North-Western European Journal of Mathematics

Banach spaces with the Blum-Hanson Property

Francois Netillard[1]

Received: June 4, 2021/Accepted: May 17, 2022/Online: May 20, 2022

Abstract

We are interested in a sufficient condition given in an article by P. Lefèvre, É. Matheron and A. Primot[2] to obtain the Blum-Hanson property and we then partially answer two questions asked in this same article on other possible conditions to have this property for a separable Banach space.

Keywords: Blum-Hanson, property (m_p), property sub-(m_p), AUS norm, Banach space, property (M^*).

MSC: 46B03, 46B06, 46B10, 46B20.

1 Introduction

These notes are essentially inspired by the article cited in the abstract[3] in which sufficient new conditions to justify that a Banach space has the Blum-Hanson property were obtained.
We recall that, for a (real or complex) Banach space X, and a contraction T on X (T is a bounded operator on X with $\|T\| \leq 1$), we say that T has the *Blum-Hanson property* if, for $x, y \in X$ such that $T^n x$ weakly converges to $y \in X$ when n tends to infinity, the mean

$$\frac{1}{N}\sum_{k=1}^{N} T^{n_k} x$$

tends toward y in norm for any increasing sequence of integers $(n_k)_{k\geq 1}$.
The space X is said to have the *Blum-Hanson property* if every contraction on X has the Blum-Hanson property.
Note, to understand the interest in this property and its historical aspect, that, when X is a Hilbert space and the linear operator T is a contraction, for all $x \in X$ such that $T^n x \xrightarrow{w} 0$, the arithmetic mean

[1]Université Bourgogne Franche-Comté, Laboratoire de Mathématiques de Besançon, UMR 6623, 16 route de Gray, 25030 Besançon Cedex, FRANCE.

[2]Lefèvre, Matheron, and Primot, 2016, "Smoothness, asymptotic smoothness and the Blum-Hanson property".

[3]Ibid.

$$\frac{1}{N}\sum_{k=1}^{N} T^{n_k}x$$

is norm convergent to 0 for any increasing sequence of integers $(n_k)_{k\geq 1}$. This result was first proved by J.R. Blum and D.L. Hanson[4] for isometries induced by measure-conserving transformations, then in two other papers[5] for arbitrary contractions. The most notable spaces having the Blum-Hanson property are the Hilbert spaces and the ℓ_p spaces for $1 \leq p < \infty$.
Note that this property is not preserved under renormings[6]. This raises the following question: "Which Banach spaces can be renormed to have the Blum-Hanson property ?", already asked before[7]. This question motivated the writing of this article.
To understand the main results of this work, we give first the following definition of an asymptotically uniformly smooth norm.

Definition 1 – Consider a Banach space $(X, \|\cdot\|)$. By following the definitions due to V. Milman[8] and the notations of two papers cited below[9], for $t \in [0,\infty)$, $x \in S_X$ and Y a closed vector subspace of X, we define the modulus of asymptotic uniform smoothness, $\overline{\rho}_X(t)$:

$$\overline{\rho}_X(t,x,Y) = \sup_{y\in S_Y}(\|x+ty\|-1).$$

Then

$$\overline{\rho}_X(t,x) = \inf_{Y\in \mathrm{cof}(X)} \overline{\rho}_X(t,x,Y) \qquad \text{and} \qquad \overline{\rho}_X(t) = \sup_{x\in S_X} \overline{\rho}_X(t,x).$$

The norm $\|\cdot\|$ is said to be *asymptotically uniformly smooth* (in short AUS) if

$$\lim_{t\to 0} \frac{\overline{\rho}_X(t)}{t} = 0.$$

Now, we can give the main property of this paper which partially answers the previous question:

[4] Blum and Hanson, 1960, "On the mean ergodic theorem for subsequences".
[5] Akcoglu, Huneke, and Rost, 1974, "A counterexample to Blum-Hanson theorem in general spaces"; Jones and Kuftinec, 1971, "A note on the Blum-Hanson theorem".
[6] Müller and Tomilov, 2007, "Quasi-similarity of power-bounded operators and Blum-Hanson property".
[7] Lefèvre, Matheron, and Primot, 2016, "Smoothness, asymptotic smoothness and the Blum-Hanson property".
[8] Milman, 1971, "Geometric theory of Banach spaces. II. Geometry of the unit ball (Russian)".
[9] Johnson et al., 2002, "Almost Fréchet differentiability of Lipschitz mappings between infinite-dimensional Banach spaces";
Lancien and Raja, 2018, "Asymptotic and Coarse Lipschitz structures of quasi-reflexive Banach spaces".

