



Stationary solutions to Smoluchowski’s coagulation equation with source

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Abstract

Existence and non-existence of integrable stationary solutions to Smoluchowski’s coagulation equation with source are investigated when the source term is integrable with an arbitrary support in $(0, \infty)$. Besides algebraic upper and lower bounds, a monotonicity condition is required for the coagulation kernel. Connections between integrability properties of the source and the corresponding stationary solutions are also studied.

Keywords: Coagulation equation, source term, stationary solution, non-existence.

MSC: 45K05.

1 Introduction

The coagulation equation with source describes the dynamics of a system of particles, in which particles interact by pairwise merging, thereby forming larger particles, and new particles are injected from the outside. Denoting the particle size distribution function of particles with size $x \in (0, \infty)$ at time $t > 0$ by $f = f(t, x) \geq 0$, the corresponding evolution equation is

$$\partial_t f(t, x) = \mathcal{C}f(t, x) + S(x), \quad (t, x) \in (0, \infty)^2, \tag{1a}$$

$$f(0, x) = f^{\text{in}}(x), \quad x \in (0, \infty), \tag{1b}$$

where S is a time-independent function accounting for the external supply of particles and the coagulation mechanism is given by the nonlinear integral operator

$$\mathcal{C}f(x) := \frac{1}{2} \int_0^x K(y, x-y)f(y)f(x-y) \, dy - \int_0^\infty K(x, y)f(x)f(y) \, dy \tag{1c}$$

for $x \in (0, \infty)$. In (1c), the coagulation kernel K is a non-negative and symmetric function and $K(x, y) = K(y, x)$ measures the rate of merging of particles with respective sizes x and y . The first integral on the right hand side of (1c) accounts for the

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formation of particles with size x as a result of the coagulation of two particles with respective sizes $y \in (0, x)$ and $x - y$, while the second one describes the disappearance of particles with size x when merging with other particles.

Since the pioneering works of Ball and Carr (1990), Leyvraz and Tschudi (1981), McLeod (1962, 1964), Melzak (1957), Spouge (1984), Stewart (1989), and White (1980), Smoluchowski’s coagulation equation (1) without source ($S \equiv 0$), originally derived in Smoluchowski (1916, 1917), has been extensively studied in the mathematical literature for various choices of the coagulation kernel K and we refer to the books Banasiak, Lamb, and Laurençot (2019) and Dubovskii (1994) and the references therein for a more detailed account. Since the addition of a source term does not change the mathematical structure of the equation, the well-posedness of Smoluchowski’s coagulation equation with source (1a)–(1b) can be proved in a similar way as that of Smoluchowski’s coagulation equation². It is however worth emphasizing that the presence of a source drastically changes the dynamics, as the continuous injection of new particles in the system somewhat balances the transfer of matter towards larger and larger particles due to coagulation. In particular, convergence to a stationary state is shown in Dubovskii (1994), Shirvani and Van Roessel (2002), and Simons (1998) for the constant coagulation kernel, a feature which leads to the question of existence and stability of stationary solutions for other choices of coagulation kernels. A thorough study of the existence issue is performed in Ferreira et al. (2019) for coagulation kernels satisfying

$$k_1(x^{\gamma+\alpha}y^{-\alpha} + x^{-\alpha}y^{\gamma+\alpha}) \leq K(x, y) \leq k_2(x^{\gamma+\alpha}y^{-\alpha} + x^{-\alpha}y^{\gamma+\alpha}) \tag{2}$$

for $(x, y) \in (0, \infty)^2$, where $(\gamma, \alpha) \in \mathbb{R}^2$ and $k_2 > k_1 > 0$. Assuming that the source term S is a non-negative bounded Radon measure on $(0, \infty)$ with compact support in $[1, L]$ for some $L > 1$, the existence of at least one non-negative measure-valued stationary solution $f(dx)$ to (1a) satisfying

$$\int_0^\infty (x^{\gamma+\alpha} + x^{-\alpha}) f(dx) < \infty \tag{3}$$

is shown in (Ferreira et al. 2019, Theorem 2.2) when $|\gamma + 2\alpha| < 1$. In addition,

$$\int_0^\infty x^\mu f(dx) < \infty, \quad \mu < \frac{1+\gamma}{2}, \quad \int_0^\infty x^{(1+\gamma)/2} f(dx) = \infty, \tag{4}$$

²Escobedo and Mischler, 2006, “Dust and self-similarity for the Smoluchowski coagulation equation”;

Kuehn and Throm, 2019, “Smoluchowski’s discrete coagulation equation with forcing”;

Shirvani and Van Roessel, 2002, “Existence and uniqueness of solutions of Smoluchowski’s coagulation equation with source terms”;

Spouge, 1985, “An existence theorem for the discrete coagulation-fragmentation equations. II. Inclusion of source and efflux terms”;

White, 1980, “A global existence theorem for Smoluchowski’s coagulation equations”.

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see Ferreira et al. (2019, Corollary 6.4), so that $f(dx)$ cannot decay too fast for large sizes (observe that the condition $|\gamma + 2\alpha| < 1$ implies that $\max\{\gamma + \alpha, -\alpha\} < (1 + \gamma)/2$). Furthermore, if $S \neq 0$ and $|\gamma + 2\alpha| \geq 1$, then there is no non-negative measure-valued stationary solution to (1a) satisfying (3), see Ferreira et al. (2019, Theorem 2.4).

The purpose of this note is twofold: on the one hand, for coagulation kernels satisfying (2), we extend the validity of the existence and non-existence results established in Ferreira et al. (2019) to source terms which are not necessarily compactly supported in $(0, \infty)$. We however restrict the analysis to source terms and stationary solutions which are absolutely continuous with respect to the Lebesgue measure on $(0, \infty)$ and, to this end, an additional monotonicity condition is required on the coagulation kernel. On the other hand, for such source terms, we provide alternative proofs for the existence and non-existence results established in Ferreira et al. (2019).

We actually begin our analysis with the following observation, already pointed out in Dubovskii (1994, Chapter 8). If K is a coagulation kernel satisfying (2) and f is a stationary solution to (1a), then $f_\theta(x) := x^\theta f(x)$, $x > 0$, is a stationary solution to (1a) with coagulation kernel $K_\theta(x, y) := (xy)^{-\theta} K(x, y)$ and $\theta := \min\{\gamma + \alpha, -\alpha\}$, and K_θ satisfies the growth condition (2) with $(|\gamma + 2\alpha|, 0)$ instead of (γ, α) . Thanks to this observation, we shall assume from now on that there are $\lambda \geq 0$ and $k_2 > k_1 > 0$ such that the coagulation kernel K satisfies

$$k_1(x^\lambda + y^\lambda) \leq K(x, y) \leq k_2(x^\lambda + y^\lambda), \quad (x, y) \in (0, \infty)^2. \quad (5)$$

We supplement (5) with the following monotonicity condition on K

$$K(x - y, y) \leq K(x, y), \quad 0 < y \leq x, \quad (6)$$

which is known to play an important role in the derivation of uniform integrability estimates such as L^p -estimates, $p > 1$, see Burobin (1983), Dubovskii (1994), Laurençot and Mischler (2002), and Mischler and Rodriguez Ricard (2003).

Before providing a precise definition of stationary solutions to (1a) along with the statements of the main results, let us introduce some notation: for $m \in \mathbb{R}$, we set $X_m := L^1((0, \infty), x^m dx)$ and $X_{0,m} := X_0 \cap X_m$, and denote their respective positive cones by X_m^+ and $X_{0,m}^+$. For $h \in X_m$, we put

$$M_m(h) = \int_0^\infty x^m h(x) dx, \quad h \in X_m.$$

We now define the notion of weak stationary solutions to the coagulation equation with source (1a) to be used in the sequel. Besides the required absolute continuity with respect to the Lebesgue measure, it is quite similar to Ferreira et al. (2019, Definition 2.1).

Definition 1 – Let $\lambda \geq 0$ and consider a coagulation kernel K satisfying (5) and $S \in X_0^+$. A stationary solution to the coagulation equation with source (1a) is a function $\varphi \in X_{0,\lambda}^+$ such that

$$\frac{1}{2} \int_0^\infty \int_0^\infty \chi_\vartheta(x,y) K(x,y) \varphi(x) \varphi(y) dy dx + \int_0^\infty S(x) \vartheta(x) dx = 0 \quad (7)$$

for all $\vartheta \in L^\infty(0, \infty)$, where

$$\chi_\vartheta(x,y) := \vartheta(x+y) - \vartheta(x) - \vartheta(y), \quad (x,y) \in (0, \infty)^2. \quad (8)$$

We now state the existence and non-existence results we establish in this paper.

Theorem 1 – Let $\lambda \geq 0$ and consider a coagulation kernel K satisfying (5) and $S \in X_0^+$.

(a) Assume further that K satisfies (6) and $S \in \bigcap_{m \in (0,1)} X_m$. If $\lambda \in [0, 1)$, then there is at least one stationary solution φ to (1a) such that

$$\varphi \in X_m, \quad 0 \leq m < \frac{1+\lambda}{2}, \quad \varphi \notin X_{(1+\lambda)/2}. \quad (9)$$

In addition, if $S \in X_m$ for some $m \in (-\infty, 0)$, then $\varphi \in X_m$.

