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Genus zero modular operad and absolute Galois group

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Abstract

In this article, we develop the geometry of canonical stratifications of the spaces $\overline{\mathcal{M}}_{0,n}$ and prepare ground for studying the action of the Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ upon strata. We define and introduce a version of a *gravity* operad constructed for a class of moduli spaces $\overline{\mathcal{M}}_{0,n}$, equipped with a hidden holomorphic involution. This additional symmetry is associated to a split quaternionic structure. We introduce a categorical framework to present this object. Interaction between the geometry, physics and the arithmetics are discussed. An important feature is that 0-divisors of the split quaternion algebra imply additional singular points, and lead to investigations concerning the geometry and mixed Hodge structures.

Keywords: genus zero modular operad, absolute Galois group, gravity operad, stacks.

MSC: 14D23, 14G32, 53Bxx, 14F20, 18D50.

1 Introduction

With the development of quantum field theory in theoretical physics, the interaction of physics and mathematics intensified. As the survey by Atiyah, Dijkgraaf, and Hitchin (2010) stressed, this interaction became a very rich source of new ideas in mathematics, in particular, in algebraic geometry.

This article is concerned with one remarkable fruit of the interaction: creation of the theory of quantum cohomology (cf. Kontsevich and Y. Manin 1994) and subsequent discovery of its connections with one of the central objects of number theory, Galois group of the field of all algebraic numbers (cf. Brito, Horel, and Robertson 2019, Combe, Y. Manin, and Marcolli 2021, Ihara 1994, and references therein).

A quantum cohomology operad has as its components cohomology spaces of moduli spaces $\overline{\mathcal{M}}_{g,n}$ (or only $g = 0$ case), and its structure morphisms correspond to

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well defined morphisms of these moduli spaces. Choosing for instance étale cohomology, the Galois group of algebraic numbers acts upon all operadic components compatibly with operadic morphisms, creating the connection between the theory of quantum cohomology and number theory.

In this paper, we equip the moduli space $\overline{\mathcal{M}}_{0,n+1}$ with an algebraic structure. This algebraic structure is motivated by physics and used in quantum theory Baez 2012; Varadarajan 1985 in general relativity and gravity Gogberashvili 2014; Kulyabov, Korolkova, and Gevorkyan 2020; Ulrych 2006. These additional algebraic structures imply symmetries.

From a more algebraic perspective, those physical requirements mean that to describe the geometry of the space we need a composition algebra. For physical reasons it is not necessary to use field extensions, quadratic extensions over real numbers are enough.

In this paper, we will equip $\overline{\mathcal{M}}_{0,n+1}$ with a given holomorphic involution θ. The underlying algebra, over which are defined the corresponding modules is a composite normed algebra and in particular a *split algebra*. Here, the hidden symmetry is associated to *split quaternion algebra*[3].

We think that working on spaces being realisations of modules over these split algebras offers an interesting model to consider, encoding also physical data. We introduce this construction in order to define what we call a *NY-gravity operad* for this model and investigate the geometry around it.

Below we briefly describe the content of our article and its place in a much wider environment.

1.1 Geometric approaches to the absolute Galois group

The study of the absolute Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ and its generalisations, Galois groups of the algebraic closures of various finitely generated fields, holds a central place in algebraic geometry and number theory. Since the works of A. Belyi, V. Drinfeld, Y. Ihara, A. Grothendieck, the main approach to it consists in considering the tower of finite coverings of $\mathbb{P}^1$ ramified only over a fixed three points, say, $\{0,1,\infty\}$, and considering actions of two different groups upon it: geometric one $\pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\})$, and algebraic one, $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, existing in view of Belyi's theorem.

The latter formed a key step allowing to bridge algebraic curves over $\mathbb{Q}$ and more combinatorial objects (the Grothendieck *dessins d'enfants*), by showing that any algebraic curve X over $\mathbb{Q}$, admits a map $X \to \mathbb{P}^1$, ramified at three points only, say $\{0,1,\infty\}$ (see for instance Schneps 1994a, and more globally Schneps 1994b).

Consider the moduli space of n-pointed curves of genus g, $\overline{\mathcal{M}}_{g,n}$, defined over $\mathbb{Q}$. There exists a canonical outer action of the absolute Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$

[3] For an introduction to the relation between linear spaces corresponding to modules over an algebra, we refer to Rosenfeld 2000; Shirokov 2002; Shurygin 1993. For manifolds defined over algebras, and verifying the Cauchy–Riemann equations, we refer to Schafer 1969, p.146.

on the algebraic fundamental group $\pi_1^{alg}(\overline{\mathcal{M}}_{g,n} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \overline{x})$, where $\overline{x}$ is a geometric point of $\overline{\mathcal{M}}_{g,n}$. Originally, this statement follows from the construction depicted in Grothendieck 1971, Exp. IX, Théorème 6.1. Namely, let V be a quasi-compact and quasi-separated scheme over $\mathbb{Q}$ (and geometrically connected). Then, there is a short exact sequence

$$1 \to \pi_1^{alg}(V \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \overline{x}) \to \pi_1^{alg}(V, \overline{x}) \to Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to 1,$$

of profinite topological groups. The canonical Galois action homomorphism holds also for stacks. In particular, for the stacky version we have:

$$Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Out}(\pi_1^{alg}(\overline{\mathcal{M}}_{g,n} \otimes \overline{\mathbb{Q}}, \overline{x}).$$

As inspired by the ideas of Grothendieck 1997 in *Esquisse d'un programme*, and then developed in an operadic framework in the works of Fresse 2017 and Brito, Horel, and Robertson 2019, the goal of this article is to start replacing in this picture the tower of coverings of $\mathbb{P}^1$ by another geometric object: the *genus zero modular operad*. More precisely, we use the categorical version, developed in our recent article Combe and Y. Manin 2021 and investigate the fruitful interaction between Quantum Cohomology and Grothendieck–Teichmüller theory.

To achieve this goal, we must suggest one more object, upon which $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ acts, in this operadic environment. The simplest version of it consists of all points in $\overline{\mathcal{M}}_{0,n}(\overline{\mathbb{Q}})$, and respective maps induced by operadic morphisms, but it is too trivial and too large for useful applications.

Its much smaller and possibly interesting version consists of moduli points of all *maximally degenerated genus zero stable curves with marked points* (strata of dimension zero).

More generally, we develop here the geometry of canonical stratifications of the spaces $\overline{\mathcal{M}}_{0,n}$, being indexed by dual graphs of the divisors, and prepare ground for studying the action of the Galois group upon strata.

1.2 Genus zero modular operad and its hidden symmetry

The components and composition morphisms of the genus zero modular operad belong to the category of smooth projective manifolds. A naive way to describe its n–ary component $\overline{\mathcal{M}}_{0,n+3}$ is this.

First, consider the moduli space of marked, pairwise distinct points, on the projective line: it can be naturally identified with an obvious open subset in $(\mathbb{P}^1)^n$ (which is the complement of the discriminant variety). Second, construct a compactification of this subset by adding as fibres "stable" curves of genus zero with marked points that can be described as degenerations of the generic stable curve.

Since we mentioned points $\{0, 1, \infty\}$, we have implicitly introduced in this description the coordinate t on $\mathbb{P}^1$ mentioned above, and thus we can extend the involution $t \mapsto 1 - t$ to induce it upon $\mathcal{M}_{0,n+3}$, and then to $\overline{\mathcal{M}}_{0,n+3}$, for $n \geq 0$.

Actually, in this context there is a better way to define the rigidification involving t. To make explicit the geometry behind it, imagine first $\mathbb{P}^1(\mathbb{C})$ as a topological sphere S^2 endowed with one complex structure and three "equators" $S^1_j \subset S^2$ in general position. For each j, we can introduce a complex coordinate t_j upon S^2 identifying S^1_j with a naturally oriented $\mathbb{P}^1(\mathbb{R})$. Then the whole symmetry group behind this rigidification will be generated by $t_j \mapsto 1-t_j$, and later it can be extended to the whole group of symmetries of the modular operad of genus zero. This is what we can call its hidden symmetry or stack symmetry. The same group is used in Ihara 1994 and Lochak, Schneps, and Scheiderer 1997 in order to treat the cycle relations in the Grothendieck–Teichmüller group.

In the main part of this article, we restrict ourselves to the study of this *hidden involution* and the *stacky* quotient operad with respect to it.

The total automorphism groups of $\overline{\mathcal{M}}_{0,n+1}$ were determined by Bruno and Mella 2013, using Kapranov's description of $\overline{\mathcal{M}}_{0,n}$ as the closure of a subscheme of a given Hilbert scheme. In our paper Combe and Y. Manin 2021 we show how one can use their results in order to encode the total structure of symmetries of genus zero modular operad in two different combinatorial structures.

Our principal results here, Theorems 1, 2, and 3, can be interpreted as showing ways to various objects upon which the absolute Galois group acts, such as points of maximum degeneration mentioned in the Abstract above. The structures connecting various components of all these objects into a unity are again of operadic type, and important role among them is played by NY–operad, taking into account also complex conjugation and split-quaternonic structure. Finally, in the two last sections we investigate the geometry around the singular points of this moduli space with symmetries and mixed Hodge structures. It is particularly interesting since additional singular points arise, on this real realisation, from the 0-divisors of the split quaternion algebra.

2 The stack category of stable labeled curves

In this new section, we are brought to the notion of *groupoids*, which is particularly well suited to the exposition of our results. Groupoids are used here in the context of (pre)stable curves.

A *prestable curve* over a scheme T is a flat proper morphism $\pi : C \to T$ whose geometric fibers are reduced one-dimensional schemes with at most ordinary double singular points. Its genus is a locally constant function on T: $g(t) := dimH^1(C_t, \mathcal{O}_t)$. Let S be a finite set. An S-labeled, prestable curve over T is a family, where $\pi : C \to T$ is a prestable curve and x_i are sections such that for any geometric point t of T, we have $x_i(t) \neq x_j(t)$.

Such a curve is *stable* if it is connected and if the normalization of each irreducible component, which has genus zero, carries at least three special points. Let

$(C, \pi, x_i | i \in S)$ be an S-pointed prestable curve. It is stable if and only if automorphism groups of its geometric fibers are finite, and if there are no infinitesimal automorphisms.

2.1 Groupoids: general setting

For the reader's convenience, let us briefly recall the notions of groupoids and stack of groupoids such as presented in Y. Manin 1999 (Chap. 5, 3.2.). Let $\mathcal{F}$ and $\mathcal{S}$ be two categories and let $b : \mathcal{F} \to \mathcal{S}$ be a functor. For $F \in Ob(\mathcal{F})$ such that $b(F) = T$, we will call F a family with base T, or a T-family.

Condition for groupoids

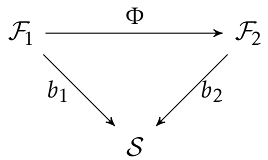
In order to form a groupoid, the data must satisfy the following condition.

First, for any base $T \in \mathcal{S}$, any morphism of families over T inducing identity on T must be an isomorphism. Designate by the symbol $\mathcal{F}_T$ this subcategory of $\mathcal{F}$. It will be called the fiber of $\mathcal{F}$ over T.

Second, for any morphism between bases $\phi : T_1 \to T_2$ there must exist a base change functor $\phi^* : \mathcal{F}_{T_2} \to \mathcal{F}_{T_1}$ such that for a pair of composable morphisms ϕ, ψ there exists a functor isomorphism $(\phi \circ \psi)^* \cong \psi^* \circ \phi^*$. The latter must satisfy the cocycle condition expressing associativity.

1-morphisms of abstract groupoids

We will be considering only morphisms between groupoids over the same category of bases $\mathcal{S}$. Such a morphism $\{b_1 : \mathcal{F}_1 \to \mathcal{S}\} \to \{b_2 : \mathcal{F}_2 \to \mathcal{S}\}$ is a functor $\Phi : \mathcal{F}_1 \to \mathcal{F}_2$ such that $b_2 \circ \Phi = b_1$.



2.2 Groupoids of S–labeled stable curves of genus g

We consider the categories $\mathcal{S} = Sch_{\mathbb{Q}}$ of schemes over $\mathbb{Q}$ and of S-labeled stable curves (genus g) $\mathcal{F} = \overline{\mathcal{M}}_{g,S}$; the genus 0 case is denoted by $\overline{\mathcal{M}}_{0,S}$. For $T \in \mathcal{S}$, an object $\mathcal{F}_T$ is a stable S–labeled curves over the scheme $T \in \mathcal{S}$. These S-pointed curves are given by n-sections $\{s_1, s_2, \ldots, s_n\}$.

