Infinite order differential equations in the space of entire functions of normal type with respect to a proximate order

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Abstract

In this paper, we consider the space of entire functions with normal type growth for a given proximate order and a continuous linear operator from such space into itself which is defined by a partial differential operator of infinite order. We will study corresponding partial differential equations with variable coefficients in the cases of regular singular type and of Korobeinik type.

Keywords: entire functions in several variables, partial differential operator of infinite order.

\textbf{MSC}: 30D15, 35R50, 32A15, 32W50.

1 Introduction

A linear operator acting continuously and locally on the holomorphic functions is represented as a partial differential operator of infinite order, more precisely a continuous morphism of the sheaf of holomorphic functions on a complex domain or complex manifold is characterized by a local partial differential operator of infinite order with the symbol being of minimal type with respect to the differential variable\textsuperscript{3}, so in the complex framework, it is quite natural to consider the infinite order PDE. Commenced first by Bernard\textsuperscript{4} and then by André\textsuperscript{5}, many people studied

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\textsuperscript{3}Ishimura, 1978, “Homomorphismes du faisceau des germes de fonctions holomorphes dans lui-même et opérateurs différentiels”;
Ishimura, 1980, “Homomorphismes du faisceau des germes de fonctions holomorphes dans lui-même et opérateurs différentiels II”.
\textsuperscript{4}Malgrange, 1956, “Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution”.
\textsuperscript{5}Martineau, 1967, “Équations différentielles d’ordre infini”.

69
Infinite order differential equations for entire functions  
R. Ishimura and X. Jin

the infinite order PDE with constant coefficients and its generalization, the convolution equation in each situation (for example Sébbar (1971), Morzhakov (1974), Berenstein and Struppa (1987), Ishimura and Okada (1994), Abanin, Ishimura, and Le Hai Khoi (2012)). Nevertheless there are not so many results concerning the variable coefficient PDE of infinite order (see Korobeinik (1959), Korobeinik (1962), Steen (1971), Ishimura (1985), Aoki (1988)).

In the preceding paper Ishimura (2007), we proved that any continuous linear operator from the space of entire functions of normal type with respect to a proximate order in itself is represented by a partial differential operator of infinite order. Recently Aoki, T., Ishimura, R., Okada, Y., Uchida, S., and Struppa, D. C. gave the complete characterization of such operator by partial differential operator as a growth condition of its symbol (arXiv:1805.00663). There we characterized also the continuous linear operator of the space of minimal type entire functions in itself as partial differential operator of infinite order.

In the present paper, we will consider infinite order partial differential equations in the space of entire functions of normal type with respect to a proximate order. We will give solvability conditions for the cases of differential equations of regular singular type and for the higher order case, of Korobeinik type.

2 Notations and recall

In this paper, we employ same notations as the preceding paper Ishimura (2007): for a point \( z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \) and setting \( \mathbb{N} := \{0, 1, 2, 3, \ldots\} \), for multi-indices \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) and \( \beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{N}^n \), we set:

\[
\begin{align*}
|z| &:= \sqrt{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2}, & |\alpha| &= \alpha_1 + \alpha_2 + \cdots + \alpha_n, \\
\bar{z} &:= (|z_1|, |z_2|, \ldots, |z_n|), & a! &= \alpha_1! \alpha_2! \cdots \alpha_n!, \\
D^\alpha &= D_z^\alpha := \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial z_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}}, & \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) &= \frac{\alpha!}{(\alpha - \beta)! \beta!}.
\end{align*}
\]

In the sequel, we will sometimes denote \( \bar{z} \leq r \) instead of \( \bar{z} \leq (r, \ldots, r) \).

We recall several notions and properties concerning the spaces of entire functions with growth order: principally, we will follow Lelong and Gruman\(^6\). Let \( w(z) \) be any real-valued continuous function defined on \( \mathbb{C}^n \) and consider the space of entire functions with growth with respect to \( w \):

\[
B_w := \left\{ f \in \mathcal{O}(\mathbb{C}^n) \left| \left\| f \right\|_w := \sup_{z \in \mathbb{C}} |f(z)e^{-w(z)}| < +\infty \right\} \right. .
\]

\(^6\)Lelong and Gruman, 1986, Entire functions of several complex variables.
3. The space $E^{\rho(r)}$

This is a Banach space with the norm $\|\cdot\|_w$. In this article, we will make use of the notion of proximate order introduced by Valiron (1923): a differentiable function $\rho(r) \geq 0$ defined for $r \geq 0$ is said to be a proximate order if it satisfies (i) $\lim_{r \to +\infty} \rho(r) = \rho$, (ii) $\lim_{r \to +\infty} \rho'(r) r \ln r = 0$. It is well-known that in the case where $\rho > 0$, there exists $r_0 > 1$ such that for $r > r_0$, the function $r^{\rho(r)}$ is strictly increasing (tending to $+\infty$), and if $\rho > 1$, $r^{\rho(r) - 1}$ is also strictly increasing (tending to $+\infty$). We have

$$\frac{d}{dr} r^{\rho(r)} = r^{\rho(r)-1} \left( \rho'(r) r \ln r + \rho(r) \right).$$

We fix a differentiable function $r = \varphi(t)$ defined for $t \geq 0$ being the inverse function of $t = r^{\rho(r)}$ for large $t$ and for any $q \in \mathbb{N}$, we set

$$A_q := A_{\rho,q} := \left( \frac{\varphi(q)^{\rho}}{e^{\rho}} \right)^{\frac{q}{\rho}}.$$  \hspace{1cm} (1)

As in Ishimura and Miyake (2007), setting $w_{\sigma}(z) := \sigma|z|^\rho |z|$ for $\sigma \geq 0$, we define the locally convex space of entire functions of type at most $\sigma > 0$ with respect to a proximate order $\rho(r)$

$$E^{\rho(r)}_{\sigma} := \lim_{\varepsilon \to 0} B_{w_{\sigma+\varepsilon}}$$

which is (by taking a decreasing sequence $(\varepsilon_j)$ tending to 0 instead of all $\varepsilon > 0$) a (FS)-space as the following asserts:

**Lemma 1** – If $s_2 > s_1 > 0$, the inclusion map $B_{w_{s_1}} \hookrightarrow B_{w_{s_2}}$ is compact.

Lemma is proven for example, as same as Martineau (1967, Lemma 1).

