



Composition operators with surjective symbol and small approximation numbers

Daniel Li¹ Hervé Queffélec² Luis Rodríguez-Piazza³

Received: November 12, 2018/Accepted: January 17, 2019/Online: February 20, 2019

Abstract

We give a new proof of the existence of a surjective symbol whose associated composition operator on $H^2(\mathbb{D})$ is in all Schatten classes, with the improvement that its approximation numbers can be, in some sense, arbitrarily small. We show, as an application, that, contrary to the 1-dimensional case, for $N \geq 2$, the behavior of the approximation numbers $a_n = a_n(C_\phi)$, or rather of $\beta_N^- = \liminf_{n \rightarrow \infty} [a_n]^{1/n^{1/N}}$ or $\beta_N^+ = \limsup_{n \rightarrow \infty} [a_n]^{1/n^{1/N}}$, of composition operators on $H^2(\mathbb{D}^N)$ cannot be determined by the image of the symbol.

Keywords: Approximation numbers, cusp map, composition operator, Hardy space, lens map, polydisk.

msc: Primary 47B33; Secondary 32A35, 46B28.

1 Introduction

We start by recalling some notations and facts.

Let \mathbb{D} be the open unit disk, H^2 the Hardy space on \mathbb{D} , and $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ a non-constant analytic self-map. It is well known⁴ that φ induces a composition operator $C_\varphi: H^2 \rightarrow H^2$ by the formula:

$$C_\varphi(f) = f \circ \varphi,$$

and the connection between the “symbol” φ and the properties of the operator $C_\varphi: H^2 \rightarrow H^2$, in particular its compactness, can be further studied⁵.

¹Univ. Artois, Laboratoire de Mathématiques de Lens (LML) EA 2462, Fédération CNRS Nord-Pas-de-Calais FR 2956, Faculté Jean Perrin, Rue Jean Souvraz, S.P. 18 F-62300 LENS, France

²Université de Lille, Laboratoire Paul Painlevé U.M.R. CNRS 8524, Fédération CNRS Nord-Pas-de-Calais FR 2956 F-59655 Villeneuve D’Ascq Cedex, France

³Universidad de Sevilla, Facultad de Matemáticas, Departamento de Análisis Matemático & IMUS, Calle Tarfia s/n, 41012 Sevilla, Spain

⁴J. Shapiro, 1993, *Composition operators and classical function theory*.

⁵Ibid.

We also recall that the n th approximation number $a_n(T)$, $n = 1, 2, \dots$, of an operator $T: H_1 \rightarrow H_2$, between Hilbert spaces H_1 and H_2 , is defined as the distance of T to operators of rank $< n$, for the operator-norm:

$$a_n(T) = \inf_{\text{rank } R < n} \|T - R\|. \quad (1)$$

The p -Schatten class $S_p(H_1, H_2)$, $p > 0$ consists of all $T: H_1 \rightarrow H_2$ such that $(a_n(T))_n \in \ell^p$. The approximation numbers have the ideal property:

$$a_n(ATB) \leq \|A\| a_n(T) \|B\|.$$

Let now, for $\xi \in \mathbb{T} = \partial\mathbb{D}$ and $h > 0$, the Carleson window $S(\xi, h)$ be defined as:

$$S(\xi, h) = \{z \in \mathbb{D}; |z - \xi| \leq h\}. \quad (2)$$

For a symbol φ , we define $m_\varphi = \varphi^*(m)$ where m is the Haar measure of \mathbb{T} and $\varphi^*: \mathbb{T} \rightarrow \overline{\mathbb{D}}$ the (almost everywhere defined) radial limit function associated with φ , namely:

$$\varphi^*(\xi) = \lim_{r \rightarrow 1^-} \varphi(r\xi).$$

Finally, we set for $h > 0$:

$$\rho_\varphi(h) = \sup_{\xi \in \mathbb{T}} m_\varphi[S(\xi, h)]. \quad (3)$$

It is known⁶ that $\rho_\varphi(h) = O(h)$ and that C_φ ⁷ is compact if and only if $\rho_\varphi(h) = o(h)$ as $h \rightarrow 0$. Simpler criteria⁸ exist when φ is injective, or even p -valent, meaning that for any $w \in \mathbb{D}$, the equation $\varphi(z) = w$ has at most p solutions.

A measure μ on \mathbb{D} is called α -Carleson, $\alpha \geq 1$, if $\sup_{|\xi|=1} \mu[S(\xi, h)] = O(h^\alpha)$.

B. MacCluer and J. Shapiro showed in MacCluer and H. Shapiro (1986, Example 3.12) the following result, paradoxical at first glance.

Theorem 1 (MacCluer-Shapiro) – *There exists a surjective and four-valent symbol $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ such that the composition operator $C_\varphi: H^2 \rightarrow H^2$ is compact.*

Observe that such a symbol φ cannot be one-valent (injective), because it would be an automorphism of \mathbb{D} , and C_φ would be invertible and therefore not compact. In Lefèvre et al. (2012, Theorem 4.1), we gave the following improved statement.

⁶J. Shapiro, 1993, *Composition operators and classical function theory*.

⁷MacCluer, 1984, "Spectra of compact composition operators on $H^p(B_N)$ ".

⁸J. Shapiro, 1993, *Composition operators and classical function theory*.

2. Background and preliminary results

Theorem 2 – For every non-decreasing function $\delta: (0, 1) \rightarrow (0, 1)$, there exists a two-valent symbol and nearly surjective (i.e. $\varphi(\mathbb{D}) = \mathbb{D} \setminus \{0\}$) symbol ϕ , and $0 < h_0 < 1$, such that:

$$m(\{z \in \mathbb{T}; |\phi^*(z)| \geq 1 - h\}) \leq \delta(h) \quad \text{for } 0 < h \leq h_0. \quad (4)$$

As a consequence, there exists a surjective and four-valent symbol $\psi: \mathbb{D} \rightarrow \mathbb{D}$ such that the composition operator $C_\psi: H^2 \rightarrow H^2$ is in every Schatten class $S_p(H^2)$, $p > 0$.

Our proof was rather technical and complicated, and based on arguments of barriers and harmonic measures.

The goal of this paper is to give a more precise statement of Theorem 2 in terms of approximation numbers $a_n(C_\varphi)$, and not only in terms of Schatten classes, and with a simpler proof. We then apply this result to show that for the polydisk \mathbb{D}^N , $N \geq 2$, the nature (boundedness, compactness, asymptotic behavior of approximation numbers) of the composition operator cannot be determined by the geometry of the image $\phi(\mathbb{D}^N)$ of its symbol ϕ . For certain asymptotic behavior of approximation numbers, this is contrary to the 1-dimensional case (see Li, Queffélec, and Rodríguez-Piazza 2015, Theorem 3.1 and Theorem 3.14).

