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Inverse of generalized Nevanlinna function that is holomorphic at infinity

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Abstract

Let $(\mathcal{H},(\cdot,\cdot))$ be a Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the linear space of bounded operators in $\mathcal{H}$. In this paper, we deal with $\mathcal{L}(\mathcal{H})$-valued function Q that belongs to the generalized Nevanlinna class $\mathcal{N}_\kappa(\mathcal{H})$, where κ is a non-negative integer. It is the class of functions meromorphic on $\mathbb{C}\setminus\mathbb{R}$, such that $Q(z)^* = Q(\bar{z})$ and the kernel $\mathcal{N}_Q(z,w) := \frac{Q(z)-Q(w)^*}{z-\bar{w}}$ has κ negative squares. A focus is on the functions $Q \in \mathcal{N}_\kappa(\mathcal{H})$ which are holomorphic at ∞. A new operator representation of the inverse function $\hat{Q}(z) := -Q(z)^{-1}$ is obtained under the condition that the derivative at infinity $Q'(\infty) := \lim_{z\to\infty} zQ(z)$ is boundedly invertible operator. It turns out that $\hat{Q}$ is the sum $\hat{Q} = \hat{Q}_1 + \hat{Q}_2$, $\hat{Q}_i \in \mathcal{N}_{\kappa_i}(\mathcal{H})$ that satisfies $\kappa_1 + \kappa_2 = \kappa$. That decomposition enables us to study properties of both functions, Q and $\hat{Q}$, by studying the simple components $\hat{Q}_1$ and $\hat{Q}_2$.

Keywords: Generalized Nevanlinna function, Pontryagin space, operator representation, generalized pole.

MSC: 47B50, 47A56, 30E99.

1 Preliminaries and introduction

1.1 Generalized Nevanlinna class, denoted by $\mathcal{N}_\kappa(\mathcal{H})$, is extensively studied class of complex functions. For example, Hermitian matrix polynomials and their inverse functions belong to $\mathcal{N}_\kappa(\mathcal{H})$. For more examples one can see, for example Luger (2015).

As usually, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{C}^+$ denote sets of positive integers, real numbers, complex numbers, and complex numbers from the upper half plane, respectively.

Definition 1 – An operator valued complex function $Q : \mathcal{D}(Q) \to \mathcal{L}(\mathcal{H})$ belongs to the class of generalized Nevanlinna functions $\mathcal{N}_\kappa(\mathcal{H})$ if it satisfies the following requirements:

- Q is meromorphic in $\mathbb{C}\setminus\mathbb{R}$,
- $Q(z)^* = Q(\bar{z})$, $z \in \mathcal{D}(Q)$,

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- Nevanlinna kernel

$$\mathcal{N}_Q(z,w) := \frac{Q(z) - Q(w)^*}{z - \bar{w}}, \quad \mathcal{N}_Q(z,\bar{z}) := Q'(z); \qquad z,w \in \mathcal{D}(Q) \cap \mathbb{C}^+,$$

has κ negative squares, i.e. for arbitrary $n \in \mathbb{N}$, $z_1,\ldots,z_n \in \mathcal{D}(Q) \cap \mathbb{C}^+$ and $h_1,\ldots,h_n \in \mathcal{H}$ the Hermitian matrix $(\mathcal{N}_Q(z_i,z_j)h_i,h_j)_{i,j=1}^n$ has at most κ negative eigenvalues, and for at least one choice of n; $z_1,\ldots,z_n$, and $h_1,\ldots,h_n$ it has exactly κ negative eigenvalues.

A generalized Nevanlinna function $Q \in \mathcal{N}_\kappa(\mathcal{H})$ is called *regular* if there exists at least one point $w_0 \in \mathcal{D}(Q) \cap \mathbb{C}^+$ such that the operator $Q(w_0)^{-1}$ is boundedly invertible.

Let $\kappa \in \mathbb{N} \cup \{0\}$ and let $(\mathcal{K},[\cdot,\cdot])$ denote a *Krein space*. That is a complex vector space on which a scalar product, i.e. a Hermitian sesquilinear form $[\cdot,\cdot]$, is defined such that the following decomposition of $\mathcal{K}$ exists

$$\mathcal{K} = \mathcal{K}_+ \dot{+} \mathcal{K}_-,$$

where $(\mathcal{K}_+,[\cdot,\cdot])$ and $(\mathcal{K}_-,-[\cdot,\cdot])$ are Hilbert spaces which are mutually orthogonal with respect to the form $[\cdot,\cdot]$. Every Krein space $(\mathcal{K},[\cdot,\cdot])$ is *associated* with a Hilbert space $(\mathcal{K},(\cdot,\cdot))$, which is defined as a direct and orthogonal sum of the Hilbert spaces $(\mathcal{K}_+,[\cdot,\cdot])$ and $(\mathcal{K}_-,-[\cdot,\cdot])$. Topology in a Krein space $\mathcal{K}$ is introduced by means of the associated Hilbert space $(\mathcal{K},(\cdot,\cdot))$. For properties of Krein spaces one can see e.g. Bognar (1974, Chapter V).

If the scalar product $[\cdot,\cdot]$ has $\kappa\,(< \infty)$ negative squares, then we call it a *Pontryagin space of index κ*. The definition of a Pontryagin space and other related concepts can be found e.g. in Iohvidov, Krein, and Langer (1982).

1.2 The following definitions of a linear relation and basic concepts related to it can be found in Arens (1961) and Sorjonen (1978). In the sequel, $\mathcal{H}$, $\mathcal{K}$, $\mathcal{M}$ are inner product spaces.

A *linear relation* from $\mathcal{H}$ into $\mathcal{K}$ is a linear manifold T of the product space $\mathcal{H} \times \mathcal{K}$. If $\mathcal{H} = \mathcal{K}$, T is said to be a *linear relation in $\mathcal{K}$*. We will use the following concepts and notations for linear relations, T and S from $\mathcal{H}$ into $\mathcal{K}$ and a linear relation R from $\mathcal{K}$ into $\mathcal{M}$.

$$D(T) := \left\{ f \in \mathcal{H} \,\middle|\, \{f,g\} \in T \text{ for some} g \in \mathcal{K} \right\},$$

$$R(T) := \left\{ g \in \mathcal{K} \,\middle|\, \{f,g\} \in T \text{ for some} f \in \mathcal{H} \right\},$$

$$\ker T := \left\{ f \in \mathcal{H} \,\middle|\, \{f,0\} \in T \right\},$$

$$T(0) := \left\{ g \in \mathcal{K} \,\middle|\, \{0,g\} \in T \right\},$$

(Cont. next page)

$$T(f) := \left\{ g \in \mathcal{K} \,\middle|\, \{f,g\} \in T \right\}, \quad f \in D(T),$$

$$
\begin{aligned}
T^{-1} &:= \Big\{\{g,f\} \in \mathcal{K}\times\mathcal{H} \,\Big|\, \{f,g\}\in T\Big\},\\
zT &:= \Big\{\{f,zg\} \in \mathcal{H}\times\mathcal{K} \,\Big|\, \{f,g\}\in T\Big\}, \quad z\in\mathbb{C},\\
S+T &:= \Big\{\{f,g+k\} \,\Big|\, \{f,g\}\in S, \{f,k\}\in T\Big\},\\
RT &:= \Big\{\{f,k\} \in \mathcal{H}\times\mathcal{M} \,\Big|\, \{f,g\}\in T, \{g,k\}\in \mathbb{R} \text{ for some} g\in\mathcal{K}\Big\},\\
T^{+} &:= \Big\{\{k,h\} \in \mathcal{K}\times\mathcal{H} \,\Big|\, [k,g]=(h,f) \text{ for all} \{f,g\}\in T\Big\},\\
T_{\infty} &:= \Big\{\{0,g\}\in T\Big\}.
\end{aligned}
$$

A linear relation is *closed* if it is a closed subset in the product space $\mathcal{H}\times\mathcal{K}$. If $T(0)=\{0\}$, we say that T is an *operator*, or *single-valued* linear relation.

Note, in definition of the adjoint linear relation T^{+}, we use the following notation for inner product spaces $(\mathcal{H},(\cdot,\cdot))$ and $(\mathcal{K},[\cdot,\cdot])$.

Let A be a linear relation in $\mathcal{K}$. We say that A is *symmetric* (*self-adjoint*) if it holds $A\subseteq A^{+}$ ($A=A^{+}$). Every point $\alpha\in\mathbb{C}$ for which $\{f,\alpha f\}\in A$, with some $f\neq 0$, is called a *finite eigenvalue*. The corresponding vectors are *eigenvectors* belonging to the eigenvalue α. A set that consists of all points $z\in\mathbb{C}$ for which the relation $(A-zI)^{-1}$ is an operator defined on the entire $\mathcal{K}$, is called the *resolvent* set $\rho(A)$.