3 The space $E^{\rho(r)}$

In this article, we consider the space of entire functions of normal type with respect to the proximate order $\rho(r)$:

**Definition 1** –

$$E^{\rho(r)} := \lim_{\sigma \to 0} B_{w_{\sigma}}$$

which is a (DFS)-space.

For any function $f = \sum_{\alpha \in \mathbb{N}^n} f_\alpha z^\alpha \in \mathcal{O}(\mathbb{C}^n)$, we set

$$H_q(f) = H_q := \sup_{|z| \leq 1} |P_q(z)| \quad \text{with} \quad P_q(z) := \sum_{|\alpha| = q} f_\alpha z^\alpha.$$

We recall Lelong and Gruman (1986, Theorem 1.23):
Theorem 1 – Let \( f \) be an entire function of finite order \( \rho > 0 \) and of proximate order \( \rho(r) \). Then its type \( \sigma \) with respect to \( \rho(r) \) is given by

\[
\frac{1}{\rho} \ln \sigma = \limsup_{q \to +\infty} \left( \frac{1}{q} \ln H_q + \ln \varphi(q) \right) - \frac{1}{\rho} - \frac{\ln \rho}{\rho}.
\]

Using the Theorem 1, we have:

Lemma 2 – An entire function \( f(z) = \sum_{\alpha} a_{\alpha} z^{\alpha} \) belongs to \( \mathcal{E}_{\rho}^{\rho(r)}(\sigma) \) if and only if

\[
\limsup_{q \to \infty} \left( H_q A_{\rho,q} \right)^{\frac{\rho}{q}} \leq \sigma.
\]

Proof. In fact, \( f \in \mathcal{E}_{\rho}^{\rho(r)}(\sigma) \) is equivalent to saying for any \( \varepsilon > 0 \), there exists \( D_\varepsilon > 0 \) such that we have

\[
|f(z)| \leq D_\varepsilon \cdot e^{w_{\sigma+\varepsilon}(z)} = D_\varepsilon \cdot e^{(\sigma+\varepsilon)|z|^{\rho(r)}}
\]

for all \( z \in \mathbb{C}^n \), that is, setting \( M_f(r) := \sup_{|z| \leq r} \ln |f(z)| \), we have

\[
\limsup_{r \to +\infty} \frac{M_f(r)}{\rho(r)} \leq \limsup_{r \to +\infty} \frac{\ln D_\varepsilon + (\sigma + \varepsilon)r_{\rho(r)}}{r_{\rho(r)}} = \sigma.
\]

In view of Theorem 1, this is also equivalent to say that we have

\[
\limsup_{q \to +\infty} \left( \frac{\rho}{q} \ln H_q + \rho \ln \varphi(q) \right) - \ln(e \rho) \leq \ln \sigma:
\]

this means (2). \( \square \)

Corollary 1 – In the hypothesis of Lemma 2, we have:

\[
\limsup_{q \to +\infty} \left( \max_{|\alpha| = q} |a_{\alpha}| A_{\rho,q} \right)^{\frac{\rho}{q}} \leq \sqrt{n}^\rho \sigma
\]

Conversely if we assume this inequality, we have \( f(z) \in \mathcal{E}_{\sqrt{n}^\rho \sigma}^{\rho(r)} \).

Proof. By using Cauchy’s inequality, we have

\[
|a_{\alpha}| = \frac{1}{\alpha!} |\partial^{\alpha} p_q(0)| \leq r^{-d} \sup_{|z| \leq r} |p_q(z)| = \sup_{|z| \leq 1} |P_q(z)|,
\]
3. The space $E^{p(r)}$

here we denoted $(r, \ldots, r)$ simply by $r$. As $|z| \leq r$ implies $|z| \leq \sqrt{n} r$ for any $r > 0$, setting $v := z/\sqrt{n}$, we can continue

$$
\sup_{|z| \leq 1} |P_q(z)| \leq \sup_{|z| \leq \sqrt{n}} |P_q(z)| = \sup_{|v| \leq 1} |P_q(\sqrt{n}v)| = \sqrt{n^q} \sup_{|z| \leq 1} |P_q(z)| = \sqrt{n^q} H_q.
$$

By the Lemma 2 on the preceding page, for any $\varepsilon > 0$, if $q = |\alpha| > 0$ is large enough, we have

$$
\left| f_\alpha \right| A_{\rho,q}^{\frac{p}{q}} \leq \sqrt{n}^{\rho} \left( H_q A_{\rho,q} \right)^{\frac{p}{q}} < \sqrt{n}^{\rho} (\sigma + \varepsilon)
$$

so we have (3).

Conversely if we have (3), for any $\varepsilon > 0$, there exists $C > 0$ such that for $|\alpha| = q \gg 1$, taking into account the fact

$$
H_q^n := \left( \frac{n + q - 1}{q} \right) = \left( 1 + \frac{q}{n - 1} \right) \left( 1 + \frac{q}{n - 2} \right) \cdots \left( 1 + \frac{q}{1} \right) \leq (q + 1)^{n-1},
$$

we have

$$
H_q = \sup_{|z| \leq 1} |P_q(z)| \leq \sup_{|z| \leq 1} \sum_{|\alpha|=q} |f_\alpha| |z|^q
$$

$$
\leq \left( \sqrt{n}^{\rho} \sigma + \varepsilon \right)^{\frac{p}{q}} \cdot \frac{H_q^n}{A_{\rho,q}} \leq \left( \sqrt{n}^{\rho} \sigma + \varepsilon \right)^{\frac{p}{q}} \cdot \frac{(q + 1)^{n-1}}{A_{\rho,q}}
$$

i.e. $\limsup_{q \to \infty} \left( H_q A_{\rho,q} \right)^{\frac{p}{q}} \leq n^\rho \sigma$ as $\limsup_{q \to \infty} ((q + 1)^{n-1})^{\frac{p}{q}} = 1$ and so we have

$$
f(z) \in E^{p(r)}_{w,\sigma}.
$$

By this Corollary, we obtain the following result:

**Proposition 1** – An entire function $f = \sum f_\alpha z^\alpha$ belongs to $E^{p(r)}$ if and only if we have

$$
\limsup_{q \to +\infty} \left( \max_{|\alpha|=q} |f_\alpha| A_{\rho,q} \right)^{\frac{1}{q}} < +\infty.
$$

(4)

As for the norm of $B_{w_\sigma}$, we have

**Proposition 2** – For $f = \sum f_\alpha z^\alpha \in B_{w_\sigma}$, there exist $C,D > 0$ such that we have

$$
\|f\|_{w_\sigma} \leq C \sum_{q \in \mathbb{N}} \left( \sum_{|\alpha|=q} |f_\alpha| D^q A_q \right).
$$
Infinite order differential equations for entire functions  
R. Ishimura and X. Jin

and for any \( \alpha \in \mathbb{N}^n \),

\[
C^{-1} D^{-|\alpha|} A_{|\alpha|} |f_{\alpha}| \leq \|f\|_{w_\sigma}. 
\]  

(5)

Proof. Ishimura and Miyake (2007, Lemma 1) assures that there exist \( C, D > 0 \) such that for any \( \alpha \), we have

\[
\|z^\alpha\|_{w_\sigma} \leq CD^q A_q \quad (|\alpha| = q)
\]

and so

\[
\|f\|_{w_\sigma} \leq \sum_{\alpha \in \mathbb{N}^n} |f_{\alpha}| \|z^\alpha\|_{w_\sigma} \leq C \sum_{q \in \mathbb{N}} \left( \sum_{|\alpha| = q} |f_{\alpha}| D^q A_q \right).
\]

As in the proof of Lelong and Gruman (1986, Lemma 9.2), by Cauchy’s formula in one variable \( \lambda \in \mathbb{C} \), for any \( q \), we have

\[
P_q(z) = \frac{1}{2\pi i} \oint_{|\lambda| = 1} f(\lambda z) \lambda^{-q+1} d\lambda:
\]

so for any \( r > 0 \), we have

\[
\sup_{|z| = r} \frac{|P_q(z)|}{|z|^q} \leq \sup_{|z| = r} \left( \left| f(z)e^{-w_\alpha(z)} \right| \frac{e^{\sigma r^{(r)}}}{r^q} \right)
\]

and then for any \( r > 0 \), we have

\[
\sup_{|z| = r} \frac{|P_q(z)|}{|z|^q} \leq \|f\|_{w_\sigma} \frac{e^{\sigma r^{(r)}}}{r^q}:
\]

as in the proof of Lelong and Gruman (1986, Lemma 9.2), the function \( \frac{e^{\sigma r^{(r)}}}{r^q} \) of \( r \) attains its minimum which, by choosing \( C, D > 0 \) large enough, is estimated by \( CD^q/A_q \); therefore we have the inequality

\[
C^{-1} D^{-q} A_q \sup_z \frac{|P_q(z)|}{|z|^q} \leq \|f\|_{w_\sigma}.
\]

As by the proof of Corollary 1 on p. 72, setting \( q = |\alpha| \), we have

\[
|f_{\alpha}| \leq \sqrt{n^q} \sup_{|z| = 1} P_q(z) = \sqrt{n^q} \sup_z \frac{|P_q(z)|}{|z|^q},
\]

taking \( C, D > 0 \) large enough, we have (5).  

4 Duality between the spaces $E^r(\rho)$ and $E^{s\ast}_0$

In this section, we will study the dual space of $E^r(\rho)$ (c.f. Lelong and Gruman (1986, Theorem 9.5 and Theorem 9.16)). In the rest of this section, we will suppose $\rho > 1$: in this case, the equation $s = r^{\rho(r) - 1}$ has the unique solution $r = \lambda(s)$ for any $s$ large enough. A proximate order $\rho^*(s)$ is said to be a conjugate proximate order of $\rho(r)$ if it satisfied for large $s$ (so for large $r$)

$$\rho^*(s) := \frac{\rho(r)}{\rho(r) - 1} \text{ i.e. } \frac{1}{\rho(r)} + \frac{1}{\rho^*(s)} = 1. \tag{6}$$

It is easy to see that any differentiable function $\rho^*(s)$ satisfying (6) for large $s$ is in fact a proximate order (see for example Lelong and Gruman (1986, Prop. 9.4)). In this section, we will take a conjugate proximate order $\rho^*(s)$. Set $\rho^* := \lim_{s \to \infty} \rho^*(s) > 1$.

As the case of $r = \varphi(t)$, let $s = \varphi^*(u)$ be a differentiable function being the inverse function of $u = s^{\rho^*(s)}$ when $u$ is large enough. We set also

$$A_q^* = A_{\rho^*,q} := \left(\frac{\varphi^*(q)(\rho^*)}{e^{\rho^*}}\right)^{\frac{q}{\rho^*}}. \tag{7}$$

Now consider the space $E_0^{\rho^*(s)}$ of entire functions of minimal type with respect to $\rho^*(s)$, which is an (FS)-space: we will show that it is the dual space of $E^r(\rho)$.

The function $z \mapsto e^{z\zeta}$ is in $E^r(\rho)$ if $\rho \geq 1$: for any $T \in (E^r(\rho))'$, we denote its Fourier-Borel transform by

$$\hat{T}(\zeta) = T_z(e^{z\zeta}).$$

**Proposition 3 –** Suppose $\rho > 1$. The map

$$T \mapsto \hat{T} : (E^r(\rho))' \simeq E_0^{\rho^*(s)} \tag{8}$$

is a continuous bijection of Fréchet spaces so it is an isomorphism.

