



Adaptive observer for age-structured population with spatial diffusion

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Abstract

We investigate the inverse problem of simultaneously estimating the state and the spatial diffusion coefficient for an age-structured population model. The time evolution of the population is supposed to be known on a subdomain in space and age. We generalize to the infinite dimensional setting an adaptive observer originally proposed for finite dimensional systems.

Keywords: Inverse problems, Observer, Unknown parameter, Population dynamics, Spatial diffusion.

msc: 92D25, 35R30, 93D15, 93B55.

1 Introduction

We consider the following system modelling the evolution of an age-structured population with spatial diffusion:

$$\left\{ \begin{array}{ll} \partial_t p(a, x, t) + \partial_a p(a, x, t) & a \in (0, a^*), x \in \Omega, t > 0, \\ = -\mu(a)p(a, x, t) + k\Delta p(a, x, t), & \\ p(a, x, t) = 0, & a \in (0, a^*), x \in \partial\Omega, t > 0, \\ p(a, x, 0) = p_0(a, x), & a \in (0, a^*), x \in \Omega, \\ p(0, x, t) = \int_0^{a^*} \beta(a)p(a, x, t) da, & x \in \Omega, t > 0. \end{array} \right. \quad (1)$$

In the above equations:

- $\Omega \subset \mathbb{R}^n$, $n \geq 1$, denotes a smooth bounded domain, k is a positive constant diffusion coefficient and Δ the laplacian with respect to the space variable x .

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- $p(a, x, t)$ denotes the distribution density of the population of age a at spatial position $x \in \Omega$ at time t ;
- p_0 denotes the initial distribution;
- $a^* \in (0, +\infty)$ is the maximal life expectancy;
- $\beta(a)$ and $\mu(a)$ are positive functions denoting respectively the birth and death rates, which are supposed to be independent of x and satisfy

$$\begin{aligned} \beta &\in L^\infty(0, a^*), & \beta &\geq 0 \text{ a.e. in } (0, a^*), \\ \mu &\in L^1_{\text{loc}}(0, a^*), & \mu &\geq 0 \text{ a.e. in } (0, a^*), \end{aligned}$$

and

$$\lim_{a \rightarrow a^*} \int_0^a \mu(s) ds = +\infty.$$

The last equation in (1) describing the birth process is the so-called *renewal equation*. We assume here homogeneous Dirichlet boundary conditions (in space) which model a hostile habitat at the boundary $\partial\Omega$.

We assume here that the diffusion coefficient k is not well known. To be more specific we shall assume that $k \in [k_0 - r_k, k_0 + r_k]$ where k_0 (an approximate value of k) and r_k (the uncertainty on k) are known.

Inverse problems for population dynamics models have been studied in several papers. Rundell *et al.*² studied the determination of the death rate for an age-structured population dynamics from the knowledge population profiles at two distinct times. Gyllenberg *et al.*³ investigate the identifiability of birth and death rates in a linear age-structured population model from data on total population size and cumulative number of births (more realistic data than those used by Rundell *et al.*⁴). Perthame and Zubelli⁵ considered the problem of determining the division (birth) rate coefficient for a size-structured model for cell division from measured stable size distribution of the population. More recently, Perasso and Razafison⁶

²Rundell, 1993, "Determining the death rate for an age-structured population from census data"; Pilant and Rundell, 1991, "Determining a coefficient in a first-order hyperbolic equation"; Engl, Rundell, and Scherzer, 1994, "A regularization scheme for an inverse problem in age-structured populations".

³Gyllenberg, Osipov, and Päiväranta, 2002, "The inverse problem of linear age-structured population dynamics".

⁴Rundell, 1993, "Determining the death rate for an age-structured population from census data"; Pilant and Rundell, 1991, "Determining a coefficient in a first-order hyperbolic equation"; Engl, Rundell, and Scherzer, 1994, "A regularization scheme for an inverse problem in age-structured populations".

⁵Perthame and Zubelli, 2007, "On the inverse problem for a size-structured population model".

⁶Perasso and Razafison, 2016, "Identifiability problem for recovering the mortality rate in an age-structured population dynamics model".

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studied the identifiability of the age-dependent mortality rate for Mc Kendrick model. Let us emphasize that all these works do not take into account the effect of spatial diffusion. On the contrary, Di Blasio and Lorenzi⁷ investigated such models from the point of view of identifiability (existence, uniqueness and continuous dependence). Traore investigated estimation problems for population dynamics with spatial diffusion to recover the state from distributed observation⁸ or boundary observation⁹ in space and full observation in age.

In this paper, we investigate the following inverse problem: Assuming the initial age distribution p_0 to be unknown, but knowing the age distribution

$$y(a, x, t) := p(a, x, t), \quad t \in (0, T), \quad a \in (a_1, a_2), \quad x \in \mathcal{O},$$

where \mathcal{O} is some given non empty open subset of Ω and $0 \leq a_1 < a_2 \leq a^*$, estimate simultaneously:

- the age distribution $p(a, x, T)$ when $T \rightarrow +\infty$, for $x \in \Omega$ and $a \in (0, a^*)$
- and the diffusion coefficient k .