A morphism $(C_1/T_1, x_{1i}|i \in S) \to (C_2/T_2, x_{2i}|i \in S)$ is a pair of compatible morphisms $\phi : T_1 \to T_2$ and $\psi : C_1 \to C_2$ such that ψ induces an isomorphism of labeled curves $C_1 \to \phi^*(C_2)$. Equivalently, the diagram

$$\begin{array}{ccc} C_1 & \xrightarrow{\psi} & C_2 \\ \downarrow & & \downarrow \\ T_1 & \xrightarrow[\phi]{} & T_2 \end{array}$$

is cartesian, and induces the bijection of the two families of S–labeled sections.

2.3 Groupoids of universal curves

Let us consider the groupoid of universal curves $\mathcal{C}_g$. The objects of $\mathcal{C}_g$ are stable curves $(C/T, x_i|i \in S)$, endowed with an additional section $\tilde{\Delta} : T \to C$ not constrained by any restrictions. The morphisms must be compatible with this additional data.

2.4 Stacks

Definition 1 (5.1 in Y. Manin 1999, 4.1 in Deligne and Mumford 1969) – A stack of groupoids is a quadruple

$$(\mathcal{F}, \mathcal{S}, b : \mathcal{F} \to \mathcal{S}, \text{Grothendieck topology } \mathcal{T} \ on\ \mathcal{S})$$

satisfying the following conditions:

1. $b : \mathcal{F} \to \mathcal{S}$ is a groupoid (such as defined in section 2.1). Each contravariant representable functor $S^{op} \to Sets$ is a sheaf on $\mathcal{T}$. The topology on $\mathcal{T}$ is given by the set of its coverings. Given an object T in $\mathcal{S}$, a covering is given by a family $\{R_i \to T\}$, being stable under change of basis (i.e. $T' \to T$), and contains all families consisting of one identity map. Moreover, one has stability under composition: the composition of a covering $\{R_i \to T\}$ with the covering of all R_i gives back a covering.

2. Given X_1, X_2 over T, the functor $T' \mapsto Iso_{T'}(X_1 \to X_2)$ is a sheaf.

3. Any family over a given base is uniquely defined by its local restrictions. Such local data can be glued if and only if they satisfy the cocycle condition[4].

The stacks over T are the objects of a 2–category (stacks/T): 1–morphisms are functors from one stack to another, and 2-morphisms are morphisms of functors. In this 2–category every 2–morphism is an isomorphism.

[4]Note that this refers to a descent situation.

Remark 1 – $\overline{\mathcal{M}}_{0,S}$ are algebraic Deligne–Mumford (DM) stacks over $\mathbb{Q}$ with respect to the étale topology (see also paragraph 3.1 and Deligne and Mumford 1969). The Grothendieck–Teichmüller set-up appears deeply intertwined within the situation where we have added an extra quadratic extension using θ, creating an analog of field extension.

3 Operads

The notation $\mathcal{M}_{g,n}$ stands for pointed curves (with respect to n disjoint sections). For general categorical background and stacks we refer to Kashiwara and Schapira 2006, Chap. 1, 4, 16, 19. In our line of sight, we aim at constructing a topological operad, obtained by endowing the moduli space $\overline{\mathcal{M}}_{0,n+1}$ with a symmetry, denoted by θ. The initial motivation for choosing as a symmetry the holomorphic involution θ was that it does not produce any problems related to orientability. However, equipping our space with such a symmetry impacts the underlying algebra.

More precisely, in physics, to express the transition from a manifold (which is a result of some given measurements) to geometry one must be able to introduce a distance between some objects and an etalon for their comparison - an etalon corresponds to a unit in the algebra. In algebraic language all the physical requirements mean that to describe the geometry we need a *composition algebra*.

Although, up to now, the division algebras such as split algebras were mostly investigated in quantum theory (see Baez 2012; Varadarajan 1985), we think that working on spaces being realisations of modules over these split algebras offers an interesting model to consider. Therefore, this paper is based on considerations of split algebras and their applications. In particular, here, the hidden symmetry is associated to a split quaternion algebra.

This algebra is a composite, normed algebra with zero divisors. Zero divisors of the split algebras play an important role in the description of mirror symmetries and super symmetries (see Baez and Huerta 2010, 2011; Kugo and Townsend 1998). This split quaternion algebra is non-commutative, associative, non-division ring, isomorphic to the ring of 2×2 real matrices. This is a model often used for space-time general relativity and gravity. We think it is an interesting model to consider.

We will study in section 3 this hidden symmetry of $\overline{\mathcal{M}}_{0,n+1}$ in a categorical framework.

Operad in a category

Let $(\mathbf{C}, \otimes, 1_C, a, l, r, \tau)$ be a symmetric monoidal category. Let **Fin** be the category of finite sets with bijections. Given any subset $X \subset Y$, we use the notation $Y/X := Y \setminus X \sqcup \{*\}$.

Definition 2 (Operad, Y. Manin and Valette 2019 Definition 4.1) – An operad $\mathcal{P}$ in $\mathbf{C}$ is a presheaf $\mathcal{P} : \mathbf{Fin}^{op} \to \mathbf{C}$ endowed with partial operadic compositions $\circ_{X \subset Y}$ or any $X \subset Y \in \mathbf{Fin}$:

$$\mathcal{P}(Y/X) \otimes \mathcal{P}(X) \to \mathcal{P}(Y),$$

for any $X \subset Y$ and a unit $\eta : 1_{\mathbf{C}} \to \mathcal{P}(\{*\})$ such that the following diagrams commute.

$$\begin{array}{ccccc}
(\mathcal{P}(Z/Y)\otimes\mathcal{P}(Y/X))\otimes\mathcal{P}(X) & \xrightarrow{\alpha} & \mathcal{P}(Z/Y)\otimes(\mathcal{P}(Y/X)\otimes\mathcal{P}(X)) & \xrightarrow{id\otimes\circ_{X\subset Y}} & \mathcal{P}(Z/Y)\otimes\mathcal{P}(Y) \\
\downarrow{\cong} & & & & \downarrow{\circ_{Y\subset Z}} \\
(\mathcal{P}((Z/Y)/(Y/X))\otimes\mathcal{P}(Y/X))\otimes\mathcal{P}(X) & \xrightarrow[\circ_{Y/X\subset Z/X}\otimes id]{} & \mathcal{P}(Z/X)\otimes\mathcal{P}(X) & \xrightarrow[\circ_{Y\subset Z}]{} & \mathcal{P}(Z)
\end{array}$$

$$\begin{array}{ccccc}
\mathcal{P}(((Z/X)/Y))\otimes(\mathcal{P}(X)\otimes\mathcal{P}(Y)) & \xrightarrow{id\otimes\tau} & \mathcal{P}(((Z/X)/Y))\otimes(\mathcal{P}(Y)\otimes\mathcal{P}(X)) & \xrightarrow{\alpha^{-1}} & (\mathcal{P}(((Z/X)/Y))\otimes\mathcal{P}(Y))\otimes\mathcal{P}(X) \\
\downarrow{\alpha^{-1}} & & & & \downarrow{\circ_{Y\subset Z/X}\otimes id} \\
(\mathcal{P}(((Z/X)/Y))\otimes\mathcal{P}(X))\otimes\mathcal{P}(Y) & & & & \mathcal{P}(Z/X)\otimes\mathcal{P}(X) \\
\downarrow{\cong} & & & & \downarrow{\circ_{X\subset Y}} \\
(\mathcal{P}(((Z/Y)/X))\otimes\mathcal{P}(X))\otimes\mathcal{P}(Y) & \xrightarrow[\circ_{X\subset Z/Y}\otimes id]{} & \mathcal{P}(Z/Y)\otimes\mathcal{P}(Y) & \xrightarrow[\circ_{Y\subset Z}]{} & \mathcal{P}(Z)
\end{array}$$

From Y. Manin and Valette 2019 we know that the skeletal category of **Fin** is the groupoid $\mathbb{S}$, whose objects are $\{1, 2, ..., n\}$ for any $n \in \mathbb{N}$. Morphisms are the elements of the symmetric groups $\mathbb{S}_n$. A presheaf on **Fin** is thus equivalent to a collection $\{\mathcal{P}(n)\}_{n\in\mathbb{N}}$ of right $\mathbb{S}_n$-modules. In these terms the above structure of operad is equivalent to partial composition products $\circ_i : \mathcal{P}(n) \otimes \mathcal{P}(m) \to \mathcal{P}(n+m-1)$, for $1 \leq i \leq n$, and a unit map $\eta : 1_{\mathbf{C}} \to \mathcal{P}(1)$ satisfying the analogous axioms.

Operad in groupoids

An operad in groupoids—also known as an operad in the category of small categories—is defined in three steps. Note that it consists of a *sequence* of small categories $\mathcal{P}(r), r \in \mathbb{N}$, each of which are equipped with a symmetric group action; a unit morphism $\eta : pt \to \mathcal{P}(1)$, and a composition product $\mu : \mathcal{P}(r) \times \mathcal{P}(n_1) \times \cdots \times \mathcal{P}(n_r) \to \mathcal{P}(n_1 + \cdots + n_r)$, is formed in the category of categories. Classical identities expressed by diagrams (mainly concerning the associativity, equivariance) hold.

More precisely, since the category of groupoids forms a *symmetric monoidal subcategory* of the category of small categories, an operad in groupoids can be defined as an operad in categories $\mathcal{P}$, of which components $\mathcal{P}(r)$ are groupoids. The composition structure of an operad in categories (resp. groupoids) can be defined by giving a collection of functors $\circ_k : \mathcal{P}(m) \times \mathcal{P}(n) \to \mathcal{P}(m+n-1)$, $k = 1, ..., m$ satisfying the equivariance, unit and associativity relations.

A morphism of operads in categories $f : \mathcal{P} \to \mathcal{Q}$ is a sequence of functors $f : \mathcal{P}(r) \to \mathcal{Q}(r)$ preserving the internal structures attached to operads. The category of operads in groupoids forms a full subcategory of the category of operads in categories. For operads in categories, we will naturally consider the operad morphisms of which all underlying functors $f : \mathcal{P}(r) \to \mathcal{Q}(r)$ are equivalences of categories.

Operads of stable S–labeled curves

For a finite set S denote by $\overline{\mathcal{M}}_{g,S}$ the Deligne–Mumford stack classifying stable curves of genus g and with marked points $(x_i)_{i\in S}$ labeled by the finite set S. For any injective map $\phi : S \to S'$ of finite sets and any g such that $\overline{\mathcal{M}}_{g,S} \neq \emptyset$, there is a natural morphism of stacks $\overline{\mathcal{M}}_{g,S'} \to \overline{\mathcal{M}}_{g,S}$ called *stable forgetting map*, see Knudsen and Mumford 1983.

Our attention focuses on the description of the genus zero modular operad (see Y. Manin 2019, section 4 for details). Take a finite set $S = \{0, 1, ..., n\}$ and equip it with the action of the symmetric group $\mathbb{S}_{n+1}$. We have the following structure morphism defined point-wise by a glueing of the respective stable curves:

$$\overline{\mathcal{M}}_{0,k+1} \times \overline{\mathcal{M}}_{0,m_1+1} \times \cdots \times \overline{\mathcal{M}}_{0,m_k+1} \to \overline{\mathcal{M}}_{0,m_1+\cdots+m_k+1} \quad (1)$$

Hidden symmetry of the DM–Stack

Consider the DM–stack $\overline{\mathcal{M}}_{0,n+1}$. Define a stable S–labeled curve $(C/T, \pi, (x_i)|i \in S, |S| = n+1)$, where:

- T is a base scheme in the category $Sch_{\mathbb{Q}}$,
- C is a scheme in the category $\overline{\mathcal{M}}_{0,n+1}$,
- $\pi : C \to T$ is a proper flat morphism.

For any geometric point $t \in T$, the sections $x_i(t)$ are smooth on C_t and for a given $i \neq j$, the section $x_i(t)$ is different from $x_j(t)$. The stack $\overline{\mathcal{M}}_{0,S}$ has as T-fiber an $n+1$-pointed curve $C \to T$ with $n+1$ sections $T \to C$ (having disjoint images). A section of C is a morphism of T-schemes defined from T to C such that composed with π one obtains the identity Id_T.

We consider the contravariant functor $\overline{\mathcal{M}}_{0,n+1}$ over $Spec\,\mathbb{Q}$, which maps a scheme T to a collection of $n+1$–pointed curves of genus 0 over T. Note that this stack description is the Lax functor one and it is not the one implemented in part 2.

The complement of the "fat diagonal" Δ (i.e. locus of colliding points), is contained in T, as the open subset over which the morphism π is smooth. This is the open set parametrizing $n+1$–pointed curves over $Spec(\mathbb{C})$, for which the curve is smooth.