**Proof.** For $r > 0$ large enough, we will find the upper bound of the function

$$h(r) = h(r,\zeta) := \sup_{|z|=r}\left(\text{Re } z \cdot \zeta - w_{\sigma}(z)\right) = r|\zeta| - \sigma r^{\rho(r)}$$

For this, we remark that for large $|\zeta|$, the equation

$$h'(r) = |\zeta| - \sigma r^{\rho(r)-1}\left(\rho'(r) r \ln r + \rho(r)\right) = 0 \tag{9}$$
has unique solution $r_0 = r(\zeta)$ ($\gg 1$): in general, it is not possible to know its exact value, but for any $\delta > 0$, if $|\zeta|$ is large enough, we have

$$\left| \left( \rho'(r_0) r_0 \ln r_0 + \rho(r_0) \right) - \rho \right| < \delta$$

so by the (9), we have

$$\frac{|\zeta|}{\sigma(\rho + \delta)} < r_0^{\rho(r_0)-1} < \frac{|\zeta|}{\sigma(\rho - \delta)}.$$

Therefore for some $c > 0$, we have

$$h(r, \zeta) \leq h(r_0, \zeta) = r_0 |\zeta| - \sigma r_0^{\rho(r_0)}$$

$$< \left( \frac{|\zeta|}{\sigma(\rho - \delta)} \right)^{\frac{1}{\rho(r_0)-1}} |\zeta| - \left( \frac{|\zeta|}{\sigma(\rho + \delta)} \right)^{\frac{1}{\rho(r_0)-1}}$$

$$= \left( \frac{1}{\sigma(\rho - \delta)} \right)^{\frac{1}{\rho(r_0)-1}} - \left( \frac{1}{\sigma(\rho + \delta)} \right)^{\frac{1}{\rho(r_0)-1}} |\zeta|^{\rho(r_0)-1}$$

$$< \left( \frac{1}{\sigma r_0} \right)^{\frac{1}{\rho(r_0)-1}} - \left( \frac{1}{\sigma r_0} \right)^{\frac{1}{\rho(r_0)-1}} + c\delta |\zeta|^{\rho(r_0)-1}$$

$$= \left( \sigma r_0 - 1 \right) \left( \frac{1}{\sigma r_0} \right)^{\frac{1}{\rho(r_0)-1}} + c\delta |\zeta|^{\rho(r_0)-1}$$

as

$$\left( \frac{1}{\sigma(\rho + \delta)} \right)^{\frac{1}{\rho(r_0)-1}} = \left( \frac{1}{\sigma r_0} \right)^{\frac{1}{\rho(r_0)-1}} + O(\delta).$$

we may replace $\delta$ by $c^{-1}(\sigma r_0 - 1) \left( \frac{1}{\sigma r_0} \right)^{\frac{r_0(\rho)}{\rho(r_0)-1}} \delta$ and then we continue

$$< (1 + \delta)(\sigma r_0 - 1) \left( \frac{1}{\sigma r_0} \right)^{\frac{r_0(\rho)}{\rho(r_0)-1}} |\zeta|^{\rho(r_0)-1} = (1 + \delta)(\sigma r_0 - 1) \left( \frac{|\zeta|}{\sigma r_0} \right)^{\frac{r_0(\rho)}{\rho(r_0)-1}}.$$

As for large $|\zeta|$ (so $r_0 \gg 1$), setting $s_0 := r_0^{\rho(r_0)-1}$, we have

$$0 = h'(r_0) = |\zeta| - \sigma r_0^{\rho(r_0)-1} \left( \rho'(r_0) r_0 \ln r_0 + \rho(r_0) \right) < |\zeta| - \sigma s_0 (\rho - \delta)$$
4. Duality between the spaces $E^{p(r)}$ and $E^{p'(s)}$

and

$$0 = h'(r_0) = |\zeta| - s_0 \rho^{p(r_0) - 1} \left( \rho'(r_0) r_0 \ln r_0 + \rho(r_0) \right) > |\zeta| - s_0 (\rho + \delta),$$

we have

$$0 < \frac{|\zeta|}{\rho (\rho + \delta)} < s_0 < \frac{|\zeta|}{\rho (\rho - \delta)}$$

i.e.

$$\frac{|\zeta|}{\rho} - s_0 = \frac{s_0 \delta}{\rho}. $$

As $\rho^*(s)$ is a proximate order, with some $\theta$ such that $0 < \theta < 1$, we have

$$\rho^*(s_0) = \rho^* \left( \frac{|\zeta|}{\rho} \right) + \rho^* \left( s_0 + \theta \cdot \left( \frac{|\zeta|}{\rho} - s_0 \right) \right) - \rho^* \left( \frac{|\zeta|}{\rho} - s_0 \right)$$

$$< \rho^* \left( \frac{|\zeta|}{\rho} \right) + \frac{1}{(\rho - \delta) \ln \left( \frac{s_0 - s_0 \delta}{\rho} \right)} \cdot s_0 \delta$$

$$= \rho^* \left( \frac{|\zeta|}{\rho} \right) + \frac{\delta}{\ln \left( \frac{|\zeta|}{\rho} - \delta \cdot \rho \right)} < \rho^* \left( \frac{|\zeta|}{\rho} \right) + \frac{\delta}{\ln \left( \frac{|\zeta|}{\rho} (1 - 2 \delta) \right)}$$

$$< \rho^* \left( \frac{|\zeta|}{\rho} \right) + \frac{\delta}{\ln \left( \frac{|\zeta|}{\rho} - 3 \delta \right)}$$

if $|\zeta|$ large enough and if $\rho - \delta > 1$ and $\ln(1 - 2 \delta) > -3 \delta$: so we have

$$\sup_{|z|=r} \left( \text{Re} z \cdot \zeta - w_\sigma(z) \right) = h(r, \zeta) < (1 + \delta) (\rho \rho - 1) \left( \frac{|\zeta|}{\rho} \right)^{p(r_0) - 1}$$

$$= (1 + \delta) (\rho \rho - 1) \left( \frac{|\zeta|}{\rho} \right)^{p'(s_0)} < (1 + \delta) (\rho \rho - 1) \left( \frac{|\zeta|}{\rho} \right)^{p'(\sigma \rho t)} + \frac{\delta}{\ln \left( \frac{|\zeta|}{\rho} - 3 \delta \right)}. \quad (10)$$