(b) If $\lambda \geq 1$ and φ is a stationary solution to (1a), then $\varphi = S \equiv 0$.

An alternative formulation of Theorem 1 (b) is that, for $\lambda \geq 1$ and $S \in X_0^+$, $S \not\equiv 0$, there is no stationary solution to (1a) in the sense of Definition 1.

Remark 1 – According to the above mentioned connection between stationary solutions to (1a) for coagulation kernels satisfying (2) and (5), existence and non-existence results of stationary solutions to (1a) for coagulation kernels satisfying (2) can be deduced from Theorem 1. Indeed, consider a coagulation kernel K satisfying (2) and $(x-y)^{-\theta} K(x-y, y) \leq x^{-\theta} K(x, y)$ for $(x, y) \in (0, \infty)^2$ and $\theta = \min\{\gamma + \alpha, -\alpha\}$. Then, given a source term $S \in \bigcap_{m \in [0,1)} X_m^+$, $S \not\equiv 0$, there is at least one stationary solution to (1a) which belongs to X_m^+ for $m \in [\theta, (1+\gamma)/2)$, but not to $X_{(1+\gamma)/2}$, when $|\gamma + 2\alpha| \in [0, 1)$ and no such solution when $|\gamma + 2\alpha| \geq 1$. This is in accordance with the results established in Ferreira et al. (2019).

As already mentioned, the outcome of Theorem 1 matches the results obtained in Ferreira et al. (2019) for source terms which are non-negative bounded Radon measures on $(0, \infty)$ with compact support in $(0, \infty)$. We here relax the assumption on the support and obtain directly integrable stationary solutions to (1a) when the source term is integrable. Also, the proof of Theorem 1 provided below relies rather on global integral estimates, while local integral estimates are at the forefront of the analysis performed in Ferreira et al. (2019). As a consequence, more precise

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information on the local behaviour of stationary solutions is obtained there, see Ferreira et al. (2019, Proposition 6.3). Finally, as already pointed out in Dubovskii (1994) and Ferreira et al. (2019), the non-integrability property stated in (9) is a striking feature of stationary solutions to (1a) as their decay at infinity is prescribed solely by the growth of the coagulation kernel and is not sensitive to the decay at infinity of the source term.

We now describe the contents of this paper. In Section 2, we derive properties of stationary solutions φ to (1a) in the sense of Definition 1, including the non-integrability property $\varphi \notin X_{(1+\lambda)/2}$ (Proposition 1) and improved integrability properties of φ for small sizes induced by that of the source term (Proposition 2). We also derive in Proposition 3 upper and lower bounds on $M_0(\varphi)$ and $M_\lambda(\varphi)$ in terms of $M_0(S)$ and $M_\lambda(S)$. Though not directly used in the subsequent analysis, these estimates, in particular (12), provide guidelines for the proof of Theorem 1 (a), see Lemma 1 and Lemma 6. Section 3 is devoted to the existence of stationary solutions (Theorem 1 (a)) and combines a dynamical approach and a compactness method, an approach which has already proved successful to construct self-similar solutions to Smoluchowski's coagulation equation³ and stationary solutions to coagulation-fragmentation equations⁴. Specifically, given a small parameter $\delta \in (0, 1)$, we consider an approximation of (1) obtained by truncating the source term ($S_\delta := S\mathbf{1}_{(0,1/\delta)}$) and adding an efflux term $-2\delta f$. We then show that the associated initial value problem is well-posed in $X_{0,1+\lambda}^+$ and construct an invariant set \mathcal{Z}_δ , which is non-empty, convex, and sequentially weakly compact in X_0 . Owing to these properties, an application of a consequence of Tychonov's fixed point theorem, see Escobedo, Mischler, and Rodriguez Ricard (2005, Theorem 1.2), ensures the existence of a stationary solution φ_δ to this approximation. A by-product of the construction of the invariant set \mathcal{Z}_δ is the derivation of estimates which do not depend on the approximation parameter δ and ensure that the family $(\varphi_\delta)_{\delta \in (0,1)}$ lies in a sequentially weakly compact subset of X_0 . We then show that the corresponding cluster points of $(\varphi_\delta)_{\delta \in (0,1)}$ as $\delta \rightarrow 0$ are stationary solutions to (1a), thereby completing the proof of Theorem 1 (a), see Section 4. We end up the paper with the non-existence of stationary solutions in the sense of Definition 1, as stated in Theorem 1 (b), which is proved in Section 5.

³Escobedo, Mischler, and Rodriguez Ricard, 2005, "On self-similarity and stationary problem for fragmentation and coagulation models";

Escobedo and Mischler, 2006, "Dust and self-similarity for the Smoluchowski coagulation equation";

Fournier and Laurençot, 2005, "Existence of self-similar solutions to Smoluchowski's coagulation equation";

Niethammer and Velázquez, 2013, "Self-similar solutions with fat tails for Smoluchowski's coagulation equation with locally bounded kernels".

⁴Escobedo, Mischler, and Rodriguez Ricard, 2005, "On self-similarity and stationary problem for fragmentation and coagulation models";

Laurençot, 2019, "Stationary solutions to coagulation-fragmentation equations".

2 Properties of stationary solutions

Let $\lambda \geq 0$ and consider a coagulation kernel K satisfying (5) and $S \in X_0^+$. We first show that non-trivial stationary solutions to (1a) do not decay too fast for large volumes, a property already observed in Dubovskii (1994, Theorem 8.1) for $\lambda = 0$ and in Ferreira et al. (2019, Corollary 6.4) for $\lambda \in [0, 1)$. The proof given below differs from that in Ferreira et al. (2019) and is closer to that in Dubovskii (1994), an additional approximation argument being needed to handle the unboundedness of K when $\lambda \in (0, 1)$.

Proposition 1 – Assume that $\lambda \in [0, 1)$ and let φ be a stationary solution to (1a). If $S \not\equiv 0$, then $\varphi \notin X_{(1+\lambda)/2}$.

Proof. We argue by contradiction and assume that $\varphi \in X_{(1+\lambda)/2}$. Then

$$J(A) := \int_A^\infty x^{(1+\lambda)/2} \varphi(x) \, dx$$

is finite for all $A \geq 0$ and

$$\lim_{A \rightarrow \infty} J(A) = 0.$$

Now, let $A > 0$ and set $\vartheta(x) := \min\{x, A\}$ for $x > 0$. We infer from (7) and the symmetry of K that

$$\begin{aligned} \int_0^\infty \vartheta_A(x) S(x) \, dx &= \frac{1}{2} \int_0^A \int_{A-x}^A (x+y-A) K(x, y) \varphi(x) \varphi(y) \, dy dx \\ &\quad + \int_0^A \int_A^\infty x K(x, y) \varphi(x) \varphi(y) \, dy dx \\ &\quad + \frac{A}{2} \int_A^\infty \int_A^\infty K(x, y) \varphi(x) \varphi(y) \, dy dx. \end{aligned} \tag{10}$$

We now study the behaviour of the terms on the right hand side of (10) as $A \rightarrow \infty$. First, since $(1 + \lambda)/2 \in (0, 1)$, it follows from (5) that, for $(x, y) \in (0, A)^2$ such that $x + y > A$,

$$\begin{aligned} (x + y - A) K(x, y) &\leq k_2 (x + y - A) (x^\lambda + y^\lambda) \\ &\leq k_2 (x + y - A)^{(1-\lambda)/2} x^\lambda (x + y - A)^{(1+\lambda)/2} \\ &\quad + k_2 (x + y - A)^{(1+\lambda)/2} (x + y - A)^{(1-\lambda)/2} y^\lambda \\ &\leq 2k_2 (xy)^{(1+\lambda)/2}. \end{aligned}$$

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Consequently,

$$\begin{aligned}
 & \frac{1}{2} \int_0^A \int_{A-x}^A (x+y-A)K(x,y)\varphi(x)\varphi(y) \, dydx \\
 & \leq k_2 \int_0^A \int_{A-x}^A (xy)^{(1+\lambda)/2} \varphi(x)\varphi(y) \, dydx \\
 & \leq k_2 \int_0^{A/2} \int_{A/2}^A (xy)^{(1+\lambda)/2} \varphi(x)\varphi(y) \, dydx \\
 & \quad + k_2 \int_{A/2}^A \int_0^A (xy)^{(1+\lambda)/2} \varphi(x)\varphi(y) \, dydx \\
 & \leq 2k_2 M_{(1+\lambda)/2}(\varphi)J(A/2).
 \end{aligned}$$

Next, using again (5) and the property $\lambda \in [0, 1)$, we find

$$\begin{aligned}
 \int_0^A \int_A^\infty xK(x,y)\varphi(x)\varphi(y) \, dydx & \leq k_2 \int_0^A \int_A^\infty (x^{1+\lambda} + xy^\lambda) \varphi(x)\varphi(y) \, dydx \\
 & \leq 2k_2 \int_0^A \int_A^\infty (xy)^{(1+\lambda)/2} \varphi(x)\varphi(y) \, dydx \\
 & \leq 2k_2 M_{(1+\lambda)/2}(\varphi)J(A)
 \end{aligned}$$

and

$$\begin{aligned}
 & \frac{A}{2} \int_A^\infty \int_A^\infty K(x,y)\varphi(x)\varphi(y) \, dydx \\
 & \leq \frac{Ak_2}{2} \int_A^\infty \int_A^\infty (x^\lambda + y^\lambda) \varphi(x)\varphi(y) \, dydx \\
 & \leq Ak_2 \int_A^\infty \int_A^\infty x^\lambda \varphi(x)\varphi(y) \, dydx \\
 & = k_2 \left(\int_A^\infty A^{(1-\lambda)/2} x^\lambda \varphi(x) \, dx \right) \left(\int_A^\infty A^{(1+\lambda)/2} \varphi(y) \, dydx \right) \\
 & \leq k_2 J(A)^2.
 \end{aligned}$$

Gathering the above estimates, we deduce from (10) that

$$\int_0^\infty \min\{x, A\}S(x) \, dx \leq 2k_2 M_{(1+\lambda)/2}(\varphi)[J(A/2) + J(A)] + k_2 J(A)^2.$$

Hence,

$$\lim_{A \rightarrow \infty} \int_0^\infty \min\{x, A\}S(x) \, dx = 0,$$

which implies that $S \equiv 0$, and a contradiction. □

We next show that the behaviour of S for small sizes governs that of stationary solutions.