Following the approach of Keel 1992; Knudsen and D. 1976 (both in section 1), $\overline{\mathcal{M}}_{0,S}$ is represented by the smooth complete variety T_S[5], together with a universal curve $(C,(x_i)_{\in S}) \to T_S$, where $(x_i)_{i\in S}$ are the universal sections. One can construct T_S by induction and identify the universal curve $(C,(x_i)_{\in S})$ to $T_{S\cup\{*\}}$. This universal curve can be obtained by blowing-up $T_{S\cup\{*\}} \times_{T_S} T_{S\cup\{*\}}$ along a subscheme of the diagonal.

Symmetry of the DM–Stack

In real algebraic geometry, a classical approach is to endow the moduli space of genus 0 curves and marked points with a antiholomorphic involution $j : \mathbb{C} \to \mathbb{C}$ such that $j^2 = Id_{\mathbb{C}}$. This method leads to real algebraic curves. However, using an antiholomorphic involution induces orientability problems, which we want to avoid.

Instead, we introduce a *holomorphic* involution. This construction will lead to a split algebra structure. More precisely, a *split quaternionic structure* is considered. It is extensively used in general gravity and also appears in supersymmetry. Recently, this type of algebraic structures, attracted some attention in the domains of relativity and particle theories. Since the vector part of the split quaternions, represents the (2+1)-Minkowski space-time, and not the usual 3-dimensional Euclidean space Gogberashvili 2014.

Consider an affine orientable symmetry group

$$G = \langle \theta | \theta^2 = Id \rangle,$$

where for any $x \in \mathbb{C}$,

$$\theta : x \mapsto 1 - x;$$

and consider the representation of this group as follows. Let C/T be an n-pointed stable curve:

$$G \to \mathrm{Aut}(C/T, (x_i)).$$

Let us consider the following example and its generalisation. Take a disc **D**. It can be partitioned by a pair of orthogonal lines into 4 identical pieces (the quadrants)

[5]Here the subscript S is meant to distinguish the scheme in the cases of $\overline{\mathcal{M}}_{0,S}$ and $\overline{\mathcal{M}}_{0,S\cup\{*\}}$

denoted $Q_i, (i = 1, 2, 3, 4)$. These quadrants are incident at a common vertex. We will denote a pair of adjacent quadrants Q_{i_1} and Q_{i_2} and their respectively diagonally opposite quadrants Q'_{i_1} and Q'_{i_2}. This vertex is precisely the fixed point under involution. The group action of G on $\mathbf{D}$ maps any marked point $x(t)$ lying in a given quadrant Q_i to a point $x'(t)$ lying in its diagonally opposite quadrant Q'_i. This example can be easily generalized to an n-dimensional sphere $\mathbf{S}^n$. This sphere can, as well, be partitioned by n mutually orthogonal sections into 2^n identical n-dimensional pieces, where $Q^n_k, k = 1, \ldots, 2^{n-1}$ and $Q'^n_k, k = 1, \ldots, 2^{n-1}$, are the partitioning pieces and Q' are the diagonally opposite pieces to the Q pieces. All the 2^n pieces are incident at a common vertex. This vertex is the fixed point under the involution. The group action of G on this sphere $\mathbf{S}^n$ maps any marked point $x(t) \in Q^n_i$ to the point $x'(t) \in Q'^n_i$ where Q^n and Q'^n are mutually diagonally opposite piece.

As a more down to earth approach, we can look at θ as an affine map acting on the set of n-sections as follows:

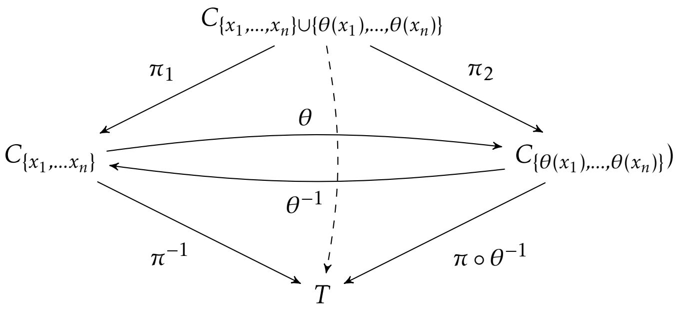
$$A_\theta : \mathbb{C}^n \to \mathbb{C}^{2n}$$

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 1 - x_1, \ldots, 1 - x_n).$$

In matrix notation one has:

$$(x_1, \ldots, x_n) \begin{bmatrix} 1 & 0 & \ldots & 0 & | & -1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & | & 0 & -1 & \ldots & 0 \\ 0 & 0 & \ldots & 1 & | & 0 & 0 & \ldots & -1 \end{bmatrix}_{(n \times 2n)} + \left(\begin{bmatrix} 0 & \ldots & 0 \\ 0 & \ldots & 0 \\ 0 & \ldots & 0 \end{bmatrix}_{1 \times n}, \begin{bmatrix} 0 & \ldots & 0 \\ 0 & \ldots & 0 \\ 0 & \ldots & 0 \end{bmatrix}_{1 \times n} \right)_{1 \times 2n}$$

Consider the graph function of $\theta \in \mathrm{Aut}(C/T, (x_i))$. Given the set of S–labeled points on C/T, consider a binary relation θ (endorelation) on $(C/T; (x_i)_{i \in S})$. This binary relation is given by the holomorphic involution $\theta : x \mapsto 1 - x$, which maps the n-tuple $(x_i)_{i \in S}$ of labeled marked points on C/T to $(\theta(x_i))_{i \in S}$. The graph of the binary relation θ from C/T to itself is formed from the pairs $(\mathbf{x}, \theta(\mathbf{x}))$ and the relation is functional and entire. For $(C/T, (x_i, \theta(x_i))_{i \in S})$, we have the following two diagrams:



,

$$\begin{array}{ccc} (C,(x_i)_{i\in S}) & \xrightarrow{\theta} & (C,(\theta(x_i))_{i\in S}) \\ & \searrow^{\pi} \quad {}^{\pi\circ\theta^{-1}}\swarrow & \\ & T & \end{array} .$$

Monad structure

As defined in section 2.4, let us consider the category $\overline{\mathcal{M}}_{0,S}$ of S–labeled stable genus 0 curves, the category $Sch_{\mathbb{Q}}$ of schemes over $\mathbb{Q}$ and the functor $b : \overline{\mathcal{M}}_{0,S} \to Sch_{\mathbb{Q}}$. We will rely on notions of monads and endofunctors, such as defined in Mac Lane 1998, Chap. VI.

Consider an involutive endofunctor $\theta : \overline{\mathcal{M}}_{0,S} \to \overline{\mathcal{M}}_{0,S}$ i.e. an endofunctor verifying the following condition:

$$\theta^2 = \theta \circ \theta = Id_{\overline{\mathcal{M}}_{0,S}}.$$

This operation defines a monad in the category $\overline{\mathcal{M}}_{0,S}$.

The transformations $\eta : I_{\overline{\mathcal{M}}_{0,S}} \xrightarrow{*} \theta$ and $\mu : \theta^2 \xrightarrow{*} \theta$ are natural, and the following diagrams commute:

$$\begin{array}{ccc} \theta^3 = \theta & \xrightarrow{\theta_\mu} & Id_{\overline{\mathcal{M}}_{0,S}} \\ {\scriptstyle \mu_\theta}\downarrow & & \downarrow{\scriptstyle \mu} \\ Id_{\overline{\mathcal{M}}_{0,S}} & \xrightarrow[\mu]{} & \theta \end{array}$$

$$\begin{array}{ccccc} \theta^3 = \theta & \xrightarrow{\eta\theta} & \theta^2 = Id_{\overline{\mathcal{M}}_{0,S}} & \xrightarrow{\theta\eta} & \theta Id_{\overline{\mathcal{M}}_{0,S}} \\ {\scriptstyle =}\downarrow & & & & \downarrow{\scriptstyle =} \\ \theta & \xrightarrow[=]{} & \theta & \xrightarrow[=]{} & \theta \end{array} .$$

In particular, we choose the θ-algebra for the monad θ to be constructed on the model of the group action $G \times \overline{\mathcal{M}}_{0,S}$, with the structure map $H : G \times \overline{\mathcal{M}}_{0,S} \to \overline{\mathcal{M}}_{0,S}$ verifying $H(g_1 g_2, x) = H(g_1, H(g_2, x))$, $H(u, x) = x$. This leads to defining the category of S–labeled stable curves of genus 0, obtained by the action of G on objects of $\overline{\mathcal{M}}_{0,S}$, denoted by $\overline{\mathcal{M}}^{\theta}_{0,S}$.

In addition, this defines a commutative diagram in the flavour of first diagram of p.11, where we replace $C_{x_1,\ldots,x_n}$ by $\overline{\mathcal{M}}_{0,S}$ (resp. $C_{\theta(x_1),\ldots,\theta(x_n)}$ by $\overline{\mathcal{M}}^{\theta}_{0,S}$) and T by $Sch_{\mathbb{Q}}$. The dashed arrow corresponds to $b : \overline{\mathcal{M}}_{0,S} \times \overline{\mathcal{M}}^{\theta}_{0,S} \to Sch_{\mathbb{Q}}$. This leads to the next proposition.

Proposition 1 – *Let $\overline{\mathcal{M}}_{0,S}$ be the category of S–labeled stable curves of genus 0 and $\overline{\mathcal{M}}^{\theta}_{0,S}$ the category of S–labeled stable curves of genus 0, obtained by the action of G on objects of $\overline{\mathcal{M}}_{0,S}$. Then, $b : \overline{\mathcal{M}}_{0,S} \times \overline{\mathcal{M}}^{\theta}_{0,S} \to Sch_{\mathbb{Q}}$ is a groupoid.*

Proof. Construct the isomorphism of categories $\theta : \overline{\mathcal{M}}_{0,S} \to \overline{\mathcal{M}}^{\theta}_{0,S}$, where any object of $\overline{\mathcal{M}}^{\theta}_{0,S}$ is the image of one object C/T by the map $\theta \in \mathrm{Aut}(C/T)$; and any morphism of S–labeled stable curves $c, c' \in \overline{\mathcal{M}}_{0,S}$ is mapped to a morphism of S–labeled stable curves in $\overline{\mathcal{M}}^{\theta}_{0,S}$: $\theta f : \theta c \to \theta c'$. The definition of this morphism induces a bijection of the two families of S–labeled curves in $\overline{\mathcal{M}}_{0,S}$ and respectively in $\overline{\mathcal{M}}^{op}_{0,S}$, in other words the following diagram is commutative:

$$\begin{array}{ccc} c & \xrightarrow{\ \theta\ } & \theta c \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \theta f} \\ c' & \xrightarrow[\ \theta\ ]{} & \theta c' \end{array}$$

Now, we show that $(b, b') : \overline{\mathcal{M}}_{0,S} \times \overline{\mathcal{M}}^{\theta}_{0,S} \to Sch_{\mathbb{Q}}$ is a groupoid. Indeed, given a base $T \in Sch_{\mathbb{Q}}$, any morphism of families over T inducing identity on T is an isomorphism for $\overline{\mathcal{M}}_{0,S}$ and, by the construction above for $\overline{\mathcal{M}}^{\theta}_{0,S}$. Hence, the *condition for groupoids* in section 2.1 is satisfied, and we have defined a groupoid. □

4 Operad for curves with θ-symmetry

In this section, we develop an operad for n-pointed genus 0 curves, endowed with the transformation θ. Throughout the paper, we call this operad *NY operad* .

The S–labeled stable curves, equipped with the bi-functor "$\otimes$" form a monoidal category. In particular, there exists a well-defined topological operad $\mathcal{P}$ for the S–labeled stable curves of genus 0 (see previous section). By symmetry, the S–labeled stable θ-curves (i.e. obtained by using the involution θ) form also a monoidal category and lead to a well-defined topological operad $\mathcal{Q}$ for the S–labeled stable θ-curves.

Define a morphism of operads in categories $\mathcal{P} \to \mathcal{Q}$. This is a sequence of functors $\mathcal{P}(r) \to \mathcal{Q}(r)$, preserving the internal structures associated to the operads. Therefore, the stable curves, indexed by the graph $(\mathbf{x}, \theta(\mathbf{x}))$ inherit a monoidal structure.

Definition 3 – Let $\overline{\mathcal{M}}_{g,S}(\theta)$ be the category fibered in groupoid of S-labeled curves with θ-symmetries over the category of schemes over $\mathbb{Q}$, $Sch/\mathbb{Q}$, whose objects for $|S| = 2n$ are $2n$-pointed stable curves $(C/T, x_1, \ldots, x_n, \theta(x_1), \ldots, \theta(x_n))$ with C of genus g. This is a subgroupoid of $\overline{\mathcal{M}}_{g,S}$.

We show that there exists a topological operad on the monoidal category $\overline{\mathcal{M}}_{g,S}(\theta)$. To construct this operad we use a monoidal functor between the classical monoidal category $\overline{\mathcal{M}}_{g,S}$ of S–labeled stable curves and the monoidal category $\overline{\mathcal{M}}_{g,S}(\theta)$ of curves with the θ-symmetry.