The notation $A \lesssim B$ means that $A \leq CB$ for some positive constant C , and $A \approx B$ that $A \lesssim B$ and $B \lesssim A$.

2 Background and preliminary results

We initiated the study of approximation numbers of composition operators on H^2 in Li, Queffélec, and Rodríguez-Piazza (2012), and proved the following basic results:

Theorem 3 – If φ is any symbol, then, for some $\delta > 0$ and $r > 0$, or $a > 0$:

$$a_n(C_\varphi) \geq \delta r^n = \delta e^{-an}.$$

Moreover, as soon as $\|\varphi\|_\infty = 1$, there exists some sequence ε_n tending to 0 such that:

$$a_n(C_\varphi) \geq \delta e^{-n\varepsilon_n}.$$

We also proved in Li, Queffélec, and Rodríguez-Piazza (2012, Theorem 5.1) that:

Proposition 1 – For any symbol φ , we have:

$$a_n(C_\varphi) \lesssim \inf_{0 < h < 1} \left[e^{-nh} + \sqrt{\frac{\rho_\varphi(h)}{h}} \right].$$

We also recall (see Li, Queffélec, and Rodríguez-Piazza 2012) that, for $\gamma > -1$, the weighted Bergman space \mathcal{B}_γ is the space of functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that:

$$\|f\|_\gamma^2 := \sum_{n=0}^{\infty} \frac{|a_n|^2}{(n+1)^{\gamma+1}} < \infty. \quad (5)$$

Equivalently, \mathcal{B}_γ is the space of analytic functions $f: \mathbb{D} \rightarrow \mathbb{C}$ such that:

$$\int_{\mathbb{D}} |f(z)|^2 (\gamma+1)(1-|z|^2)^\gamma dA(z) < \infty, \quad (6)$$

where dA is the normalized area measure on \mathbb{D} , and then:

$$\int_{\mathbb{D}} |f(z)|^2 (\gamma+1)(1-|z|^2)^\gamma dA(z) \approx \|f\|_\gamma^2. \quad (7)$$

The case $\gamma = 0$ corresponds to the usual Bergman space \mathcal{B}^2 , and the limiting case $\gamma = -1$ to the Hardy space H^2 . We wish to note in passing (we will make use of that elsewhere) that the proof of Theorem 5.1 in Li, Queffélec, and Rodríguez-Piazza (2012) easily gives the following result.

Proposition 2 – Let $\gamma > -1$ and φ a symbol inducing a bounded composition operator $C_\varphi: \mathcal{B}_\gamma \rightarrow H^2$. Then:

$$a_n(C_\varphi: \mathcal{B}_\gamma \rightarrow H^2) \lesssim \inf_{0 < h < 1} \left((n+1)^{(\gamma+1)/2} e^{-nh} + \sup_{0 < t \leq h} \sqrt{\frac{\rho_\varphi(t)}{t^{2+\gamma}}} \right).$$

Proof. Take $E = z^n \mathcal{B}_\gamma$; this is a subspace of \mathcal{B}_γ of codimension $\leq n$. Let $f \in E$ with $\|f\|_\gamma = 1$. Writing $f = z^n g$ with $\|g\|_\gamma^2 \leq (n+1)^{\gamma+1}$ and splitting the integral into two parts, we have, for $0 < h < 1$:

$$\|C_\varphi f\|_{H^2}^2 = \int_{\mathbb{D}} |f|^2 dm_\phi \leq (1-h)^{2n} \int_{(1-h)\mathbb{D}} |g|^2 dm_\phi + \int_{\mathbb{D} \setminus (1-h)\mathbb{D}} |f|^2 dm_\phi.$$

For the first integral, we have:

$$\int_{(1-h)\mathbb{D}} |g|^2 dm_\phi \leq \int_{\mathbb{D}} |g|^2 dm_\phi = \|C_\phi g\|_{H^2}^2 \leq \|C_\phi\|_{\mathcal{B}_\gamma \rightarrow H^2}^2 \|g\|_\gamma^2. \quad (8)$$

For the second integral, we have:

$$\int_{\mathbb{D} \setminus (1-h)\mathbb{D}} |f|^2 dm_\phi \leq \|J: \mathcal{B}_\gamma \rightarrow L^2(\mu_h)\|^2,$$

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where μ_h is the restriction of m_φ to the annulus $\{z \in \mathbb{D}; 1-h < |z| < 1\}$ and J the canonical injection of \mathcal{B}_γ into $L^2(\mu_h)$. Hence Stegenga's version of the Carleson embedding theorem for \mathcal{B}_γ (Stegenga 1980, Theorem 1.2; see Hastings 1975 for the unweighted case; see also Duren and Schuster 2004, p. 62 or Zhu 2007, p. 167) gives us:

$$\int_{\mathbb{D} \setminus (1-h)\mathbb{D}} |f|^2 dm_\phi \lesssim \sup_{0 < t \leq h} \frac{\rho_\phi(t)}{t^{2+\gamma}}. \quad (9)$$

Putting (8) and (9) together, that gives:

$$\|C_\varphi f\|_{H^2} \lesssim e^{-nh} (n+1)^{(\gamma+1)/2} + \sup_{0 < t \leq h} \sqrt{\frac{\rho_\phi(t)}{t^{2+\gamma}}}.$$

In other terms, using the Gelfand numbers c_k :

$$c_{n+1}(C_\phi: \mathcal{B}_\gamma \rightarrow H^2) \lesssim (n+1)^{(\gamma+1)/2} e^{-nh} + \sup_{0 < t \leq h} \sqrt{\frac{\rho_\phi(t)}{t^{2+\gamma}}}.$$

As $a_{n+1} = c_{n+1}$ and as we can ignore the difference between a_n and a_{n+1} , that finishes the proof. \square

As an application, we mention the following result. We refer to Li, Queffelec, and Rodríguez-Piazza (2013, Section 4.1) for the definition of the cusp map, denoted χ .

Theorem 4 – Let $\chi: \mathbb{D} \rightarrow \mathbb{D}$ be the cusp map and $\Phi: \mathbb{D}^N \rightarrow \mathbb{D}^N$ the diagonal map defined by:

$$\Phi(z_1, z_2, \dots, z_N) = (\chi(z_1), \chi(z_1), \dots, \chi(z_1)). \quad (10)$$

Then, the composition operator C_Φ maps $H^2(\mathbb{D}^N)$ to itself and:

$$a_n(C_\Phi) \lesssim e^{-d\sqrt{n}} \quad (11)$$

where d is a positive constant depending only on N .

Remark 1 – We have to compare with Bayart et al. (2018, Theorem 6.2) where, for:

$$\Psi(z_1, \dots, z_N) = (\chi(z_1), \dots, \chi(z_N)),$$

it is shown that, for constants $b \geq a > 0$ depending only on N :

$$e^{-b(n^{1/N}/\ln n)} \lesssim a_n(C_\Psi) \lesssim e^{-a(n^{1/N}/\ln n)}.$$

Note also that for $N = 1$, the estimate of Theorem 4 is very crude.