Proposition 2 – *Let φ be a stationary solution to (1a). If $w \in C((0, \infty))$ is a non-negative and non-increasing function and $S \in L^1((0, \infty), w(x)dx)$, then $\varphi \in L^1((0, \infty), w(x)dx)$. In particular, if $S \in X_m$ for some $m \in (-\infty, 0)$, then $\varphi \in X_m$.*

Proof. Proposition 2 being obvious when $\varphi \equiv 0$, we may thus assume that $\varphi \not\equiv 0$. Consider $\varepsilon \in (0, 1)$ and set $w_\varepsilon(x) := w(x + \varepsilon)$ for $x > 0$. Owing to the monotonicity of w , there holds $w(x + \varepsilon) \leq w(\varepsilon)$ for $x > 0$ and

$$-\chi_{w_\varepsilon}(x, y) = w(x + \varepsilon) + w(y + \varepsilon) - w(x + y + \varepsilon) \geq w(x + \varepsilon) \geq 0, \quad (x, y) \in (0, \infty)^2.$$

We may then take $\vartheta = w_\varepsilon$ in (7) and use the above inequality, the symmetry of K , and (5) to obtain

$$\begin{aligned} \int_0^\infty w_\varepsilon(x)S(x) dx &= -\frac{1}{2} \int_0^\infty \int_0^\infty \chi_{w_\varepsilon}(x, y)K(x, y)\varphi(y)\varphi(x) dydx \\ &\geq k_1 \int_0^\infty \int_0^\infty \chi_{w_\varepsilon}(x, y)y^\lambda \varphi(y)\varphi(x) dydx \\ &\geq k_1 M_\lambda(\varphi) \int_0^\infty w(x + \varepsilon)\varphi(x) dx. \end{aligned}$$

We then let $\varepsilon \rightarrow 0$ in the previous inequality and deduce from Fatou's lemma that

$$\int_0^\infty w(x)S(x) dx \geq k_1 M_\lambda(\varphi) \int_0^\infty w(x)\varphi(x) dx,$$

thereby completing the proof, since $M_\lambda(\varphi)$ is finite and positive. \square

We end up this section with upper and lower bounds on the moments of order zero and λ of stationary solutions to (1a).

Proposition 3 – *Let φ be a stationary solution to (1a). Then*

$$k_1 M_0(\varphi)M_\lambda(\varphi) \leq M_0(S) \leq k_2 M_0(\varphi)M_\lambda(\varphi). \quad (11)$$

Assume further that $\lambda \in [0, 1)$ and $S \in X_\lambda$. Then

$$\frac{2^\lambda}{k_2} M_\lambda(S) \leq M_\lambda(\varphi)^2 \leq \frac{2^{1-\lambda}}{k_1(2-2^\lambda)} M_\lambda(S). \quad (12)$$

Proof. First, it follows from (7) with the choice $\vartheta \equiv 1$ that

$$M_0(S) = \frac{1}{2} \int_0^\infty \int_0^\infty K(x, y)\varphi(x)\varphi(y) dydx.$$

Combining the above identity with (5) readily gives (11).

2. Properties of stationary solutions

Next, the bounds (12) formally follow from (5) and (7) with $\vartheta(x) = x^\lambda$, $x > 0$. This function being not bounded, an approximation is needed. Specifically, let $A > 0$ and set $\vartheta_A(x) = \min\{x^\lambda, A^\lambda\}$ for $x > 0$. We infer from (7) and the symmetry of K that

$$\begin{aligned} \int_0^\infty \vartheta_A(x)S(x) dx &= \frac{1}{2} \int_0^A \int_0^{A-x} [x^\lambda + y^\lambda - (x+y)^\lambda] K(x,y)\varphi(x)\varphi(y) dy dx \\ &\quad + \frac{1}{2} \int_0^A \int_{A-x}^A [x^\lambda + y^\lambda - A^\lambda] K(x,y)\varphi(x)\varphi(y) dy dx \\ &\quad + \int_0^A \int_A^\infty x^\lambda K(x,y)\varphi(x)\varphi(y) dy dx \\ &\quad + \frac{A^\lambda}{2} \int_A^\infty \int_A^\infty K(x,y)\varphi(x)\varphi(y) dy dx. \end{aligned} \quad (13)$$

We now identify the limit as $A \rightarrow \infty$ of each term on the right hand side of (13). We first recall the following algebraic inequalities

$$2^\lambda(2 - 2^\lambda) \frac{(xy)^\lambda}{(x+y)^\lambda} \leq x^\lambda + y^\lambda - (x+y)^\lambda \leq \frac{(xy)^\lambda}{(x+y)^\lambda}, \quad (x,y) \in (0, \infty)^2, \quad (14)$$

see Dongen and Ernst (1985, Eq. (9)), and

$$2^{\lambda-1} (x^\lambda + y^\lambda) \leq (x+y)^\lambda \leq x^\lambda + y^\lambda, \quad (x,y) \in (0, \infty)^2, \quad (15)$$

which are valid due to $\lambda \in [0, 1)$. We deduce from (5), (14), and (15) that

$$\begin{aligned} 0 &\leq \mathbf{1}_{(0,A)}(x)\mathbf{1}_{(0,A-x)}(y) [x^\lambda + y^\lambda - (x+y)^\lambda] K(x,y)\varphi(x)\varphi(y) \\ &\leq k_2(xy)^\lambda \frac{x^\lambda + y^\lambda}{(x+y)^\lambda} \varphi(x)\varphi(y) \leq 2^{1-\lambda}(xy)^\lambda \varphi(x)\varphi(y). \end{aligned}$$

Since $\varphi \in X_\lambda$ and

$$\lim_{A \rightarrow \infty} \mathbf{1}_{(0,A)}(x)\mathbf{1}_{(0,A-x)}(y) = 1, \quad (x,y) \in (0, \infty)^2,$$

Lebesgue's dominated convergence theorem entails that

$$\begin{aligned} \lim_{A \rightarrow \infty} \frac{1}{2} \int_0^A \int_0^{A-x} [x^\lambda + y^\lambda - (x+y)^\lambda] K(x,y)\varphi(x)\varphi(y) dy dx \\ = \frac{1}{2} \int_0^\infty \int_0^\infty [x^\lambda + y^\lambda - (x+y)^\lambda] K(x,y)\varphi(x)\varphi(y) dy dx. \end{aligned}$$

Next, by (5),

$$\begin{aligned} 0 &\leq \mathbf{1}_{(0,A)}(x)\mathbf{1}_{(A-x,A)}(y) [x^\lambda + y^\lambda - A^\lambda] K(x,y)\varphi(x)\varphi(y) \\ &\leq k_2 [x^\lambda + y^\lambda - A^\lambda] (x^\lambda + y^\lambda) \varphi(x)\varphi(y) \\ &\leq k_2 (x^\lambda y^\lambda + y^\lambda x^\lambda) \varphi(x)\varphi(y) = 2k_2(xy)^\lambda \varphi(x)\varphi(y). \end{aligned}$$

Since $\varphi \in X_\lambda$ and

$$\lim_{A \rightarrow \infty} \mathbf{1}_{(0,A)}(x) \mathbf{1}_{(A-x,A)}(y) = 0, \quad (x, y) \in (0, \infty)^2,$$

we use again Lebesgue's dominated convergence theorem to obtain

$$\lim_{A \rightarrow \infty} \frac{1}{2} \int_0^A \int_{A-x}^A [x^\lambda + y^\lambda - A^\lambda] K(x, y) \varphi(x) \varphi(y) \, dy dx = 0.$$