Setting $t := \frac{|\zeta|}{\rho}$, for $0 < \delta \ll 1$, if $t > 0$ is large enough, we have

$$\frac{h(r, \zeta)}{(1 + \delta) (\rho \rho - 1)} < \ln t^{p'(t) + \frac{2 \delta}{\ln t - 3 \delta}} = \left( p'(t) + \frac{2 \delta}{\ln t - 3 \delta} \right) \ln t$$

$$< p'(t) \ln t + 3 \delta = \ln t^{p'(t)} + 3 \delta < \ln t^{p'(\sigma \rho t)} + 4 \delta$$

as by, for example, Lelong and Gruman (1986, Theorem 1.18), we have $\frac{t^{p'(t)}}{t^{p'(\sigma \rho t)}} < e^\delta$: thus we have

$$\frac{h(r, \zeta)}{(1 + \delta) (\rho \rho - 1)} < e^{4 \delta} t^{p'(\sigma \rho t)}.$$
Infinite order differential equations for entire functions  
R. Ishimura and X. Jin

Therefore for large \(|\zeta|\), continuing (10), we have

\[
\sup_z \left( \text{Re} z \cdot \zeta - w_\alpha(z) \right) = h(r, \zeta) < (1 + \delta)(\sigma_0 - 1)e^{4\delta \frac{|\zeta|}{\sigma_0}}
= (1 + \delta)e^{4\delta \sigma^*|\zeta|^{\rho^*(|\zeta|)}} = (1 + \delta)e^{4\delta \sigma^*w^*(\zeta)} \tag{11}
\]

here we set \(\sigma^* := \frac{\sigma_0 - 1}{(\sigma_0)^{\sigma^*}}\) (which is decreasing as a function of \(\sigma \geq 1\) and tending to 0 if \(\sigma\) tends to \(\infty\)). Thus for any \(\epsilon > 0\), there exists a constant \(C = C_\epsilon > 0\) such that we have

\[
|\hat{T}(\zeta)| = |T_z(e^{z\cdot\zeta})| \leq \|T\|_{(B_{w_\alpha})'} \cdot \|e^{z\cdot\zeta}\|_{w_\alpha} = \|T\|_{(B_{w_\alpha})'} \cdot \sup_z \exp \left( \text{Re} z \cdot \zeta - w_\alpha(z) \right)
\leq C\|T\|_{(B_{w_\alpha})'} \cdot \exp \left( \sigma^* + \epsilon \right)w^*(\zeta)
\]

where \(\|T\|_{(B_{w_\alpha})'} := \sup_{\|f\|_{w_\alpha} = 1} |T(f)|\) is the operator norm of \(T\) in \((B_{w_\alpha})'\); so \(T \mapsto \hat{T}\) is a continuous map \((\mathcal{E}_{\rho(r)})' \to E_0^{\rho^*(s)}\) which is evidently injective.

Conversely taking into account of Corollary 1 on p. 72 for \(\sigma^*\) instead of \(\rho\) and in the case \(\sigma = 0\), an entire function \(F(\zeta) = \sum F_\alpha \zeta^\alpha\) belongs to \(E_0^{\rho^*(s)}\) if and only if we have

\[
\limsup_{q \to \infty} \left( \max_{|\alpha| = q} |F_\alpha| A_{\rho^*,q}^{\frac{1}{q}} \right) = 0 ;
\]

by (7), for any \(\epsilon > 0\), there exists \(C > 0\) such that for any \(q \geq 0\), we have

\[
\max_{|\alpha| = q} |F_\alpha| \leq \frac{\epsilon^q}{\varphi^*(q)^q}.
\]

Now take any \(F(\zeta) = \sum F_\alpha \zeta^\alpha \in E_0^{\rho^*(s)}\) and set for any \(f(z) = \sum f_\alpha z^\alpha \in \mathcal{E}_{\rho(r)}\),

\[
Tf := \sum_{\alpha \in \mathbb{N}^n} \alpha! f_\alpha F_\alpha \tag{12}
\]

we prove that this is in fact convergent and so well-defined. By Proposition 1 on p. 73, taking into account (1), there exist \(C' > 0\) and \(D > 0\) such that for any \(q \geq 0\), we have

\[
\max_{|\alpha| = q} |f_\alpha| \leq \frac{C' D^q}{\varphi(q)^q}.
\]
5. Partial differential equations of infinite order in $\mathcal{E}^{\rho(r)}$

Set $r = r_\theta = \varphi(q)$ and $s = s_\theta = \varphi^*(q)$ i.e. $q^{1/r} = r, q^{1/s} = s$, then for any $\alpha$ with $|\alpha| = q$, we have

\[
\frac{\alpha!}{\varphi(q)^{|\alpha|} \varphi^*(q)^{|\alpha|}} \leq \frac{q^q}{(rs)^q} = 1
\]

and thus (12) is convergent, that is $T$ is in fact, an element of $(\mathcal{E}^{\rho(r)})'$ and we have

\[
\hat{T}(\zeta) = T_z(e^{\zeta}) = T \left( \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \zeta^\alpha z^\alpha \right) = \sum_{\alpha \in \mathbb{N}^n} \zeta^\alpha F_\alpha = F(\zeta)
\]

and thus $T : (\mathcal{E}^{\rho(r)})' \to E_0^{\rho^{(s)}}$ is a continuous bijection and so by a Theorem of Banach, it is in fact an isomorphism of Fréchet spaces. □

5 Partial differential equations of infinite order in $\mathcal{E}^{\rho(r)}$

We proved that any continuous linear operator from $\mathcal{E}^{\rho(r)}$ into itself is represented by an infinite order partial differential operator of the form

\[
P = P(z, D_z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha(z) D_z^\alpha
\]

such as with a $\sigma > 1$, $\sum_{\alpha \in \mathbb{N}^n} \|a_\alpha\|_{w, \sigma} |\alpha|^{1-\frac{1}{\sigma}} \zeta^\alpha$ is holomorphic at 0\(^7\). We call

\[
P(z, \zeta) := \sum_{|\alpha| = 0}^{\infty} a_\alpha(z) \zeta^\alpha
\]

the symbol of $P$: Expanding each coefficient

\[
a_\alpha(z) := \sum_{\beta \in \mathbb{N}^n} a_\alpha^\beta z^\beta,
\]

we define at least formally the transpose of $P = P(\zeta, D_\zeta)$:

\[
t^P := (P(\zeta, D_\zeta))^* := \sum_{\beta \in \mathbb{N}^n} \left( \sum_{\alpha \in \mathbb{N}^n} a_\alpha^\beta \zeta^\alpha \right) D_\zeta^\beta.
\]

\(^7\)Ishimura and Miyake, 2007, “Endomorphisms of the space of entire functions with proximate order and infinite order differential operators”.