Proof (Proof of Theorem 4 on the previous page). Take $\gamma = N - 2$. As in Li, Queffélec, and Rodríguez-Piazza (n.d.[a], Section 4), we have thanks to the Cauchy-Schwarz inequality, and the fact that $\sum_{|\alpha|=n} 1 \approx (n+1)^{N-1}$, a factorization:

$$C_\Phi = JC_\chi M,$$

where $M: H^2(\mathbb{D}^N) \rightarrow \mathcal{B}_\gamma$ is defined by $Mf = g$ with:

$$g(z) = f(z, z, \dots, z) = \sum_{n=0}^{\infty} \left(\sum_{|\alpha|=n} a_\alpha \right) z^n, \quad z \in \mathbb{D}, \quad (12)$$

for

$$f(z_1, z_2, \dots, z_N) = \sum_{\alpha} a_\alpha z_1^{\alpha_1} \cdots z_N^{\alpha_N},$$

and where $J: H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D}^N)$ is the canonical injection given by:

$$(Jh)(z_1, z_2, \dots, z_N) = h(z_1). \quad (13)$$

This corresponds to a diagram:

$$H^2(\mathbb{D}^N) \xrightarrow{M} \mathcal{B}_\gamma \xrightarrow{C_\chi} H^2(\mathbb{D}) \xrightarrow{J} H^2(\mathbb{D}^N), \quad (14)$$

where $C_\chi: \mathcal{B}_\gamma = \mathcal{B}_{N-2} \rightarrow H^2(\mathbb{D})$ is a bounded operator. Indeed, we have the behavior⁹:

$$|1 - \chi^*(e^{i\theta})| \approx \frac{1}{\ln(1/|\theta|)},$$

and this implies, with c an absolute constant:

$$\begin{aligned} m_\chi[S(\xi, h)] &\lesssim m_\chi[S(1, h)] = m(\{|\chi^*(e^{i\theta}) - 1| < h\}) \\ &\lesssim m(\{c/\ln(1/|\theta|) < h\}) \leq e^{-c/h}; \end{aligned} \quad (15)$$

in particular $\rho_\chi(h) \leq e^{-c/h} = O(h^N)$, so m_χ is an N -Carleson measure and the Stegenga-Carleson theorem¹⁰ says that the operator $C_\chi: \mathcal{B}_{N-2} \rightarrow H^2(\mathbb{D})$ is bounded.

Now Proposition 2 on p. 4 with (15) give:

$$a_n(C_\chi: \mathcal{B}_\gamma \rightarrow H^2) \lesssim \inf_{0 < h < 1} \left[(n+1)^{(N-1)/2} e^{-nh} + e^{-c/h} h^{-N/2} \right].$$

Adjusting $h = 1/\sqrt{n}$, we get $a_n(C_\chi: \mathcal{B}_\gamma \rightarrow H^2) \lesssim e^{-d\sqrt{n}}$ for some positive constant d . Finally, the factorization $C_\Phi = JC_\chi M$ and the ideal property of approximation numbers give the result. \square

2. Background and preliminary results

In the case of lens maps, Proposition 2 on p. 4 gives very poor estimates. We avoid using this theorem in Li, Queffélec, and Rodríguez-Piazza (n.d.[a], Section 4), when $N = 2$, using the semi-group property of those lens maps. The same proof gives for arbitrary $N \geq 2$ the following result.

Theorem 5 – Let λ_θ the lens map with parameter θ , $0 < \theta < 1$, and let $\Phi: \mathbb{D}^N \rightarrow \mathbb{D}^N$ be the diagonal map defined by:

$$\Phi(z_1, z_2, \dots, z_N) = \left(\lambda_\theta(z_1), \lambda_\theta(z_1), \dots, \lambda_\theta(z_1) \right). \quad (16)$$

Then:

- 1) if $\theta > 1/N$, C_Φ is unbounded on $H^2(\mathbb{D}^N)$;
- 2) if $\theta = 1/N$, C_Φ is bounded and not compact on $H^2(\mathbb{D}^N)$;
- 3) if $\theta < 1/N$, C_Φ is compact on $H^2(\mathbb{D}^N)$ and moreover:

$$a_n(C_\Phi) \lesssim e^{-d\sqrt{n}} \quad (17)$$

for a constant $d > 0$ depending only on θ and N .

Remark 2 – In Bayart et al. (2018, Theorem 6.1), it is shown that, for:

$$\Psi(z_1, \dots, z_N) = \left(\lambda_\theta(z_1), \dots, \lambda_\theta(z_N) \right),$$

we have, for constants $b \geq a > 0$, depending only on θ and N :

$$e^{-bn^{1/(2N)}} \lesssim a_n(C_\Psi) \lesssim e^{-an^{1/(2N)}}.$$

Proof (Proof of Theorem 5). That had been proved, for $N = 2$ in Li, Queffélec, and Rodríguez-Piazza (n.d.[a], Theorem 4.2 and Theorem 4.4). For convenience of the reader, we sketch the proof.

Assume first $\theta \leq 1/N$, and write $\lambda_\theta = \lambda_{N\theta} \circ \lambda_{1/N}$, where we set, for convenience, $\lambda_1(z) = z$, so $C_{\lambda_1} = Id$. As in the proof of Theorem 4 on p. 5 (see Li, Queffélec, and Rodríguez-Piazza n.d.(a), Section 4), we have a factorization:

$$C_\Phi = JC_{\lambda_{N\theta}}C_{\lambda_{1/N}}M,$$

where M and J are defined in (12) and (13).

⁹Li, Queffélec, and Rodríguez-Piazza, 2013, “Estimates for approximation numbers of some classes of composition operators on the Hardy space”, Lemma 4.2.

¹⁰Stegenga, 1980, “Multipliers of the Dirichlet space”, Theorem 1.2.

This corresponds to a diagram (recall that $\gamma = N - 2$):

$$H^2(\mathbb{D}^N) \xrightarrow{M} \mathcal{B}_\gamma \xrightarrow{C_{\lambda_{1/N}}} H^2(\mathbb{D}) \xrightarrow{C_{\lambda_{N\theta}}} H^2(\mathbb{D}) \xrightarrow{J} H^2(\mathbb{D}^N).$$

The second arrow is bounded, since we know¹¹ that the pullback measure $m_{\lambda_{1/N}}$ is N -Carleson, so that $C_{\lambda_{1/N}}$ maps \mathcal{B}_{N-2} to $H^2(\mathbb{D})$ by the Stegenga–Carleson embedding theorem¹².