Finally, using once more (5),

$$\begin{aligned} 0 &\leq \int_0^A \int_A^\infty x^\lambda K(x, y) \varphi(x) \varphi(y) \, dy dx \leq k_2 \int_0^A \int_A^\infty x^\lambda (x^\lambda + y^\lambda) \varphi(x) \varphi(y) \, dy dx \\ &\leq 2k_2 \int_0^A \int_A^\infty (xy)^\lambda \varphi(x) \varphi(y) \, dy dx \leq 2k_2 M_\lambda(\varphi) \int_A^\infty y^\lambda \varphi(y) \, dy \end{aligned}$$

and

$$\begin{aligned} 0 &\leq A^\lambda \int_A^\infty \int_A^\infty K(x, y) \varphi(x) \varphi(y) \, dy dx \leq k_2 A^\lambda \int_A^\infty \int_A^\infty (x^\lambda + y^\lambda) \varphi(x) \varphi(y) \, dy dx \\ &\leq 2k_2 \int_A^\infty \int_A^\infty (xy)^\lambda \varphi(x) \varphi(y) \, dy dx \leq 2k_2 M_\lambda(\varphi) \int_A^\infty y^\lambda \varphi(y) \, dy, \end{aligned}$$

from which we deduce that

$$\begin{aligned} \lim_{A \rightarrow \infty} \int_0^A \int_A^\infty x^\lambda K(x, y) \varphi(x) \varphi(y) \, dy dx &= 0, \\ \lim_{A \rightarrow \infty} \frac{A^\lambda}{2} \int_A^\infty \int_A^\infty K(x, y) \varphi(x) \varphi(y) \, dy dx &= 0, \end{aligned}$$

recalling that $\varphi \in X_\lambda$. Collecting the above information, we may take the limit $A \rightarrow \infty$ in (13) and obtain, since $S \in X_\lambda$,

$$M_\lambda(S) = \frac{1}{2} \int_0^\infty \int_0^\infty [x^\lambda + y^\lambda - (x+y)^\lambda] K(x, y) \varphi(x) \varphi(y) \, dy dx. \quad (16)$$

Now, we infer from (5), (14), (15), and (16) that

$$2^\lambda (1 - 2^{\lambda-1}) k_1 M_\lambda(\varphi)^2 \leq M_\lambda(S) \leq 2^{-\lambda} k_2 M_\lambda(\varphi)^2,$$

from which (12) follows. □

3. Approximation

3 Approximation

Throughout this section, we assume that $\lambda \in [0, 1)$ and that the coagulation kernel K satisfies (5) and (6). Also, let S be a source term satisfying

$$S \in \bigcap_{m \in [0,1)} X_m^+, \quad S \not\equiv 0. \quad (17)$$

Since $S \in X_0$, it follows from a refined version of the de la Vallée-Poussin theorem⁵, see Lê (1977) or Banasiak, Lamb, and Laurençot (2019, Theorem 7.1.6), that there is a function $\Phi \in C^1([0, \infty))$ depending only on S which satisfies the following properties: Φ is convex, $\Phi(0) = \Phi'(0) = 0$, Φ' is a concave function which is positive on $(0, \infty)$,

$$\lim_{r \rightarrow \infty} \Phi'(r) = \lim_{r \rightarrow \infty} \frac{\Phi(r)}{r} = \infty, \quad (18a)$$

and

$$L_\Phi(S) := \int_0^\infty \Phi(S(x)) \, dx < \infty. \quad (18b)$$

For $\delta \in (0, 1)$, we define

$$S_\delta = S \mathbf{1}_{(0, 1/\delta)} \in X_{0, 1+\lambda}^+. \quad (19)$$

We shall then prove the existence of a stationary solution to the following approximation of (1)

$$\partial_t f(t, x) = \mathcal{C}f(t, x) + S_\delta(x) - 2\delta f(t, x), \quad (t, x) \in (0, \infty)^2, \quad (20a)$$

$$f(0, x) = f^{\text{in}}(x), \quad x \in (0, \infty), \quad (20b)$$

which is a coagulation equation with a truncated source term and an additional efflux term. In (20a), the coagulation operator $\mathcal{C}f$ is still given by (1c).

Proposition 4 – *There is $\delta_0 \in (0, 1)$ depending only on λ, k_1, k_2 , and S such that, for $\delta \in (0, \delta_0)$, there is at least one stationary solution $\varphi_\delta \in X_{0, 1+\lambda}^+$ to (20a) which satisfies the following properties: for all $\vartheta \in L^\infty(0, \infty)$,*

$$\begin{aligned} \frac{1}{2} \int_0^\infty \int_0^\infty \chi_\vartheta(x, y) K(x, y) \varphi_\delta(x) \varphi_\delta(y) \, dy dx + \int_0^\infty S_\delta(x) \vartheta(x) \, dx \\ = 2\delta \int_0^\infty \varphi_\delta(x) \vartheta(x) \, dx, \end{aligned} \quad (21)$$

⁵De La Vallée Poussin, 1915, “Sur l’intégrale de Lebesgue”.

the function χ_ϑ being defined in (8), and there are positive constants $\gamma_1 > 0$ and $\gamma_2 > 0$ depending only on λ, k_1, k_2 , and S such that

$$0 < \gamma_1 \leq M_\lambda(\varphi_\delta) \leq \gamma_2, \quad \int_0^\infty \Phi(\varphi_\delta(x)) \, dx \leq \gamma_2, \quad (22)$$

and, for each $\mu \in [0, (1 + \lambda)/2)$, there is a positive constant $\gamma_3(\mu) > 0$ depending only on λ, k_1, k_2, S , and μ such that

$$M_\mu(\varphi_\delta) \leq \gamma_3(\mu). \quad (23)$$

As in Ferreira et al. (2019), the proof of Proposition 4 relies on a dynamical approach. As already outlined in the Introduction, it amounts to prove that the coagulation equation with source and efflux terms (20a)–(20b) is well-posed in an appropriately defined subset of X_0 , which is here chosen to be $X_{0,1+\lambda}^+$, and generates a semi-flow $\Psi_\delta(\cdot, f^{\text{in}})$ on that set endowed with the weak topology of X_0 , while leaving invariant a closed convex and weakly compact subset \mathcal{Z}_δ . We then deduce from an application of Tychonov's fixed point theorem, see Escobedo, Mischler, and Rodriguez Ricard (2005, Theorem 1.2), that the semi-flow Ψ_δ has a fixed point in \mathcal{Z}_δ , which is obviously a stationary solution to (20a). To set up the stage for the proof of Proposition 4, we first state the well-posedness of (20a)–(20b) in $X_{0,1+\lambda}^+$.

Proposition 5 – *Let $\delta \in (0, 1)$. Given $f^{\text{in}} \in X_{0,1+\lambda}^+$, there is a unique weak solution $f_\delta = \Psi_\delta(\cdot, f^{\text{in}})$ to (20a)–(20b) satisfying*

$$f_\delta \in C([0, \infty), X_0^+), \quad f_\delta(0) = f^{\text{in}}, \quad (24)$$

$$f_\delta \in W^{1,\infty}((0, T), X_0) \cap L^\infty((0, T), X_{1+\lambda}), \quad T > 0, \quad (25)$$

and

$$\begin{aligned} \frac{d}{dt} \int_0^\infty f_\delta(t, x) \vartheta(x) \, dx &= \frac{1}{2} \int_0^\infty \int_0^\infty \chi_\vartheta(x, y) K(x, y) f_\delta(t, x) f_\delta(t, y) \, dy dx \\ &\quad + \int_0^\infty S_\delta(x) \vartheta(x) \, dx - 2\delta \int_0^\infty f_\delta(t, x) \vartheta(x) \, dx \end{aligned} \quad (26)$$

for all $t > 0$ and $\vartheta \in L^\infty(0, \infty)$. Moreover, if $R > 0$ and $(f_j^{\text{in}})_{j \geq 1}$ is a sequence in $\{h \in X_{0,1+\lambda}^+ : M_{1+\lambda}(h) \leq R\}$ which converges weakly in X_0 to f^{in} , then $(\Psi_\delta(\cdot, f_j^{\text{in}}))_{j \geq 1}$ converges to $\Psi_\delta(\cdot, f^{\text{in}})$ in $C([0, T], X_{0,w})$ for any $T > 0$, where $X_{0,w}$ denotes the space X_0 endowed with its weak topology.

Since the proof of Proposition 5 follows the same lines as that of similar results for coagulation-fragmentation equations and stronger versions of most of the estimates involved in it are derived in Sections 3.1–3.2, we omit the proof here

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and refer instead to Banasiak, Lamb, and Laurençot (2019), Dubovskii (1994), Escobedo, Mischler, and Rodriguez Ricard (2005), and Stewart (1989) and the references therein. Let us also mention here that the well-posedness of the discrete coagulation-fragmentation equations with source and efflux terms is specifically studied in Kuehn and Throm (2019) and Spouge (1985).

In the following, C and $(C_i)_{i \geq 1}$ denote positive constant depending only on λ , k_1 , k_2 , and S . Dependence upon additional parameters will be indicated explicitly. Also, for $m \in \mathbb{R}$ and $x \in (0, \infty)$, we set $\vartheta_m(x) := x^m$ and $\chi_m := \chi_{\vartheta_m}$.

3.1 Moment estimates

We begin with a bound on the moment of order λ which depends, neither on $\delta \in (0, 1)$, nor on $t > 0$.