Theorem 1 – *Let Y be a separable Banach space whose norm is AUS. Then Y has an equivalent norm with the Blum-Hanson property.*

Remark 1 – A Banach space Y which has an AUS norm is an Asplund space. Consequently, Y is separable if and only if its dual is separable.

2 Banach space with property (m_p)

N. Kalton and D. Werner introduced the property (m_p)[10]:

Definition 2 – A Banach space X has property (m_p), where $1 \leq p \leq \infty$ if, for any $x \in X$ and every weakly null sequence $(x_n) \subset X$, it holds that:

$$\limsup_{n\to\infty} \|x + x_n\| = (\|x\|^p + \limsup \|x_n\|^p)^{\frac{1}{p}}.$$

For $p = \infty$, the right-hand side is of course to be interpreted as $\max(\|x\|, \limsup \|x_n\|)$.

Example 1 – ℓ_p has property (m_p), c_0 has property m_∞.

Remark 2 – We shall say that X has property *sub*-(m_p) if, for any $x \in X$ and every weakly null sequence $(x_n) \subset X$, it holds that:

$$\limsup_{n\to\infty} \|x + x_n\| \leq (\|x\|^p + \limsup \|x_n\|^p)^{\frac{1}{p}}.$$

As before, for $p = \infty$, the right-hand side is of course to be interpreted as $\max(\|x\|, \limsup \|x_n\|)$.

P. Lefèvre, É. Matheron and A. Primot[11] obtained the following property which was a corollary of one of the main theorems of their paper. It is this property which allowed us in particular to obtain Theorem 1.

Proposition 1 – [12] *For any $p \in (1, \infty]$, property sub-(m_p) implies Blum-Hanson property.*

Example 2 – [13]
We recall the definition of the James space J_p. This is the real Banach space of all sequences $x = (x(n))_{n\in\mathbb{N}}$ of real numbers satisfying $\lim_{n\to\infty} x(n) = 0$, endowed with the norm

[10] Kalton and Werner, 1995, "Property (M), M-ideals, and almost isometric structure of Banach spaces".
[11] Lefèvre, Matheron, and Primot, 2016, "Smoothness, asymptotic smoothness and the Blum-Hanson property".
[12] Ibid.

$$\|x\|_{J_p} = \sup\left\{\left(\sum_{i=1}^{n-1}|x(p_{i+1})-x(p_i)|^p\right)^{\frac{1}{p}} : 1 \le p_1 < p_2 < \cdots < p_n\right\}.$$

This is a quasi-reflexive Banach space which is isomorphic to its bidual.
Historically, R.C. James has focused exclusively on $J = J_2$[14], and I.S. Edelstein and B.S. Mityagin[15] are apparently the first to have observed that we could generalize the definition to $p \ge 1$ arbitrary and to have observed the quasi-reflexivity of J_p for any $p > 1$.
There exists an equivalent norm $|\cdot|$ on J_p[16] (Corollary 2.4 of the cited paper for the proof) such that, for all $x, y \in J_p$ verifying $\max\{i \in \mathbb{N} : x(i) \neq 0\} < \min\{i \in \mathbb{N} : y(i) \neq 0\}$, it holds that

$$|x+y|^p \le |x|^p + |y|^p.$$

Thus, $\tilde{J_p} := (J_p, |\cdot|)$ has the sub-(m_p) property, and therefore the Blum-Hanson property.

We now introduce a notion that is essentially dual to sub-(m_p).

