by working in the intersection ring $H_*(\mathbf{S}_1^1 \times \mathbf{S}_2^1 \times \mathbf{S}_3^1, \mathbf{Z})$:

Lemma 7.1. *The embedding $\mathcal{T}_1 \subset \mathbf{S}_1^1 \times \mathbf{S}_2^1 \times \mathbf{D}_3^2$ is a cable knot of type $(P_1^1 J_1^2, P_1^2 J_1^1, Q_1)$ around $\mathbf{S}_1^1 \times \mathbf{S}_2^1 \times \{0\}$.*

For $k \in \{2, \dots, G\}$, denote by $(U_1^{k-1}, U_2^{k-1}) \in (M_{k-1})^2$ the dual basis of (V_1^{k-1}, V_2^{k-1}) and define $X_{i,k-1} := \chi^{U_i^{k-1}}$. Then, $X_1 = X_{1,k-1}^{R_{k-1}} X_{2,k-1}^{S_{k-1}}$ and $X_2 = X_{2,k-1}^{T_{k-1}}$, which implies that $X^{A_k} = X_{1,k-1}^{R_{k-1}A_k^1} X_{2,k-1}^{S_{k-1}A_k^1 + T_{k-1}A_k^2}$. One should notice that these exponents already appeared in Proposition 6.1.

Remark that, when $(X_{1,k-1}, X_{2,k-1})$ vary on small 2-tori $\mathbf{S}_1^1(\rho_1) \times \mathbf{S}_2^1(\rho_2)$, the images by $(p_X \circ \nu_k)_*$ of $[\mathbf{S}_1^1(\rho_1) \times \{*\}]$ and $[\{*\} \times \mathbf{S}_2^1(\rho_2)]$ define the basis associated to $W_{k-1} = (p_X)_*(\pi_1(\mathcal{T}_{k-1}))$ with respect to (V_1, V_2). So, $\mathcal{T}_k$ is a cable knot around $\mathcal{T}_{k-1}$ having the same type as the knot $\mathcal{T}_1$ associated to a quasi-ordinary singularity having as first characteristic exponent $(R_{k-1}A_k^1, S_{k-1}A_k^1 + T_{k-1}A_k^2)$. Using Lemma 7.1 and Proposition 6.1, we get:

Proposition 7.2. *The pair* $(\mathbf{S}_1^1 \times \mathbf{S}_2^1 \times \mathbf{D}_3^2, \mathcal{S}_G \cap (\mathbf{S}_1^1 \times \mathbf{S}_1^1 \times \mathbf{D}_3^2))$ *is diffeomorphic to the pair* $(\mathbf{T}_0^2 \times \mathbf{D}^2, j_G(\mathbf{T}^2))$, *where* $j_G(\mathbf{T}^2)$ *is an iterated knot of type* $(\alpha_1, \dots, \alpha_G)$, *where* $\alpha_k = (P_k^1 J_k^2, P_k^2 J_k^1, Q_k)$.

In order to describe the embedding $\mathcal{S}_G \cap \partial(\mathbf{D}_1^2 \times \mathbf{D}_2^2 \times \mathbf{D}_3^2) \subset \partial(\mathbf{D}_1^2 \times \mathbf{D}_2^2 \times \mathbf{D}_3^2)$, it is enough now to describe the pairs $(\mathbf{D}_1^2 \times \mathbf{S}_2^1 \times \mathbf{D}_3^2, \mathcal{S}_G \cap (\mathbf{D}_1^2 \times \mathbf{S}_2^1 \times \mathbf{D}_3^2))$ and $(\mathbf{S}_1^1 \times \mathbf{D}_2^2 \times \mathbf{D}_3^2, \mathcal{S}_G \cap (\mathbf{S}_1^1 \times \mathbf{D}_2^2 \times \mathbf{D}_3^2))$. Indeed, then one glues them along $(\mathbf{S}_1^1 \times \mathbf{S}_2^1 \times \mathbf{D}_3^2, \mathcal{S}_G \cap (\mathbf{S}_1^1 \times \mathbf{S}_2^1 \times \mathbf{D}_3^2))$ and caps the result by $\mathbf{D}_1^2 \times \mathbf{D}_2^2 \times \mathbf{S}_3^1$. As $A_1^1 > 0$ (see Section 4), for r_1, r_2, r_3 sufficiently small and for any point $* \in \mathbf{S}_2^1(r_2)$, the polycylinder $\mathbf{D}_1^2(r_1) \times \mathbf{D}_3^2(r_3)$ is a Milnor ball for $(\mathcal{S}_G \cap (\mathbf{D}_1^2(r_1) \times \{*\} \times \mathbf{D}_3^2(r_3)), \{0\} \times \{*\} \times \{0\})$. We deduce:

Proposition 7.3. *The pair* $(\mathbf{D}_1^2 \times \mathbf{S}_2^1 \times \mathbf{D}_3^2, \mathcal{S} \cap (\mathbf{D}_1^2 \times \mathbf{S}_2^1 \times \mathbf{D}_3^2))$ *is diffeomorphic to the pair* $(d_2(p_2)(C(p_2 \circ p)), d_2(p_2) \circ c(j_G)(C(p_2 \circ p \circ j_G)))$ *associated to the sequence of morphisms* $\mathbf{T}^2 \xrightarrow{j_G} \mathbf{T}_0^2 \times \mathbf{D}^2 \xrightarrow{p} \mathbf{T}_0^2 \xrightarrow{p_2} \mathbf{S}_2^1$ *using diagram* (1) *of Section* 3.

Let $H \in \{0, \dots, G\}$ be maximal such that $A_k^2 = 0,\ \forall k \leqslant H$. If $H = 0$, the pair $(\mathbf{S}_1^1 \times \mathbf{D}_2^2 \times \mathbf{D}_3^2, \mathcal{S}_G \cap (\mathbf{S}_1^1 \times \mathbf{D}_2^2 \times \mathbf{D}_3^2))$ can be described by an obvious analog of Proposition 7.3. Otherwise, then one has first to construct a classical 1-dimensional iterated torus knot $\mathbf{S}_1^1 \overset{h_H}{\hookrightarrow} \mathbf{S}_1^1 \times \mathbf{D}_3^2$ which describes the embedded boundary of the curve germ having as Newton–Puiseux expansion $X_1^{A_1^1} + \dots + X_H^{A_H^1}$ (see [7]). Then, make the base-change by the canonical projection $\mathbf{S}_1^1 \times \mathbf{D}_2^2 \to \mathbf{S}_1^1$ and place a copy of the construction described in Proposition 7.3 in a tubular neighborhood of the preimage of $h_H(\mathbf{S}_1^1)$ by the base change. One has to be careful to permute the indices and to use as starting data $A_{H+1}, \dots, A_G$, conveniently renormalized using $A_1^1, \dots, A_H^1$, following the comments preceeding Proposition 7.2.

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