In¹⁰, the authors answered the above question in the case where the diffusion coefficient k is known. To do so, they constructed an observer for system (1), *i.e.* a new evolution system using the available measurements as inputs and whose dynamics is suitably chosen to make its state $\hat{p}(t)$ converge (asymptotically in time) to the state of the initial system $p(t)$. The design of this observer crucially uses the fact that the initial system has a finite number of unstable modes (corresponding to eigenvalues with non negative real parts), and the infinite dimensional observer is then constructed by designing a Luenberger observer for the finite dimensional unstable part of the system. The main contribution of this work is to extend this approach to the case where the diffusion coefficient k is unknown. This is far from being obvious as the eigenvalues of the infinite dimensional system are then unknown. However, we can take advantage from the fact that eigenfunctions are known and this allows us to design a new observer following an idea proposed by Kreisselmeier in a finite dimensional setting in¹¹ (see also¹²). Let us emphasize

⁷Di Blasio and Lorenzi, 2011, "Direct and inverse problems in age-structured population diffusion";
Di Blasio and Lorenzi, 2013, "An identification problem in age-dependent population diffusion".

⁸Traore, 2010, "Null controllability and application to data assimilation problem for a linear model of population dynamics".

⁹Traore, 2007, "Approximate controllability and application to data assimilation problem for a linear population dynamics model".

¹⁰Ramdani, Tucsnak, and Valein, 2016, "Detectability and state estimation for linear age-structured population diffusion models".

¹¹Kreisselmeier, 1979, "The generation of adaptive law structures for globally convergent adaptive observers".

¹²Kreisselmeier, 1977, "Adaptive observers with exponential rate of convergence".

that this observer requires more measurements than the observer proposed in¹³, as it uses the projected outputs not only on the unstable modes but also on a finite number of stable modes (see (3)).

For the sake of clarity, the main results are stated in Section 2, and their proofs are given in Section 3.

2 Statement of the main results

Using a semigroup formulation, we first rewrite problem (1) in the abstract form (throughout the paper, the dot denotes the derivative with respect to time)

$$\begin{cases} \dot{p}(t) = Ap(t), & t \in (0, T), \\ p(0) = p_0, \end{cases} \quad (1)$$

where $A : \mathcal{D}(A) \rightarrow X$ is the generator of a C_0 -semigroup on a Hilbert space $X := L^2((0, a^*) \times \Omega)$ defined by

$$\begin{aligned} \mathcal{D}(A) = \left\{ \varphi \in X \cap L^2((0, a^*), H_0^1(\Omega)) \mid -\frac{\partial \varphi}{\partial a} - \mu\varphi + k\Delta\varphi \in X; \right. \\ \left. \varphi(a, \cdot)|_{\partial\Omega} = 0 \text{ for almost all } a \in (0, a^*); \right. \\ \left. \varphi(0, x) = \int_0^{a^*} \beta(a)\varphi(a, x) da \text{ for almost all } x \in \Omega \right\} \\ A\varphi = -\partial_a\varphi - \mu\varphi + k\Delta\varphi, \quad \forall \varphi \in \mathcal{D}(A), \end{aligned}$$

(see Chan and Guo¹⁴, for more details). Similarly, the available observation can also be reformulated using a bounded observation operator $C \in \mathcal{L}(X, Y)$, where $Y := L^2((a_1, a_2) \times \mathcal{O})$, defined by $C\varphi = \varphi|_{(a_1, a_2) \times \mathcal{O}}$ ($\varphi \in X$):

$$y(t) = Cp(t), \quad t \in (0, T).$$

We recall here some results about the spectrum of A , we refer to¹⁵ for more details:

- The operator A has compact resolvent and its spectrum is constituted of a countable (infinite) set of isolated eigenvalues with finite algebraic multiplicity.

¹³Ramdani, Tucsnak, and Valein, 2016, “Detectability and state estimation for linear age-structured population diffusion models”.

¹⁴Chan and Guo, 1989, “On the semigroups of age-size dependent population dynamics with spatial diffusion”.

¹⁵Chan and Guo, 1989, “On the semigroups of age-size dependent population dynamics with spatial diffusion”;

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- The eigenvalues of A are given by

$$\sigma(A) = \left\{ \lambda_i^0 - k\lambda_j^D \mid i, j \in \mathbb{N}^* \right\}, \quad (2)$$

where $(\lambda_n^D)_{n \geq 1}$ denotes the increasing positive sequence of eigenvalues of the Dirichlet Laplacian and $(\lambda_n^0)_{n \geq 1}$ denotes the sequence of eigenvalues of the free diffusion operator ($k = 0$), which are the solutions of the characteristic equation

$$F(\lambda) := \int_0^{a^*} \beta(a) e^{-\lambda a - \int_0^a \mu(s) ds} da = 1.$$

- A has a real dominant eigenvalue λ_1 :

$$\lambda_1 = \lambda_1^0 - k\lambda_1^D > \operatorname{Re}(\lambda), \quad \forall \lambda \in \sigma(A), \lambda \neq \lambda_1.$$

- The eigenspace associated to an eigenvalue λ of A is given by

$$\operatorname{Span} \left\{ e^{-\lambda_i^0 a - \int_0^a \mu(s) ds} \varphi_j^D(x) \mid \lambda_i^0 - k\lambda_j^D = \lambda \right\}$$

where $(\varphi_n^D)_{n \geq 1}$ denotes an orthonormal basis of $L^2(\Omega)$ constituted of eigenfunctions of the Dirichlet Laplacian.