Definition 3 – Let X be a separable Banach space and $q \in (1,\infty)$. We say that X^* has property sup-$(m_q)^*$ if, for any $x^* \in X^*$ and any weak*-null sequence (x_n^*) in X^*, we have:

$$\liminf_{n\to\infty}\|x^* + x_n^*\|^q \ge \|x^*\|^q + \liminf_{n\to\infty}\|x_n^*\|^q.$$

The following is an easy adaptation of the proof of Proposition 2.6 from an article by G. Godefroy, N.J. Kalton and G. Lancien[17].

Proposition 2 – *Let X be a separable Banach space. Let $p \in (1,\infty)$ and q be its conjugate exponent. Assume that X^* has property sup-$(m_q)^*$, then X has property sub-(m_p).*

Proof. Let $x \in X$ and (x_n) be a weakly null sequence in X and denote $s = \limsup_n \|x_n\|$. Pick $y_n^* \in X^*$ so that $\|y_n^*\| = 1$ and $y_n^*(x+x_n) = \|x+x_n\|$. After extracting a subsequence, we may assume that (y_n^*) is weak* converging to $x^* \in B_{X^*}$. Denote $x_n^* = y_n^* - x^*$ and assume also, as we may, that $\lim_n \|x_n^*\| = t$. Since X^* has sup-$(m_q)^*$, we have that $\|x^*\|^q + t^q \le 1$. Therefore

$$\begin{aligned}\limsup_n \|x+x_n\| &= \limsup_n (x^* + x_n^*)(x+x_n) \le x^*(x) + st \\ &\le (\|x^*\|^q + t^q)^{1/q}(\|x\|^p + s^p)^{1/p} \le (\|x\|^p + s^p)^{1/p}. \qquad \square\end{aligned}$$

This concludes our proof.

[13] García-Lirola and Petitjean, 2021, "On the weak maximizing properties"; Netillard, 2018, "Coarse Lipschitz embeddings of James spaces".
[14] James, 1950, "Bases and reflexivity of Banach spaces".
[15] Edelstein and Mityagin, 1970, "Homotopy type of linear groups of two classes of Banach spaces".
[16] Netillard, 2018, "Coarse Lipschitz embeddings of James spaces".
[17] Godefroy, Kalton, and Lancien, 2001, "Szlenk indices and uniform homeomorphisms".

3 Main results

We give some definitions that will be used later.

Definition 4 – Given an FDD (E_n), (x_n) is said to be a block sequence with respect to (H_i) if there exists a sequence of integers $0 = m_1 < m_2 < \cdots$ such that $x_n \in \bigoplus_{j=m_n}^{m_{n+1}-1} E_j$.

Definition 5 – Let $1 \leq q \leq p \leq \infty$ and $C < \infty$. A (finite or infinite) FDD (E_i) for a Banach space Z is said to satisfy $C-(p,q)$ estimates if for all $n \in \mathbb{N}$ and block sequences $(x_i)_{i=1}^n$ with respect to (E_i):

$$C^{-1}\left(\sum_1^n \|x_i\|^p\right)^{\frac{1}{p}} \leq \left\|\sum_1^n x_i\right\| \leq C\left(\sum_1^n \|x_i\|^q\right)^{\frac{1}{q}}.$$

For the central theorem for this work, we now recall the definition of the Szlenk index.

Definition 6 – Let X be a Banach space and K be a weak*-compact subset of X^*. For $\epsilon > 0$, let $\mathcal{V}$ be the set of all weak*-open subsets of K such that the norm diameter (for the norm of X^*) of V is less than ϵ, and

$$s_\epsilon K = K \setminus \bigcup\{V : V \in \mathcal{V}\}.$$

As a remark, $s_\epsilon^\alpha B_{X^*}$ is defined inductively for any ordinal α by

$$s_\epsilon^{\alpha+1} B_{X^*} = s_\epsilon(s_\epsilon^\alpha B_{X^*})$$

and

$$s_\epsilon^\alpha B_{X^*} = \bigcap_{\beta<\alpha} s_\epsilon^\beta B_{X^*} \text{ if } \alpha \text{ is a limit ordinal.}$$

We define $Sz(X,\epsilon)$ to be the least ordinal α so that $s_\epsilon^\alpha B_{X^*} = \emptyset$ if such an ordinal exists. Otherwise we write $Sz(X,\epsilon) = \infty$ by convention.
We will then denote $Sz(X)$ the Szlenk index of X, defined by

$$Sz(X) = \sup_{\epsilon>0} Sz(X,\epsilon).$$

Remark 3 – For a detailed report about the Szlenk index, one can refer to the article by G. Lancien quoted below[18].
Note that the Szlenk index was introduced by W. Szlenk[19] to show that there is no universal reflexive space for the class of separable reflexive spaces.