For $\theta < 1/N$, we have $N\theta < 1$ and $C_{\lambda_{N\theta}}$ is compact and, for some constant $b = b(\theta)$, we have $a_n(C_{\lambda_{N\theta}}) \lesssim e^{-b\sqrt{n}}$ ¹³. Hence C_Φ is compact and $a_n(C_\Phi) \lesssim e^{-b\sqrt{n}}$.

Now, for $\theta \geq 1/N$, we consider the reproducing kernels:

$$K_{a_1, \dots, a_N}(z_1, \dots, z_N) = \prod_{j=1}^N \frac{1}{1 - \bar{a}_j z_j}.$$

We have:

$$\|K_{a_1, \dots, a_N}\|^2 = \prod_{j=1}^N \frac{1}{1 - |a_j|^2}$$

and:

$$C_\Phi^*(K_{a_1, \dots, a_N}) = K_{\lambda_\theta(a_1), \dots, \lambda_\theta(a_N)},$$

so:

$$\|C_\Phi^*(K_{a_1, \dots, a_N})\|^2 = \left(\frac{1}{1 - |\lambda_\theta(a_1)|^2} \right)^N.$$

Since:

$$1 - |\lambda_\theta(a_1)|^2 \approx 1 - |\lambda_\theta(a_1)| \approx (1 - |a_1|)^\theta,$$

we see that $\|C_\Phi^*(K_{a_1, \dots, a_N})\|/\|K_{a_1, \dots, a_N}\|$ is not bounded for $\theta > 1/N$, so C_Φ is then not bounded; and it does not converge to 0 for $\theta = 1/N$, so C_Φ is then not compact. \square

¹¹Lefèvre et al., 2013b, “Some new properties of composition operators associated with lens maps”, Lemma 3.3.

¹²Stegenga, 1980, “Multipliers of the Dirichlet space”, Theorem 1.2.

¹³Lefèvre et al., 2013b, “Some new properties of composition operators associated with lens maps”, Theorem 2.1.

3. Surjectivity

3 Surjectivity

Let us come back to our surjectivity issues.

Let us first remark that Theorem 2 on p. 3 gives the following result.

Theorem 6 – *For every non-decreasing function $\delta: (0, 1) \rightarrow (0, 1)$, there exists a surjective and four-valent symbol ψ , and $0 < h_0 < 1$, such that, for $0 < h \leq h_0$:*

$$m(\{z \in \mathbb{T}; |\phi^*(z)| \geq 1 - h\}) \leq \delta(h). \quad (18)$$

Proof. Just observe that the passage from “ φ two-valent and nearly surjective” to “ ψ four-valent and surjective” is harmless: for this, consider the Blaschke product:

$$B(z) = \left(\frac{z - a}{1 - az} \right)^2,$$

where $0 < a < 1$, and take $\psi = B \circ \varphi$; we observe that $B(\mathbb{D} \setminus \{0\}) = \mathbb{D}$ since $a^2 = B\left(\frac{2a}{1+a^2}\right)$, and, for $z \in \mathbb{D}$:

$$\frac{1 - |B(z)|}{1 - |z|} \geq \frac{1 - \left| \frac{z-a}{1-az} \right|^2}{1 - |z|^2} = \frac{1 - a^2}{|1 - az|^2} \geq \frac{1 - a^2}{4},$$

so that:

$$m(|\psi^*| > 1 - h) = m(1 - |B \circ \varphi^*| < h) \leq m(1 - |\varphi^*| \leq \kappa_a h),$$

with $\kappa_a = 4/(1 - a^2)$. Hence, this map ψ is surjective, four-valent, and satisfies (18), as well, up to a change of $\delta(h)$ to $\delta(h/\kappa_a)$ for φ at the beginning. \square

3.1 A more precise statement

Our new statement is as follows.

Theorem 7 – *For every positive sequence $(\varepsilon_n)_n$ with limit 0, there exists a surjective and four-valent symbol φ such that:*

$$a_n(C_\varphi) \lesssim e^{-n\varepsilon_n}.$$

Consequently, there exists a surjective and four-valent symbol $\varphi: \mathbb{D} \rightarrow \mathbb{D}$ such that the composition operator $C_\varphi: H^2 \rightarrow H^2$ is in every Schatten class $S_p(H^2)$, $p > 0$.

Proof. Observe first that $\|\varphi\|_\infty = 1$ when φ is surjective, so that, in view of Theorem 3 on p. 3, we cannot dispense with the numbers ε_n , even if they can tend to 0 arbitrarily slowly.

Now, we can choose $\delta: (0, 1) \rightarrow (0, 1)$ non-decreasing such that $\delta(\varepsilon_n) \leq e^{-n\varepsilon_n}$ for all n , and then, using Theorem 6 on the previous page, we get a surjective and four-valent symbol φ , satisfying for all h small enough:

$$\rho_\varphi(h) \leq h\delta^2(h).$$

Proposition 1 on p. 3 gives:

$$a_n(C_\varphi) \lesssim \inf_{0 < h < 1} \left[e^{-nh} + \delta(h) \right].$$

Adjusting $h = \varepsilon_n$, we get $a_n(C_\varphi) \lesssim e^{-n\varepsilon_n}$.

To get the second part of the theorem, just take $\varepsilon_n = n^{-1/2}$. \square

3.2 A simplified proof of Theorem 2 on p. 3

We give here the announced simplified proof of Theorem 2 on p. 3. This proof is based on the following key lemma, in which $\mathcal{H}(\mathbb{D})$ denotes the set of holomorphic functions on \mathbb{D} .

Lemma 1 (Lefèvre et al. 2013a, Lemma 2.11) – *There exists a numerical constant C such that, if $f \in \mathcal{H}(\mathbb{D})$ satisfies, for some $\alpha \in \mathbb{R}$:*

$$\begin{cases} \operatorname{Im}[f(0)] < \alpha \\ f(\mathbb{D}) \subseteq \{z \in \mathbb{C}; 0 < \Re z < \pi\} \cup \{z \in \mathbb{C}; \operatorname{Im} z < \alpha\}, \end{cases}$$

then:

$$m(\{\operatorname{Im} f^* > y\}) \leq C e^{\alpha - y}, \quad \text{for } y \geq \alpha.$$

Proof (Proof of Theorem 2 on p. 3). Let $g: (0, \infty) \rightarrow (0, \infty)$ be a continuous decreasing function such that:

$$\lim_{t \rightarrow 0^+} g(t) = +\infty, \quad g(\pi) = \pi, \quad \lim_{t \rightarrow +\infty} g(t) = 0.$$

Then let Ω be the simply connected region defined by:

$$\Omega = \{x + iy; x > 0, g(x) < y < g(x) + 4\pi\},$$

and $f: \mathbb{D} \rightarrow \Omega$ be a Riemann map such that $f(0) = \pi + 3i\pi$. Observe that we can apply Lemma 1 to f with $\alpha = 5\pi$ since $\operatorname{Im} f(0) = 3\pi$ and if $f(z) = x + iy$ with $x \geq \pi$; hence:

$$\operatorname{Im} f(z) = y < g(x) + 4\pi \leq g(\pi) + 4\pi = 5\pi.$$

3. Surjectivity

Finally, consider the symbol $\varphi = e^{-f}$. It is nearly surjective: $\phi(\mathbb{D}) = \mathbb{D} \setminus \{0\}$, and two-valent, as easily checked.