79
For a formal power series $f(z) := \sum \nu f_\nu z^\nu \in \mathbb{C}[z]$, we see formally

$$P(z,D_z)f(z) = \sum_{\mu \in \mathbb{N}_n} \left( \sum_{\nu \in \mathbb{N}_n} \left( \sum_{\lambda \leq \mu, \nu} \frac{\nu!}{\lambda!} a_{\nu-\lambda}^\mu \right) f_\nu \right) z^\mu :$$

thus by the identification $f \mapsto \left( f_\gamma \right): \mathbb{C}[[z]] \xrightarrow{\sim} \mathbb{C}^{\mathbb{N}_n}$, the operator $P: \mathbb{C}[[z]] \to \mathbb{C}[[z]]$ is identified with the infinite matrix which we call its characteristic matrix

$$C_P := \left( c^\mu_{\nu,\gamma} \right)_{\mu,\nu} := \left( \sum_{\lambda \leq \mu, \nu} \frac{\nu!}{\lambda!} a_{\nu-\lambda}^\mu \right)_{\mu,\nu} : \mathbb{C}^{\mathbb{N}_n} \to \mathbb{C}^{\mathbb{N}_n}.$$ (13)

Similarly for $F(\zeta) = \sum \nu F_\nu \zeta^\nu \in \mathbb{C}[[\zeta]]$, we have

$$f P(\zeta,D_\zeta)F(\zeta) = \sum_{\mu \in \mathbb{N}_n} \left( \sum_{\nu \in \mathbb{N}_n} \left( \sum_{\lambda \leq \mu, \nu} \frac{\nu!}{\lambda!} a_{\nu-\lambda}^\mu \right) F_\nu \right) \zeta^\mu :$$

so we have

$$C_{P,F} = \left( \sum_{\lambda \leq \mu, \nu} \frac{\nu!}{\lambda!} a_{\nu-\lambda}^\mu \right)_{\mu,\nu} = \left( \frac{\nu!}{\mu!} c^\mu_{\nu,\gamma} \right)_{\mu,\nu}.$$ (14)

Now we will consider the partial differential equation in $g^{\mu(r)}$:

$$P(z,D_z)f = \sum_{|\alpha|=0}^{\infty} a_\alpha(z) D_\alpha^z f(z) = g.$$ By (13), expanding $f = \sum \nu f_\nu z^\nu$, $g = \sum_{\mu \in \mathbb{N}_n} g_\mu z^\mu$, this means formally

$$\sum_{\nu \in \mathbb{N}_n} c^\mu_{\nu,\gamma} f_\nu \equiv \sum_{\nu \in \mathbb{N}_n} \left( \sum_{\lambda \leq \mu} \frac{\nu!}{\lambda!} a_{\nu-\lambda}^\mu \right) f_\nu = g_\mu \quad \text{(for any} \ \mu \in \mathbb{N}_n).$$ (15)

In this article, as in Ishimura (2007), we will study the following two cases:

- partial differential operator of infinite order $P = \sum_{|\alpha|=0}^{\infty} a_\alpha(z) D_\alpha^z$ of
  - (I) regular singular type, that is, each $a_\alpha(z)$ is divisible by $z^\alpha$

$$a_\alpha(z) = \sum \beta a^\beta_\alpha z^\beta :$$

80
5. Partial differential equations of infinite order in $E^{p(r)}$

(II) Korobeinik type, namely, each $a_\alpha(z)$ is a polynomial of order $\leq \alpha$

$$a_\alpha(z) = \sum_{\beta \leq \alpha} a_\alpha^{\beta} z^\beta.$$  

This second type operator was first studied by Korobeinik (1959) in one variable case; we remark also Korobeinik (1962) and van der Steen (1971) studied the operators of first type, always in one variable case.

(O) We will call $P$ an operator of Euler type if it is regular singular type and Korobeinik type at the same time i.e. with a constant $a_\alpha \in \mathbb{C}$, $P$ has the form

$$P = \sum_{|\alpha|=0}^{\infty} a_\alpha z^\alpha D_z^\alpha.$$  

**Theorem 2** – Let a continuous linear operator $P : E^{p(r)} \to E^{p(r)}$ be of regular singular type as differential operator. Suppose the following conditions hold: there exist $C, R, \kappa > 0$ such that

1. for all $\mu \in \mathbb{N}^n$, we have

$$C^{-1} \kappa^{|\mu|} < \left| \sum_{\lambda \leq \mu} \frac{\mu!}{\lambda!} a_{\mu-\lambda} \right|.$$  

2. whenever $v < \mu$, we have

$$\left| \frac{\sum_{\lambda \leq v} \frac{\mu!}{\lambda!} a_{\mu-\lambda}}{\sum_{\lambda \leq \mu} \frac{\mu!}{\lambda!} a_{\mu-\lambda}} \right| \leq C \cdot \frac{A|\nu| R|\mu-\nu|}{A|\mu|}.$$  

Then $P : E^{p(r)} \to E^{p(r)}$ is surjective; so is an isomorphism of Fréchet spaces.

**Proof.** Assume that $P f = g \in E^{p(r)}$ and expand $f = \sum_{v} f_v z^v, g = \sum_{\mu} g_\mu z^\mu$: taking into account that in this case, (15) is just $\sum_{v \in \mathbb{N}^n} \left( \sum_{\lambda \leq v} \frac{\mu!}{\lambda!} a_{\mu-\lambda} \right) f_v = \sum_{v \leq \mu} c_\nu f_\nu = g_\mu$, the condition (1) assures that the coefficients $(f_\mu)$ are determined uniquely from $(g_\mu)$ and so, in view of Proposition 1 on p. 73, it suffices to show that there exist $D, K > 0$ such that for all $\mu \in \mathbb{N}^n$, we have

$$|f_\mu| \leq D \cdot \frac{K|\mu|}{A|\mu|}.$$  

(16)
we argue by induction on \( q := |\mu| \). First we remark, by the condition (1) and Proposition 1 on p. 73, there exist \( D_0, L > 0 \) such that for all \( \mu \in \mathbb{N}^n \), we have

\[
\left| \frac{g_\mu}{c_\mu} \right| \leq D_0 \frac{L |\mu|}{A |\mu|}. \tag{17}
\]