For the sake of clarity let us recall a few notions and notations concerning monoidal functors and Lax monoidal functors.

Definition 4 (Monoidal functor) – A monoidal functor $\Phi = (F_1, F_2, F_0) : \mathbf{C} \to \mathbf{C}'$ between monoidal categories $\mathbf{C}$ and $\mathbf{C}'$ consists of the following items:

- An ordinary functor $F_1 : \mathbf{C} \to \mathbf{C}'$ between categories;

- For objects a, b in $\mathbf{C}$ morphisms:

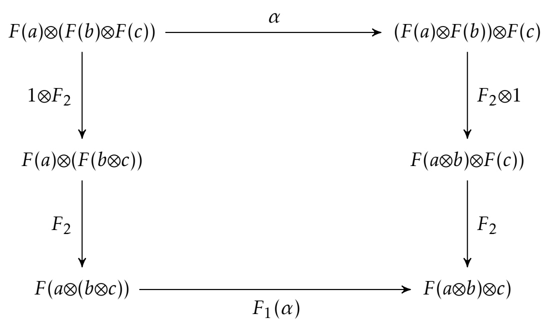
$$F_2(a,b) : F(a) \otimes F(b) \to F(a \otimes b) \tag{2}$$

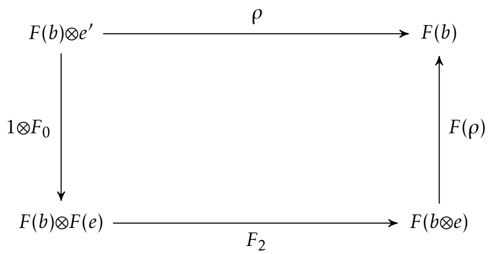
  in $\mathbf{C}$ which are natural in a and b

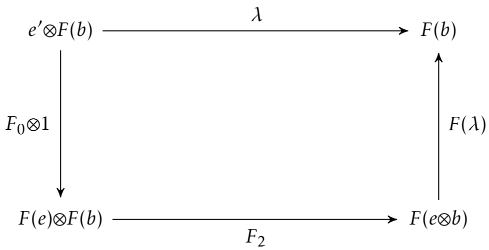
- For the units e and e', a morphism in $\mathbf{C}'$

$$F_0 : e' \to Fe \tag{3}$$

Together these must make all the following three diagrams involving the structural maps α, λ and ρ commute in $\mathbf{C}'$.







Proposition 2 – *Lax monoidal functors send monoids to monoids: if* $F : (\mathbf{C}, \otimes) \to (\mathbf{C}', \otimes)$ *is a lax monoidal functor and*

$$A \in \mathbf{C}, \mu_A : A \otimes A \to A, i_A : I \to A \tag{4}$$

is a monoid object in $\mathbf{C}$*, the object* $F(A)$ *is naturally equipped with the structure of a monoid in* $\mathbf{C}'$ *by setting*

$$i_{F(A)} : I_{\mathbf{C}'} \to F(I_{\mathbf{C}}) \xrightarrow{F(i_A)} F(A) \tag{5}$$

and

$$\mu_{F(A)} : F(A) \otimes F(A) \to F(A \otimes A) \xrightarrow{F(\mu_A)} F(A). \tag{6}$$

This construction defines functor,

$$Mon(f) : Mon(\mathbf{C}) \to Mon(\mathbf{C}').$$

Lemma 1 – *Let* $\overline{\mathcal{M}}_{g,S}(\theta)$ *be the monoidal category of curves with* θ*-symmetry defined above. Then, there exists an operadic structure on the monoidal category* $\overline{\mathcal{M}}_{g,S}(\theta)$*.*

Proof. Let $\Phi : \overline{\mathcal{M}}_{g,S} \to \overline{\mathcal{M}}_{g,S}(\theta)$ be a monoidal functor between the categories $\overline{\mathcal{M}}_{g,S}$ and $\overline{\mathcal{M}}_{g,S}(\theta)$ defined above.

First consider the category $\overline{\mathcal{M}}_{g,S}$. By applying definition 2, we have a presheaf: $\mathcal{P} : Fin^{op} \to \overline{\mathcal{M}}_{g,S}$ that defines an operad structure.

Using the functor defined in equation 6, Prop. 2, we define $Mon(f): Mon(\overline{\mathcal{M}}_{g,S}) \to Mon(\overline{\mathcal{M}}_{g,S}(\theta))$, mapping the monoid in $\overline{\mathcal{M}}_{g,S}$ to the monoid in $\overline{\mathcal{M}}_{g,S}(\theta)$. Using the relations in the diagrams of definition 4 we have that

$$\Phi(\mathcal{P}): Fin^{op} \to \overline{\mathcal{M}}_{g,S}(\theta)$$

is a presheaf. In fact, from Prop 2, it follows that the composition operation is conserved in $\overline{\mathcal{M}}_{g,S}(\theta)$, i.e. one has:

$$\Phi(\mathcal{P}(Y/X) \otimes \mathcal{P}(X)) = \Phi(\mathcal{P}(Y/X)) \otimes \Phi(\mathcal{P}(X)).$$

Therefore, the morphism $\Phi(\mathcal{P}(Y/X) \otimes \mathcal{P}(X)) \to \Phi(\mathcal{P}(Y))$ turns out to be:

$$\Phi(\mathcal{P}(Y/X)) \otimes \Phi(\mathcal{P}(X)) \to \Phi(\mathcal{P}(Y)),$$

which defines what we wanted. □

Therefore, we may formulate the following definition of the topological NY operad.

Definition 5 (NY Operad) – Let $\overline{\mathcal{M}}_{g,S}$ and $\overline{\mathcal{M}}_{g,S}(\theta)$ be monoidal categories of labeled stable curves, defined above. Let $\Phi : \overline{\mathcal{M}}_{g,S} \to \overline{\mathcal{M}}_{g,S}(\theta)$ the monoidal functor. Let $\mathcal{P}$ be the presheaf giving the operad structure on $\{\overline{\mathcal{M}}_{0,n+1}\}_{n\geq 3}$. We define the $NY = \{NY(n+1)\}_{n\geq 3}$ to be the operad in the category $\overline{\mathcal{M}}_{g,S}(\theta)$, given by the presheaf $\mathcal{P}' = \Phi \circ \mathcal{P} =: Fin^{op} \to \overline{\mathcal{M}}_{g,S}(\theta)$ and endowed with the partial operadic composition $\circ_{X\subset Y}$:

$$\mathcal{P}'(Y/X) \otimes \mathcal{P}'(X) \to \mathcal{P}'(Y),$$

for any $X \subset Y$ and a unit $\eta : 1_{\mathbf{C}} \to \mathcal{P}'(\{*\})$.

Remark 2 – The projective manifold parametrising stable curves of genus 0 with θ-symmetry contains an extra nodal point. This is the fixed point under θ, denoted Fix_θ.

This leads us to explicitly consider the smooth stratum, parametrising stable curves of genus 0 with θ-symmetry. It is given by:

$$\{(z_1, \dots, z_n, \theta(z_1), \dots, \theta(z_n), 0, 1, \infty) \in \mathbb{P}^{2n+3} | z_i \in \mathbb{P}^1 \setminus \{0, \frac{1}{2}, 1, \infty\}, z_i \neq z_j\}.$$

Note that it corresponds to a subspace of $\mathcal{M}_{0,2n+3}$ (with no presupposed symmetries). It is, in particular, necessary to remove $\frac{1}{2}$ from $\mathbb{P} \setminus \{0,1,\infty\}$ for each point $z_i \in \mathbb{P} \setminus \{0,1,\infty\}$.

Naturally, the composition operation for the NY operad is inherited from the one of $\{\overline{\mathcal{M}}_{0,2n+3}\}_{n\geq 0}$:

$$\overline{\mathcal{M}}_{0,2n+3} \circ_i \overline{\mathcal{M}}_{0,2m+3} \to \overline{\mathcal{M}}_{0,2(n+m+2)+3}.$$

This remark is illustrated by the following investigation.

∗ The codimension 0 stratum of NY is obtained by cutting out the fixed point under the transformation θ, which forms a singularity. The smooth stratum is given by the set of n-tuples:

$$\{(x_1, x_2, ..., x_{2n}, 0, 1, \infty) \in \mathbb{P}^{2n+3} | x_i \in \mathbb{P} \setminus \{0, \frac{1}{2}, 1, \infty\}, x_i \neq x_j\}.$$

The codimension 0 stratum of $NY(n)$ is contained in $\mathcal{M}_{0,2n+3}$, defined as:

$$\mathcal{M}_{0,2n+3} = \{(x_1, x_2, ..., x_{2n}, 0, 1, \infty) \in \mathbb{P}^{2n+3} \mid x_i \in \mathbb{P} \setminus \{0, 1, \infty\}, x_i \neq x_j\}.$$

∗ The codimension 1 stratum:

$$(\mathcal{M}_{0,2n+3} \setminus \bigcup_{i=1}^{n}\{x_i = \theta(x_i) = \frac{1}{2}\}) \times (\mathcal{M}_{0,2m+3} \setminus \bigcup_{i=1}^{m}\{x_i = \theta(x_i) = \frac{1}{2}\}),$$

is isomorphic to a subspace of the codimension 1 stratum $\mathcal{M}_{0,2n+3} \times \mathcal{M}_{0,2m+3}$.

The composition map is illustrated in the following way.

Example 1 – In the simplest case i.e. when $n = 1$, we have the moduli space of 5-pointed curves with two θ-symmetric points given by:

$$\{(x_1, \theta(x_1), 0, 1, \infty) \in \mathbb{P}^5 \mid x_i \in \mathbb{P}^1 - \{0, \frac{1}{2}, 1, \infty)\},$$

where the transformation θ, implies $\theta(x_1) = 1 - x_1 \in \mathbb{P}^1 \setminus \{0, \frac{1}{2}, 1, \infty\}$. This is a sub-moduli space of $\mathcal{M}_{0,5}$. We now endow this space with a composition map " ∘ " to form an algebra structure (and thus the NY operad).

Let us compose this space, for instance, with the moduli space of 7-pointed curves with four pairwise θ-symmetric points:

$$\{(y_1, y_2, \theta(y_1), \theta(y_2), 0, 1, \infty) \in \mathbb{P}^7 \mid y_i \in \mathbb{P}^1 \setminus \{0, \frac{1}{2}, 1, \infty\}, y_1 \neq y_2\}.$$

The composition is given by using the subspace of the codimension 1 stratum $\mathcal{M}_{0,5} \times \mathcal{M}_{0,7}$ and results in the following space:

$$\{(z_1, z_2, z_3, z_4, \theta(z_1), \theta(z_2), \theta(z_3), \theta(z_4), 0, 1, \infty) \in \mathbb{P}^{11} \mid z_i \in \mathbb{P}^1 \setminus \{0, \frac{1}{2}, 1, \infty\}, z_i \neq z_j\}.$$

This gives a moduli space of 11-pointed curves with two θ-symmetric 4-tuples.

The generalisation to other subspaces or higher arities can be done following the same recipe in terms of codimension 1 strata.

5 The DM–stack with hidden symmetry

It has been previously shown that the monoidal category of DM–stacks, enriched by the θ-symmetry, gives a collection of objects, forming an operad denoted $NY = \{NY(n+1)\}_{n\geq 1}$. We use this NY operad to construct a new type of gravity operad, in the flavour of Getzler 1995, Ginzburg and Kapranov 1994.

5.1 Stratification of $\overline{\mathcal{M}}_{0,n}$ with θ-symmetry

We show properties of the stratification of $\overline{\mathcal{M}}_{0,n}$ with symmetry. Consider an object $(C/T,(x_i)_{i\in S})$, where C belongs to the category $\overline{\mathcal{M}}_{0,n}$ discussed in section 2.1, and T belongs to the category $Sch_{\mathbb{Q}}$. The scheme $T \in Sch_{\mathbb{Q}}$ is decomposed into a disjoint union of strata, where each stratum $D(\tau)$ is indexed by a stable S-tree τ. The stratum $D(\tau)$ is a locally closed, reduced and irreducible subscheme of T, and the parametrizing curves are of the combinatorial type τ. The codimension of the stratum $D(\tau)$ equals to the cardinality of the set of edges of τ (i.e. the number of singular points of a curve of type τ). Note that this subscheme depends only on the n-isomorphism classes of τ.

The graph, τ, is the dual graph. It is obtained by blowing up the colliding points in the fat diagonal Δ (where Δ is given by $\{x_i = x_j | i \neq j\}$). The fat diagonal forms an $\mathcal{A}_{n-1}$ singularity. By Hironaka 1964, one can blow-up the singular locus in such a way that it gives a divisor with normal crossings. This divisor is usually called D and lies in the total space. To avoid any type of confusion, we denote by D^τ a divisor with normal crossings, lying in the total space C and with combinatorial data encoded by the graph τ.