For $0 < h \leq 1/2$, we have for $\xi \in \mathbb{T}$ and $|\phi^*(\xi)| > 1 - h$:

$$e^{-2h} \leq 1 - h < |\phi^*(\xi)| = \exp\left(-\Re f^*(\xi)\right);$$

hence $\Re f^*(\xi) < 2h$.

But if $2h > x = \Re f^*(\xi)$, we have $g(x) > g(2h)$. As $f^*(\xi) = x + iy \in \overline{\Omega}$, we get $\Im f^*(\xi) = y \geq g(x) > g(2h)$. Lemma 1 on the preceding page now gives:

$$m(\{\xi; |\varphi^*(\xi)| > 1 - h\}) \leq m(\{\xi; \Im f^*(\xi) > g(2h)\}) \leq C e^{5\pi - g(2h)}. \quad (19)$$

It is now enough to adjust g so as to have $e^{g(t)} \geq C e^{5\pi}/\delta(t/2)$ for t small enough to get (4) from (19). \square

For sake of completeness, we give the proof of Lemma 1 on the preceding page.

Proof (Proof of Lemma 1 on the preceding page). We now prove Lemma 1 on the preceding page. If $e^{y-\alpha} < 2$, there is nothing to prove, since then:

$$m(\Im f^* > y) \leq 1 \leq 2e^{\alpha-y}.$$

We can hence assume that $e^{y-\alpha} \geq 2$. First, we make a comment. If the Riemann mapping theorem is very general and flexible, it gives very few informations on the parametrization $t \mapsto f^*(e^{it})$ when $f: \mathbb{D} \rightarrow \Omega$ is a conformal map, except in some specific cases (lens maps, cusps, etc.: see Li, Queffélec, and Rodríguez-Piazza 2013). Here, the Kolmogorov weak type inequality provides a substitute. Write:

$$f = u + iv$$

and set:

$$f_1 = -if + i\frac{\pi}{2} - \alpha = v - \alpha + i\left(\frac{\pi}{2} - u\right)$$

and:

$$F_1 = 1 + e^{f_1} = (1 + e^{v-\alpha} \sin u) + ie^{v-\alpha} \cos u.$$

If $v < \alpha$, then $\Re F_1 > 1 - |\sin u| \geq 0$. If $v \geq \alpha$, then $0 < u < \pi$ and $\Re F_1 \geq 1$. Hence F_1 maps \mathbb{D} to the right half-plane $\mathbb{C}_0 = \{z; \Re z > 0\}$. Finally, let $F = U + iV: \mathbb{D} \rightarrow \mathbb{C}_0$ be defined by:

$$F = F_1 - i\Im F_1(0),$$

so that $V(0) = 0$. By the Kolmogorov inequality for the conjugation map $U \mapsto V$, and the harmonicity of U , we have, for all $\lambda > 0$ (a designating an absolute constant):

$$m(|F^*| > \lambda) \leq \frac{a}{\lambda} \|U^*\|_1 = \frac{a}{\lambda} \int_{\mathbb{T}} U^* dm = \frac{a}{\lambda} U(0). \quad (20)$$

Next, we claim that:

$$|\operatorname{Im} F_1(0)| < 1 \quad \text{and} \quad U(0) < 2. \quad (21)$$

Indeed, $v(0) < \alpha$ by hypothesis, so that $|\operatorname{Im} F_1(0)| = e^{v(0)-\alpha} |\cos u(0)| < 1$, and $U(0) = 1 + e^{v(0)-\alpha} \sin u(0) < 2$. Suppose now that, for some $y > \alpha$ and $z \in \mathbb{D}$, we have $v(z) > y$. Then, $0 < u(z) < \pi$ by our second assumption, and this implies $\Re e^{f_1(z)} = e^{v(z)-\alpha} \sin u(z) > 0$, so that, using $|1+w| \geq |w|$ if $\Re w > 0$ and (21), and remembering that $e^{y-\alpha} \geq 2$:

$$\begin{aligned} |F(z)| &= \left| 1 + e^{f_1(z)} - i \operatorname{Im} F_1(0) \right| \geq \left| 1 + e^{f_1(z)} \right| - 1 \\ &\geq \left| e^{f_1(z)} \right| - 1 = e^{v(z)-\alpha} - 1 > e^{y-\alpha} - 1 \geq \frac{1}{2} e^{y-\alpha}. \end{aligned}$$

Taking radial limits and using (20) and (21), we get:

$$m(\operatorname{Im} f^* > y) \leq m(|F^*| > e^{y-\alpha}/2) \leq 4a e^{\alpha-y}.$$

This ends the proof of Lemma 1 on p. 10 with $C = \max(2, 4a)$. \square

4 Application to the multidimensional case

In this section, we apply Theorem 6 on p. 9 and Theorem 7 on p. 9 to show that, for $N \geq 2$, the image of the symbol cannot determine the behavior of the approximation numbers, or rather of $\beta_N(C_\phi)$, of the associated composition operator $C_\phi: H^2(\mathbb{D}^N) \rightarrow H^2(\mathbb{D}^N)$.

Recall that for an operator $T: H_1 \rightarrow H_2$, we set:

$$\beta_N^-(T) = \liminf_{n \rightarrow \infty} [a_n(T)]^{1/n^{1/N}} \quad \text{and} \quad \beta_N^+(T) = \limsup_{n \rightarrow \infty} [a_n(T)]^{1/n^{1/N}}, \quad (22)$$

and write $\beta_N(T)$ when $\beta_N^-(T) = \beta_N^+(T)$.

Theorem 8 – For $N \geq 2$, there exist pairs of symbols $\Phi_1, \Phi_2: \mathbb{D}^N \rightarrow \mathbb{D}^N$, such that $\Phi_1(\mathbb{D}^N) = \Phi_2(\mathbb{D}^N)$ and:

- 1) C_{Φ_1} is not bounded, but C_{Φ_2} is compact, and even $\beta_N(C_{\Phi_2}) = 0$;
- 2) C_{Φ_1} is bounded but not compact, so $\beta_N(C_{\Phi_1}) = 1$, and C_{Φ_2} is compact, with $\beta_N(C_{\Phi_2}) = 0$;

4. Application to the multidimensional case

- 3) C_{Φ_1} is compact, with $\beta_N^-(C_{\Phi_1}) > 0$ and $\beta_N^+(C_{\Phi_1}) < 1$, and C_{Φ_2} is compact, with $\beta_N(C_{\Phi_2}) = 0$;
- 4) C_{Φ_1} is compact, with $\beta_N(C_{\Phi_1}) = 1$, and C_{Φ_2} is compact, but with $\beta_N(C_{\Phi_2}) = 0$.