It is convenient to deal with the following representation of generalized Nevanlinna functions.

Theorem 1 – *A function $Q:\mathcal{D}(Q)\to\mathcal{L}(\mathcal{H})$ is a generalized Nevanlinna function of some index κ, denoted by $Q\in\mathcal{N}_{\kappa}(\mathcal{H})$, if and only if it has a representation of the form*

$$Q(z)=Q(z_0)^{*}+(z-\bar{z}_0)\Gamma_{z_0}^{+}\Big(I+(z-z_0)(A-z)^{-1}\Big)\Gamma_{z_0}, \quad z\in\mathcal{D}(Q), \tag{1}$$

where, A is a self-adjoint linear relation in some Pontryagin space $(\mathcal{K},[\cdot,\cdot])$ of index $\tilde{\kappa}\geq\kappa$; $\Gamma_{z_0}:\mathcal{H}\to\mathcal{K}$ is a bounded operator; $z_0\in\rho(A)\cap\mathbb{C}^{+}$ is a fixed point of reference. (Then, obviously $\rho(A)\subseteq\mathcal{D}(Q)$.) This representation can be chosen to be minimal, that is

$$\mathcal{K}=\text{c.l.s.}\Big\{\Gamma_z h : z\in\rho(A), h\in H\Big\}, \tag{2}$$

where

$$\Gamma_z=\Big(I+(z-z_0)(A-z)^{-1}\Big)\Gamma_{z_0}. \tag{3}$$

If realization (1) is minimal, then $Q\in\mathcal{N}_{\kappa}(\mathcal{H})$ if and only if the negative index of the Pontryagin space $\tilde{\kappa}$ equals κ. In the case of minimal representation $\rho(A)=\mathcal{D}(Q)$ and the triple $(\mathcal{K},A,\Gamma_{z_0})$ is uniquely determined (up to isomorphism).

Such operator representations were developed by M. G. Krein and H. Langer, see e.g. Krein and Langer (1973, 1977) and later converted to representations in terms of linear relations (multivalued operators), see e.g. Dijksma, Langer, and Snoo H. S. V. (1993) and Hassi, Snoo H. S. V., and Woracek (1998).

In this note, a point $\alpha \in \mathbb{C}$ is called a *finite generalized pole* of Q if it is an eigenvalue of the representing relation A in the minimal representation (1). It means that it may be isolated singularity, i.e. an ordinary pole, as well as an embedded singularity of Q. The latter may be the case only if $\alpha \in \mathbb{R}$.

1.3 In this paper, we focus on the class of functions $Q \in \mathcal{N}_\kappa(\mathcal{H})$ that are holomorphic at ∞, i.e. there exists

$$Q'(\infty) := \lim_{z\to\infty} zQ(z). \tag{4}$$

That is equivalent to

$$Q(z) = \Gamma^+(A-z)^{-1}\Gamma, \tag{5}$$

where A is a bounded self-adjoint operator in some Pontrjagin space $\mathcal{K}$, and $\Gamma : \mathcal{H} \to \mathcal{K}$ is a bounded operator, see Lemma 3 below. We also assume that drivative $Q'(\infty)$ is boundedly invertible. In this study, $\lim_{z\to\infty} zQ(z)$ refers to convergence in the Banach space of bounded operators $\mathcal{L}(\mathcal{H})$. By $z \to \infty$ we denote the limit if Q is holomorphic at ∞, and by $z \hat{\to} \infty$ we denote the non-tangential limit, which we use if singularities of Q exist (on the real axis) in every neighborhood of ∞, see Krein and Langer (1977). The same convention applies to limits toward finite points in complex plane.

The following well known decomposition easily follows from Daho and Langer (1985, Proposition 3.3) for matrix functions. See Luger (2006, Section 5.1) for operator valued functions.

Lemma 1 – *If $Q \in \mathcal{N}_\kappa(\mathcal{H})$ and α is a finite generalized pole of Q, then it holds*

$$Q(z) = Q_\alpha(z)+H_\alpha(z), \tag{6}$$

where $Q_\alpha \in \mathcal{N}_{\kappa_1}(\mathcal{H})$ is holomorphic at $\infty, H_\alpha \in \mathcal{N}_{\kappa_2}(\mathcal{H})$ is holomorphic at $\alpha, \kappa_1 + \kappa_2 = \kappa$. Then Q_α admits representation

$$Q_\alpha(z) = \Gamma_\alpha^+(A_\alpha - z)^{-1}\Gamma_\alpha,$$

with a bounded operator A_α. Operator A_α has the same root manifold at α as the representing relation A of Q in (1).

Remark 1 – The decomposition (6) can be tweaked if necessary so that it holds

$$Q(z) = \tilde{Q}(z) + \tilde{H}(z),$$

where $\tilde{Q}(z) = \Gamma^+(\tilde{A} - z)^{-1}\Gamma \in \mathcal{N}_{\kappa_1}(\mathcal{H})$, self-adjoint extension $\tilde{A}$ of A_α has the same root manifold at α as A_α, and $\Gamma^+\Gamma$ is a boundedly invertible operator. Then the equality $\kappa = \kappa_1 + \kappa_2$ does not have to be preserved because the number of negative squares of $\tilde{H}(z)$ may be greater than the number of negative squares of $H_\alpha(z)$.

Indeed, if $\Gamma_{\alpha}^{+}\Gamma_{\alpha}$ is not already boundedly invertible operator in decomposition (6) of Q, then we can add the term $\frac{B}{\beta - z}$ to $Q_{\alpha}(z)$, where B is a positive operator, $\Gamma_{\alpha}^{+}\Gamma_{\alpha} + B$ is boundedly invertible operator and $\beta \in \mathbb{R} \setminus \{\alpha\}$. Also we will subtract the same term from $H_{\alpha}(z)$. Functions $\tilde{Q}(z) := Q_{\alpha}(z) + \frac{B}{\beta - z}$ and $\tilde{H}(z) := H_{\alpha}(z) - \frac{B}{\beta - z}$, will have claimed properties. □

1.4 The following is the summary of the main results of the paper.

In Proposition 3 we prove that function Q, which is holomorphic at ∞ and has invertible operator $Q'(\infty)$, has $\ker Q := \bigcap_{z \in D(Q)} \ker Q(z) = \{0\}$.

The task of finding representation of $\hat{Q}(z) := -Q(z)^{-1}$ in terms of representing relation A of Q has been studied in several papers, see e.g. Langer and Luger (2000) and Luger (2002). In Theorem 2, we give an operator representation of $\hat{Q}$, when function Q is holomorphic at infinity and $Q'(\infty)$ is boundedly invertible operator. According to Remark 1, those assumptions do not restrict generality in research of local properties of the function $Q \in \mathcal{N}_{\kappa}(\mathcal{H})$.

Theorem 2 enables us to prove many properties of $\hat{Q}$ and Q. For example, in Theorem 3 we prove that function Q which is holomorphic at ∞ and has $Q'(\infty)$ boundedly invertible, is a regular function. In Proposition 5 we prove that for such Q the inverse function $\hat{Q}$ must have a pole at ∞. In Theorem 4 we prove that $\hat{Q}(z) := -Q(z)^{-1}$ is the sum $\hat{Q} = \hat{Q}_1 + \hat{Q}_2$, $\hat{Q}_i \in \mathcal{N}_{\kappa_i}(\mathcal{H})$, where both functions $\hat{Q}_i$ are represented in terms of the representing operator A of Q, and it holds $\kappa_1 + \kappa_2 = \kappa$. One of the functions, say $\hat{Q}_1$, is a polynomial of degree one, and $\hat{Q}_2$ has representation of the form (5). Therefore, we can call functions $\hat{Q}_1$ and $\hat{Q}_2$, *polynomial*, and *resolvent part of* $\hat{Q}$, respectively. Negative index κ_1 of $\hat{Q}_1$ is equal to the number of negative eigenvalues of the self-adjoint operator $\Gamma^{+}\Gamma = -\lim_{z \to \infty} zQ(z)$. The set of zeros of Q coincides with the set of poles of $\hat{Q}_2$.

In Example 1, we show how the above results can be applied to find representing operators A and Γ of Q in some cases. In Example 2, we show how to implement formulae given in Theorem 4 to a concrete function Q, in order to obtain a decomposition $\hat{Q} = \hat{Q}_1 + \hat{Q}_2$ with nice properties described in that theorem.