The divisor D^τ corresponds to the given stratum $D(\tau)$ of $\overline{\mathcal{M}}_{0,n}$ (the base space). It is decomposed into a sum of irreducible components which are closed integral subschemes of codimension 1, on the blown-up algebraic variety. We have

$$D^\tau = \sum_{n\in I} n_i D_{\tau,i},$$

where the n_i are integers. Each of the irreducible components are isomorphic to $\mathbb{P}^1$. The normal crossing condition implies that each irreducible component is non singular and, whenever r irreducible components $D_{\tau,1},\dots,D_{\tau,r}$ meet at a point P, the local equations $f_1,\dots,f_r$ of the $D_{\tau,i}$ form part of a regular system of parameters at P. It is also possible to define locally the divisor by $\{(U_i,f_i)\}$, where f_i are holomorphic functions and U_i are open subsets.

The closure of a stratum $\overline{D}(\tau)$ in T is formed from the union of subschemes $D(\sigma)$, where $\tau > \sigma$, such that τ and σ have the same set of tails. The tails correspond to the set of marked points on the irreducible connected component. In the geometric realization of the graph, the tails—also called *flags*—correspond to a set of numbered half-edges, which are incident to only one vertex. This set is denoted by $F_\tau(v)$, where v is a given vertex in the graph τ, to which the half-edges are incident. In our case, where the genus of the curve is zero, the condition $\tau > \sigma$ is uniquely specified by the *splitting* data, which can be described as a certain type of Whitehead move.

Let us expose roughly the splitting procedure. Choose a vertex v of τ and a partition of the set of flags, incident to v: $F_\tau(v) = F'_\tau(v) \cup F''_\tau(v)$ such that both subsets are invariant under the involution $j_\tau : F_\tau \to F_\tau$. To obtain σ, replace the vertex v by two vertices v' and v'' connected by an edge e, where the flags verify $F'_\tau(v') = F'_\tau(v) \cup \{e'\}$, $F'_\tau(v'') = F''_\tau(v) \cup \{e''\}$, where e', e'' are the two halves of the edge e. The remaining vertices, flags and incidence relations stay the same for τ and σ. For more details, see Y. Manin 1999 Chap. III § 2.7, p.90.

Consider the stratification of the scheme T by graphs (trees, in fact), such as depicted in Y. Manin 1999 in Chap. III §3. Let τ be a tree. If S is a finite set, then $\mathcal{T}((S))$ is the set of isomorphism classes of trees τ, whose external edges are labeled by the elements of S. The set of trees is graded by the number of edges:

$$\mathcal{T}((S)) = \bigcup_{i=0}^{|S|-3} \mathcal{T}_i((S)),$$

where $\mathcal{T}_i((S))$ is a tree with i edges. The tree $\mathcal{T}_0((S))$ is the tree with one vertex and the set of flags equals to S.

Not only do those trees allow a stratification of $\overline{\mathcal{M}}_{0,n}$, but due to Etingof et al. 2010; Keel 1992; Singh 2004, one may derive relations for computing the cohomology ring $H^*(\overline{\mathcal{M}}_{0,n}(\mathbb{C}))$. In particular, $\overline{\mathcal{M}}_{0,n}$ has no odd homology and its Chow groups are finitely generated and free abelian.

We recall briefly the Keel presentation Keel 1992. Let D_S be a component of a divisor lying in $\overline{\mathcal{M}}_{0,n}$, where $S \subset \{1, 2, ..., n\}$. In Keel's presentation (see Keel 1992 section 1), the n-th Chow ring turns out to be isomorphic to the quotient of the polynomials ring, generated by degree 2 elements (the D_S), modulo an ideal in which the D_S are subject to the following relations:

- $D_S = D_{\{0,1,...,n\}\setminus S}$.
- For distinct elements $i, j, k, l \in \{0, 1, ..., n\}$

$$\sum_{i,j\in S; k,l\notin S} D_S = \sum_{i,k\in S; j,l\notin S} D_S.$$

- If $S \cap T \notin \{0, S, T\}$ and $S \cup T \neq \{0, 1, ..., n\}$ then $D_S D_T = 0$.

In Dotsenko 2020, it is shown that $H^*(\overline{\mathcal{M}}_{0,n})$ is Koszul, which is useful to determine various homotopy invariants for the DM–compactification.

We proceed to a brief discussion concerning the stratification of the S-stable curves of genus 0 with θ-symmetry $\overline{\mathcal{M}}_{0,S}(\theta)$, i.e. indexed by the graph $(x,\theta(x))$ and compare it to the strata in the classical $\overline{\mathcal{M}}_{0,S}$ case. A list of the first strata occurring in the base space for $\overline{\mathcal{M}}_{0,S}(\theta)$ is given. The word "codim" stands for codimension:

Codim n			
$n=0$	$(x_1,..,x_n)$:	$x_i \neq x_j$ for $i \neq j$,	$x_i \in \mathbb{P}^1 \setminus \{0,1,\frac{1}{2},\infty\}$.
$n=1$	$(x_1,..,x_n)$:	$x_i \neq x_j$ for $i \neq j$,	$x_i \in \mathbb{P}^1 \setminus \{0,1,\infty\}$.
$n=1$	$(x_1,..,x_n)$:	$x_{i_1} = x_{i_2}$,	$x_i \in \mathbb{P}^1 \setminus \{0,1,\frac{1}{2},\infty\}$,
		$x_{i_k} \neq x_{i_j}$ for $k,j \notin \{1,2\}$,	$x_i \in \mathbb{P}^1 \setminus \{0,1,\frac{1}{2},\infty\}$.
$n=2$	$(x_1,..,x_n)$:	$x_{i_1} = x_{i_2} = x_{i_3}$,	$x_i \in \mathbb{P}^1 \setminus \{0,1,\infty\}$.
$n=2$	$(x_1,..,x_n)$:	$x_{i_2} = x_{i_3}$,	$x_i \in \mathbb{P}^1 \setminus \{0,1,\infty\}$,
			$x_{i_1} \in \mathbb{P}^1 \setminus \{0,1,\frac{1}{2},\infty\}$.
$n=3$	$(x_1,..,x_n)$:	$x_{i_1} = x_{i_2} = x_{i_3} = x_{i_4}$,	$x_i \in \mathbb{P}^1 \setminus \{0,1,\infty\}$.
$n=3$	$(x_1,..,x_n)$:	$x_{i_2} = x_{i_3} = x_{i_4}$,	$x_i \in \mathbb{P}^1 \setminus \{0,1,\infty\}$,
			$x_{i_1} \in \mathbb{P}^1 \setminus \{0,1,\frac{1}{2},\infty\}$.
$n=4$	$(x_1,..,x_n)$:	$x_{i_1} = x_{i_2}$,	$x_i \in \mathbb{P}^1 \setminus \{0,1,\frac{1}{2},\infty\}$.
$n=4$	$(x_1,..,x_n)$:	$x_{i_1} = ... = x_{i_5}$,	$x_i \in \mathbb{P}^1 \setminus \{0,1,\infty\}$.

Table 1

This discussion, allows further considerations concerning the fibers of the proper flat map $\pi : C \to T$, where C is the $|S|$-stable curve. To a given divisor $D^\tau = \sum_{n\in I} n_i D_i^\tau$, locally defined by a chart $\{(U_i,f_i)\}$, the symmetry θ maps D^τ to the corresponding divisor: $D^{\theta(\tau)} = \sum_{n\in I} n_i D_i^{\theta(\theta)}$ defined by $\{(U_i,\theta(f_i))\}$. So, this defines a pair of divisors $(D^\tau, D^{\theta(\tau)})$, being isomorphic. The intersection (D_τ, D_τ^θ) is non-empty if and only if $f_i = \theta(f_i)$.

Proposition 3 – *Any pair of divisors D^τ and $D^{\theta(\tau)}$ indexing a given stratum of the parametrising space of $\overline{\mathcal{M}}_{0,S}$ and $\overline{\mathcal{M}}_{0,S}^\theta$ are a point reflection of each other and their union in $\overline{\mathcal{M}}_{0,S} \times \overline{\mathcal{M}}_{0,S}^\theta$ forms a connected set if and only if there exists a holomorphic function such that $f_i = \theta(f_i)$.*

Proof. One direction is easy: if $D^{\theta(\tau)}$ and D^τ are disjoint, then there are no points verifying $f_i = \theta(f_i)$. This occurs in the case where the sections are different from $\frac{1}{2}$, so lying in $\mathbb{P}^1 \setminus \{0,1,\frac{1}{2},\infty\}$. Reciprocally, if there exists a chart for which $f_i = \theta(f_i)$, then the set $D^\tau \cup D^{\theta(\tau)}$ is connected. It is clear that $D^{\theta(\tau)}$ is a point reflection of D^τ: each component of the divisor D^τ, being given by the charts an $\{(U_i,f_i)\}$, where U_i is in $\mathbb{C}$ and f_i is a holomorphic function, therefore the charts for $D^{\theta(\tau)}$ given by $\{\theta(U_i),\theta(f_i)\}$ are given by the point the point reflection θ, so concerning the geometry of the divisor $D^{\theta(\tau)}$ it is a point reflection of D^τ. □

Proposition 4 – *The graphs, indexing the strata of the scheme parametrising $\overline{\mathcal{M}}_{0,n}(\theta)$, are connected and invariant under a symmetry group of order 2.*

Proof. The hyperplane arrangements giving the set of colliding points, for $\overline{\mathcal{M}}_{0,n}$ and $\overline{\mathcal{M}}^{\theta}_{0,n}$ are symmetric to each other by the point reflection θ. The divisors of $\overline{\mathcal{M}}_{0,n}$ and $\overline{\mathcal{M}}^{\theta}_{0,n}$ being defined exactly in the same way (i.e. in local coordinates we have $f_1...f_i = 0$, and for their symmetrical image $\theta(f_1)...\theta(f_i) = 0$) the divisors for $\overline{\mathcal{M}}_{0,n}$ and $\overline{\mathcal{M}}^{\theta}_{0,n}$ are isomorphic to each other. These divisors and glued together along the irreducible component corresponding to the blown-up fixed point Fix_{θ}. The graphs are thus glued along a point and symmetrical to each other. So, this implies that the dual graphs indexing $\overline{\mathcal{M}}_{0,n}(\theta)$ are connected and invariant under a symmetry group of order 2. □

5.2 The stacky $\overline{\mathcal{M}}_{0,S}(\theta)$

Theorem 1 – *The category fibered in groupoid $\overline{\mathcal{M}}_{0,S}(\theta)$ over $Sch_{\mathbb{Q}}$ of S-pointed stable curves with θ-symmetry is a stack.*

Proof. In order to show that $\overline{\mathcal{M}}_{0,S}(\theta)$ is a stack, let us first equip the base space $\mathcal{S}$ with the étale topology $\mathcal{T}$. We need to verify the three conditions of definition 1.

1. The first condition is to show that the contravariant functor from $\mathcal{S}^{op}$ to the category of sets Set is a sheaf.

   We know that if $\mathcal{S}$ has a Grothendieck topology, then $\overline{\mathcal{M}}_{0,n}$ is a stack. The modification of this stack into $\overline{\mathcal{M}}_{0,(x,\theta(x))}$ implies a slight modification of the data. Indeed, $\mathcal{F} : \mathcal{S}^{op} \to Set$ is a sheaf. Properties of the category Set allow to consider the direct sum $Set \oplus Set$. So, we are dealing with the section $\mathcal{S}^{op} \to Set \oplus Set$, which turns out to be a direct sum of sheafs: $\mathcal{F} + \mathcal{F} : \mathcal{S}^{op} \to Set \oplus Set$, hence a sheaf. So, we have a sheaf $\mathcal{S}^{op} \to Set$.

2. The second condition to check is that for any open T' in T, the functor $T' \mapsto Iso_{T'}(X_1 \oplus X_1^{\theta}, X_2 \oplus X_2^{\theta})$ is a sheaf.

   By hypothesis, we know that for any open T', the functor $T' \mapsto Iso_{T'}(X_1, X_2)$ is a sheaf. Clearly, the functor $T' \mapsto Iso_{T'}(X_1^{\theta}, X_2^{\theta})$ is also a sheaf. So, the map from $T' \mapsto Iso_{T'}(X_1, X_2) \oplus Iso_{T'}(X_1^{\theta}, X_2^{\theta})$ is a sheaf and by elementary properties of $\oplus$, we have $Iso_{T'}(X_1, X_2) \oplus Iso_{T'}(X_1^{\theta}, X_2^{\theta}) = Iso_{T'}(X_1 \oplus X_1^{\theta}, X_2 \oplus X_2^{\theta})$.