Proof. Let $\sigma: \mathbb{D} \rightarrow \mathbb{D}$ be a surjective symbol such that $\rho_\sigma(h) \leq h^N e^{-2/h^2}$ given by Theorem 6 on p. 9. By Proposition 2 on p. 4, we have, with $\gamma = N - 2$:

$$a_n(C_\sigma: \mathcal{B}_\gamma \rightarrow H^2) \lesssim \inf_{0 < h < 1} (n^{(N-1)/2} e^{-nh} + e^{-1/h^2}),$$

and, with $h = 1/n^{1/3}$, we get $a_n(C_\sigma: \mathcal{B}_\gamma \rightarrow H^2) \lesssim e^{-dn^{2/3}}$.

We choose the exponent $2/3$ for fixing the ideas, but every exponent $\alpha > 1/2$, with $\alpha < 1$, (i.e. $a_n(C_\sigma: \mathcal{B}_\gamma \rightarrow H^2) \lesssim e^{-dn^\alpha}$) would be suitable.

1) We take $\Phi_1(z_1, z_2, z_3, \dots, z_N) = (z_1, z_1, \dots, z_1)$. The composition operator C_{Φ_1} is not bounded because if $f_n(z_1, \dots, z_N) = \left(\frac{z_1 + z_2}{2}\right)^n$, then $\|f_n\|_2^2 = 4^{-n} \sum_{k=0}^n \binom{n}{k}^2 = 4^{-n} \binom{2n}{n} \approx 1/\sqrt{n}$, though $(C_{\Phi_1} f_n)(z_1, \dots, z_N) = z_1^n$ and $\|C_{\Phi_1} f_n\|_2 = 1$.

We define Φ_2 by:

$$\Phi_2(z_1, z_2, \dots, z_N) = (\sigma(z_1), \sigma(z_1), \dots, \sigma(z_1)).$$

Since σ is surjective, we have $\Phi_2(\mathbb{D}^N) = \Phi_1(\mathbb{D}^N)$. Now, as in the proof of Theorem 4 on p. 5, we have $C_{\Phi_2} = JC_\sigma M$, so:

$$a_n(C_{\Phi_2}) \leq a_n(C_\sigma: \mathcal{B}_{N-2} \rightarrow H^2) \lesssim e^{-dn^{2/3}},$$

by the ideal property. Hence $[a_n(C_{\Phi_2})]^{1/n^{1/N}} \lesssim e^{-dn^{\frac{2}{3} - \frac{1}{N}}}$ and therefore $\beta_N(C_{\Phi_2}) = 0$ since $\frac{2}{3} - \frac{1}{N} > 0$.

2) We consider the lens map $\lambda = \lambda_{1/N}$ of parameter $1/N$. We define:

$$\begin{cases} \Phi_1(z_1, \dots, z_N) = (\lambda(z_1), \lambda(z_1), \dots, \lambda(z_1)) \\ \Phi_2(z_1, \dots, z_N) = (\lambda[\sigma(z_1)], \lambda[\sigma(z_1)], \dots, \lambda[\sigma(z_1)]) \end{cases}$$

Since σ is surjective, we have $\Phi_1(\mathbb{D}^N) = \Phi_2(\mathbb{D}^N)$ and we saw in Theorem 5 on p. 7 that C_{Φ_1} is bounded but not compact.

On the other hand, we have the factorization $C_{\Phi_2} = JC_\sigma C_\lambda M$. Hence C_{Φ_2} is compact, and, as in 1), $\beta_N(C_{\Phi_2}) = 0$.

3) For this item, the map σ does not suffice, and we will use another surjective symbol $s: \mathbb{D} \rightarrow \mathbb{D}$. By Theorem 6 on p. 9, there exists such a map s with:

$$\rho_s(t) \leq t^2 e^{-2/t^2} \tag{23}$$

and

$$\rho_s(t) \leq t \delta^2(t) \quad (24)$$

for t small enough, where $\delta: (0, 1) \rightarrow (0, 1)$ is a non-decreasing function such that $\delta(\varepsilon_n) \leq e^{-n\varepsilon_n}$ and:

$$\varepsilon_n = n^{-\frac{1}{4N-7}}. \quad (25)$$

By the proof of Theorem 7 on p. 9, (24) implies that:

$$a_n(C_s) \leq e^{-n\varepsilon_n}. \quad (26)$$

We also consider a lens map $\lambda = \lambda_\theta$, with parameter $\theta < 1/N$, and we set:

$$\begin{cases} \Phi_1(z_1, \dots, z_N) = \left(\lambda(z_1), \lambda(z_1), \frac{z_3}{2}, \dots, \frac{z_N}{2} \right) \\ \Phi_2(z_1, \dots, z_N) = \left(\lambda[s(z_1)], \lambda[s(z_1)], \frac{s(z_3)}{2}, \dots, \frac{s(z_N)}{2} \right). \end{cases}$$

Since s is surjective, we have $\Phi_1(\mathbb{D}^N) = \Phi_2(\mathbb{D}^N)$.

a) Let us prove that $\beta_N^-(C_{\Phi_1}) > 0$ and $\beta_N^+(C_{\Phi_1}) < 1$.

Note that:

$$C_{\Phi_1} = C_u \otimes C_{v_3} \otimes \dots \otimes C_{v_N},$$

where $u: \mathbb{D}^2 \rightarrow \mathbb{D}^2$ is defined by $u(z_1, z_2) = (\lambda(z_1), \lambda(z_1))$ and $v_j: \mathbb{D} \rightarrow \mathbb{D}$ is defined by $v_j(z_j) = z_j/2$. In fact, if $f \in H^2(\mathbb{D}^2)$ and $g_j \in H^2(\mathbb{D})$, $3 \leq j \leq N$, we have:

$$\begin{aligned} & [C_{\Phi_1}(f \otimes g_3 \otimes \dots \otimes g_N)](z_1, z_2, z_3, \dots, z_N) \\ &= (f \otimes g_3 \otimes \dots \otimes g_N)(u(z_1, z_2), v_3(z_3), \dots, v_N(z_N)) \\ &= f[\lambda(z_1), \lambda(z_1)] g_3[v_3(z_3)] \dots g_N[v_N(z_N)] \\ &= (C_u f)(z_1, z_2) (C_{v_3} g_3)(z_3) \dots (C_{v_N} g_N)(z_N) \\ &= [(C_u \otimes C_{v_3} \otimes \dots \otimes C_{v_N})(f \otimes g_3 \otimes \dots \otimes g_N)](z_1, z_2, z_3, \dots, z_N), \end{aligned}$$

hence the result since $H^2(\mathbb{D}^2) \otimes H^2(\mathbb{D}) \otimes \dots \otimes H^2(\mathbb{D})$ is dense in $H^2(\mathbb{D}^N)$. That proves in particular that C_{Φ_1} is compact since C_u and C_{v_3}, \dots, C_{v_N} are (by Theorem 5 on p. 7 for C_u).