Lemma 1 – *There is $C_1 > 0$ such that, if*

$$M_\lambda(f^{\text{in}}) \leq C_1 := \sqrt{\frac{2M_\lambda(S)}{(1 - 2^{\lambda-1})k_1}}, \quad (27)$$

then

$$M_\lambda(f_\delta(t)) \leq C_1, \quad t \geq 0.$$

Proof. Let $t > 0$. It follows from (26) that

$$\begin{aligned} \frac{d}{dt} M_\lambda(f_\delta(t)) - \frac{1}{2} \int_0^\infty \int_0^\infty \chi_\lambda(x, y) K(x, y) f_\delta(t, x) f_\delta(t, y) \, dy \, dx \\ = M_\lambda(S_\delta) - 2\delta M_\lambda(f_\delta(t)). \end{aligned}$$

Arguing as in Fournier and Laurençot (2005, Lemma 3.1, Step 1), we infer from (5) and the symmetry of K_δ that

$$\begin{aligned} & \frac{1}{2} \int_0^\infty \int_0^\infty [x^\lambda + y^\lambda - (x+y)^\lambda] K(x, y) f_\delta(t, x) f_\delta(t, y) \, dy \, dx \\ &= \int_0^\infty \int_0^\infty x [x^{\lambda-1} - (x+y)^{\lambda-1}] K(x, y) f_\delta(t, x) f_\delta(t, y) \, dy \, dx \\ &\geq k_1 \int_0^\infty \int_0^\infty xy^\lambda [x^{\lambda-1} - (x+y)^{\lambda-1}] f_\delta(t, x) f_\delta(t, y) \, dy \, dx \\ &\geq k_1 \int_0^\infty \int_x^\infty xy^\lambda [x^{\lambda-1} - (2x)^{\lambda-1}] f_\delta(t, x) f_\delta(t, y) \, dy \, dx \\ &\geq (1 - 2^{\lambda-1}) k_1 \int_0^\infty \int_x^\infty x^\lambda y^\lambda f_\delta(t, x) f_\delta(t, y) \, dy \, dx \\ &\geq \frac{(1 - 2^{\lambda-1}) k_1}{2} M_\lambda(f_\delta(t))^2. \end{aligned}$$

Consequently, using also (19),

$$\frac{d}{dt}M_\lambda(f_\delta(t)) + \frac{(1-2^{\lambda-1})k_1}{2}M_\lambda(f_\delta(t))^2 \leq M_\lambda(S),$$

from which we deduce by the comparison principle that

$$M_\lambda(f_\delta(t)) \leq \max\{M_\lambda(f^{\text{in}}), C_1\}, \quad t \geq 0.$$

Lemma 1 then follows, thanks to (27). \square

The next step is the derivation of two bounds on the moment of order zero, the first one depending on $\delta \in (0, 1)$ but not on $t > 0$, while the second one depends mildly on $t > 0$ but not on $\delta \in (0, 1)$.

Lemma 2 – *If*

$$M_0(f^{\text{in}}) \leq \frac{M_0(S)}{2\delta}, \quad (28)$$

then

$$M_0(f_\delta(t)) \leq \frac{M_0(S)}{2\delta}, \quad t \geq 0. \quad (29)$$

In addition,

$$\frac{k_1}{t} \int_0^t M_0(f_\delta(s))M_\lambda(f_\delta(s)) ds \leq \frac{M_0(f^{\text{in}})}{t} + M_0(S), \quad t > 0. \quad (30)$$

Proof. Let $t > 0$. By (26),

$$\frac{d}{dt}M_0(f_\delta(t)) + \frac{1}{2} \int_0^\infty \int_0^\infty K(x, y)f_\delta(t, x)f_\delta(t, y) dy dx = M_0(S_\delta) - 2\delta M_0(f_\delta(t)),$$

which entails, together with (5) and (19), that

$$\frac{d}{dt}M_0(f_\delta(t)) + 2\delta M_0(f_\delta(t)) + k_1 M_0(f_\delta(t))M_\lambda(f_\delta(t)) \leq M_0(S). \quad (31)$$

It first follows from (31) that

$$\frac{d}{dt}M_0(f_\delta(t)) + 2\delta M_0(f_\delta(t)) \leq M_0(S).$$

Hence,

$$M_0(f_\delta(t)) \leq e^{-2\delta t}M_0(f^{\text{in}}) + \frac{M_0(S)}{2\delta}(1 - e^{-2\delta t}) \leq \max\left\{M_0(f^{\text{in}}), \frac{M_0(S)}{2\delta}\right\}$$

from which we deduce (29) after using (28). We next integrate (31) with respect to time over $(0, t)$ and discard the first two non-negative terms in the left hand side of the resulting inequality divided by t to obtain (30). \square

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We now derive bounds for moments of order up to $(1 + \lambda)/2$. To this end, we need the following lemma.

Lemma 3 – Consider $\theta \in [0, 1/2]$, $m \in (0, 1)$, and $\sigma \in [0, (m + 2\theta)/2]$.

If $g \in L^1((1, \infty), x^\sigma dx)$ is non-negative almost everywhere in $(1, \infty)$, then

$$\left(\int_1^\infty x^\sigma g(x) dx \right)^2 \leq \frac{\kappa(\theta, m, \sigma)}{2} \int_1^\infty \int_1^\infty [x^m + y^m - (x + y)^m] (xy)^\theta g(x)g(y) dy dx,$$

where

$$\kappa(\theta, m, \sigma) := \frac{2^{1-m}\pi^2}{3(1-m)} 4^{(2-m)/(m+2\theta-2\sigma)}.$$

Proof. We argue as in the proof of Banasiak, Lamb, and Laurençot (2019, Lemma 8.2.14) and define $\zeta := 2/(m + 2\theta - 2\sigma) > 0$ and $x_i := i^\zeta$, $i \geq 1$. Setting

$$\mathcal{I} := \frac{1}{2} \int_1^\infty \int_1^\infty [x^m + y^m - (x + y)^m] (xy)^\theta g(x)g(y) dy dx \geq 0,$$

Lemma 3 is obviously true if $\mathcal{I} = \infty$. We then assume that $\mathcal{I} < \infty$ and observe that

$$\begin{aligned} \mathcal{I} &= \int_1^\infty \int_1^\infty [x^{m-1} - (x + y)^{m-1}] x^{\theta+1} y^\theta g(x)g(y) dy dx \\ &\geq (1 - m) \int_1^\infty \int_1^\infty (x + y)^{m-2} (xy)^{\theta+1} g(x)g(y) dy dx \\ &\geq (1 - m) \sum_{i=1}^\infty \int_{x_i}^{x_{i+1}} \int_{x_i}^{x_{i+1}} (x + y)^{m-2} (xy)^{\theta+1} g(x)g(y) dy dx \\ &\geq (1 - m) 2^{m-2} \sum_{i=1}^\infty x_{i+1}^{m-2} \int_{x_i}^{x_{i+1}} \int_{x_i}^{x_{i+1}} (xy)^{\theta+1} g(x)g(y) dy dx \\ &= (1 - m) 2^{m-2} \sum_{i=1}^\infty x_{i+1}^{m-2} J_i^2, \end{aligned} \tag{32}$$

where

$$J_i := \int_{x_i}^{x_{i+1}} x^{\theta+1} g(x) dx, \quad i \geq 1.$$

Next, since $\sigma < 1 + \theta$, we infer from the Cauchy-Schwarz inequality that

$$\begin{aligned} \int_1^\infty x^\sigma g(x) dx &= \sum_{i=1}^\infty \int_{x_i}^{x_{i+1}} x^\sigma g(x) dx \leq \sum_{i=1}^\infty x_i^{\sigma-1-\theta} J_i \\ &\leq \left(\sum_{i=1}^\infty x_i^{2\sigma-2-2\theta} x_{i+1}^{2-m} \right)^{1/2} \left(\sum_{i=1}^\infty x_{i+1}^{m-2} J_i^2 \right)^{1/2}. \end{aligned}$$

Hence,

$$\left(\int_1^\infty x^\sigma g(x) dx \right)^2 \leq \left(\sum_{i=1}^\infty x_i^{2\sigma-2-2\theta} x_{i+1}^{2-m} \right) \sum_{i=1}^\infty x_{i+1}^{m-2} J_i^2. \quad (33)$$

Owing to the definition of $(x_i)_{i \geq 1}$ and ζ ,

$$\sum_{i=1}^\infty x_i^{2\sigma-2-2\theta} x_{i+1}^{2-m} \leq \sum_{i=1}^\infty i^{(2\sigma-2-2\theta)\zeta} (2i)^{(2-m)\zeta} = 2^{(2-m)\zeta} \frac{\pi^2}{6}.$$

Combining (32) and (33) gives

$$\mathcal{I} \geq \frac{6(1-m)2^{m-2}}{2^{(2-m)\zeta} \pi^2} \left(\int_1^\infty x^\sigma g(x) dx \right)^2,$$

as claimed. \square

Thanks to Lemma 3, we are now in a position to estimate moments of order $m \in (0, 1)$. As in Lemma 2, two estimates are derived, one which depends on $\delta \in (0, 1)$ but not on $t > 0$, the other one being independent of $\delta \in (0, 1)$ with a mild dependence upon $t > 0$.