3. The last property is the so-called cocycle condition. Let $\{T_i \xrightarrow{\phi_i} T\}_i$ be an étale cover of T, where ϕ_i are étale maps and let F be a family over T. Then, applying to F the base change functors ϕ^*, we get localized families F_i over T_i, and similarly localized families F_{ij} over $T_{ij} := T_i \times_T T_j$, F_{ijk} over T_{ijk}, (etc).

They come along with the descent data, i.e isomorphisms $f_{ij} : pr^*_{ji,i}F_i \xrightarrow{\cong} pr^*_{ji,j}F_j$, which turn to satisfy the cocycle condition: $f_{ki} = f_{kj} \circ f_{ji}$ on T_{kji}. The family F is compatible with the direct sum operation.

Therefore, we have over T the family: $F \oplus F^\theta = (C/T, (x_i)) \oplus (C/T, (\theta(x_i)))$. The base change functors ϕ^* give localized families $F_i \oplus F_i^\theta$ over T_i (more generally, $F_{ij} \oplus F_{ij}^\theta$ over $T_{ij} := T_i \times_T T_j$, $F_{ijk} \oplus F_{ijk}^\theta$ over T_{ijk}, etc).

We have $(f_{ij}, f_{ij}^\theta) : (pr^*_{ji,i}F_i, pr^*_{ji,i}F_i^\theta) \xrightarrow{(\cong,\cong)} (pr^*_{ji,j}F_j, pr^*_{ji,j}F^\theta j)$, satisfying the cocycle condition: $(f_{ki}, f_{ki}^\theta) = (f_{kj} \circ f_{ji}, f_{kj}^\theta \circ f_{ji}^\theta)$. The converse property comes from 2). □

The construction of $\overline{\mathcal{M}}_{0,S}(\theta)$ being very close and in some sense inherited from $\overline{\mathcal{M}}_{0,S}$, we can state the following.

Corollary 1 – *The category fibered in groupoid $\overline{\mathcal{M}}_{0,S}(\theta)$ over $Sch_{\mathbb{Q}}$ of S-pointed stable curves with θ-symmetry is an algebraic stack of Deligne–Mumford type.*

5.3 The NY Gravity operad

In this section, we introduce the NY gravity operad $Grav_{NY}$. The terminology of *gravity* was chosen here because of the geometric structure of the space $\overline{\mathcal{M}}_{0,n}$ with θ-symmetry. In particular, this is equivalent to introducing a split quaternionic structure, given by θ. Such a structure can be interpreted as the realisation of a module over a ring extending the complex numbers: the ring of split quaternion numbers. This type of model serves in gravity and general relativity, as one can see for instance in Kulyabov, Korolkova, and Gevorkyan 2020 and Ulrych 2006.

Actually, whenever we want to take into account action of symmetries upon complexes (de Rham, Hodge etc) and respective mixed structures, we are bound to pass to one of the several derived environments. For example, one can use a quite general "dendroidal" formalism of Cisinski and Moerdijk 2013, perhaps extended by various versions of graph categories developed in Borisov and Y. Manin 2007, or even Feynman categories.

For the NY gravity operad, we take inspiration in a similar construction as in Getzler 1995. In the latter case, the gravity operad $Grav = \{Grav(n)\}_{n\geq 0}$ with n-arity $Grav(n) = sH_*(\mathcal{M}_{0,n})$ where s denotes the suspension, has a mixed Hodge structure.

Grav lies in the symmetric monoidal category of mixed Hodge complexes. The monoidal structure is given by the graded tensor product. Every object $Grav(n) = sH_*(\mathcal{M}_{0,n})$ in this category, carries a unique mixed Hodge structure, compatible with the Poincaré residue map.

The reason mixed Hodge structures enter the game follows from Prop. 8.2.2 in Deligne 1971. This proposition states that for any integer k, the k^{th} cohomology group of a given complex algebraic variety is endowed with a functorial mixed Hodge structure. We will investigate similar structures for our NY gravity operad.

As was described in Rem. 2, Fix_θ denotes the set of fixed points under the involution θ. The locus, in this context, is a point. Since this gives a singular (nodal) point (or an irreducible curve if we have proceeded to a blow-up), it is necessary that we remove it, in order to define the NY Gravity operad.

Definition 6 – Let $Grav_{NY}(n)$ be the stable cyclic $\mathbb{S}$-module defined as follows:

$$Grav_{NY}(n) = \begin{cases} sH_*(\mathcal{M}_{0,n}(\theta) \setminus Fix_\theta), & n \geq 3 \\ 0, & n < 3 \end{cases}$$

Due to the split quaternionic structure of θ, we have the following implication.

Lemma 2 (Splitting lemma) – *The stack $\mathcal{M}_{0,S}(\theta)$ can be decomposed into the pair $\overline{\mathcal{M}}_{0,S} \times \overline{\mathcal{M}}_{0,S}^{\theta}$.*

Proof. The proof follows immediately from the previous discussions. □

We briefly discuss the logarithmic forms along a normal crossing divisor and recall notations from previous sections, namely about the coordinates of the normal crossing divisor. We define a divisor D as $\{f_1 \dots f_r = 0\}$ in $\overline{\mathcal{M}}_{0,n}$, for a fixed positive integer r. A meromorphic differential form on $\overline{\mathcal{M}}_{0,n}$ has a logarithmic form on the divisor if it can be written as a linear combination of forms of the type:

$$\frac{\partial f_{i_1}}{f_{i_1}} \wedge \cdots \wedge \frac{\partial f_{i_r}}{f_{i_r}} \wedge \eta,$$

with $i_1 \leq ... \leq i_r$ and where η is a holomorphic form on $\overline{\mathcal{M}}_{0,n}$. On each closure of a stratum $\overline{D(\sigma)}$, define a complex of sheaves of logarithmic forms : $\Omega^\bullet_{\overline{D(\sigma)}}(log\overline{D(\sigma)})$. If $j_\sigma : D(\sigma) \to \overline{D(\sigma)}$ denotes the natural open immersion, we have a quasi-isomorphism:

$$(j_\sigma)_\star(\mathbb{C}_{D(\sigma)}) \cong \Omega^\bullet_{\overline{D(\sigma)}}(log\overline{D(\sigma)}).$$

This induces an isomorphism of cohomology groups.

Theorem 2 – *The stable cyclic $\mathbb{S}$-module $\{Grav_{NY}(n)\}_{n\geq 3}$ admits a natural structure of cyclic operad and is denoted by $Grav_{NY}$.*

Proof. It is known that the collection of stable curves with labeled points forms a topological operad. As shown in equation (1), we have a well defined operation composition for this:

$$\overline{\mathcal{M}}_{0,l} \times \overline{\mathcal{M}}_{0,m_1} \times \cdots \times \overline{\mathcal{M}}_{0,m_l} \to \overline{\mathcal{M}}_{0,m_1+\cdots+m_l}.$$

By θ-symmetry, the same holds for the collection $\{\overline{\mathcal{M}}^{\theta}_{0,n}\}_{n\geq 3}$. Combining both morphisms together, we get the following:

$$\overline{\mathcal{M}}_{0,l}\times\overline{\mathcal{M}}^{\theta}_{0,l}\times\overline{\mathcal{M}}_{0,m_1}\times\overline{\mathcal{M}}^{\theta}_{0,m_1}\times\cdots\times\overline{\mathcal{M}}_{0,m_l}\times\overline{\mathcal{M}}^{\theta}_{0,m_l}$$
$$\to\overline{\mathcal{M}}_{0,m_1+\cdots+m_l}\times\overline{\mathcal{M}}^{\theta}_{0,m_1+\cdots+m_l}.$$

To construct the NY gravity operad, we restrict our considerations to the smooth stratum and remove the fixed point Fix_θ.

To define the composition map for $Grav_{NY}$,we proceed by using the Poincaré residue morphism(see Deligne 1971). We need the following embedding :

$$(\mathcal{M}_{0,l}\times\mathcal{M}^{\theta}_{0,l})\setminus Fix_\theta\times(\mathcal{M}_{0,m}\times\mathcal{M}^{\theta}_{0,m})\setminus Fix_\theta$$
$$\to\overline{\mathcal{M}}_{0,m+l}\times\overline{\mathcal{M}}^{\theta}_{0,m+l},$$

The Poincaré residue associated to that embedding is given by:

$$Res: H^*(\mathcal{M}_{0,m+l}\times\mathcal{M}^{\theta}_{0,m+l})\to H^*(\mathcal{M}_{0,m}\times\mathcal{M}^{\theta}_{0,m}\times\mathcal{M}_{0,l}\times\mathcal{M}^{\theta}_{0,l}).$$

The composition operation is now discussed. With the residue morphism, defined in the paragraph above, for $\tau>\sigma$ we have the residue formula:

$$Res^{\sigma}_{\tau}: H^{\bullet}(D(\sigma)\sqcup D(\sigma^{\theta}))\to H^{\bullet-1}(D(\tau)\sqcup D(\tau^{\theta}))(-1), \tag{7}$$

where we have added a right Tate twist (-1) and $D(\sigma^\theta)$ (resp. $D(\tau^\theta)$) is the stratum in $\overline{\mathcal{M}}^{\theta}_{0,m+l}$.

Let $\sigma\sqcup\sigma^\theta\in\mathcal{T}((n))\oplus\mathcal{T}((n))$ index a stratum in the moduli space $\overline{\mathcal{M}}_{0,m+l}\times\overline{\mathcal{M}}^{\theta}_{0,m+l}\setminus Fix_\theta$. We choose σ (resp. σ^θ) to have one internal edge e (resp. e_θ) joining two internal vertices v' and v'' (resp. v'_θ and v''_θ). The stratum $D(\sigma)$ in $\overline{\mathcal{M}}_{0,m+l}$ can be decomposed as follows:

$$D(\sigma)\cong\mathcal{M}_{0,F(v')}\times\mathcal{M}_{0,F(v'')},$$

where in the flag $F(v)$ (resp. $F(v'')$) there are m (resp. l) incident edges to v' (resp. v''). Proceed similarly for $D(\sigma^\theta)$. Given σ with one edge, we get the residue morphism:

$$H^{a+b-1}(\mathcal{M}_{0,m+l}\times\overline{\mathcal{M}}^{\theta}_{0,m+l}\setminus Fix_\theta)(-1)\to$$

$$H^{a-1}(\mathcal{M}_{0,F(v')}\times\mathcal{M}_{0,F(v'_\theta)})(-1)\otimes H^{b-1}(\mathcal{M}_{0,F(v'')}\times\mathcal{M}_{0,F(v''_\theta)})(-1),$$

which is obtained from equation 7 by using the Künneth formula, a Tate twist (-1) and multiplying by the Koszul sign $(-1)^{a-1}$. Finally, using the Poincaré duality, leads to the definition of a composition product $\circ_i$ for $Grav_{NY}$.

It thus remains to verify if the $Grav_{NY}$ satisfies the equivariance and associativity axioms. The equivariance axiom is, indeed, not modified. The main argument is that $t_i \in \mathbb{S}_i$ acts on the set of marked points $\{1,...,i\}$ and, thus on the set obtained under θ. So, the following diagram remains commutative:

$$\begin{array}{ccc} \mathcal{P}(k)\otimes\mathcal{P}(r_1)\otimes\cdots\otimes\mathcal{P}(r_k) & \xrightarrow{id\otimes(t_1\otimes...\otimes t_k)} & \mathcal{P}(k)\otimes\mathcal{P}(r_1)\otimes\cdots\otimes\mathcal{P}(r_k) \\ \downarrow \mu & & \downarrow \mu \\ \mathcal{P}(r_1+....+r_k) & \xrightarrow[t_1\oplus...\oplus t_k]{} & \mathcal{P}(r_1+....+r_k) \end{array}.$$

The associativity axioms of a cyclic operad also holds. Therefore, on $Grav_{NY}$ there exists a natural cyclic operad structure. □

Remark 3 – An explicit presentation of $Grav_{NY}(n)$, can be given by using the interpretation

$$Grav_{NY}(n) = sH^*(\mathbb{P}^{2n+3} \setminus (\Delta \bigcup_{i=1}^{n} \{x_i = \theta(x_i) = \frac{1}{2}\})/PSL_2(\mathbb{C})),$$

where Δ is the fat diagonal given by the colliding points i.e. $\{x_i = x_j\}$.

Then, it is enough to apply the works of Deligne, Goresky, and MacPherson (2000), Eisenbud, Popescu, and Yuzvinsky (2003), and Yuzvinsky (1999), to compute the cohomology of linear spaces minus hyperplanes.