2 Representation $Q(z) = S + \Gamma^{+}(A - z)^{-1}\Gamma$

2.1 We will frequently need the following proposition in this paper.

Proposition 1 – *(i) Let function $Q \in \mathcal{N}_{\kappa}(\mathcal{H})$ be represented by a self-adjoint linear relation A in representation (1), which is not necessarily minimall. If for any point $z_0 \in \rho(A)$ it holds*

$$R(\Gamma_{z_0}) \subseteq D(A), \tag{7}$$

then the same inclusion holds for every $z \in \rho(A)$. We can define linear relation

$$\Gamma := (A-z)\Gamma_z, \quad z \in \rho(A), \tag{8}$$

that satisfies $D(\Gamma) = \mathcal{H}$, $\Gamma(0) = A(0)$. Then function Q has representation of the form

$$Q(z) = S + \Gamma^+(A-z)^{-1}\Gamma \in \mathcal{N}_\kappa(\mathcal{H}), \quad S = S^* \in \mathcal{L}(\mathcal{H}). \tag{9}$$

(ii) Conversely, if A in representation (9) of Q is a self-adjoint linear relation in Pontryagin space $\mathcal{K}$, and $\Gamma \subseteq \mathcal{H} \times \mathcal{K}$, $D(\Gamma) = \mathcal{H}$, is a linear relation that satisfies $A(0) = \Gamma(0)$, then for any point $z_0 \in \rho(A)$ and operator

$$\Gamma_{z_0} := (A-z_0)^{-1}\Gamma, \tag{10}$$

function Q satisfies (1).

(iii) It holds

$$\Gamma_z := \left(I + (z-z_0)(A-z)^{-1}\right)\Gamma_{z_0} = (A-z)^{-1}\Gamma, \quad \forall z \in \rho(A). \tag{11}$$

Representation (1) is minimal if and only if representation (9) is minimal.

Note, case $S = 0$ is not excluded in Proposition 1.

Proof. (i) For function Q given by (1), it holds

$$\Gamma_z = \left(I + (z-w)(A-z)^{-1}\right)\Gamma_w, \quad \forall z, w \in \rho(A),$$

see the proof in Dijksma, Langer, and Snoo H. S. V. (1993), which obviously can be repeated when $Q \in N_\kappa(\mathcal{H})$. If we substitute w by z_0 in the above equation, then from assumption (7) it follows

$$R(\Gamma_z) \subseteq D(A), \quad \forall z \in \rho(A).$$

In the following few steps we use properties of linear relations listed in Arens (1961, Theorem 1.2). Note, Γ_z are single-valued linear relations defined on the entire $\mathcal{H}$ which simplifies verification of the following steps. Therefore

$$(A-z)\left(\Gamma_{z_0} + (z-z_0)(A-z)^{-1}\Gamma_{z_0}\right) = (A-z)\Gamma_z.$$

According to $(A-z)(A-z)^{-1} \supseteq I$ it holds

$$(A-z)\Gamma_{z_0} + (z-z_0)\Gamma_{z_0} \subseteq (A-z)\Gamma_z \;\Rightarrow\; (A-z_0)\Gamma_{z_0} \subseteq (A-z)\Gamma_z, \quad \forall z \in \rho(A).$$

By the same token, the converse inclusion $(A-z)\Gamma_z \subseteq (A-z_0)\Gamma_{z_0}$, $\forall z \in \rho(A)$ holds. Therefore,

$$(A-z)\Gamma_z = (A-z_0)\Gamma_{z_0}, \quad \forall z \in \rho(A),$$

and we can define linear relation Γ by (8). According to (8) it holds $\Gamma(0) = A(0)$, and therefore $(A-z)^{-1}\Gamma$ is also an operator, $\forall z \in \rho(A)$.

Thus, Γ is an invariant of Q, i.e. Γ is a characteristic of the function Q (independent of $z \in \rho(A)$). That makes relation Γ and representation (5) particularly interesting.

Let us now show that linear relation Γ^{+} is an operator. If we assume the contrary, then it holds

$$\{0,g\} \in \Gamma^{+} \Rightarrow [k,0] = (h,g), \quad \forall \{h,k\} \in \Gamma.$$

Since $D(\Gamma) = \mathcal{H}$, it follows $g = 0$. Therefore, Γ^{+} is single-valued.

From (8), for $z_0 \in \rho(A)$, we get $\Gamma = (A-z_0)\Gamma_{z_0}$ and $\Gamma_{z_0} = (A-z_0)^{-1}\Gamma$. Then we substitute $\Gamma_{z_0}^{+}$ and Γ_{z_0} into (1) and easily derive

$$Q(z) = Q(\bar{z}_0) + (z-\bar{z}_0)\Gamma^{+}(A-\bar{z}_0)^{-1}(A-z)^{-1}\Gamma.$$

By means of the resolvent equation we get

$$Q(z) = Q(\bar{z}_0) - \Gamma^{+}(A-\bar{z}_0)^{-1}\Gamma + \Gamma^{+}(A-z)^{-1}\Gamma.$$

By substituting here

$$S := Q(\bar{z}_0) - \Gamma^{+}(A-\bar{z}_0)^{-1}\Gamma,$$

we get the first equation of (9).

From the first equation of (9) and from $Q(z)^{*} = Q(\bar{z})$ it follows $S = S^{*}$.

(ii) Conversely, assume (9) holds with linear relation A. From (9), for $z = z_0$, we get $S = S^{*} = Q(z_0)^{*} - \Gamma^{+}(A-\bar{z}_0)^{-1}\Gamma$. Substituting S into (9) and applying resolvent equation we obtain

$$Q(z) = Q(z_0)^{*} + (z-\bar{z}_0)\Gamma^{+}(A-\bar{z}_0)^{-1}(A-z)^{-1}\Gamma.$$

Now (10) gives

$$Q(z) = Q(z_0)^{*} + (z-\bar{z}_0)\Gamma_{z_0}^{+}(A-z)^{-1}\Gamma.$$

According to resolvent equation it holds

$$(A-z)^{-1} = \left(I + (z-z_0)(A-z)^{-1}\right)(A-z_0)^{-1}, \quad \forall z \in \rho(A). \tag{12}$$

Therefore

$$Q(z) = Q(z_0)^* + (z - \bar{z}_0)\Gamma_{z_0}^+\Big(I + (z - z_0)(A - z)^{-1}\Big)(A - z_0)^{-1}\Gamma.$$

Substituting here Γ_{z_0} from (10) gives (1).

(iii) From (12) and (10) it follows

$$(A - z)^{-1}\Gamma = \Big(I + (z - z_0)(A - z)^{-1}\Big)\Gamma_{z_0} =: \Gamma_z, \quad \forall z \in \rho(A).$$

This proves (11). Minimality of a representation is defined in terms the of vectors $\Gamma_z h$ by (2). According to (11) we conclude that represention (9) is minimal if and only if

$$\mathcal{K} = \text{c.l.s.}\Big\{(A - z)^{-1}\Gamma h : z \in \rho(A), h \in \mathcal{H}\Big\}.$$

This proves (iii). □

Note, the first statement of the proposition is well known for matrix functions represented by operators. This was proven in Krein and Langer (1977) for scalar, and in Langer and Luger (2000) for matrix valued function Q. In both cases one additional assumption on Q was made so that A was linear operator from the start.

By definition, ∞ is generalized pole of Q if and only if 0 is generalized pole of the function $\tilde{Q}(\zeta) = Q(\frac{-1}{\zeta})$, see Borogovac and Luger (2014, Remark 3.13.). This is equivalent to $A(0) \neq \{0\}$, where A is representing relation of Q. In that case ∞ is called an eigenvalue of A and nonzero vectors from $A(0)$ are called *eigenvectors at* ∞, see Luger (2002).

The following statement is well known for closed linear relations in Hilbert space $\mathcal{H}$, see e.g. Langer and Textorius (1977). We will state it here in our setting, for convenience of the reader.

Lemma 2 – *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert and Krein space, respectively, and let linear relation $T \subseteq \mathcal{H} \times \mathcal{K}$ has closed $T(0)$. Then it holds:*

$$T = \tilde{T} \dotplus T_\infty,$$

where $\dotplus$ denotes direct sum of subspaces, $\tilde{T}$ is an operator with $D(\tilde{T}) = D(T)$ and $T_\infty := \{\{0, g\} \in T\}$.