Lemma 4 – *Let $m \in (0, 1)$ and $\mu \in [0, (m + \lambda)/2)$. If*

$$M_m(f^{\text{in}}) \leq \frac{M_m(S)}{2\delta}, \quad (34)$$

then

$$M_m(f_\delta(t)) \leq \frac{M_m(S)}{2\delta}, \quad t \geq 0. \quad (35)$$

Moreover, there is $C_2(m, \mu) > 0$ such that

$$\frac{1}{t} \int_0^t \left(\int_1^\infty x^\mu f_\delta(s, x) dx \right)^2 ds \leq C_2(m, \mu) \left(\frac{M_m(f^{\text{in}})}{t} + M_m(S) \right). \quad (36)$$

Proof. Let $t > 0$. By (26),

$$\begin{aligned} \frac{d}{dt} M_m(f_\delta(t)) - \frac{1}{2} \int_0^\infty \int_0^\infty \chi_m(x, y) K(x, y) f_\delta(t, x) f_\delta(t, y) dy dx \\ = M_m(S_\delta) - 2\delta M_m(f_\delta(t)). \end{aligned} \quad (37)$$

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We infer from (5), the inequality $x^\lambda + y^\lambda \geq 2(xy)^{\lambda/2}$, $(x, y) \in (0, \infty)^2$, and Lemma 3 (with $(\theta, m, \sigma) = (\lambda/2, m, \mu)$) that

$$\begin{aligned} & \frac{1}{2} \int_0^\infty \int_0^\infty [x^m + y^m - (x+y)^m] K(x, y) f_\delta(t, x) f_\delta(t, y) \, dy dx \\ & \geq k_1 \int_0^\infty \int_0^\infty [x^m + y^m - (x+y)^m] (xy)^{\lambda/2} f_\delta(t, x) f_\delta(t, y) \, dy dx \\ & \geq k_1 \int_1^\infty \int_1^\infty [x^m + y^m - (x+y)^m] (xy)^{\lambda/2} f_\delta(t, x) f_\delta(t, y) \, dy dx \\ & \geq \frac{2k_1}{\kappa(\lambda/2, m, \mu)} \left(\int_1^\infty x^\mu f_\delta(t, x) \, dx \right)^2. \end{aligned}$$

Consequently, using (37) along with (19), we obtain

$$\frac{d}{dt} M_m(f_\delta(t)) + \frac{1}{C_2(m, \mu)} \left(\int_1^\infty x^\mu f_\delta(t, x) \, dx \right)^2 + 2\delta M_m(f_\delta(t)) \leq M_m(S). \quad (38)$$

A first consequence of (38) is that

$$\frac{d}{dt} M_m(f_\delta(t)) + 2\delta M_m(f_\delta(t)) \leq M_m(S).$$

After integration, we obtain

$$M_m(f_\delta(t)) \leq e^{-2\delta t} M_m(f^{\text{in}}) + \frac{M_m(S)}{2\delta} (1 - e^{-2\delta t}) \leq \max \left\{ M_m(f^{\text{in}}), \frac{M_m(S)}{2\delta} \right\}$$

and use (34) to deduce (35). We next integrate (38) with respect to time over $(0, t)$ and discard the non-negative contributions of the first and third terms in the left hand side of the resulting inequality divided by t to obtain (36). \square

We next derive estimates in $X_1 \cap X_{1+\lambda}$ which strongly depend on δ .

Lemma 5 – *There is $C_3 > 0$ such that, if f^{in} satisfies (28) along with*

$$M_1(f^{\text{in}}) \leq \frac{M_\lambda(S)}{2\delta^{2-\lambda}} \quad \text{and} \quad M_{1+\lambda}(f^{\text{in}}) \leq \frac{C_3}{\delta^{(4+\lambda-\lambda^2)/(1-\lambda)}}, \quad (39)$$

then

$$M_1(f_\delta(t)) \leq \frac{M_\lambda(S)}{2\delta^{2-\lambda}} \quad \text{and} \quad M_{1+\lambda}(f_\delta(t)) \leq \frac{C_3}{\delta^{(4+\lambda-\lambda^2)/(1-\lambda)}}, \quad t \geq 0.$$

Proof. Let $t > 0$. It first follows from (19) and (26) that

$$\frac{d}{dt} M_1(f_\delta(t)) + 2\delta M_1(f_\delta(t)) = M_1(S_\delta) \leq \frac{M_\lambda(S)}{\delta^{1-\lambda}}.$$

Hence,

$$M_1(f_\delta(t)) \leq e^{-2\delta t} M_1(f^{\text{in}}) + \frac{M_\lambda(S)}{2\delta^{2-\lambda}} (1 - e^{-2\delta t}) \leq \max \left\{ M_1(f^{\text{in}}), \frac{M_\lambda(S)}{2\delta^{2-\lambda}} \right\},$$

which, together with (39), readily gives the claimed estimate on $M_1(f_\delta)$. We next infer from (5), (19), and (26) that

$$\begin{aligned} & \frac{d}{dt} M_{1+\lambda}(f_\delta(t)) + 2\delta M_{1+\lambda}(f_\delta(t)) \\ &= \frac{1}{2} \int_0^\infty \int_0^\infty \chi_{1+\lambda}(x, y) K(x, y) f_\delta(t, x) f_\delta(t, y) \, dy dx + M_{1+\lambda}(S_\delta) \\ &\leq \frac{k_2}{2} \int_0^\infty \int_0^\infty \chi_{1+\lambda}(x, y) (x^\lambda + y^\lambda) f_\delta(t, x) f_\delta(t, y) \, dy dx + \frac{M_\lambda(S)}{\delta} \\ &= k_2 \int_0^\infty \int_0^\infty x^\lambda \chi_{1+\lambda}(x, y) f_\delta(t, x) f_\delta(t, y) \, dy dx + \frac{M_\lambda(S)}{\delta}. \end{aligned}$$

For $(x, y) \in (0, \infty)^2$, it follows from Banasiak, Lamb, and Laurençot (2019, Lemma 7.4.4) that

$$\chi_{1+\lambda}(x, y) = (x + y)^{1+\lambda} - x^{1+\lambda} - y^{1+\lambda} \leq (1 + \lambda) \frac{x^{1+\lambda} y + x y^{1+\lambda}}{x + y},$$

from which we deduce that

$$\begin{aligned} x^\lambda \chi_{1+\lambda}(x, y) &\leq (1 + \lambda) \frac{x^{1+2\lambda} y + x^{1+\lambda} y^{1+\lambda}}{x + y} \\ &\leq (1 + \lambda) \left[\frac{x}{x + y} + \frac{x^{1-\lambda}}{(x + y)^{1-\lambda}} \frac{y^\lambda}{(x + y)^\lambda} \right] x^{2\lambda} y \\ &\leq 4x^{2\lambda} y. \end{aligned}$$

Therefore,

$$\frac{d}{dt} M_{1+\lambda}(f_\delta(t)) + 2\delta M_{1+\lambda}(f_\delta(t)) \leq 4k_2 M_{2\lambda}(f_\delta(t)) M_1(f_\delta(t)),$$

and we use the just established bound on $M_1(f_\delta(t))$ to obtain

$$\frac{d}{dt} M_{1+\lambda}(f_\delta(t)) + 2\delta M_{1+\lambda}(f_\delta(t)) \leq 2k_2 \frac{M_\lambda(S)}{\delta^{2-\lambda}} M_{2\lambda}(f_\delta(t)).$$

Now, since $2\lambda \in [0, 1 + \lambda)$, it follows from (29) and Hölder's inequality that

$$\begin{aligned} M_{2\lambda}(f_\delta(t)) &\leq M_{1+\lambda}(f_\delta(t))^{2\lambda/(1+\lambda)} M_0(f_\delta(t))^{(1-\lambda)/(1+\lambda)} \\ &\leq \left(\frac{M_0(S)}{2\delta} \right)^{(1-\lambda)/(1+\lambda)} M_{1+\lambda}(f_\delta(t))^{2\lambda/(1+\lambda)}. \end{aligned}$$

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Combining the above two inequalities gives

$$\frac{d}{dt}M_{1+\lambda}(f_\delta(t)) + 2\delta M_{1+\lambda}(f_\delta(t)) \leq C_3^{(1-\lambda)/(1+\lambda)} \delta^{-(3-\lambda^2)/(1+\lambda)} M_{1+\lambda}(f_\delta(t))^{2\lambda/(1+\lambda)},$$

with

$$C_3 := (2k_2 M_\lambda(S))^{(1+\lambda)/(1-\lambda)} \frac{M_0(S)}{2}.$$

We finally use Young's inequality to derive

$$\frac{d}{dt}M_{1+\lambda}(f_\delta(t)) + 2\delta M_{1+\lambda}(f_\delta(t)) \leq \delta M_{1+\lambda}(f_\delta(t)) + C_3 \delta^{-(2\lambda+3-\lambda^2)/(1-\lambda)}.$$

Hence,

$$\frac{d}{dt}M_{1+\lambda}(f_\delta(t)) + \delta M_{1+\lambda}(f_\delta(t)) \leq C_3 \delta^{-(2\lambda+3-\lambda^2)/(1-\lambda)},$$

from which we deduce

$$\begin{aligned} M_{1+\lambda}(f_\delta(t)) &\leq e^{-\delta t} M_{1+\lambda}(f^{\text{in}}) + \frac{C_3}{\delta^{(4+\lambda-\lambda^2)/(1-\lambda)}} (1 - e^{-\delta t}) \\ &\leq \max \left\{ M_{1+\lambda}(f^{\text{in}}), \frac{C_3}{\delta^{(4+\lambda-\lambda^2)/(1-\lambda)}} \right\}. \end{aligned}$$

Combining (39) with the above inequality completes the proof. \square

We end up this section with a lower bound on the moment of order λ in the spirit of that established in Proposition 3 which depends, neither on $\delta \in (0, 1)$, nor on $t > 0$, provided the former is small enough.