The main ingredient of our argument is the following result, which is deduced from a work of H. Knaust, E. Odell and T. Schlumprecht[20] (Corollary 5.3) and is already cited in an article by G. Lancien[21] (in the proof of Theorem 4.15). However, in this last paper, we do not find the detailed proof of this property, that we include now.

Proposition 3 – *Let Y be a separable Banach space such that $Sz(Y) \leq \omega$, where ω denote the first infinite ordinal.*
Then Y can be renormed so as to have property sub-(m_q) for some value $q \in (1,\infty)$.

Proof. According to Corollary 5.3 of the article by H. Knaust, E. Odell and T. Schlumprecht cited above[22], $\mathrm{Sz}(Y) \leq \omega$ implies that there exists a Banach space Z with a boundedly complete FDD (E_i) (in particular Z is isometric to a dual space X^*) with the following properties.

1. There exists $p \in (1,\infty)$ such that (E_i) satisfies $1-(p,1)$ estimates.
2. Y^* is isomorphic (norm and weak*) to a weak*-closed subspace F of $Z = X^*$.

Let us denote $S : Y^* \to F$ this isomorphism. Then, there exists a subspace G of X such that $G^\perp = F$ and S is the adjoint of an isomorphism T from X/G onto Y. Let now q be the conjugate exponent of p. It is thus enough to prove that $E = X/G$ has sub-(m_q). Since $X^* = Z$ satisfies $1-(p,1)$ estimates with respect to the boundedly complete FDD (E_i), it is immediate that X^* has sup-$(m_p)^*$. Now this property passes clearly to its weak*-closed subspace F and E^* has sup-$(m_p)^*$. Finally, we deduce from Proposition 2 that E has sub–(m_q). This finishes our proof. □

Thanks to this Proposition, we obtain the Theorem 1.

Proof. The proof is immediate by applying the Proposition 1. □

We will now talk about property (M^*).
It was studied by N.J. Kalton and D. Werner[23] :

Definition 7 – A Banach space X has property (M^*) if, for $u^*, v^* \in S_{X^*}$ and $(x_n^*) \subseteq X^*$ a weak* null sequence, it holds that

[18] Lancien, 2006, "A survey on the Szlenk index and some of its applications".

[19] Szlenk, 1968, "The non existence of a separable reflexive space universal for all reflexive Banach spaces".

[20] Knaust, Odell, and Schlumprecht, 1999, "On asymptotic structure, the Szlenk index and UKK properties in Banach spaces".

[21] Lancien, 2012, "A short course on non linear geometry of Banach spaces".

[22] Knaust, Odell, and Schlumprecht, 1999, "On asymptotic structure, the Szlenk index and UKK properties in Banach spaces".

[23] Kalton and Werner, 1995, "Property (M), M-ideals, and almost isometric structure of Banach spaces".

$$\limsup_n \|u^* + x_n^*\| = \limsup_n \|v^* + x_n^*\|.$$

Remark 4 – It has been shown[24] that, if X is a separable Banach space having property (M^*), then its dual is separable.

The following Proposition follows from an article by S. Dutta and A.Godard[25] (Proposition 2.2 of this article).

Proposition 4 – *Let X be a separable Banach space with property (M^*). Then X is asymptotically uniformly smooth for a norm $\|\cdot\|_M$.*

Corollary 1 – *Let X be a separable Banach space with property (M^*). Then X has an equivalent norm with the Blum-Hanson property.*

Proof. It follows from the Proposition 4 and from the Theorem 1. □

Acknowledgments

The author is grateful to Gilles Lancien and Audrey Fovelle for helpful discussions. He would also like to thank the referee for helpful remarks.