Proof. Because $T(0) \subseteq \mathcal{K}$ is closed subspace of the Hilbert space $(\mathcal{K}, (\cdot,\cdot))$ associated with Krein space $(\mathcal{K}, [\cdot,\cdot])$, we can uniquely and orthogonaly decompose $(\mathcal{K}, (\cdot,\cdot))$ by means of $T(0)$. Thus, for every $\{f, g\} \in T$ we have, $\{f, g\} = \{f, g_1 \,(\dotplus)\, g_0\}$, where $(\dotplus)$ is direct and orthogonal sum in the Hilbert space $(\mathcal{K}, (\cdot,\cdot))$, and $g_0 \in T(0)$ and $g_1 \in \mathcal{K} \,(-)\, T(0)$ are uniquely determined vectors. We define

$$\tilde{T} := \Big\{\{f, g_1\} \,\Big|\, \{f, g\} \in T\Big\},$$

and T_∞ is as above. Then we have

$$T = \tilde{T}\ (\dot{+})\ T_\infty \subseteq \mathcal{H} \times \mathcal{K},$$

where $(\dot{+})$ denotes direct orthogonal sum in the Hilbert space associated with $\mathcal{H} \times \mathcal{K}$.

Because the sum $g_1\ (\dot{+})\ g_0$ does not have to be orthogonal in the Krein space $(\mathcal{K}, [\cdot,\cdot])$, we write

$$T = \tilde{T} \dot{+} T_\infty.$$

It is easy to verify that $\tilde{T} = T\ (-)\ T_\infty$ is single-valued. □

Corollary 1 – *If representing relation A of $Q \in \mathcal{N}_\kappa(\mathcal{H})$ satisfies condition (7), then A can be replaced in (1) by its operator part $\tilde{A}$. If representation (1) is minimal, it will remain minimal with self-adjoint operator $\tilde{A}$. The function Q does not have generalized pole at ∞.*

Proof. Because A is closed linear relation, it is easy to verify that $A(0)$ is closed. According to Lemma 2 it holds

$$A = \tilde{A} \dot{+} A_\infty.$$

According to Proposition 1 (i) there exists a linear relation

$$\Gamma := (A-z)\Gamma_z, \quad z \in \rho(A),$$

with $\Gamma(0) = A(0)$. Because $\Gamma(0)$ is closed, according to Lemma 2 it holds

$$\Gamma = \tilde{\Gamma} \dot{+} \Gamma_\infty.$$

Because $\Gamma(0) = A(0) = \ker(A-z)^{-1}$, it holds

$$\Gamma_z = (A-z)^{-1}\Gamma = (\tilde{A}-z)^{-1}\tilde{\Gamma}, \quad \forall z \in \rho(A). \tag{13}$$

Let $z_0 \in \rho(A) \setminus \mathbb{R}$ be the point of reference in (1). Let us now prove that we can replace $(A-z)^{-1}\Gamma_{z_0}$ by $(\tilde{A}-z)^{-1}\Gamma_{z_0}$ in (1). We start from (3) written in the form

$$(A-z)^{-1}\Gamma_{z_0} = \frac{\Gamma_z - \Gamma_{z_0}}{z - z_0}, \quad \forall z \in \rho(A).$$

According to (13) and the resolvent equation we have

$$(A-z)^{-1}\Gamma_{z_0} = \frac{(\tilde{A}-z)^{-1}\tilde{\Gamma} - (\tilde{A}-z_0)^{-1}\tilde{\Gamma}}{z - z_0} = (\tilde{A}-z)^{-1}(\tilde{A}-z_0)^{-1}\tilde{\Gamma} = (\tilde{A}-z)^{-1}\Gamma_{z_0}.$$

This proves

$$(A-z)^{-1}\Gamma_{z_0} = (\tilde{A}-z)^{-1}\Gamma_{z_0}.$$

Therefore, we can substitute $(\tilde{A}-z)^{-1}\Gamma_{z_0}$ for $(A-z)^{-1}\Gamma_{z_0}$ into (3) and (1), and values of Γ_z and $Q(z)$ will not change. Thus,

$$\Gamma_z = \left(I + (z-z_0)(\tilde{A}-z)^{-1}\right)\Gamma_{z_0}.$$
$$Q(z) = Q(z_0)^* + (z-\bar{z}_0)\Gamma_{z_0}^+\left(I + (z-z_0)(\tilde{A}-z)^{-1}\right)\Gamma_{z_0}, \quad z \in \mathcal{D}(Q).$$

According to definition of minimality (2), we conclude that minimal representation (1) remains minimal when $\tilde{A}$ replaces A. Because of the uniqueness of the minimal representation (1) it must be $A = \tilde{A}$. Therefore, $\tilde{A}$ must be a self-adjoint operator, as the unique representing operator of a generalized Nevanlinna function. Because the function Q is represented by operator $\tilde{A}$, we conclude that Q cannot have generalized pole at ∞. □

2.2 By definition a function Q has a non-tangential limit at ∞ if and only if the function $\tilde{Q}(\zeta) = Q(\frac{-1}{\zeta})$ has a non-tangential limit at 0. By the same token a function Q is holomorphic at ∞ if and only if the function $\tilde{Q}(\zeta) = Q(\frac{-1}{\zeta})$ is holomorphic at 0. The following proposition, that corresponds to Krein and Langer (1977, Satz 1.4) holds.

Proposition 2 – *Let $Q \in \mathcal{N}_\kappa(\mathcal{H})$ satisfies non-tangential version of (4):*

$$\exists Q'(\infty) := \lim_{z \hat{\to} \infty} zQ(z), \tag{14}$$

where the limit denotes convergence in the Banach space of bounded operators. Then $Q'(\infty) \in \mathcal{L}(\mathcal{H})$, and Q has minimal representation (1) with a self-adjoint operator A.

Proof. Because $\mathcal{L}(\mathcal{H})$ is a Banach space with respect to norm topology, we conclude that $Q'(\infty)$, given by (14), is a bounded operator. Under assumption that limit (14) exists, it holds

$$\lim_{\zeta \hat{\to} 0} \tilde{Q}(\zeta) := \lim_{z \hat{\to} \infty} Q(z) = 0.$$

If we define $\tilde{Q}(0) := \lim_{\zeta \hat{\to} 0} \tilde{Q}(\zeta) = 0$, then

$$\tilde{Q}'(0) := \lim_{\zeta \hat{\to} 0} \frac{\tilde{Q}(\zeta) - \tilde{Q}(0)}{\zeta} = \lim_{z \hat{\to} \infty} zQ(z) =: Q'(\infty).$$

According to Borogovac and Luger (2014, Defintion 3.1 (B)), $\zeta = 0$ is not a generalized pole of $\tilde{Q}$, i.e. ∞ is not a generalized pole of Q. Therefore, the representing relation A satisfies $A(0) = 0$. Hence, Q is represented by the self-adjoint operator A in (1). □

2. Representation $Q(z) = S + \Gamma^{+}(A - z)^{-1}\Gamma$

Lemma 3 – *A function $Q \in \mathcal{N}_{\kappa}(\mathcal{H})$ is holomorphic at ∞ if and only if $Q(z)$ has minimal representation (5)*

$$Q(z) = \Gamma^{+}(A - z)^{-1}\Gamma, \quad z \in \mathcal{D}(Q),$$

with a bounded self-adjoint operator A in a Pontryagin space $\mathcal{K}$, and bounded operator $\Gamma : \mathcal{H} \to \mathcal{K}$. In this case

$$Q'(\infty) := \lim_{z\to\infty} zQ(z) = -\Gamma^{+}\Gamma.$$

Proof. If $Q(z)$ is holomorphic at ∞, then it satisfies (14). According to Proposition 2, Q is represented by an operator A. From the assumption of holomorphy at ∞ it follows that operator A has bounded spectrum. According to Langer (1982, Corollary 2), A is bounded. Then condition (7) is satisfied. According to Proposition 1 (i), Q has minimal representation (9). Then, from existence of limit (14), it follows $S = 0$.

Conversely, if A is bounded operator in representation (5), then it has bounded spectrum, and therefore, Q is holomorphic at infinity.