Lemma 6 – *There are $C_4 > 0$ and $\delta_0 \in (0, 1)$ depending only on λ , k_1 , k_2 , and S and such that, if $\delta \in (0, \delta_0)$ and*

$$M_\lambda(f^{\text{in}}) \geq C_4 := \sqrt{\frac{M_\lambda(S)}{4^{1-\lambda} k_2}}, \quad (40)$$

then

$$M_\lambda(f_\delta(t)) \geq C_4 > 0, \quad t \geq 0.$$

Proof. Let $t > 0$. Owing to (26),

$$\begin{aligned} \frac{d}{dt}M_\lambda(f_\delta(t)) - \frac{1}{2} \int_0^\infty \int_0^\infty \chi_\lambda(x, y) K(x, y) f_\delta(t, x) f_\delta(t, y) dy dx \\ = M_\lambda(S_\delta) - 2\delta M_\lambda(f_\delta(t)). \end{aligned}$$

On the one hand, by (5), (14), and (15),

$$-\chi_\lambda(x, y)K(x, y) \leq k_2 \left[x^\lambda + y^\lambda - (x + y)^\lambda \right] (x^\lambda + y^\lambda) \leq 2^{1-\lambda} k_2 (xy)^\lambda$$

for $(x, y) \in (0, \infty)^2$. On the other hand, it follows from (17) and (19) that there is $\delta_1 \in (0, 1)$ depending only on S such that

$$M_\lambda(S_\delta) \geq \frac{M_\lambda(S)}{2}, \quad \delta \in (0, \delta_1).$$

Consequently, for $\delta \in (0, \delta_1)$,

$$\frac{d}{dt} M_\lambda(f_\delta(t)) + F_\delta(M_\lambda(f_\delta(t))) \geq \frac{M_\lambda(S)}{2}, \quad (41)$$

with

$$F_\delta(z) := 2^{-\lambda} k_2 z^2 + 2\delta z, \quad z \in \mathbb{R}.$$

Since F_δ is increasing and maps $[0, \infty)$ onto $[0, \infty)$, there is a unique $z_\delta > 0$ such that $F_\delta(z_\delta) = M_\lambda(S)/2$, which is here explicitly given by

$$z_\delta := \frac{\sqrt{k_2 M_\lambda(S) + 2^{1+\lambda} \delta^2} - 2^{(1+\lambda)/2} \delta}{2^{(1-\lambda)/2} k_2}.$$

We then infer from (41) and the comparison principle that

$$M_\lambda(f_\delta(t)) \geq \min\{M_\lambda(f^{\text{in}}), z_\delta\}, \quad t \geq 0. \quad (42)$$

Moreover, since

$$\lim_{\delta \rightarrow 0} z_\delta = \sqrt{\frac{M_\lambda(S)}{2^{1-\lambda} k_2}} > 2^{(\lambda-1)/2} \sqrt{\frac{M_\lambda(S)}{2^{1-\lambda} k_2}} = C_4,$$

there is $\delta_0 \in (0, \delta_1)$ such that $z_\delta \geq C_4$ for $\delta \in (0, \delta_0)$. This property, together with (40) and (42) completes the proof. \square

3.2 Uniform integrability

The next step is devoted to uniform integrability estimates.

Lemma 7 – *There is $C_5 > 0$ such that, if $\delta \in (0, \delta_0)$ and f^{in} satisfies (40) as well as*

$$\int_0^\infty \Phi(f^{\text{in}}(x)) \, dx \leq C_5, \quad (43)$$

the function Φ being defined in (18), then

$$\int_0^\infty \Phi(f_\delta(t, x)) \, dx \leq C_5, \quad t \geq 0.$$

3. Approximation

Proof. Since $K(x-y, y) \leq K(x, y)$ for $(x, y) \in (0, \infty)^2$ by (6), it follows from Banasiak, Lamb, and Laurençot (2019, Lemma 8.2.18) that

$$\begin{aligned} J_\delta(t) &:= \int_0^\infty \Phi'(f_\delta(t, x)) \mathcal{C} f_\delta(t, x) \, dx \\ &\leq -\frac{1}{2} \int_0^\infty \int_x^\infty K(x, y) [f_\delta \Phi'(f_\delta) - \Phi(f_\delta)](t, x) f_\delta(t, y) \, dy \, dx \\ &\quad - \frac{1}{2} \int_0^\infty \int_0^\infty K(x, y) f_\delta(t, x) \Phi'(f_\delta(t, x)) f_\delta(t, y) \, dy \, dx. \end{aligned}$$

By (5), $K(x, y) \geq k_1 y^\lambda$, $(x, y) \in (0, \infty)^2$, and the properties of Φ guarantee that $r\Phi'(r) \geq \Phi(r) \geq 0$, $r \geq 0$, so that we further obtain

$$\begin{aligned} J_\delta(t) &\leq -\frac{1}{2} \int_0^\infty \int_0^\infty K(x, y) \Phi(f_\delta(t, x)) f_\delta(t, y) \, dy \, dx \\ &\leq -\frac{k_1}{2} \int_0^\infty \int_0^\infty y^\lambda \Phi(f_\delta(t, x)) f_\delta(t, y) \, dy \, dx \\ &= -\frac{k_1}{2} M_\lambda(f_\delta(t)) \int_0^\infty \Phi(f_\delta(t, x)) \, dx. \end{aligned}$$

Hence, owing to (27) and Lemma 6,

$$J_\delta(t) \leq -2C_6 \int_0^\infty \Phi(f_\delta(t, x)) \, dx \quad \text{with} \quad C_6 := \frac{\min\{2, k_1 C_4\}}{4} \in (0, 1).$$

We then infer from (19), (20a), and the non-negativity of Φ' that

$$\begin{aligned} \frac{d}{dt} \int_0^\infty \Phi(f_\delta(t, x)) \, dx &= \int_0^\infty \Phi'(f_\delta(t, x)) \partial_t f_\delta(t, x) \, dx \\ &= J_\delta(t) + \int_0^\infty \Phi'(f_\delta(t, x)) S_\delta(x) \, dx \\ &\leq -2C_6 \int_0^\infty \Phi(f_\delta(t, x)) \, dx + \int_0^\infty \Phi'(f_\delta(t, x)) S(x) \, dx. \end{aligned} \quad (44)$$

Recalling that the properties of Φ implies that

$$s\Phi'(r) \leq \Phi(r) + \Phi(s) \quad \text{and} \quad \Phi(sr) \leq \max\{1, s^2\}\Phi(r), \quad (r, s) \in [0, \infty)^2,$$

see Banasiak, Lamb, and Laurençot (2019, Proposition 7.1.9 (b) & (d)), we find

$$\begin{aligned} \int_0^\infty \Phi'(f_\delta(t, x)) S(x) \, dx &= C_6 \int_0^\infty \Phi'(f_\delta(t, x)) \frac{S(x)}{C_6} \, dx \\ &\leq C_6 \int_0^\infty \Phi(f_\delta(t, x)) \, dx + C_6 \int_0^\infty \Phi\left(\frac{S(x)}{C_6}\right) \, dx \\ &\leq C_6 \int_0^\infty \Phi(f_\delta(t, x)) \, dx + \frac{1}{C_6} \int_0^\infty \Phi(S(x)) \, dx. \end{aligned}$$

Combining the above inequality with (18b) and (44) leads us to the differential inequality

$$\frac{d}{dt} \int_0^\infty \Phi(f_\delta(t, x)) dx + C_6 \int_0^\infty \Phi(f_\delta(t, x)) dx \leq \frac{L_\Phi(S)}{C_6},$$

from which we deduce that

$$\begin{aligned} \int_0^\infty \Phi(f_\delta(t, x)) dx &\leq e^{-C_6 t} \int_0^\infty \Phi(f^{\text{in}}(x)) dx + \frac{L_\Phi(S)}{C_6^2} (1 - e^{-C_6 t}) \\ &\leq \max \left\{ \int_0^\infty \Phi(f^{\text{in}}(x)) dx, \frac{L_\Phi(S)}{C_6^2} \right\}. \end{aligned}$$

Lemma 7 is now a straightforward consequence of (43) and the above inequality with $C_5 = L_\Phi(S)/C_6^2$. \square

3.3 Proof of Proposition 4

We fix $\delta \in (0, \delta_0)$ and consider the subset \mathcal{Z}_δ of $X_0 = L^1(0, \infty)$ defined by: $h \in \mathcal{Z}_\delta$ if and only if $h \in X_0^+$ satisfies

$$C_4 \leq M_\lambda(h) \leq C_1, \quad \int_0^\infty \Phi(h(x)) dx \leq C_5, \quad (45a)$$

$$M_m(h) \leq \frac{M_m(S)}{2\delta}, \quad m \in [0, 1), \quad (45b)$$

$$M_1(h) \leq \frac{M_\lambda(S)}{2\delta^{2-\lambda}}, \quad M_{1+\lambda}(h) \leq \frac{C_3}{\delta^{(4+\lambda-\lambda^2)/(1-\lambda)}}. \quad (45c)$$

