North-Western European Journal of Mathematics

On a moment estimate of sum of weakly dependent random variables using simple random walk on graph

Susanna Spektor[1]

Received: September 17, 2018/Accepted: December 7, 2020/Online: December 28, 2020

Abstract

We obtained a moment estimate for the sum of Rademacher random variables under condition that they are dependent in the way that their sum is zero.

Keywords: random walk, concentration inequality.

MSC: 05C81, 60A10.

1 Introduction

Let $X_1, \dots, X_N$ be a sequence of independent real valued random variables and let $\Sigma = \sum_{i=1}^{N} X_i$. The estimate of moments of Σ, that is of the quantities $\|\Sigma\|_p = \mathbb{E}(\Sigma^p)^{1/p}$, appear often in many areas of mathematics. The growth of moments is closely related to the behavior of the tails of Σ.

Probabilists have been interested in the moments of sums of random variables since the early part of last century. Khinchine's 1923 paper appears to make the first significant contribution to this problem[2]. It provides inequalities for the moments of a sum of Rademacher random variables. In 1970, Rosenthal generalised Khinchine's result to the case of positive or mean-zero random variables[3]. Further refinements to these bounds have been made by Latala and Hitczenko, Montogomery-Smith and Oleszkiewiez in more recent times[4]. Nowadays, it appears that in the different applications of mathematics, statistics, computer science and engineering similar estimates for the case when random variables are not independent are important (see for example B. Pass (2008) and P. Doukhan (2007)). The main motivation of

[1] Professor, PSB, Sheridan College Institute of Technology and Advanced Learning,

[2] A. Khinchine, 1923, "Uber dyadische bruche".

[3] H.P. Rosental, 1980, "On the subspaces of L_p, $(p > 2)$ spanned by sequences of independent random variables".

[4] P. Hitczenko, 1997, "Moment inequalities for sums of certain independent symmetric random variables";

R. Latala, 1997, "Estimation of moments of sums of independent real random variables".

the current work came from an idea, represented by authors in A.B. Kashlak (2020), on using a concentration inequality for dependent Rademacher random variables to obtained a computation-free approach to permutation testing.

Our aim in the present work is to obtain a concentration inequality for Rademacher random variables under condition that their sum is zero. More precisely, we would like to find a bound on the sum of random variables, $\Sigma = \sum_{i=1}^{2n} X_i$, in the case when $X_i = a_i\varepsilon_i$, where $a \in \mathbb{R}^{2n}$ and $\varepsilon_i, i = 1,\dots,2n$ are independent Rademacher random variables, that is variables satisfying the following condition: $\mathbb{P}(\varepsilon_i = 1) = \mathbb{P}(\varepsilon_i = -1) = \frac{1}{2}$. As usual, for $\varepsilon \in \{\pm 1\}^{2n}$ by $\varepsilon_1,\dots,\varepsilon_{2n}$ we denote coordinates of ε. We will put an additional assumption on the Rademacher random variables, namely

$$S = \sum_{i=1}^{2n} \varepsilon_i = 0. \tag{1}$$

To shorter notation, by $\mathbb{E}_{S=0}$ we denote an expectation with condition that (1) holds, i.e $\mathbb{E}_{S=0}\left|\sum_{i=1}^{2n} a_i\varepsilon_i\right|^p = \mathbb{E}\left(\left|\sum_{i=1}^{2n} a_i\varepsilon_i\right|^p \mid S = \sum_{i=1}^{2n}\varepsilon_i = 0\right)$.

Consider the following set

$$\Omega = \left\{\varepsilon \in \{-1,1\}^{2n} \mid \sum_{i=1}^{2n} \varepsilon_i = 0\right\} = \left\{\varepsilon \in \{-1,1\}^{2n} \mid \mathrm{card}\{i : \varepsilon_i = 1\} = n\right\}. \tag{2}$$

Thus, for ε taken uniformly at random in Ω the sequence of its coordinates is a sequence of a weekly dependent Rademacher random variables.

For $\varepsilon \in \Omega$ we put into correspondence a subset of the group Π_{2n} of all permutations of set $\{1,\dots,2n\}$ as

$$\sigma \in \Pi_{2n} \longleftrightarrow A_\sigma = \{\varepsilon \in \Omega \mid \varepsilon_i = 1 \text{ if } \sigma(i) \le n;\ \varepsilon_i = -1 \text{ if } \sigma(i) > n\}.$$

It is easy to see that this correspondence brings uniform measure on Π_{2n} to the uniform measure on Ω.

Define $f : \Pi_{2n} \longrightarrow \mathbb{R}$ by

$$f(\sigma) := \left|\sum_{i=1}^{n} a_{\sigma(i)} - \sum_{i=n+1}^{2n} a_{\sigma(i)}\right|. \tag{3}$$

Note, that $\mathbb{E}_{S=0}\left|\sum_{i=1}^{n} a_i\varepsilon_i\right|^p = \mathbb{E}f^p$, where $\mathbb{E}f^p$ is an expectation when one takes a permutation σ uniformly at random on Π_{2n}. Thus, it is enough to estimate p-th moments of f.