To prove the last statement of the lemma, we use Neumann series of resolvent of the bounded operator A.

$$Q'(\infty) := \lim_{z\to\infty} zQ(z) = \lim_{z\to\infty} z\Gamma^{+}\left(\sum_{i=0}^{\infty} -\frac{A^{i}}{z^{i+1}}\right)\Gamma = -\Gamma^{+}\Gamma. \qquad \square$$

The concept

$$\ker Q := \bigcap_{z\in D(Q)} \ker Q(z)$$

was introduced in Dijksma, Langer, and Snoo H. S. V. (1993). For matrix function $Q \in N_{\kappa}^{n\times n}$, represented by (1) it was proven

$$\ker Q = \ker \Gamma_{z_0} \cap \ker Q(z_0)^{*}.$$

Proposition 3 – *If $Q \in N_{\kappa}(\mathcal{H})$ is holomorphic at infinity and $Q'(\infty)$ is invertible, then*

$$\ker Q = \{0\}.$$

Proof. According to Lemma 3 we can assume that Q is minimally represented by bounded operator A. Recall, for $z, w \in \rho(A) = \mathcal{D}(Q)$ it holds

$$\Gamma_z = \left(I + (z - w)(A - z)^{-1}\right)\Gamma_w.$$

Obviously,

$$\Gamma_w h = 0 \;\Rightarrow\; \Gamma_z h = 0,$$

If we reverse roles of z and w, then the converse implication holds. Hence, it holds

$$\ker \Gamma_z = \ker \Gamma_w.$$

If $Q(z)$ is holomorphic at ∞, according to Lemma 3, Q has representation (5) with bounded operator A. Therefore, condition (7) is satisfied. According to Proposition 1 (iii) we have

$$\Gamma_z = (A - z)^{-1}\Gamma, \quad \forall z \in \mathcal{D}(Q).$$

Then we have:

$$(5) \Rightarrow Q(z)h = \Gamma^+\Gamma_z h, \quad \forall h \in \mathcal{H}, \ \forall z \in \mathcal{D}(Q).$$

If we assume $h \in \ker Q$, then according to definition of $\ker Q$ we have

$$\begin{aligned} h \in \ker Q \ &\Leftrightarrow \ h \in \ker zQ(z), \quad \forall z \in \mathcal{D}(Q) \\ &\Leftrightarrow \ 0 = \lim_{z \hat{\to} \infty} zQ(z)h = -\Gamma^+\Gamma h = Q'(\infty)h \ \Leftrightarrow \ h = 0. \end{aligned}$$

This proves the statement. □

We cannot here claim that $Q(z)$ is a regular function. We will prove it in the following section.

3 Inverse of $\Gamma^+(A - z)^{-1}\Gamma$

Lemma 4 – *Let bounded operators $\Gamma : \mathcal{H} \to \mathcal{K}$ and $\Gamma^+ : \mathcal{K} \to \mathcal{H}$ be introduced as usually, see Section 1. Assume also that $\Gamma^+\Gamma$ is a boundedly invertible operator in the Hilbert space $(\mathcal{H}, (\cdot,\cdot))$. Then for operator*

$$P := \Gamma(\Gamma^+\Gamma)^{-1}\Gamma^+ \tag{15}$$

the following statements hold:

(i) P is orthogonal projection in Pontryagin space $(\mathcal{K}, [\cdot,\cdot])$.

(ii) Scalar product does not degenerate on $\Gamma(\mathcal{H})$ and therefore it does not degenerate on $\Gamma(\mathcal{H})^{[\perp]} = \ker \Gamma^+$.

(iii) $\ker \Gamma^+ = (I - P)\mathcal{K}$.

(iv) Pontryagin space $\mathcal{K}$ can be decomposed as a direct orthogonal sum of Pontryagin spaces i.e.

$$\mathcal{K} = (I - P)\mathcal{K} \, [+] \, P\mathcal{K}. \tag{16}$$

Proof. (i) Obviously $P^2 = P$.

According to well known properties of adjoint operators, see e.g. Iohvidov, Krein, and Langer (1982, p. 34), it is easy to verify $[(\Gamma^{+}\Gamma)^{-1}]^{*} = (\Gamma^{+}\Gamma)^{-1}$ and then to verify $[Px,y] = [x,Py]$, i.e. $P^{[*]} = P$. This proves (i).

(ii) If $\Gamma h \neq 0$ and $[\Gamma h, \Gamma g] = 0$, $\forall g \in \mathcal{H}$, then $(\Gamma^{+}\Gamma h, g) = 0$, $\forall g \in \mathcal{H}$. Then we have $\Gamma^{+}\Gamma h = 0 \Rightarrow h = 0 \Rightarrow \Gamma h = 0$. This is a contradiction that proves (ii).

(iii) It is sufficient to prove $\ker \Gamma^{+} = \ker P$.

$$P := \Gamma(\Gamma^{+}\Gamma)^{-1}\Gamma^{+} \Rightarrow \ker \Gamma^{+} \subseteq \ker P.$$

Conversely, because $\Gamma^{+}\Gamma$ is boundedly invertible $R(\Gamma^{+}) = \mathcal{H}$. Then

$$y \in \ker P \;\Rightarrow\; 0 = \left[\Gamma(\Gamma^{+}\Gamma)^{-1}\Gamma^{+}y, x\right] = \left((\Gamma^{+}\Gamma)^{-1}\Gamma^{+}y, \Gamma^{+}x\right), \quad \forall \Gamma^{+}x \in \mathcal{H}.$$
$$R(\Gamma^{+}) = \mathcal{H} \;\Rightarrow (\Gamma^{+}\Gamma)^{-1}\Gamma^{+}y = 0 \;\Rightarrow\; \Gamma^{+}y = 0 \;\Rightarrow\; y \in \ker \Gamma^{+}.$$

(iv) This statement follows directly from (iii) and (ii). □

Assume now that function Q is given by (5) and that projection P is given by (15). We define

$$\tilde{A} := (I-P)A_{|(I-P)\mathcal{K}}.$$

Then

$$(\tilde{A} - zI_{|(I-P)\mathcal{K}})^{-1} : (I-P)\mathcal{K} \to (I-P)\mathcal{K}.$$

Note that it is customary to omit the identity mapping in resolvents. Therefore, we will frequently write $(\tilde{A}-z)^{-1}$ rather than $(\tilde{A} - zI_{|(I-P)\mathcal{K}})^{-1}$. It holds

$$(I-P)(\tilde{A}-z)^{-1}(I-P) = \begin{pmatrix} (\tilde{A} - zI_{|(I-P)\mathcal{K}})^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

In the sequel, we will use notation from the left hand side of this equation because it makes the following proofs easier to write.

Theorem 2 – *Assume that function $Q \in \mathcal{N}_{\kappa}(\mathcal{H})$ is holomorphic at ∞, and that*

$$Q'(\infty) := \lim_{z\to\infty} zQ(z)$$

is boundedly invertible. Then there exists the inverse function

$$\hat{Q}(z) := -Q(z)^{-1},$$

and $\hat{Q}(z)$ has the following representation on $\mathcal{D}(Q) \cap \mathcal{D}(\hat{Q})$

$$\hat{Q}(z) = (\Gamma^{+}\Gamma)^{-1}\Gamma^{+}\left\{A(I-P)(\tilde{A}-z)^{-1}(I-P)A - (A - zI)\right\}\Gamma(\Gamma^{+}\Gamma)^{-1}, \tag{17}$$

where operator Γ was defined by (8) and projection P was defined by equation (15).

Proof. According to Lemma 3, function Q has minimal representation (5) with bounded operator A. For projection P defined in Lemma 4, we have the following decomposition with respect to (16)

$$A - zI = \begin{pmatrix} (I-P)(A-zI)(I-P) & (I-P)AP \\ PA(I-P) & P(A-zI)P \end{pmatrix}.$$

Let us denote

$$\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} := (A-z)^{-1}.$$

By solving operator equations derived from the identity

$$\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}\begin{pmatrix} \tilde{A}-z(I-P) & (I-P)AP \\ PA(I-P) & P(A-zI)P \end{pmatrix} = \begin{pmatrix} I-P & 0 \\ 0 & P \end{pmatrix}$$

we get

$$W = \left\{P(A-zI)P - PA(I-P)(\tilde{A}-z)^{-1}(I-P)AP\right\}^{-1}.$$

It is easy to verify the following equalities:

$$\Gamma^+ P = \Gamma^+, \quad P\Gamma = \Gamma, \quad \Gamma^+(I-P) = 0, \quad (I-P)\Gamma = 0.$$

It follows

$$Q(z) = \Gamma^+\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}\Gamma = (\Gamma^+(I-P), \Gamma^+ P)\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}\begin{pmatrix} (I-P)\Gamma \\ P\Gamma \end{pmatrix}$$
$$\Rightarrow \ Q(z) = (0, \Gamma^+)\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}\begin{pmatrix} 0 \\ \Gamma \end{pmatrix} = \Gamma^+\begin{pmatrix} 0 & 0 \\ 0 & W \end{pmatrix}\Gamma.$$

Therefore, we do not need to find operators X, Y, Z. By substituting W here, we get

$$Q(z) = \Gamma^+\left\{P(A-zI)P - PA(I-P)(\tilde{A}-z)^{-1}(I-P)AP\right\}^{-1}\Gamma. \tag{18}$$

By substituting expressions (18) and (17) for Q and $\hat{Q}$, respectively, into the following product, we verify

$$\begin{aligned} Q(z)\hat{Q}(z) &= \Gamma^+\left\{P(A-zI)P - PA(I-P)(\tilde{A}-z)^{-1}(I-P)AP\right\}^{-1}\Gamma(\Gamma^+\Gamma)^{-1}\Gamma^+ \\ &\quad \times\left\{A(I-P)(\tilde{A}-z)^{-1}(I-P)A - (A-zI)\right\}\Gamma(\Gamma^+\Gamma)^{-1} \\ &= \Gamma^+\left\{P(A-zI)P - PA(I-P)(\tilde{A}-z)^{-1}(I-P)AP\right\}^{-1} \\ &\quad \times\left\{PA(I-P)(\tilde{A}-z)^{-1}(I-P)AP - P(A-zI)P\right\}\Gamma(\Gamma^+\Gamma)^{-1} \\ &= \Gamma^+(-P)\Gamma(\Gamma^+\Gamma)^{-1} = -I. \end{aligned}$$

□

The remaining statements of this paper are consequences of Theorem 2.