Example 2 – We give an example on the construction of the $Grav_{NY}(n)$ operad . Consider the stratum of codimension 1. It is isomorphic to a product $\mathcal{M}_{0,2r+3} \times \mathcal{M}_{0,2s+3}$ with $(2r+2)+(2s+2) = 2n+3$. Using the residue morphisms, we have that:

$$H^*(\mathcal{M}_{0,2n+3} \setminus \cup_{i=1}^{n} H_i) \to H^{*-1}(\mathcal{M}_{0,2r+3} \setminus \cup_{i=1}^{r} H_i \times \mathcal{M}_{0,2s+3} \setminus \cup_{i=1}^{s} H_i)$$

where $H_i = \{x_i = \theta(x_i) = \frac{1}{2}\}$ are hyperplanes. After dualization, the operad structure on the collection of graded vector spaces $H_{*-1}(\mathcal{M}_{0,2n+3} \setminus H_i)$ is given.

5.4 Comparison of the NY gravity operad with gravity operad and *Hycom* operad

We study the relations between the NY gravity operad and the gravity operad. Namely, a relation is given, using the comparison theorem in this section.

The relations and the presentation for the Gravity operad are as follows. The degree 1 subspace of $Grav(k)$ is one dimensional for each $k \geq 2$, and is spanned by the operation:

$$\{a_1,...,a_n\} = \sum_{1\geq i<j\geq n} (-1)^{\epsilon(i,j)}\{a_i,a_j\}a_1....\hat{a}_i....\hat{a}_j....a_n,$$

$\epsilon(i,j) = (|a_1|+...+|a_{i-1}|)|a_i| + (|a_1|+...+|a_{j-1}|)|a_j| + |a_i||a_j|$. Using the presentation of Getzler 1995, the operations $\{a_1,...,a_k\}$ generate the gravity operad $Grav(k)$. All relations among them follow from the generalized Jacobi identity :

$$\sum_{1\leq i<j\leq k} (-1)^{\epsilon(i,j)}\{\{a_i,a_j\},a_1,...,\hat{a}_i,...,\hat{a}_j,\dots,a_k,b_1,...,b_l\} = \{\{a_1,...,a_k\},b_1,...,b_l\}, \tag{8}$$

where the right hand side is 0 if $l=0$.

The gravity operad is related to the hypercommutative operad (in short $Hycom$ operad) by Koszul duality, in the sense of Ginzburg and Kapranov 1994, i.e. we have $Hycom^! \cong Grav$. By definition,

$$Hycom(n) = \begin{cases} H_*(\overline{\mathcal{M}}_{0,n}), & n\geq 3; \\ 0, & n<3. \end{cases}$$

Supposing that $V \subset Hycom$ is the cyclic $\mathbb{S}$-submodule spanned by the fundamental classes. Then,

$$[\overline{\mathcal{M}}_{0,n}] \in H_{2(n-3)}(\overline{\mathcal{M}}_{0,n}) \subset Hycom(n).$$

The operad $Hycom$ is Koszul with generators V.

The properties of $Hycom$ are that it is quadratic with generators $\mathcal{V}$ and relations R, where $\mathcal{V}((n))$ is spanned by an element of degree $2(n-3)$ and weight $2(3-n)$. We have that $\mathcal{V}((n))$ is identified with $H_{2(n-3)}(\overline{\mathcal{M}}_{0,n})$. Relations R—where $R((n))$ has dimension $\binom{n-1}{2}$-1—are given by the following generalized associativity equation:

$$\sum_{S_1\sqcup S_2} \pm((a,b,x_{S_1}),c,x_{S_2}) = \sum_{S_1\sqcup S_2} \pm(a,(b,c,x_{S_1}),x_{S_2}), \tag{9}$$

where $a,b,c,x_1,...,x_n$ lie in $Hycom$ and $S_1 \sqcup S_2 = \{1,...,n\}$. The symbol $\pm$ stands for the Quillen sign convention for $\mathbb{Z}_2$-graded vector spaces: it equals +1 if all the variables are of even degree.

Now, we can present the *Comparison theorem*.

Theorem 3 (Comparison theorem) – *Consider the NY Gravity operad $Grav_{NY}$. Then, for $n\geq 3$ we have:*

$$Grav_{NY}(n) = sH_*(\mathcal{M}_{0,n} \setminus Fix_\theta) \otimes H_*(\mathcal{M}^\theta_{0,n} \setminus Fix_\theta)$$

where Fix_θ is the set of fixed points of the automorphism θ.

Proof. To compare the gravity operad and the NY Gravity operad, notice that it is necessary to remove the fixed points under the involution θ (one needs smooth strata). Removing Fix_θ from $\mathcal{M}_{0,n}(\theta)$ leaves us to consider the smooth stratum of this space, and implies a modification regarding the classical *Grav* operad. By the Künneth formula we have:

$$\begin{aligned} Grav_{NY}(n) &= sH_*(\mathcal{M}_{0,n} \times \mathcal{M}^{\theta}_{0,n} \setminus Fix_\theta) \\ &= s(H_*(\mathcal{M}_{0,n} \setminus Fix_\theta) \otimes H_*(\mathcal{M}^{\theta}_{0,n} \setminus Fix_\theta) \\ &= sH_*(\mathcal{M}_{0,n} \setminus Fix_\theta) \otimes H_*(\mathcal{M}^{\theta}_{0,n} \setminus Fix_\theta) \end{aligned}$$

□

Remark 4 –

We compare the NY operads' components to those of the gravity operad. This is done using the *excision theorem* and Mayer–Vietoris' exact sequence. We rely on notations of the previous examples. It follows from the excision theorem that there exists a relative homology of $H_k(\mathbb{P}^{2n+3} \setminus \cup H_i, \mathcal{M}_{0,2n+3} \setminus \cup_{i=1}^{n} H_i)$ to $H_k(\mathbb{P}^{2n+3}, \mathcal{M}_{0,2n+3})$.

The excision theorem leads to the Mayer–Vietoris (long) exact sequence. Consider $\mathcal{M}_{0,2n+3}$ and the following subsets of $\mathcal{M}_{0,2n+3}$:

- $\mathcal{M}^{\frac{1}{2}}_{0,2n+3}$ is the moduli space $\mathcal{M}_{0,2n+3}$ from which the hyperplane arrangement $\cup_{i=1}^{n} H_i$ is removed.
- Let Tub be the tubular neighborhood of the hyperplane arrangement $\cup_{i=1}^{n} H_i$.

The union of the interiors of those subspaces cover $\mathcal{M}_{0,2n+3}$, so we can define the following long exact sequence:

$$\begin{aligned} \cdots \to H_{k+1}(\mathcal{M}_{0,2n+3}) \to H_k(Tub \setminus \cup_{i=1}^{n} H_i) \to H_k(\mathcal{M}^{\frac{1}{2}}_{0,2n+3}) \oplus H_k(Tub) \\ \to H_k(\mathcal{M}_{0,2n+3}) \to H_{k-1}(Tub \setminus \cup_{i=1}^{n} H_i) \to \dots \end{aligned}$$

Applying the suspension onto the long exact sequence, we have the following:

$$\begin{aligned} \cdots \to sH_{k+1}(\mathcal{M}_{0,2n+3}) \to sH_k(Tub \setminus \cup_{i=1}^{n} H_i) \to sH_k(\mathcal{M}^{\frac{1}{2}}_{0,2n+3}) \oplus sH_k(Tub) \\ \to sH_k(\mathcal{M}_{0,2n+3}) \to sH_{k-1}(Tub \setminus \cup_{i=1}^{n} H_i) \to \dots \end{aligned}$$

and the long exact sequence looks in the following way:

$$\begin{aligned} \cdots \to H_{2n+4}(\mathcal{M}_{0,2n+3}) \to 0 \to H_{2n+3}(\mathcal{M}^{\frac{1}{2}}_{0,2n+3}) \to H_{2n+2}(\mathcal{M}_{0,n}) \to \mathbb{Z} \to \\ H_{2n+2}(\mathcal{M}^{\frac{1}{2}}_{0,2n+3}) \to H_{2n+2}(\mathcal{M}_{0,2n+3}) \to 0 \to \dots \end{aligned}$$

6 The neighbourhood of the fat diagonal and Hodge structure

This last section is devoted to geometric considerations of $\overline{\mathcal{M}}_{0,S}(\theta)$, in particular in the neighborhood of the fat diagonal.

Previously, we have introduced the NY Gravity operad, which we think describes an interesting geometric construction modeling an object coming from physics origins. In algebraic language all the physical requirements mean that to describe the geometry we need a specific algebra. Here we have chosen the algebra of split quaternion numbers, which fits nicely to space-time related problems, relativity and gravity.

1) The presence of supplementary singular points attracts our attention to how the geometry is perturbed. Those singular points appear in the realisation of the module over split quaternion numbers and in particular when it occurs to the 0-divisors of the underlying ring.

2) We propose to look at the neighbourhood of the fat diagonal for $\mathcal{M}_{0,n}(\theta)$ from a Hodge structure aspect. Our NY gravity operad being directly related to mixed Hodge structures, it is natural to consider the Hodge structure in a mostly geometric way, via tools coming from L^2-cohomology, and which are closely based on results concerning the KZ equations for $\mathcal{M}_{0,n}(\mathbb{C})$ (see Ginzburg and Kapranov 1994), and to the Stasheff polytope approach Kapranov 1993.

6.1 The fat diagonal of $\mathcal{M}_{0,n}$ with hidden symmetry

Let M be a smooth manifold. Let

$$[0,1]\times M/\sim,$$

where $(0,x)\sim(0,x')$ and $x,x'\in M$ be a (topological) cone. Let us stratify the thick diagonal in the following (rather unconventional) way:

$$\Delta^{(n)}\subset\Delta^{(n-1)}\subset\cdots\subset\Delta^{(1)}, \tag{10}$$

where:

- $\Delta^{(1)}$ is an n–tuple of Δ, where there exists one x_i lying in a small neighborhood of Fix_θ, and different from all the other x_j in Δ;
- $\Delta^{(j)}$ is the n–tuple, where the set $\{x_{i_1}=\cdots=x_{i_j}\}$ lie in a small neighborhood of Fix_θ, and those x_i are different from the remaining x_k in Δ.

Proposition 5 – *Let Δ_θ be the fat diagonal of $\overline{\mathcal{M}}_{0,n}(\theta)$. Then, Δ_θ has a $\mathbb{Z}_2$-symmetry about the locus in Δ_θ corresponding to Fix_θ. Locally, the locus Fix_θ defines the apex of a cone which can be decomposed into two symmetrical subcones.*

Proof. The space $\overline{\mathcal{M}}_{0,n}(\theta)$ being equipped with the split quaternions structure, it subdivides into a pair of isomorphic (and symmetric) subspaces. The singular locus (i.e. fat diagonal) of this spaces obeys naturally to the same property. Therefore, Δ_θ is divided into symmetrical subspaces, which we call respectively Δ_+ and Δ_- Consider the stratum $\Delta^{(1)}$, defined in formulae 10. Then, in the stratum Δ^1_+ there exists one point x_i tending to Fix_θ. In Δ^1_-, we have its symmetric version (recall examples of section 3). So, in a small neighborhood of Fix_θ, we have a topological cone of apex Fix_θ defined over the intersection of a small sphere with center in Fix_θ and $\Delta^1_+ \cup \Delta^1_-$. By induction we can see that the same holds for strata of higher codimension. We have thus defined a cone of apex Fix_θ which can be sliced into a pair of symmetrical subcones. □

The fat diagonal of the parametrizing scheme $\overline{\mathcal{M}}_{0,n} \times \overline{\mathcal{M}}^{\theta}_{0,n}$ is endowed with additional critical points, compared to the classical case (i.e. with no symmetry). Going back to the stable curve, these additional points will be blown-up, adding new irreducible components (isomorphic to $\mathbb{P}^1$). This data modifies the geometric aspect around the divisor of $\overline{\mathcal{M}}_{0,n} \times \overline{\mathcal{M}}^{\theta}_{0,n}$, compared to the one of the stable curve $\overline{\mathcal{M}}_{0,n}$. Indeed, those points, lying in Fix_θ, are conjugated to complex germs of the type $(z^{2k}, 0)$, where $1 \leq k \leq n$. Therefore, not only this modifies the stratification of the stable curves, (comparing it to the one of $\overline{\mathcal{M}}_{0,n}$), but also implies differences in the metrics around the divisors.

6.2 The neighborhood of the fat diagonal in $\mathcal{M}_{0,n}(\theta)$

We consider the neighborhood of the fat diagonal in $\mathcal{M}_{0,n}(\theta)$. One approach is done using Kähler geometry; the second one using Riemannian geometry. Both approaches describe the split quaternionic structure and lead in the end to the cohomology ring of the investigated space.