In the present paper we obtained the following result.

Theorem 1 – *Let f defined as above. Then, for $p > 0$,*

$$(\mathbb{E}f^p)^{1/p} \leq \mathbb{E}|f| + Cp\frac{2n-1}{2n}\|a\|_2,$$

where $C > 0$ is a constant independent of p and n.

The paper is organized as follows. In the next section we provide the necessary known tools and definitions. In Section 3, we will establish the bound on the p-th moment of $\sum_{i=1}^{2n} a_i\varepsilon_i$ under condition that $\sum_{i=1}^{2n} \varepsilon_i = 0$.

2 Preliminaries

Let $G(V,E)$ be a connected undirected finite graph, where V stays for a set of vertices and E is a set of edges. A *simple random walk* is a sequence of vertices $v_0, v_1, \ldots, v_t$, where $v_i \sim v_{i+1}$ (that is $\{v_i, v_{i+1}\} \in E$) for $i = 0, 1, \ldots, t-1$. That is, given an initial vertex v_0, select randomly an adjacent vertex v_1, and move to this neighbor. Then, select randomly a neighbor v_2 of v_1, and move to it, etc. The probability it moves from vertex v_i to v_{i+1} (assuming it sits at v_i) is given by

$$p(v_i, v_{i+1}) = \begin{cases} \dfrac{1}{deg(v_i)}, & \text{if } v_i \sim v_{i+1} \\ 0, & \text{otherwise,} \end{cases} \tag{4}$$

where $deg(v_i)$ denotes the degree of vertex v_i. This is a walk using a transition probability matrix, $P = (p(v_i, v_{i+1}))_{v_i, v_{i+1} \in V}$. The transition probability (4) (see for example G. Grimmett (2020)) has a reversible equilibrium probability distribution $\mu(v_i)$. That is,

$$\mu(v_i)p(v_i, v_{i+1}) = \mu(v_{i+1})p(v_{i+1}, v_i)$$

and $\mu(v_i)$ is proportional to $deg(v_i)$.

Let I be the $V \times V$ identity matrix. The discrete Laplacian is the matrix $L = P - I$ with its eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots$, ordered in non-increasing order. The smallest eigenvalue, $\lambda_1 > 0$, is called the *spectral gap* of the random walk.

For $f : V \longrightarrow \mathbb{R}$ define

$$|||f|||_\infty^2 = \frac{1}{2}\sup_{v_i \in V} \sum_{v_{i+1} \in V} |f(v_i) - f(v_{i+1})|^2 p(v_i, v_{i+1}). \tag{5}$$

We will use the following concentration inequality (see A. Aida (1994) or M. Ledoux (2001)):

Theorem 2 – *Assume that (p,μ) is reversible on the finite graph $G(V,E)$, and let $\lambda_1 > 0$ be the spectral gap. Then,*

$$\mu\left(f > \int f\,d\mu + t\right) \le 3\exp\left(\frac{-t\sqrt{\lambda_1}}{2|||f|||_\infty^2}\right). \tag{6}$$

For purpose of our work we specialize now to $V = \Pi_{2n}$, the group of all permutations σ of the set $\{1,\ldots,2n\}$, and to $E = \{(\sigma,\sigma\tau) \mid \tau \text{ is a transposition on } \Pi_{2n}\}$. We will be using a random walk with positive probability to stay at the same vertex. We will follow P. Diaconis (1981) and consider the identical permutation as transposition. The transition probability $p(\sigma,\sigma\tau)$ on $G = (\Pi_{2n}, E)$ is

$$p(\sigma,\sigma\tau) = \frac{2}{(2n)^2}, \tag{7}$$

and reversible equilibrium distribution μ on Π_{2n} is a unique invariant measure for p (see for example S. Chatterjee (2009) for these facts). Also, as proved in P. Diaconis (1981), the spectral gap of the random transposition walk on Π_{2n} is $\lambda_1 = \frac{2}{2n} = \frac{1}{n}$. Thus, the concentration inequality (6) for simple random walk on $G(\Pi_{2n},E)$ can be rewritten as

$$\mu(\{\sigma : f(\sigma) - \mathbb{E}f \ge t\}) \le 3\exp\left(\frac{-t}{2|||f|||_\infty^2\sqrt{n}}\right). \tag{8}$$

3 Proof of Theorem 1

We are going to use inequality (8). We calculate first

$$|||f|||_\infty^2 = \frac{1}{2}\sup_{\sigma\in\Pi_{2n}}\sum_{all\,\tau}|f(\sigma) - f(\sigma\tau)|^2 p(\sigma,\sigma\tau),$$

where $p(\sigma,\sigma\tau)$ is defined in (7).