Theorem 3 – *Let* $Q \in \mathcal{N}_{\kappa}(\mathcal{H})$.

(i) Q is holomorphic at ∞ *and* $Q'(\infty)$ *is boundedly invertible if and only if*

$$\hat{Q}(z) = \tilde{\Gamma}^{+}(\tilde{A}-z)^{-1}\tilde{\Gamma} + \hat{S} + \hat{G}z, \forall z \in \mathcal{D}(Q) \cap \mathcal{D}(\hat{Q}) \tag{19}$$

where $\tilde{A}$ *is a self-adjoint bounded operator in the Pontryagin space* $(I-P)\mathcal{K}$, $\hat{S}$ *and* $\hat{G}$ *are self-adjoint bounded operators in the Hilbert space* $\mathcal{H}$, *and* $\hat{G}$ *is boundedly invertible.*

(ii) In that case function $Q \in \mathcal{N}_{\kappa}(\mathcal{H})$ *is regular.*

Proof. (i) ($\Rightarrow$) The assumptions are the same as in Theorem 2. Therefore, representation (17) holds. If we substitute

$$\hat{S} = -(\Gamma^{+}\Gamma)^{-1}\Gamma^{+}A\Gamma(\Gamma^{+}\Gamma)^{-1}, \quad \hat{G} = (\Gamma^{+}\Gamma)^{-1} \tag{20}$$

$$\tilde{\Gamma} := (I-P)A\Gamma(\Gamma^{+}\Gamma)^{-1}, \tag{21}$$

into representation (17) we get representation (19). Operator $\tilde{A}$ is bounded because it is a restriction of the bounded operator A. The statements about $\hat{S}$ and $\hat{G}$ are easy verification.

($\Leftarrow$) Now we assume that (19) holds. Obviously:

$$\lim_{z\to\infty}\frac{\hat{Q}(z)}{z} = \lim_{z\to\infty}(-zQ(z))^{-1}.$$

On the other hand, because $\tilde{A}$ is bounded we can apply Neumann series of the resolvent $(\tilde{A}-z)^{-1}$. We have

$$\lim_{z\to\infty}\frac{\hat{Q}(z)}{z} = \lim_{z\to\infty}\left(\frac{\tilde{\Gamma}^{+}(\tilde{A}-z)^{-1}\tilde{\Gamma}+\hat{S}}{z}+\hat{G}\right)$$
$$= \lim_{z\to\infty}\left(\tilde{\Gamma}^{+}\sum_{i=0}^{\infty}-\frac{\tilde{A}^{i}}{z^{i+2}}\tilde{\Gamma}+\frac{\hat{S}}{z}\right)+\hat{G} = \hat{G}.$$

Therefore,

$$\lim_{z\to\infty}(-zQ(z))^{-1} = \hat{G}.$$

Because $\hat{G}$ is bounded, $\lim_{z\to\infty} zQ(z)$ is boundedly invertible.

(ii) This statement holds because, according to (19), operator $\hat{Q}(z)$ is obviously bounded for every $z \in \mathcal{D}(Q) \cap \mathcal{D}(\hat{Q})$. □

It is usually very difficult to find representing operator for a given function $Q \in \mathcal{N}_\kappa(\mathcal{H})$. The construction used in cited papers is abstract and not applicable in concrete situations. Theorem 2 gives us a new simple relationships between representing operators A, Γ and Γ^+. That might help us to find those operators in some cases, like e.g. in the following case.

Example 1 – Given function

$$Q(z) = -\begin{bmatrix} 0 & z^{-1} \\ z^{-1} & z^{-2} \end{bmatrix}.$$

It is easy to verify that function $Q(z)$ is holomorphic at infinity, and that it holds

$$Q'(\infty) := \lim_{z\to\infty} zQ(z) = -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

According to Lemma 3, $Q(z)$ admits minimal representation (5). Hence,

$$Q(z) = \Gamma^+(A - zI)^{-1}\Gamma \wedge -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -\Gamma^+\Gamma.$$

In addition,

$$Q(z)^{-1} = \begin{bmatrix} 1 & -z \\ -z & 0 \end{bmatrix} =: L(z).$$

i.e. the inverse function is a polynomial. Therefore, the resolvent part of $\hat{Q}$ in representation (17) must be equal to zero. It holds,

$$(\Gamma^+\Gamma)^{-1}\Gamma^+(A - zI)\Gamma(\Gamma^+\Gamma)^{-1} = \begin{bmatrix} 1 & -z \\ -z & 0 \end{bmatrix}$$

$$\Rightarrow \Gamma^+(A - zI)\Gamma = \begin{bmatrix} 0 & -z \\ -z & 1 \end{bmatrix} \Rightarrow \Gamma^+A\Gamma = \Gamma^* JA\Gamma = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Here J denotes a fundamental symmetry in $\mathcal{K}$. Because function Q has a single pole of order two at $z = 0$, the representing operator has the single eigenvalue of order two at $z = 0$. All those information enable us to make an easy educated guess

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \Gamma^+. \qquad \square$$

We will refer to this example for a different reason in Theorem 4.

Proposition 4 – *Let $Q(z)$, $\hat{Q}(z)$, Γ, Γ^+ be the same as in Theorem 2. Then for all $z \in \mathcal{D}(Q) \cap \mathcal{D}(\hat{Q})$ it holds*

$$\hat{Q}(z)\Gamma^+ = (\Gamma^+\Gamma)^{-1}\Gamma^+\left\{-I + A(I-P)(\tilde{A} - z)^{-1}(I-P)\right\}(A - zI). \tag{22}$$

Proof. In the following derivations we will frequently use $\Gamma^+ P = \Gamma^+$ and $P\Gamma = \Gamma$. From (17) it follows

$$
\begin{aligned}
\hat{Q}(z)\Gamma^+ &= (\Gamma^+\Gamma)^{-1}\Gamma^+\Big\{A(I-P)(\tilde{A}-z)^{-1}(I-P)A-(A-zI)\Big\}\Gamma(\Gamma^+\Gamma)^{-1}\Gamma^+ \\
&= (\Gamma^+\Gamma)^{-1}\Gamma^+\Big\{A(I-P)(\tilde{A}-z)^{-1}(I-P)(A-zI)P-(A-zI)P\Big\} \\
&= (\Gamma^+\Gamma)^{-1}\Gamma^+\Big\{A(I-P)(\tilde{A}-z)^{-1}(I-P)(A-zI)(P-I) \\
&\qquad\qquad + A(I-P)(\tilde{A}-z)^{-1}(I-P)(A-zI)-(A-zI)P\Big\} \\
&= (\Gamma^+\Gamma)^{-1}\Gamma^+\Big\{-A(I-P)+A(I-P)(\tilde{A}-z)^{-1}(I-P)(A-zI)-(A-zI)P\Big\} \\
&= (\Gamma^+\Gamma)^{-1}\Gamma^+\Big\{-(A-zI)+A(I-P)(\tilde{A}-z)^{-1}(I-P)(A-zI)\Big\} \\
&= (\Gamma^+\Gamma)^{-1}\Gamma^+\Big\{-I+A(I-P)(\tilde{A}-z)^{-1}(I-P)\Big\}(A-zI). \qquad \square
\end{aligned}
$$

Note, if $x_0\, x_1, \ldots, x_{k-1}$ is a Jordan chain of A at the eigenvalue $\alpha \in \mathbb{C}$, then it holds

$$(A-zI)\Big(x_0+(z-\alpha)x_1+\cdots+(z-\alpha)^{k-1}x_{k-1}\Big) = -(z-\alpha)^k x_{k-1}.$$

This formula together with (22) enables us to prove that if α is not a zero of Q, then the function

$$\eta(z) := \hat{Q}(z)\Gamma^+\Big(x_0+(z-\alpha)x_1+\cdots+(z-\alpha)^{k-1}x_{k-1}\Big) = (\Gamma^+\Gamma)^{-1}\Gamma^+(z-\alpha)^k x_{k-1}$$

is a pole cancellation functions of Q at α, cf. Borogovac and Luger (2014, Remark 3.7).