Assuming the condition (2), take any \( K > \max(L, R, C) \) and \( D > D_0 \) so that for all \( \mu \in \mathbb{N}^n \), we have

\[
D_0 \left( \frac{L}{K} \right)^{|\mu|} + C \left( \frac{K^n}{(K-R)^n} - 1 \right) < 1.
\]

For \( q = |\mu| \), we have

\[
\frac{1}{|c_\mu|} \sum_{\nu < \mu} |c_\nu f_\nu| \leq \sum_{\nu < \mu} C \cdot A_{|\nu|} R^{|\nu| - |\mu|} \cdot D \cdot \frac{K^{|\nu|}}{A_{|\nu|}} = CD \left( R^q \sum_{\nu \leq \mu} \left( \frac{K}{R} \right)^{|\nu|} - K^q \right)
\]

\[
= CD \frac{A_q}{A_q} \left( R^q \sum_{\nu_1 \leq \mu_1} \left( \frac{K}{R} \right)^{\nu_1} \cdots \sum_{\nu_n \leq \mu_n} \left( \frac{K}{R} \right)^{\nu_n} - K^q \right)
\]

\[
= CD \frac{A_q}{A_q} \left( R^q \cdot \frac{\left( \frac{K}{R} \right)^{\mu_1 + 1} - 1 - \left( \frac{K}{R} \right)^{\mu_n + 1} - 1}{\frac{K}{R} - 1} - K^q \right)
\]

\[
\leq CD \frac{A_q}{A_q} \left( R^q \cdot \left( \frac{K}{R} \right)^{q + n} - K^q \right) = CD \frac{K^q}{A_q} \left( \frac{K^n}{(K-R)^n} - 1 \right).
\]

Therefore, by (17), we have

\[
|f_\mu| \leq \left| \frac{g_\mu}{c_\mu} \right| + \frac{1}{|c_\mu|} \sum_{\nu < \mu} |c_\nu f_\nu| \leq D_0 \cdot \frac{L^q}{A_q} + CD \cdot \frac{K^q}{A_q} \left( \frac{K^n}{(K-R)^n} - 1 \right) < D \cdot \frac{K^q}{A_q},
\]

that is (16) as desired. \( \square \)

In particular, when \( P \) is of Euler type, we have:

**Corollary 2** – If \( P \) is of Euler type, then \( P : \mathcal{E}_1^{\rho(r)} \rightarrow \mathcal{E}_1^{\rho(r)} \) is an isomorphism if and only if there exist \( C, \kappa > 0 \) such that for any \( \mu \in \mathbb{N}^n \), we have

\[
C^{-1} K^{|\mu|} < \sum_{\lambda \leq \mu} \frac{\mu_\lambda}{\lambda!} a_{\mu-\lambda}.
\]
5. Partial differential equations of infinite order in $\mathcal{E}^{\rho(r)}$

Proof. We need only to prove the "only if" part: in fact, taking $g = \sum_{\mu} \frac{1}{A_{|\mu|}}$ and $f := P^{-1}g$, by Proposition 1 on p. 73, with some $C, \kappa > 0$, we have

$$\frac{1}{|c_{\mu}|} = |f_{\mu}|A_{|\mu|} \leq C\kappa^{-q}.$$  \hfill \Box

Example 1 – Suppose $n = 1, \rho > 1$. In the following cases, $P = \sum a_\alpha(z)D^\alpha$ satisfies the condition of Theorem 2 on p. 81:

(i) with $k > 0$, set

$$a_\alpha^\beta := \begin{cases} \frac{k^\alpha}{\alpha!} & \beta = \alpha \\ \frac{(-k)^\alpha}{\alpha!} & \beta = \alpha + 1 \\ 0 & \text{otherwise} \end{cases}.$$  

(ii) with $k \neq -1$, set

$$a_\alpha^\beta := \begin{cases} \frac{k^\alpha}{\alpha!} & \beta = \alpha \\ \frac{(-1)^\alpha}{\alpha!(\beta - \alpha)!} & \beta > \alpha \\ 0 & \text{otherwise} \end{cases}.$$  

Theorem 3 – Suppose $\rho > 1$. Let a continuous linear operator $P : \mathcal{E}^{\rho(r)} \rightarrow \mathcal{E}^{\rho(r)}$ be of Korobetnik type as partial differential operator. Suppose the following conditions hold:

1. there exist $C, \kappa > 0$ such that for all $\mu \in \mathbb{N}^n$, we have

$$C^{-1}k_{|\mu|} < \sum_{\lambda \leq \mu} \frac{\mu!}{\lambda!} a_{\mu - \lambda}^{|\mu - \lambda|},$$  

2. for any $\delta > 0$, there exists $M = M_\delta > 0$ such that whenever $|\mu| \geq M$ and $\nu < \mu$, we have

$$\left| \sum_{\lambda \leq \nu} \frac{1}{\lambda!} a_{\nu - \lambda}^{|\mu - \lambda|} \frac{A_{|\lambda|}}{A_{|\mu|}} A_{|\nu|}^* \right| \leq A_{|\nu|}^* |\mu - \nu|.$$  

Then $P : \mathcal{E}^{\rho(r)} \rightarrow \mathcal{E}^{\rho(r)}$ is surjective; so is an epimorphism of Fréchet spaces.
Infinite order differential equations for entire functions  
R. Ishimura and X. Jin