Recall some notions. Let $Conf_{0,n}$ be the configuration space of n marked points on the complex plane. For $1 \leq j \neq k \leq n$, let $w_{jk} = \frac{dlog(x_j - x_k)}{2\pi\imath}$ be the logarithmic differential form. The cohomology ring $H^*(Conf_{0,n}, \mathbb{Z})$ is the graded commutative ring with generators $[w_{jk}]$, and relations:

- $w_{jk} = w_{kj}$
- $w_{ij}w_{jk} + w_{jk}w_{ki} + w_{ki}w_{ij} = 0$

The cohomology ring $H^*(\mathcal{M}_{0,n+1}, \mathbb{C})$ may be identified with the kernel of the differential $\imath$ on $H^*(Conf_{0,n}, \mathbb{C})$ whose action on the generators is $\imath w_{jk} = 1$.

The Kähler logarithmic derivatives approach is of interest to us, because not only it is related to the KZ equations but also to L^2–cohomology and thus in a certain sense to the Hodge structure. We wish to highlight the bridges between those seemingly different view points. The generators $[w_{jk}]$ of the cohomology ring $H^*(Conf_{0,k+1})$ are induced by $w_{jk} = \frac{dlog(x_j - x_k)}{2\pi\imath}$ and those logarithmic derivatives are also a way to have some generators for the L^2–cohomology.

An incomplete Riemannian metric over a given smooth manifold (for instance $\mathcal{M}_{0,n}$) can be obtained using a projective embedding. Recall its construction. If $p : \mathbb{C}^{n+1} - 0 \to \mathbb{P}^n$ is the projection where $(x_1, \dots, x_n) \in \mathbb{C}^{n+1}$, then $\mathbb{P}^n$ is endowed with a natural Kähler metric ν, called Fubini-Study metric and defined by:

$$p^*\nu = \frac{i}{2\pi}\partial\bar{\partial} log\| x_1|^2 + |x_2|^2 + \cdots + |x_n|^2|.$$

This gives a point-wise norm on smooth forms ω of type (p,q) on $\mathcal{M}_{0,n}$, and defines an L^2 norm $\|\omega\|_2$. One defines thus a simplicial complex by setting $\mathcal{F}^{p,q}(\mathcal{M}_{0,n}) := \{\omega \| \|\omega\|_2 < \infty, \|\bar{\partial}\omega\|_2 < \infty\}$. A Dolbeault type of complex, $(\mathcal{F}^{p,*}(\mathcal{M}_{0,n+1}), \bar{\partial})$ for each $p \geq 0$ can be defined, from this definition. The existence of such a complex allows the definition of an q-th L^2-cohomology group. It is usually denoted by $H_2^{p,q}(\mathcal{M}_{0,n})$, in the literature. See for more details Pardon and Stern (1989, 1991).

A subtile change of variables leads to define locally a Kähler metric on the smooth part of the moduli space $\mathcal{M}_{0,n+1}$, in the neighborhood of the fat diagonal $\{x_i = x_j | i \neq j\}$. This construction allows to define a norm with respect to the metric. Using this tool, we can apply results from Demailly (1982), Donnelly and Fefferman (1983), and Ohsawa (1987) to describe the L^2-cohomology.

Choose a ball B_r, centered at 0 and of radius r, such that $\sum_{i=1}^{n} |x_i|^2 \ll 1$. Consider the Euclidean distance between a pair of points x_i and x_j, where those points lie in the interior of the ball of radius r. Set

$$F = -log(dist(x_i, x_j)), \tag{11}$$

and

$$F_k := -log(dist(x_i, x_j)) - \frac{1}{k} log(-log \sum |x_i|^2) \tag{12}$$

where $x_i, x_j \in B_r$ and $k > 1$.

The latter strictly pluri-subharmonic function defines a Kähler metric $h_k := -i\partial\bar{\partial}F_k$ on the regular part of $(\mathcal{M}_{0,n+1}(\theta) \setminus Fix_\theta) \cap B_r$. The metric $h_k := -i\partial\bar{\partial}F_k$ on $\mathcal{M}_{0,n+1}(\theta) \setminus Fix_\theta$ is complete and decreases monotonically to $h := -i\partial\bar{\partial}F$. Independently of k, $\langle \partial F_k, \partial F_k \rangle$ is bounded, where $\langle \cdot, \cdot \rangle_k^{\frac{1}{2}}$ denotes the pointwise norm on 1-forms with respect to h_k.

The statements below (Prop. 6 and Prop. 7) are a direct adaptation of Lem. 2.2 and Lem. 2.3 from Pardon and Stern (1991) to our setting.

Proposition 6 – *Let $\mathcal{M}_{0,n+1}(\theta) \setminus Fix_\theta$ be endowed with a Kähler metric, which is given by the potential function $F : \mathcal{M}_{0,n+1}(\theta) \setminus Fix_\theta \to \mathbb{R}$ such that $\langle \partial F, \partial F \rangle$ is bounded. Then,* *the $L_2 - \overline{\partial}$-cohomology with respect to ω, $H^{p,q}_{(2)}(\mathcal{M}_{0,n+1}(\theta)\setminus Fix_\theta,\omega) = 0$ for $p+q \neq n+1$. In fact, if $\langle \partial F, \partial F\rangle^{\frac{1}{2}} \leq B$, and ϕ is a (p,q)-form on $\mathcal{M}_{0,n+1}(\theta)\setminus Fix_\theta$ with $\overline{\partial}\phi = 0$, $q > 0$ and $p+q \neq 0$, then there is a $(p,q-1)$-form ν such that $\overline{\partial}\nu = \phi$ and $\|\nu\| \leq 4B\|\phi\|$.*

Globally speaking, statements in Lem. 2.2 and Lem. 2.3 from Pardon and Stern (1991) hold for any complex manifold. This is interesting for us here since $\mathcal{M}_{0,n+1}(\theta)\setminus Fix_\theta$ is still considered as complex variety although it is a realisation of the corresponding module over the split quaternion numbers.

Proposition 7 – *Let $\mathcal{M}_{0,n+1}(\theta)\setminus Fix_\theta$ be endowed with a decreasing sequence of complete hermitian metrics h_k, $k \geq 1$, which converges pointwise to a hermitian metric h. If $H^{n+1,q}_{(2)}(\mathcal{M}_{0,n+1}(\theta)\setminus Fix_\theta, h_k)$ vanishes with an estimate that is independent of k, then $H^{n+1,q}_{(2)}(\mathcal{M}_{0,n+1}(\theta)\setminus Fix_\theta, h)$ vanishes with an estimate.*

Here, $H^{n+1,q}_{(2)}(\mathcal{M}_{0,n+1}(\theta)\setminus Fix_\theta, h_k)$ denotes $L_2 - \overline{\partial}$-cohomology with respect to the metric h_k.

Hence, these results provide interesting results concerning the Kähler geometry for the realisation of the split quaternionic structure, in our context. We now continue our investigations, but from an Euclidean geometry point of view. Note that the approach below holds no longer for the parametrizing scheme T, but for the universal curve, where colliding points have already been blown-up.

6.3 Riemannian geometry in the neighbourhood of the divisor

In this subsection, given parametrizing scheme T, consider the universal curve and study the Riemannian geometry around the divisor Δ_θ. We highlight the geometric differences in terms of Riemannian geometry around the divisors of $\overline{\mathcal{M}}_{0,n+1}$ and of $\overline{\mathcal{M}}_{0,n+1}(\theta)$. The creation of additional singularities due to the split quaternionic structure model implies modifications around the geometry, which we want to understand better.

To avoid any notation confusion: let D^τ be a given divisor of the $n+1$-stable curve $\overline{\mathcal{M}}_{0,n+1}$, parametrized by a subscheme $D(\tau)$ in T. For this subscheme in T, let D_τ be the divisor of $NY(n+1)$. The notation D is used for a normal crossing divisor.

The aim of the next proposition is to give an idea of the geometric differences occurring in $\overline{\mathcal{M}}_{0,n+1}$ and $\overline{\mathcal{M}}_{0,n}\times\overline{\mathcal{M}}^{\theta}_{0,n}$. We introduce the classical filtration: $\Delta_{(0)} \subset \Delta_{(1)} \subset \ldots \Delta_{(n)}$, where an i-th (smooth) stratum is given by $\Delta_{(i)} - \Delta_{(i-1)}$ (note that this is a different stratification than in the previous section). The analogous stratification is done for the fat diagonal Δ_θ of $\overline{\mathcal{M}}_{0,n}\times\overline{\mathcal{M}}^{\theta}_{0,n}$.

In order to understand the neighbourhood–geometry of Δ_θ in $\overline{\mathcal{M}}_{0,n}(\theta)$ we apply some results of Hsiang–Pati Hsiang and Pati 1985 and Nagase Nagase 1989. In the following part, we assume that we intersect the fat diagonal with a n-ball of radius

1 centered at 0. It is known that this intersection behaves as a topological cone defined over the boundary of this intersection. We claim the following:

Proposition 8 – *Let us stratify the singular locus $\Delta(\theta)$ as above. Let D_+ and D_- be the connected components of D_θ symmetric to each other about the component D_{Fix_θ} corresponding to Fix_θ. Suppose that $(r,\Theta,y) \in (0,1]\times[0,1]\times Y$ and $\tilde{g}(y)$ is Riemannian metric on a manifold Y. Then, on each stratum of Δ_θ there exists an isolated (conical) singularity of type A_k. For a suitable system of coordinates, we describe the Riemannian metric in the neighbourhood of D_θ as follows:*

$$\begin{cases} \textit{The neighborhoof of} \quad D_+: & dr^2 + r^2 d\Theta^2 + r^{2c}\tilde{g}(y). \\ \textit{The neighborhoof of} \quad D_-: & dr^2 + r^2 d\Theta^2 + r^{2c}(ds^2 + h^2(r,s)d\phi), \\ & h \textit{ is a smooth function.} \\ \textit{The neighborhoof of} \quad D_{Fix_\theta}: & dr^2 + r^2 d\Theta^2 + r^2\tilde{g}(y). \end{cases}$$

Remark 5 – In the statement above, the notation for the Riemannian metric $\tilde{g}(y)$ is used in a large sense, i.e. depends completely on the neighbourhood of the divisor.

For the reader's convenience, let us recall the construction of Hsiang–Pati and Nagase metric around D as in Hsiang and Pati 1985; Nagase 1989. Roughly speaking the Hsiang–Pati/ Nagase metric is of the type $dr^2 + r^2 d\Theta^2 + r^{2c}\tilde{g}(y)$ where $(r,\Theta,y) \in (0,1]\times[0,1]\times Y$ and $\tilde{g}(y)$ is Riemannian metric. However, we distinguish the Hsiang–Pati metric from the Nagase, for the following technical reason. The Hsiang–Pati metric is defined around the divisor's components (but not around intersections of the divisors' components) and the Nagase metric is meant to be used on the remain complementary part (i.e. around the intersection of the divisors' components).

The metrics are given as follows.

1. Let Y be a compact polygon in $\mathbb{R}^2$ with standard metric $\tilde{g}$.
   Let $W_{HP} = (0,1]\times[0,1]\times Y \ni (r,\Theta,y)$, be endowed with the Riemannian metric:
   $$g_{HP} := dr^2 + r^2 d\Theta^2 + r^{2c}\tilde{g}(y).$$
2. Let $W_N = (0,1]\times[0,1]^3$ be endowed with the Riemannian metric:
   $$g_N := dr^2 + r^2 d\Theta^2 + r^{2c}(ds^2 + h^2(r,s)d\phi),$$
   with $h(r,s) = \frac{f(r)}{l(\frac{s}{f(r)})}$, where $f(r)$ is a smooth function on $[0,1]$ such that $f'(r) \geq 0, \forall r \geq 0$ and $l(x)$ is a smooth function on $[0,\infty)$ such that $l'(x) \geq 0$ and $l''(x) \geq 0$ for any $x \geq 0$. We define these two functions as :
   - $f(r) = r^b$, if r is small and $r > 0, b > 0$.
   - $f(r) = \frac{1}{2}$ if r is large and $r \leq 1$.

- $l(x) = 1$, if $0 \leq x \leq 1 - \epsilon$.
- $l(x) = x$ if $1 + \epsilon \leq x$.

The W_{HP} and W_N are models obtained up to quasi-isometry of the subsets defining a tubular neighborhood around the divisor. By quasi-isometry we mean that for Riemannian manifolds (Y_i, g_i), the diffeomorphism $f : (Y_1, g_1) \to (Y_2, g_2)$ satisfies for a positive constant $a > 0$ the inequality: $C^{-1}g_1 \leq f^* g_2 \leq a g_1$.

We can now conclude with the proof of Prop. 8.

Proof. Using the classical stratification for Δ_θ we may notice that on each stratum, there exists a point lying in Fix_θ. In particular, for each codimension k stratum there exists an isolated singularity of type A_{k-1}. Blowing-up this point, we can describe the geometry around its irreducible components, using the Hsiang–Pati and Nagase metrics. Now, following some technical computations, we find out that the coefficient for the metric defined around the locus D_{Fix_θ} is $c = 1$. □

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