According to Luger (2002, Proposition 2.1), for a regular function $Q \in \mathcal{N}_\kappa(\mathcal{H})$ with representing relation A, the inverse $\hat{Q}$ admits representation

$$\hat{Q}(z) = \hat{Q}(\bar{z}_0)+(z-\bar{z}_0)\hat{\Gamma}^+\Big(I+(z-z_0)(\hat{A}-z)^{-1}\Big)\hat{\Gamma} \tag{23}$$

where $\hat{\Gamma} := -\Gamma_{z_0}Q(z_0)^{-1}$ and it holds

$$(\hat{A}-z)^{-1} = (A-z)^{-1}-\Gamma_z Q(z)^{-1}\Gamma_{\bar{z}}^+, \quad \forall z \in \rho(A)\cap\rho(\hat{A}). \tag{24}$$

The following proposition gives us one more relationship between representations (17) and (23).

Proposition 5 – *Let $Q \in \mathcal{N}_\kappa(\mathcal{H})$ be holomorphic at ∞ and let $Q'(\infty)$ be boundedly invertible. If $\hat{A}$ is the representing linear relation in (23), then $\hat{A}$ satisfies*

$$\hat{A}(0) = R(P) = R(\Gamma).$$

and $\hat{A}(0)$ is not degenerate.

Proof. Function $Q \in \mathcal{N}_\kappa(\mathcal{H})$ that admits representation (5) is a special case of the function that admits representation (1). Let us select a (non-real) point of reference $z_0 \in \mathcal{D}(Q) \cap \mathcal{D}(\hat{Q})$, so that $Q(z_0)$ is boundedly invertible. Let us introduce Γ_{z_0} by (10). Then according to Proposition 1 (ii) function Q given by (5) admits representation (1) with the same representing self-adjoint operator A and $Q(z_0)^* = \Gamma^+(A-\bar{z}_0)^{-1}\Gamma$. From (24), for $z = z_0$ we get

$$(\hat{A}-z_0)^{-1} = (A-z_0)^{-1} - \Gamma_{z_0} Q(z_0)^{-1}\Gamma_{\bar{z}_0}^+. \tag{25}$$

From (10), it follows

$$\Gamma_{z_0} = (A-z_0)^{-1}\Gamma \ \wedge \ \Gamma_{\bar{z}_0}^+ = \Gamma^+(A-z_0)^{-1}.$$

Substituting this into (25) gives

$$\begin{aligned}(\hat{A}-z_0)^{-1} &= (A-z_0)^{-1} - (A-z_0)^{-1}\Gamma Q(z_0)^{-1}\Gamma^+(A-z_0)^{-1}\\ &= (A-z_0)^{-1}\left(I - \Gamma Q(z_0)^{-1}\Gamma^+(A-z_0)^{-1}\right).\end{aligned}$$

By substituting here the expression for $Q(z_0)^{-1}\Gamma^+$ from (22) we get

$$\begin{aligned}(\hat{A}-z_0)^{-1} &= (A-z_0)^{-1}\left(I + P(-I + A(I-P)(\tilde{A}-z_0)^{-1}(I-P))\right)\\ &= (A-z_0)^{-1}\left(I - P + PA(I-P)(\tilde{A}-z_0)^{-1}(I-P)\right).\end{aligned}$$

Hence

$$(\hat{A}-z_0)^{-1} = (A-z_0)^{-1}\left(I + PA(I-P)(\tilde{A}-z_0)^{-1}\right)(I-P). \tag{26}$$

From this we conclude $\ker(\hat{A}-z_0)^{-1} \supseteq R(P)$ and, therefore $\hat{A}(0) \supseteq R(\Gamma)$.

In order to prove $\ker(\hat{A}-z_0)^{-1} \subseteq R(\Gamma)$, assume the contrary, that there exists $0 \neq (I-P)y \in \ker(\hat{A}-z_0)^{-1}$. Because, $z_0 \in \rho(A)$ and A is single-valued, from (26) it follows

$$\left(I + PA(I-P)(\tilde{A}-z_0)^{-1}\right)(I-P)y = 0.$$

Then, it must be

$$-(I-P)y = PA(I-P)(\tilde{A}-z_0)^{-1}(I-P)y = 0,$$

which is a contradiction. Therefore, $\ker(\hat{A}-z_0)^{-1} = R(\Gamma)$. □

Note, since the non-real point $z_0 \in \mathcal{D}(Q) \cap \mathcal{D}(\hat{Q})$ was arbitrarily selected, all formulae derived in the proof of Proposition 5 hold for all non-real points $z \in \mathcal{D}(Q) \cap \mathcal{D}(\hat{Q})$.

One consequence of Proposition 5 is that function $\hat{Q}$ must have a generalized pole at ∞. This means that regular function $\hat{Q}$ does not have a derivative at ∞.

4 Properties of $\hat{Q}$

The following theorem is also a consequence of Theorem 2.

Theorem 4 – *Assume that function $Q \in \mathcal{N}_\kappa(\mathcal{H})$ is holomorphic at ∞, i.e. $Q(z) := \Gamma^+(A-z)^{-1}\Gamma$, and assume that operator*

$$Q'(\infty) := \lim_{z\to\infty} zQ(z)$$

is boundedly invertible. Then for functions

$$\hat{Q}_1(z) = \hat{S} + z\hat{G} \in \mathcal{N}_{\kappa_1}(\mathcal{H}), \tag{27}$$

and

$$\hat{Q}_2(z) := \tilde{\Gamma}^+(\tilde{A}-z)^{-1}\tilde{\Gamma} \in \mathcal{N}_{\kappa_2}(\mathcal{H}), \tag{28}$$

where operators $\hat{S}$, $\hat{G}$ and $\tilde{\Gamma}$ are given by equations (20) and (21), the inverse function $\hat{Q}(z)$ has decomposition

$$\hat{Q}(z) = \hat{Q}_1(z) + \hat{Q}_2(z). \tag{29}$$

That decomposition has the following properties:

- *(i) It must be $\hat{Q}_1 \not\equiv 0$ while function $\hat{Q}_2$ may be zero function in some cases. $\hat{Q}_1$ has only one generalized pole, it is at ∞, while $\hat{Q}_2$ is holomorphic at ∞.*
- *(ii) Finite generalized zeros of Q, coincide with generalized poles of $\hat{Q}_2$ including multiplicities.*
- *(iii) $\hat{Q}_1 \in \mathcal{N}_{\kappa_1}(\mathcal{H})$, where negative index κ_1 is equal to the number of negative eigenvalues of the bounded self-adjoint operator $-Q'(\infty)$ in the Hilbert space $\mathcal{H}$ and that is equal to negative index of $P\mathcal{K}$.*
- *(iv) $\kappa_1 + \kappa_2 = \kappa$.*

Proof. (i) According to above definitions of $\hat{Q}_1$ and $\hat{Q}_2$, and (19), it holds $\hat{Q}(z) = \hat{Q}_1(z) + \hat{Q}_2(z)$. According to Proposition 5, $\hat{Q}$ has generalized pole at ∞. Since representing operator $\tilde{A}$ of $\hat{Q}_2$ is bounded operator, according to Lemma 3 $\hat{Q}_2$ is holomorphic at ∞. Therefore, $\hat{Q}_1 \not\equiv 0$ and it must have generalized pole at ∞. According to Example 1 it is possible to have $\hat{Q}_2 \equiv 0$.

(ii) The statement follows immediately from (i) and formula (29).

(iii) Note, representation (27) of $\hat{Q}_1$ is not a typical operator representation of a generalized Nevanlinna function, because $A - zI$ is not a resolvent.

We know $\hat{Q} \in \mathcal{N}_\kappa(\mathcal{H})$ and $\kappa_1 + \kappa_2 \geq \kappa$. Let us denote by κ' and κ'' negative indexes of subspaces $P\mathcal{K}$ and $(I-P)\mathcal{K}$, respectively. Then, according to (16) $\kappa' + \kappa'' = \kappa$.