Proof. In the case, the characteristic matrix $C_{tP} = \left( \frac{\nu!}{\mu!} c^\nu_\mu \right)_{\mu,\nu}$ (see (14)) being "lower diagonal", $tP$ is injective: so by virtue of the closed range theorem, we only need to show that $tP(E_0^\sigma(s))$ is closed in $E_0^\sigma(s)$: let $\tilde{T}^i \in E_0^\sigma(s)$ be a sequence so that the sequence $\tilde{S}^i := tP(\tilde{T}^i)$ is a Cauchy sequence in $E_0^\sigma(s)$. We will show that the sequence $(\tilde{T}^i)$ is a Cauchy sequence in $E_0^\sigma(s)$ i.e. for any $\sigma > 0$ and any $\epsilon > 0$, there exists $N > 0$ such that for any $i, j > N$, we have $\|\tilde{T}^i - \tilde{T}^j\|_{w_\sigma} \leqslant \epsilon$. Now expanding $\tilde{T}^i(\zeta) = \sum_{\mu \in \mathbb{N}_n} \tilde{T}^i_\mu \zeta^\mu$, and $\tilde{S}^i(\zeta) = \sum_{\mu \in \mathbb{N}_n} \tilde{S}^i_\mu \zeta^\mu$, taking into account of Proposition 2 on p. 73, we will show by induction, that for any $\epsilon > 0$, there exists $N > 0$ such that for any $i, j > N$ and any $\mu \in \mathbb{N}_n$, we have

$$\left| \tilde{T}^i_\mu - \tilde{T}^j_\mu \right| \leqslant \frac{\epsilon |\mu| + 1}{A^*_|\mu|}. \quad (18)$$

Remark for any $\mu \in \mathbb{N}_n$, inductively, we have

$$\tilde{T}^i_\mu - \tilde{T}^j_\mu = \frac{1}{c^\mu_\mu} \left( \left( \sum_{\nu < \mu} \frac{\nu!}{\mu!} c^\nu_\mu \right) - \sum_{\nu < \mu} \frac{\nu!}{\mu!} c^\nu_\mu \left( \tilde{T}^i_\nu - \tilde{T}^j_\nu \right) \right).$$

by this, we first remark that for $M = M_\delta$ being fixed, there exists $C_M > 1$ such that whenever $|\mu| < M$, we have

$$\left| \tilde{T}^i_\mu - \tilde{T}^j_\mu \right| \leqslant C_M \epsilon |\mu| \max_{\nu < \mu} \frac{A^*_|\nu|}{A^*_|\mu|} \left| \tilde{S}^i_\nu - \tilde{S}^j_\nu \right|.$$  

The sequence $(\tilde{S}^i)$ being a Cauchy sequence, we may assume that taking $N > 0$ large enough, for any $i, j > N$ and $\mu \in \mathbb{N}_n$, we have

$$\left| \tilde{S}^i_\mu - \tilde{S}^j_\mu \right| \leqslant \frac{\epsilon \kappa |\mu|}{2C_M A^*_|\mu|}, \quad (19)$$

so by (1), for all $i, j > N$ and $\mu \in \mathbb{N}_n$, we have then

$$\frac{1}{c^\mu_\mu} \left| \tilde{S}^i_\mu - \tilde{S}^j_\mu \right| \leqslant \frac{\epsilon |\mu| + 1}{2C_M A^*_|\mu|} \leqslant \frac{\epsilon |\mu| + 1}{2A^*_|\mu|}. \quad (20)$$

In the condition (1), we may of course assume $0 < \kappa < 1$ and so from (19), if $|\mu| < M$, we have

$$\left| \tilde{T}^i_\mu - \tilde{T}^j_\mu \right| \leqslant \frac{\epsilon |\mu| + 1}{2A^*_|\mu|} \leqslant \frac{\epsilon |\mu| + 1}{A^*_|\mu|}.$$  

\[84\]
5. Partial differential equations of infinite order in \( \mathcal{E}^{\rho(r)} \)

thus we have (18) in the case \( |\mu| < M \). For the case \( |\mu| \geq M \), remark the following

**Lemma 3** – If \( \mu \in \mathbb{N}^n \), then

\[
\sum_{\nu<\mu} \frac{\nu!}{\mu!} \leq 2^n.
\]

For the proof, we only remark that in the case where \( n = 1 \), if \( \mu = 0, 1 \), it is evident
and if \( \mu \geq 2 \), we have

\[
\sum_{\nu=0}^{\mu} \frac{\nu!}{\mu!} \leq 1 + \frac{1}{\mu} + \frac{1}{\mu(\mu - 1)} = 1 + \frac{2}{\mu} \leq 2.
\]

In the hypothesis (2), take \( \delta > 0 \) so that \( \delta < \varepsilon/2^n+1 \) and apply the Lemma: then for \( \nu = |\mu| \geq M \), we have

\[
\left| \frac{1}{c^{\mu}} \sum_{\nu<\mu} \frac{\nu!}{\mu!} c^{\nu} (\hat{T}_i^{\nu} - \hat{T}_j^{\nu}) \right| \leq \sum_{|\nu|=0}^{q-1} \frac{\nu!}{\mu!} \cdot \delta^{q-|\nu|} \cdot \frac{A^*_{|\nu|}}{A^*_{|\nu|}} \cdot \frac{\varepsilon^{q+1}}{2A^*_{|\nu|}}
\]

this and (21) for the case \( |\mu| < M \), combining with (20), imply the desired estimate (18).

\[ \square \]

**Example 2** – Let \( n = 1, \rho > 1 \). In the following cases, \( P = \sum a_\alpha(z)D^\alpha \) satisfies the condition of Theorem 3 on p. 83:

(i) with \( k > 0 \), define

\[
a_\alpha^\beta := \begin{cases} \frac{k^\alpha}{\alpha!} & \beta = \alpha \\ \frac{(-k)^{\alpha-1}}{(\alpha - 1)!} & \beta = \alpha - 1 \\ 0 & \text{otherwise} \end{cases}
\]

(ii) with \( k \neq -1 \), define

\[
a_\alpha^\beta := \begin{cases} \frac{k^\alpha}{\alpha!} & \beta = \alpha \\ \frac{(-1)^\beta}{\beta!} \left( \frac{1}{(\alpha - \beta)!} \right)^2 & \beta < \alpha \\ 0 & \text{otherwise} \end{cases}
\]
References


References


## Contents

1. Introduction .................................................. 69
2. Notations and recall ......................................... 70
3. The space $E^{\rho(r)}$ ........................................ 71
4. Duality between the spaces $E^{\rho(r)}$ and $E^{\rho(r)}_0$ ... 75
5. Partial differential equations of infinite order in $E^{\rho(r)}$ ... 79

References ......................................................... 86

Contents ............................................................ i