Consider $f(\sigma) := |g(\sigma)| = \left|\sum_{i=1}^n a_{\sigma(i)} - \sum_{i=n+1}^{2n} a_{\sigma(i)}\right|$. Since $\tau(i,j)$ is a random transposition with i,j chosen uniformly from the set $\{1,\ldots,2n\}$, we obtain

$$g(\sigma) - g(\sigma\tau) = 2(a_i - a_j)h(i,j),$$

where

$$h(i,j) = \begin{cases} 1 & \text{if } j \le n < i \le 2n \\ -1 & \text{if } i \le n < j \le 2n \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $|f(\sigma) - f(\sigma\tau)|^2 = 4(a_i - a_j)^2 h^2(i,j)$. And we can calculate

$$
\begin{aligned}
|||f|||_\infty^2 &= \frac{1}{n^2} \sum_{\tau(i,j)} (a_i - a_j)^2 h^2(i,j) \\
&= \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j=n+1}^{2n} (a_i - a_j)^2 h^2(i,j) \\
&= \frac{2}{n^2} \left(n\|a\|_2^2 - 2 \sum_{i=1}^{n} \sum_{j=n+1}^{2n} a_i a_j \right)
\end{aligned}
$$

Since

$$
-\sum_{i=1}^{n} \sum_{j=n+1}^{2n} a_i a_j \le \sum_{i=1}^{n} \sum_{j=n+1}^{2n} \frac{a_i^2 + a_j^2}{2} = \frac{n}{2}\|a\|_2^2,
$$

the last equation can be bounded by

$$
|||f|||_\infty^2 \le \frac{4}{n}\|a\|_2^2. \tag{9}
$$

Now, using (8), (9) and an upper bound $\Gamma(x) \le x^{x-1}$, for all $x \ge 1$ (see for example G. D. Anderson (1997)), we obtain

$$
\begin{aligned}
\mathbb{E}|f - \mathbb{E}f|^p = \int_0^\infty \mu((f(\sigma) - \mathbb{E}f)^p \ge t^p) dt^p &\le 6p \int_0^\infty e^{-t/(4\|a\|_2)} t^{p-1} dt \\
&= 6\,p\,4^p \Gamma(p) \|a\|_2^p \le 4^p\, 6p^p \|a\|_2^p.
\end{aligned}
$$

Hence

$$
(\mathbb{E}f^p)^{1/p} \le \mathbb{E}|f| + 24p\|a\|_2. \tag{10}
$$

Let us note now that under condition (1) the random variables are invariant under the shifts. Replacing a_i with $a_i - \frac{1}{2n}\sum_{i=1}^{2n} a_i$ for $i = 1, \ldots 2n$ in (10) would give us the desired result.

Remark: Note that $\mathbb{E}|f| \le \left(\mathbb{E}|f|^2\right)^{1/2}$, where $\mathbb{E}|f|^2$ can be directly calculated (see S. Spektor (2016)).

Acknowledgments

I would like to thank the anonymous referee for pointing out on the idea of optimizing result by replacing scalars with their differences with average.

References

A. Aida, D. S. (1994). "Moment Esimates derived from Poincaré and logarithmic Sobolev inequalities". *Math., Research Letters, 1*, pp. 75–86 (cit. on p. 3).

A. Khinchine (1923). "Uber dyadische bruche". *Mathematische Zeitschrift, 18*, pp. 109–116 (cit. on p. 1).

A.B. Kashlak S. Myroshnychenko, S. S. (2020). "Analytic permutation testing via Kahane-Khintchine inequalities". *arXiv:2001.01130v1* (cit. on p. 2).

B. Pass, S. S. (2008). "Khinchine type inequality for k-dependent Rademacher random variables". *Statistics and Probability Letters, 132*, pp. 35–39 (cit. on p. 1).

G. D. Anderson, S. L. Q. (1997). "A monotoneity property of the gamma function". *Proc. Amer. Math. Soc., 125*, pp. 3355–3362 (cit. on p. 5).

G. Grimmett, D. S. (2020). *Probability and Random Processes*. Oxford University Press (cit. on p. 3).

H.P. Rosental (1980). "On the subspaces of L_p, $(p > 2)$ spanned by sequences of independent random variables". *Israel Journal of Mathematics, 8*, pp. 273–303 (cit. on p. 1).

M. Ledoux (2001). *The concentration of measure phenomenon*. Amer. Math. Soc (cit. on p. 3).

P. Diaconis, M. S. (1981). "Generating a random permutation with random transpositions". *Z. Wahrsch Verw. Gebiete, 57 (2)*, pp. 159–179 (cit. on p. 4).

P. Doukhan, M. N. (2007). "Probability and moment inequalities for sums of weakly dependent random variables, with applications". *Stochastic Processes and their Applications, Elsevier, 117 (7)*, pp. 878–903 (cit. on p. 1).

P. Hitczenko S.J. Montgomery-Smith, K. O. (1997). "Moment inequalities for sums of certain independent symmetric random variables". *Studia Mathematicia, 123*, pp. 15–45 (cit. on p. 1).

R. Latala (1997). "Estimation of moments of sums of independent real random variables". *The Annals of Probability, 25*, pp. 1502–1513 (cit. on p. 1).

S. Chatterjee, M. L. (2009). "An observation about submatrices". *Elect. Comm. in Probab., 14*, pp. 495–500 (cit. on p. 4).

S. Spektor (2016). "Khinchine inequality for dependent random variables". *Canad. Math. Bull, 59*, pp. 204–210 (cit. on p. 5).

Contents