References

Akcoglu, M., J. Huneke, and H. Rost (1974). “A counterexample to Blum-Hanson theorem in general spaces”. *Pacific J. Math.* **50**, pp. 305–308 (cit. on p. 104).

Blum, J. and D. Hanson (1960). “On the mean ergodic theorem for subsequences”. *Bull. Amer. Math. Soc.* **66**, pp. 308–311 (cit. on p. 104).

Dutta, S. and A. Godard (2008). “Banach spaces with property (M) and their Szlenk indices”. *MedJM* **5**, pp. 211–220 (cit. on p. 109).

Edelstein, I. and B. Mityagin (1970). “Homotopy type of linear groups of two classes of Banach spaces”. *Functional Anal. Appl.* **4**, pp. 221–231 (cit. on p. 106).

García-Lirola, L. and C. Petitjean (2021). “On the weak maximizing properties”. *Banach Journal of Mathematical Analysis 15* **3** (cit. on p. 106).

Godefroy, G., N. Kalton, and G. Lancien (2001). “Szlenk indices and uniform homeomorphisms”. *Trans. Amer. Math. Soc.* **353 (10)**, pp. 3895–3918 (cit. on p. 106).

James, R. (1950). “Bases and reflexivity of Banach spaces”. *Ann. of Math. (2)* **52**, pp. 518–527 (cit. on p. 106).

[24] Kalton and Werner, 1995, “Property (M), M-ideals, and almost isometric structure of Banach spaces”.

[25] Dutta and Godard, 2008, “Banach spaces with property (M) and their Szlenk indices”.

Johnson, W. et al. (2002). "Almost Fréchet differentiability of Lipschitz mappings between infinite-dimensional Banach spaces". *Proc. Lond. Math. Soc. (3)* **84 (3)**, pp. 711–746 (cit. on p. 104).
Jones, L. and V. Kuftinec (1971). "A note on the Blum-Hanson theorem". *Proc. Amer. Math. Soc.* **30**, pp. 202–203 (cit. on p. 104).
Kalton, N. and D. Werner (1995). "Property (M), M-ideals, and almost isometric structure of Banach spaces". *J. Rein Angew. Math.* **461**, pp. 137–178 (cit. on pp. 105, 108, 109).
Knaust, H., E. Odell, and T. Schlumprecht (1999). "On asymptotic structure, the Szlenk index and UKK properties in Banach spaces". *Positivity* **3**, pp. 173–199 (cit. on p. 108).
Lancien, G. (2006). "A survey on the Szlenk index and some of its applications". *RACSAM Rev. R. Acad. Cienc. Exactas Fìs Nat. Ser. A Mat.* **100**, pp. 209–235 (cit. on p. 108).
Lancien, G. (2012). "A short course on non linear geometry of Banach spaces". *Topics in Functional and Harmonic Analysis, Theta Series in Advanced Mathematics*, pp. 77–102 (cit. on p. 108).
Lancien, G. and M. Raja (2018). "Asymptotic and Coarse Lipschitz structures of quasi-reflexive Banach spaces". *Houston J. of Math.* **44** (3), pp. 927–940 (cit. on p. 104).
Lefèvre, P., É. Matheron, and A. Primot (2016). "Smoothness, asymptotic smoothness and the Blum-Hanson property". *Israel J. Math.* **211** (1), pp. 271–309 (cit. on pp. 103–105).
Milman, V. (1971). "Geometric theory of Banach spaces. II. Geometry of the unit ball (Russian)". *Uspehi Mat. Nauk* **26**, pp. 73–149 (cit. on p. 104).
Müller, V. and Y. Tomilov (2007). "Quasi-similarity of power-bounded operators and Blum-Hanson property". *J. Funct. Anal.* **246**, pp. 385–399 (cit. on p. 104).
Netillard, F. (2018). "Coarse Lipschitz embeddings of James spaces". *Bull. Belg. Math. Soc. Simon Stevin* **25 (1)**, pp. 71–84 (cit. on p. 106).
Szlenk, W. (1968). "The non existence of a separable reflexive space universal for all reflexive Banach spaces". *Studia Math* **30**, pp. 53–61 (cit. on p. 108).

Contents