By the supermultiplicativity of singular numbers of tensor products (see Li, Queffélec, and Rodríguez-Piazza n.d.(a), Lemma 3.2), it ensues that:

$$a_{nN}(C_{\Phi_1}) \geq a_{n^2}(C_u) \prod_{j=3}^N a_n(C_{v_j}) = a_{n^2}(C_u) \left(\frac{1}{2}\right)^{n(N-2)}.$$

4. Application to the multidimensional case

By Li, Queffélec, and Rodríguez-Piazza (n.d.[a], Remark at the end of Section 4), we have $a_{n^2}(C_u) \gtrsim e^{-bn}$ for some positive constant $b = b(\theta)$. Indeed, if $J = J_2: H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D}^2)$ is the canonical injection defined by $(Jh)(z_1, z_2) = h(z_1)$ and $Q: H^2(\mathbb{D}^2) \rightarrow H^2(\mathbb{D})$ is defined by $(Qf)(z_1) = f(z_1, 0)$, we have $C_\lambda = QC_uJ$. Hence $a_k(C_u) \gtrsim a_k(C_\lambda) \gtrsim e^{-b\sqrt{k}}$.

Therefore we get:

$$a_{n^N}(C_{\Phi_1}) \gtrsim e^{-cn}$$

for some positive constant depending only on θ and N . It follows that $\beta_N^-(C_{\Phi_1}) > 0$.

To see that $\beta_N^+(C_{\Phi_1}) < 1$, we need the following lemma, whose proof is postponed.

Lemma 2 – *Let $S: H_1 \rightarrow H_1$ and $T: H_2 \rightarrow H_2$ be two operators between Hilbert spaces and A, B a pair of positive numbers. Then, whenever:*

$$a_{[n^A]}(S) \leq e^{-cn} \quad \text{and} \quad a_{[n^B]}(T) \leq e^{-cn},$$

where $[\cdot]$ stands for the integer part, we have, for some constant integer $M = M(A, B) > 0$:

$$a_{M[n^{A+B}]}(S \otimes T) \leq e^{-cn}.$$

Let $S = C_u$ and $T = C_{v_3} \otimes \dots \otimes C_{v_N}$. For c small enough, we have $a_{n^{N-2}}(T) \leq C(1/2)^n \leq e^{-cn}$ and, using (17), $a_{n^2}(S) \leq e^{-dn} \leq e^{-cn}$. Hence, with $A = 2$, $B = N - 2$, Lemma 2 gives:

$$a_{Mn^N}(C_{\Phi_1}) \lesssim e^{-cn}.$$

Therefore $\beta_N^+(C_{\Phi_1}) \leq e^{-c/M^{1/N}} < 1$.

b) Define $\Psi: \mathbb{D}^N \rightarrow \mathbb{D}^N$ by:

$$\Psi(z_1, z_2, z_3, \dots, z_N) = (s(z_1), s(z_1), s(z_3), \dots, s(z_N)).$$

If $\tau_1: \mathbb{D}^2 \rightarrow \mathbb{D}^2$ is defined by $\tau_1(z_1, z_2) = (s(z_1), s(z_1))$ and the map $\tau_2: \mathbb{D}^{N-2} \rightarrow \mathbb{D}^{N-2}$ by $\tau_2(z_3, \dots, z_N) = (s(z_3), \dots, s(z_N))$, we have:

$$C_\Psi = C_{\tau_1} \otimes C_{\tau_2}.$$

As in the proof of Theorem 4 on p. 5, we have the factorization:

$$\tau_1: H^2(\mathbb{D}^2) \xrightarrow{M} \mathcal{B}_0 = \mathcal{B}^2 \xrightarrow{C_s} H^2(\mathbb{D}) \xrightarrow{J} H^2(\mathbb{D}^2).$$

Hence $a_n(C_{\tau_1}) \leq \|M\| \|J\| a_n(C_s: \mathcal{B}^2 \rightarrow H^2)$.

By Proposition 2 on p. 4, we have:

$$a_n(C_s: \mathcal{B}^2 \rightarrow H^2) \lesssim \inf_{0 < h < 1} \left(\sqrt{n} e^{-nh} + \sup_{0 < t \leq h} \sqrt{\frac{\rho_s(t)}{t^2}} \right);$$

so (23) implies that $a_n(C_s: \mathcal{B}^2 \rightarrow H^2) \lesssim \inf_{0 < h < 1} (\sqrt{n} e^{-nh} + e^{-1/h^2})$ and, taking $h = n^{-1/3}$, we get, with some c small enough:

$$a_n(C_s: \mathcal{B}^2 \rightarrow H^2) \lesssim e^{-cn^{2/3}}.$$

It follows that $a_n(C_{\tau_1}) \lesssim e^{-cn^{2/3}}$ and hence:

$$a_{\lfloor n^{3/2} \rfloor}(C_{\tau_1}) \lesssim e^{-cn}. \quad (27)$$

On the other hand, Bayart et al. (2018, Theorem 5.5) says that:

$$a_n(C_{\tau_2}) \leq 2^{N-3} \|C_s\|^{N-2} \inf_{n_3 \dots n_N \leq n} (a_{n_3}(C_s) + \dots + a_{n_N}(C_s)).$$

Taking $n_3 = \dots = n_N = n^{\frac{1}{N-2}}$, we get, using (26):

$$a_n(C_{\tau_2}) \leq K^N N \exp\left(-n^{\frac{1}{N-2}} \varepsilon_{n^{\frac{1}{N-2}}}\right).$$

Using (25), that gives:

$$a_n(C_{\tau_2}) \lesssim \exp\left(-n^{\frac{1}{N-2}(1-\frac{1}{4N-7})}\right) = \exp\left(-n^{\frac{4}{4N-7}}\right),$$

or:

$$a_{\lfloor n^{N-\frac{7}{4}} \rfloor}(C_{\tau_2}) \lesssim e^{-n} \leq e^{-cn}. \quad (28)$$

Now, (27) and (28) allow to use Lemma 2 on the previous page with $A = 3/2$ and $B = N - 7/4$, and we get:

$$a_M \left[\lfloor n^{N-\frac{1}{4}} \rfloor \right](C_\Psi) \lesssim e^{-cn}.$$

Equivalently:

$$a_k(C_\Psi) \lesssim \exp\left(-c'k^{\frac{4}{4N-1}}\right)$$

and:

$$\left(a_k(C_\Psi)\right)^{1/k^{1/N}} \lesssim \exp\left(-c'k^{\frac{4}{4N-1}-\frac{1}{N}}\right) = \exp\left(-c'k^{\frac{1}{N(4N-1)}}\right),$$

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which gives $\beta_N(C_\Psi) = 0$.