For any $f, g \in \mathcal{H}$ we have

$$\left(\frac{\hat{Q}_1(z) - \hat{Q}_1(w)^*}{z - \bar{w}} f, g\right) = \left((\Gamma^+\Gamma)^{-1} f, g\right).$$

Hence, κ_1 equals number of negative eigenvalues of $(\Gamma^+\Gamma)^{-1}$. Since $(\Gamma^+\Gamma)^{-1}$ is bounded, hence defined on the whole $\mathcal{H}$, we can consider $f = \Gamma^+\Gamma f_0$ and $g = \Gamma^+\Gamma g_0$, where f_0 and g_0 run through entire $\mathcal{H}$ when f and g run through $\mathcal{H}$. Therefore

$$\left((\Gamma^+\Gamma)^{-1} f, g\right) = [\Gamma f_0, \Gamma g_0].$$

Because $R(\Gamma) = R(P)$, we conclude that $\kappa_1 = \kappa'$. Real number $\alpha < 0$ is an eigenvalue of $\Gamma^+\Gamma = -Q'(\infty)$ if and only if $\alpha^{-1} < 0$ is an eigenvalue of $(\Gamma^+\Gamma)^{-1}$. Hence, statement (iii) follows.

(iv)

$$\kappa_1 = \kappa' \Rightarrow \kappa' + \kappa_2 \geq \kappa = \kappa' + \kappa'' \Rightarrow \kappa_2 \geq \kappa''$$

Because $\tilde{A}$, the representing operator of $\hat{Q}_2$, is self-adjoint operator in $(I-P)\mathcal{K}$, it must be $\kappa_2 \leq \kappa''$. Therefore, $\kappa_2 = \kappa''$ and

$$\kappa_1 + \kappa_2 = \kappa.$$

That proves (iv). □

In the following example we will show how Theorem 4 can be applied to a concrete generalized Nevanlinna functions.

Example 2 – Let

$$Q(z) = \begin{bmatrix} \frac{-(1+z)}{z^2} & \frac{1}{z} \\ \frac{1}{z} & \frac{1}{1+z} \end{bmatrix}.$$

The function Q has representation (5)

$$Q(z) = \Gamma^+(A - z)^{-1}\Gamma,$$

where the space $\mathbb{K} = \mathbb{C}^3$. In that representation fundamental symmetry, and representing operators of Q are:

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0.5 & -1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\Gamma^+ = \Gamma^* J = \begin{bmatrix} 1 & 0.5 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

Here, $\Gamma^* : \mathbb{C}^3 \to \mathbb{C}^2$ is adjoint operator of Γ with respect to Hilbert spaces $\mathbb{C}^2$ and $\mathbb{C}^3$. It is easy to see that this representation is minimal. From the shape of the fundamental symmetry J we conclude $\kappa = 2$, i.e. $Q \in \mathcal{N}_2(\mathbb{C}^2)$. We have

$$\hat{Q}(z) = \begin{bmatrix} \frac{z^2}{2(1+z)} & -\frac{z}{2} \\ -\frac{z}{2} & \frac{-(1+z)}{2} \end{bmatrix} \in N_2(\mathbb{C}^2).$$

Limit (14) gives

$$\Gamma^+\Gamma = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}, \quad (\Gamma^+\Gamma)^{-1} = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & -0.5 \end{bmatrix}.$$

This means that conditions of Theorem 4 are satisfied.

Let us calculate $\hat{Q}_1(z)$. By substituting matrices $(\Gamma^+\Gamma)^{-1}, \Gamma^+, \Gamma$ into formulae for $\hat{G}$ and $\hat{S}$, we obtain

$$\hat{Q}_1(z) = \begin{bmatrix} \frac{-1+z}{2} & -\frac{z}{2} \\ -\frac{z}{2} & -\frac{1+z}{2} \end{bmatrix}.$$

Let us now find $\hat{Q}_2(z)$ by means of formulae (28). In order to do that, we have first to find matrices for projections P and $(I-P)$. By means of formula (15) we get

$$P = \begin{bmatrix} 0.75 & 0.125 & 0.25 \\ 0.5 & 0.75 & -0.5 \\ 0.5 & -0.25 & 0.5 \end{bmatrix}, \quad I - P = \begin{bmatrix} 0.25 & -0.125 & -0.25 \\ -0.5 & 0.25 & 0.5 \\ -0.5 & 0.25 & 0.5 \end{bmatrix}.$$

Obviously, range $(I-P) = 1$, i.e. $\dim(I-P)\mathbb{K} = 1$. We also have

$$(I-P)A(I-P) - z(I-P) = \begin{bmatrix} -0.25 & 0.125 & 0.25 \\ 0.5 & -0.25 & -0.5 \\ 0.5 & -0.25 & -0.5 \end{bmatrix} - z \begin{bmatrix} 0.25 & -0.125 & -0.25 \\ -0.5 & 0.25 & 0.5 \\ -0.5 & 0.25 & 0.5 \end{bmatrix},$$

$$\tilde{\Gamma} := (I-P)A\Gamma(\Gamma^+\Gamma)^{-1} = \begin{bmatrix} 0.25 & 0 \\ -0.5 & 0 \\ -0.5 & 0 \end{bmatrix}, \quad \tilde{\Gamma}^+ = \tilde{\Gamma}^* J = \begin{bmatrix} -0.5 & 0.25 & 0.5 \\ 0 & 0 & 0 \end{bmatrix}.$$

Obviously, $\tilde{\Gamma}$, and $\tilde{\Gamma}^+$, each have only one linearly independent row, column, respectively. Therefore, operators $\tilde{\Gamma}, \tilde{\Gamma}^+$ can be represented by equivalent matrices, i.e. we can write

$$\tilde{\Gamma} := \begin{bmatrix} 0.25 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{\Gamma}^+ = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Accordingly, we will write in the equivalent matrix form

$$(I-P)A(I-P)-z(I-P)=\begin{bmatrix}-0.25-0.25z & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

Then, the matrix form of the operator

$$(I-P)(\tilde{A}-z)^{-1}(I-P)=\begin{pmatrix}(\tilde{A}-z)^{-1} & 0\\ 0 & 0\end{pmatrix}$$

is

$$\begin{bmatrix}\frac{-4}{1+z} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

Now, according to (28) we calculate

$$\hat{Q}_2(z):=\tilde{\Gamma}^{+}(\tilde{A}-z)^{-1}\tilde{\Gamma}=\begin{bmatrix}-0.5 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\frac{-4}{1+z} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0.25 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}.$$

Thus

$$\hat{Q}_2(z)=\begin{bmatrix}\frac{1}{2(1+z)} & 0\\ 0 & 0\end{bmatrix}.$$

We obtained the decomposition (29) of $\hat{Q}(z)$:

$$\begin{bmatrix}\frac{z^2}{2(1+z)} & -\frac{z}{2}\\ -\frac{z}{2} & \frac{-(1+z)}{2}\end{bmatrix}=\begin{bmatrix}\frac{-1+z}{2} & -\frac{z}{2}\\ -\frac{z}{2} & -\frac{1+z}{2}\end{bmatrix}+\begin{bmatrix}\frac{1}{2(1+z)} & 0\\ 0 & 0\end{bmatrix}.$$

There are many decompositions of the function $\hat{Q}$. For this decomposition, we know that the following claims hold:

- Because Hermitian matrix $\Gamma^{+}\Gamma$ has one simple negative eigenvalue, according to Theorem 4 (iii) the function $\hat{Q}_1$ has negative index $\kappa_1=1$.
- Because, $\kappa=2$, according to Theorem 4 (iv), it must be $\kappa_2=1$.
- According to Theorem 4 (ii), $z=-1$ is zero of the function Q. Indeed, it is a pole of $\hat{Q}_2$ with pole cancellation function $\eta(z)=\begin{bmatrix}1+z\\ 0\end{bmatrix}$, according to Borogovac and Luger (2014, Definition 3.1). □

In this example we have demonstrated how to use formulae given in Theorem 4 to obtain decomposition (29). The example was selected to be as simple as possible to make it readable. In more complicated cases, the calculation of

$$\hat{Q}_1(z) = \hat{S} + z\hat{G}$$

remains simple, while calculation of $\hat{Q}_2(z)$ can get very involved .

Fortunately, Theorem 4 enables us to avoid the difficult calculation of $\hat{Q}_2$ given by formula (28). Instead, we can obtain $\hat{Q}_2$ by formula $\hat{Q}_2(z) := \hat{Q}(z) - \hat{Q}_1(z)$.

In general case, it is an interesting task to decompose a generalized Nevanlinna function into a sum that preserves the number of negative squares, i.e. $Q = Q_1 + Q_2$ and $\kappa = \kappa_1 + \kappa_2$.

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