On the one hand, given $f^{\text{in}} \in \mathcal{Z}_\delta$ and $t \geq 0$, it follows from Lemma 1, Lemma 6, and Lemma 7 that $f_\delta(t) = \Psi_\delta(t, f^{\text{in}})$ satisfies (45a) and from Lemma 2 and Lemma 4 that it satisfies (45b). Furthermore, $\Psi_\delta(t, f^{\text{in}})$ satisfies (45c) due to Lemma 5. Consequently, $\Psi_\delta(t, f^{\text{in}}) \in \mathcal{Z}_\delta$ for all $t \geq 0$, so that \mathcal{Z}_δ is a positive invariant set for the semi-flow Ψ_δ . On the other hand, \mathcal{Z}_δ is non-empty since $C_4 < C_1$ by (27) and (40). Moreover, owing to the superlinearity (18a) of Φ , the Dunford-Pettis theorem ensures that \mathcal{Z}_δ is a closed convex and sequentially weakly compact subset of $L^1(0, \infty)$. Since Ψ_δ is a semi-flow on \mathcal{Z}_δ endowed with its weak topology by Proposition 5, it follows from Escobedo, Mischler, and Rodriguez Ricard (2005, Theorem 1.2) that there is $\varphi_\delta \in \mathcal{Z}_\delta$ such that $\Psi_\delta(t, \varphi_\delta) = \varphi_\delta$ for all $t \geq 0$; that is, $\varphi_\delta \in \mathcal{Z}_\delta$ is a stationary solution to (20a). Since $\Psi_\delta(t, \varphi_\delta) = \varphi_\delta$ for all $t \geq 0$, we infer from (30) that

$$k_1 M_0(\varphi_\delta) M_\lambda(\varphi_\delta) \leq \frac{M_0(\varphi_\delta)}{t} + M_0(S)$$

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for all $t > 0$. Hence, taking the limit $t \rightarrow \infty$,

$$k_1 M_0(\varphi_\delta) M_\lambda(\varphi_\delta) \leq M_0(S),$$

from which we deduce, thanks to the lower bound for $M_\lambda(\varphi_\delta)$ in (45),

$$k_1 C_4 M_0(\varphi_\delta) \leq M_0(S). \quad (46)$$

Similarly, for $\mu \in (0, (1 + \lambda)/2)$, it follows from (36) with $m = (2\mu + 1 - \lambda)/2 \in (0, 1)$ that

$$\left(\int_1^\infty x^\mu \varphi_\delta(x) \, dx \right)^2 \leq C_2((2\mu + 1 - \lambda)/2, \mu) \left(\frac{M_{(2\mu+1-\lambda)/2}(\varphi_\delta)}{t} + M_{(2\mu+1-\lambda)/2}(S) \right)$$

for all $t > 0$. Letting $t \rightarrow \infty$ gives

$$\left(\int_1^\infty x^\mu \varphi_\delta(x) \, dx \right)^2 \leq C_2((2\mu + 1 - \lambda)/2, \mu) M_{(2\mu+1-\lambda)/2}(S).$$

Together with (46), the above estimate entails that

$$\begin{aligned} M_\mu(\varphi_\delta) &= \int_0^1 x^\mu \varphi_\delta(x) \, dx + \int_1^\infty x^\mu \varphi_\delta(x) \, dx \\ &\leq M_0(\varphi_\delta) + \sqrt{C_2((2\mu + 1 - \lambda)/2, \mu) M_{(2\mu+1-\lambda)/2}(S)} \\ &\leq C_7(\mu) := \frac{M_0(S)}{k_1 C_4} + \sqrt{C_2((2\mu + 1 - \lambda)/2, \mu) M_{(2\mu+1-\lambda)/2}(S)}. \end{aligned} \quad (47)$$

Collecting the estimates (45a), (46), and (47) gives (22) and (23) and completes the proof of Proposition 4.

4 Existence

Proof (of Theorem 1 (a)). Since Φ is superlinear at infinity by (18a), it follows from (22), (23), and the Dunford-Pettis theorem that $(\varphi_\delta)_{\delta \in (0, \delta_0)}$ is relatively sequentially weakly compact in X_0 . In turn, this compactness property and (23) imply that $(\varphi_\delta)_{\delta \in (0, \delta_0)}$ is actually relatively sequentially weakly compact in X_μ for any $\mu \in [0, (1 + \lambda)/2)$. Consequently, using a diagonal process, there are a subsequence $(\varphi_{\delta_j})_{j \geq 2}$ of $(\varphi_\delta)_{\delta \in (0, \delta_0)}$ and

$$\varphi \in \bigcap_{\mu \in [0, (1+\lambda)/2)} X_\mu^+ \quad (48)$$

such that, as $j \rightarrow \infty$,

$$\varphi_{\delta_j} \rightharpoonup \varphi \text{ in } X_\mu, \quad \mu \in \left[0, \frac{1+\lambda}{2}\right). \quad (49)$$

Since $\lambda \in [0, (1 + \lambda)/2)$, it readily follows from (49) that $([(x, y) \mapsto \varphi_{\delta_j}(x)\varphi_{\delta_j}(y)])_{j \geq 2}$ converges weakly to $[(x, y) \mapsto \varphi(x)\varphi(y)]$ in $X_{0,\lambda} \times X_{0,\lambda}$ as $j \rightarrow \infty$. It is then straightforward to pass to the limit $j \rightarrow \infty$ in the identity (21) satisfied by φ_{δ_j} and deduce that φ satisfies (7), thereby completing the proof of Theorem 1 (a), recalling that the other integrability properties of φ listed there follow from Propositions 1 and 2, see also (48). \square

5 Non-existence

Proof (of Theorem 1 (b)). The proof relies on the same device as those of Propositions 1 and 3. For $A > 0$ and $x > 0$, we set $\vartheta_A(x) = \min\{x, A\}$. We infer from (7) and the symmetry of K that

$$\begin{aligned} \int_0^\infty \vartheta_A(x)S(x) \, dx &= \frac{1}{2} \int_0^A \int_{A-x}^A (x+y-A)K(x,y)\varphi(x)\varphi(y) \, dydx \\ &\quad + \int_0^A \int_A^\infty xK(x,y)\varphi(x)\varphi(y) \, dydx \\ &\quad + \frac{A}{2} \int_A^\infty \int_A^\infty K(x,y)\varphi(x)\varphi(y) \, dydx. \end{aligned} \quad (50)$$

We are left with identifying the limit as $A \rightarrow \infty$ of each term on the right hand side of (50). We first infer from (5) that

$$\begin{aligned} 0 &\leq \mathbf{1}_{(0,A)}(x)\mathbf{1}_{(A-x,A)}(y)(x+y-A)K(x,y)\varphi(x)\varphi(y) \\ &\leq k_2(x+y-A)(x^\lambda+y^\lambda)\varphi(x)\varphi(y) \\ &\leq k_2(x^\lambda y+y^\lambda x)\varphi(x)\varphi(y). \end{aligned}$$

Since $\varphi \in X_1 \cap X_\lambda \subset X_0 \cap X_\lambda$ due to $\lambda \geq 1$ and

$$\lim_{A \rightarrow \infty} \mathbf{1}_{(0,A)}(x)\mathbf{1}_{(A-x,A)}(y) = 0, \quad (x, y) \in (0, \infty)^2,$$

it follows from Lebesgue's dominated convergence theorem that

$$\lim_{A \rightarrow \infty} \frac{1}{2} \int_0^A \int_{A-x}^A (x+y-A)K(x,y)\varphi(x)\varphi(y) \, dydx = 0.$$

Acknowledgments

Next, using once more (5),

$$\begin{aligned} 0 &\leq \int_0^A \int_A^\infty xK(x,y)\varphi(x)\varphi(y) \, dydx \leq k_2 \int_0^A \int_A^\infty x(x^\lambda + y^\lambda)\varphi(x)\varphi(y) \, dydx \\ &\leq 2k_2 \int_0^A \int_A^\infty xy^\lambda\varphi(x)\varphi(y) \, dydx \leq 2k_2M_1(\varphi) \int_A^\infty y^\lambda\varphi(y) \, dy \end{aligned}$$

and

$$\begin{aligned} 0 &\leq A \int_A^\infty \int_A^\infty K(x,y)\varphi(x)\varphi(y) \, dydx \leq Ak_2 \int_A^\infty \int_A^\infty (x^\lambda + y^\lambda)\varphi(x)\varphi(y) \, dydx \\ &\leq k_2 \int_A^\infty \int_A^\infty (x^\lambda y + xy^\lambda)\varphi(x)\varphi(y) \, dydx \leq 2k_2M_1(\varphi) \int_A^\infty y^\lambda\varphi(y) \, dy, \end{aligned}$$

from which we deduce that

$$\begin{aligned} \lim_{A \rightarrow \infty} \int_0^A \int_A^\infty xK(x,y)\varphi(x)\varphi(y) \, dydx &= 0, \\ \lim_{A \rightarrow \infty} \frac{A}{2} \int_A^\infty \int_A^\infty K(x,y)\varphi(x)\varphi(y) \, dydx &= 0, \end{aligned}$$

recalling that $\varphi \in X_\lambda$. Collecting the above information, we may take the limit $A \rightarrow \infty$ in (50) and conclude that

$$\lim_{A \rightarrow \infty} \int_0^\infty \min\{x, A\}S(x) \, dx = 0.$$

Hence, $S \equiv 0$ which, together with (11), implies that $\varphi \equiv 0$ as well. \square

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