To end the proof, it suffices to remark that $C_{\Phi_2} = C_\Psi \circ C_{\Phi_1}$, since $\Phi_2 = \Phi_1 \circ \Psi$, and hence $\beta_N^+(C_{\Phi_2}) \leq \beta_N^+(C_\Psi) = 0$, so $\beta_N(C_{\Phi_2}) = 0$.

4) We use a Shapiro-Taylor map. This one-parameter map ς_θ , $\theta > 0$, was introduced by J. Shapiro and P. Taylor in 1973¹⁴ and was further studied, with a slightly different definition, in Lefèvre et al. (2008, Section 5). J. Shapiro and P. Taylor proved that $C_{\varsigma_\theta}: H^2 \rightarrow H^2$ is always compact, but is Hilbert-Schmidt if and only if $\theta > 2$. Let us recall their definition.

For $0 < \varepsilon < 1$, we set $V_\varepsilon = \{z \in \mathbb{C}; \Re z > 0 \text{ and } |z| < \varepsilon\}$. For $\varepsilon = \varepsilon_\theta > 0$ small enough, one can define:

$$f_\theta(z) = z(-\ln z)^\theta,$$

for $z \in V_\varepsilon$, where $\ln z$ will be the principal determination of the logarithm. Let now g_θ be the conformal mapping from \mathbb{D} onto V_ε , which maps $\mathbb{T} = \partial\mathbb{D}$ onto ∂V_ε , defined by $g_\theta(z) = \varepsilon \phi_0(z)$, where ϕ_0 is given by:

$$\phi_0(z) = \frac{\left(\frac{z-i}{iz-1}\right)^{1/2} - i}{-i\left(\frac{z-i}{iz-1}\right)^{1/2} + 1}.$$

Then, we define:

$$\varsigma_\theta = \exp(-f_\theta \circ g_\theta).$$

We proved in Li, Queffélec, and Rodríguez-Piazza (2013, Section 4.2) (though it is not sharp) that:

$$a_n(C_{\varsigma_\theta}) \gtrsim \frac{1}{n^{\theta/2}}. \quad (29)$$

We define $\Phi_1: \mathbb{D}^N \rightarrow \mathbb{D}^N$ as:

$$\Phi_1(z_1, z_2, \dots, z_N) = (\varsigma_\theta(z_1), 0, \dots, 0). \quad (30)$$

If $J = J_N: H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D}^N)$ is the canonical injection defined by $(Jh)(z_1, \dots, z_N) = h(z_1)$ and $Q = Q_N: H^2(\mathbb{D}^N) \rightarrow H^2(\mathbb{D})$ is defined by $(Qf)(z_1) = f(z_1, 0, \dots, 0)$, then $C_{\Phi_1} = JC_{\varsigma_\theta}Q$; hence C_{Φ_1} is compact. On the other hand, we also have $QC_{\Phi_1}J = C_{\varsigma_\theta}$, which implies that $a_n(C_{\Phi_1}) \gtrsim a_n(C_{\varsigma_\theta}) \gtrsim n^{-\theta/2}$. It follows that:

$$\beta_N(C_{\Phi_1}) \geq \lim_{n \rightarrow \infty} (n^{-\theta/2})^{1/n^{1/N}} = 1,$$

and hence $\beta_N(C_{\Phi_1}) = 1$.

Now, if:

$$\Phi_2(z_1, \dots, z_N) = (\zeta_\theta[\sigma(z_1)], 0, \dots, 0),$$

since σ is surjective, we have $\Phi_1(\mathbb{D}^N) = \Phi_2(\mathbb{D}^N)$. Moreover, we have $C_{\Phi_2} = JC_{\zeta_\theta \circ \sigma} Q = JC_\sigma C_{\zeta_\theta} Q$, so $a_n(C_{\Phi_2}) \lesssim a_n(C_\sigma)$. Since $\rho_\sigma(h) \leq h^{N+1} e^{-2/h^2}$, Proposition 1 on p. 3 gives, with $h = 1/n^{1/3}$:

$$a_n(C_\sigma) \lesssim e^{-cn^{2/3}},$$

so $[a_n(C_{\Phi_2})]^{1/n^{1/N}} \lesssim \exp(-cn^{\frac{2}{3} - \frac{1}{N}})$ and $\beta_N(C_{\Phi_2}) = 0$. \square

Proof (Proof of Lemma 2 on p. 15). In Li, Queffélec, and Rodríguez-Piazza (n.d.[a]), we observed that the singular numbers of $S \otimes T$ are the non-increasing rearrangement of the numbers $s_j t_k$, where s_j and t_k denote respectively the j -th and the k -th singular number of S and T . We can assume $s_1 = t_1 = 1$. Using this observation, we will majorize the number of pairs (j, k) such that $s_j t_k > e^{-cn}$. Let (j, k) be such a pair. Since $s_j \leq s_1 = 1$, we have $t_k \geq e^{-cn}$ so that $k \leq [n^B] \leq n^B$. Hence, for some $2 \leq l \leq n$, we have $(l-1)^B < k \leq l^B$. Then, due to the assumption on T , $t_k < e^{-c(l-1)}$ and $s_j \geq e^{-cn} t_k^{-1} \gtrsim e^{-c(n-l+1)}$, implying that $j \lesssim (n-l+1)^A$, thanks to the assumption on S . As a consequence, since the number of integers k such that $(l-1)^B < k \leq l^B$ is dominated by l^{B-1} , the number ν_n of pairs (j, k) such that $s_j t_k > e^{-cn}$ is dominated by:

$$\sum_{l=1}^n (n-l+1)^A l^{B-1} \sim n^{A+B} \int_0^1 t^A (1-t)^B dt,$$

by a Riemann sum argument. Next, let $M \in \mathbb{N}$ big enough to have:

$$\sum_{l=1}^n (n-l+1)^A l^{B-1} \leq M n^{A+B} - 1, \quad \text{for all } n.$$

By definition, $a_{M[n^{A+B}]}(S \otimes T) \leq a_{\nu_n+1}(S \otimes T) \leq e^{-cn}$, giving the result. \square

Acknowledgments

This paper was made when the two first-named authors visited the University of Sevilla in February 2018. It is their pleasure to thank this university and all colleagues therein for their warm welcome. The third-named author is partially supported by the project MTM2015-63699-P (Spanish MINECO and FEDER funds).

¹⁴J. Shapiro and Taylor, 1973, "Compact, nuclear, and Hilbert-Schmidt composition operators on H^2 ".

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