- Every vertical strip of the complex plane contains a finite number of eigenvalues of A .
- The semigroup e^{tA} generated on X by A is compact for $t \geq a^*$, which implies in particular that the exponential stability of e^{tA} is equivalent to the condition $\omega_0(A) := \sup \{ \operatorname{Re} \lambda \mid \lambda \in \sigma(A) \} < 0$ (see Zabczyk¹⁶, Section 2).

We denote by M the number of eigenvalues of A (counted without multiplicities) with positive real part and we assume that A has no eigenvalue of real part equal to zero:

$$\dots \leq \operatorname{Re} \lambda_{M+1} < 0 < \operatorname{Re} \lambda_M \leq \dots \leq \operatorname{Re} \lambda_2 < \lambda_1.$$

To solve our estimation problem, we shall first construct an observer for the finite dimensional system in \mathbb{C}^M corresponding to the unstable eigenvalues. To design this observer, we need to use an observation coming not only from the M unstable modes, but also from some additional stable ones. More precisely let us choose $N \in \mathbb{N}^*$ such that

$$\operatorname{Re} \lambda_{N+1} < -3\lambda_1. \quad (3)$$

¹⁶Zabczyk, 1975/76, "Remarks on the algebraic Riccati equation in Hilbert space".

Remark 1 – According to formula (2), the eigenvalues $\lambda_n = \lambda_i^0 - k\lambda_j^D$ verify

$$\operatorname{Re} \lambda_n \leq \operatorname{Re} \lambda_i^0 - (k_0 - r_k)\lambda_j^D, \quad \forall k \in [k_0 - r_k, k_0 + r_k].$$

Therefore, such a value N (but not necessarily the optimal one) can be determined without knowing the exact value of k .

In the sequel, we also need to define $\alpha > 0$ such that

$$\operatorname{Re} \lambda_{N+1} < -\alpha < -3\lambda_1. \quad (4)$$

According to (2), the eigenvalues of A depend linearly on the diffusion coefficient k and, hence, N also depends a priori on k . We will assume that the (finite) number of eigenvalues of A of real part greater than $-3\lambda_1$ is constant when k varies in $[k_0 - r_k, k_0 + r_k]$. This assumption is not crucial but is made for the sake of simplicity. Then, let us consider a curve Γ^N in the complex plane enclosing the set of eigenvalues $\Sigma^N := \{\lambda_1, \dots, \lambda_N\}$ but no other elements of the spectrum of A . We denote by P^N the projection operator defined by

$$P^N := -\frac{1}{2\pi i} \int_{\Gamma^N} (\xi - A)^{-1} d\xi.$$

We set $X^N = P^N(X)$ and $X_-^N = (\operatorname{Id} - P^N)(X)$, and then P^N provides the following decomposition of X

$$X = X^N \oplus X_-^N.$$

Moreover X^N and X_-^N are invariant subspaces under A (since A and P^N commute) and the spectra of the restricted operators $A|_{X^N}$ and $A|_{X_-^N}$ are respectively Σ^N and $\sigma(A) \setminus \Sigma^N$ (see¹⁷). We also set

$$A^N := A|_{\mathcal{D}(A) \cap X^N} : \mathcal{D}(A) \cap X^N \rightarrow X^N \quad \text{and} \quad A_-^N := A|_{\mathcal{D}(A) \cap X_-^N} : \mathcal{D}(A) \cap X_-^N \rightarrow X_-^N.$$

Throughout the paper, we make the following assumption about A^N :

(H₁) The restriction A^N of operator A to X^N is diagonalizable.

This assumption deals with an important particular case that has also been considered in references¹⁸. Due to the explicit formula for the eigenvalues (see (2)), this assumption can be easily checked in some cases (for instance when the number of

¹⁷Kato, 1995, *Perturbation theory for linear operators*.

¹⁸Barbu and Triggiani, 2004, “Internal stabilization of Navier-Stokes equations with finite-dimensional controllers”;

Ramdani, Tucsnak, and Valein, 2016, “Detectability and state estimation for linear age-structured population diffusion models”.

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unstable eigenvalues is small), even though the diffusion coefficient is not known with accuracy.

Let us emphasize that under this assumption, we have

$$X^N = \bigoplus_{n=1}^N \text{Ker}(A - \lambda_n) = \text{Span}\{\varphi_n, 1 \leq n \leq N\},$$

where $(\varphi_n)_{1 \leq n \leq N}$ denotes a basis of X^N constituted of eigenfunctions of A^N associated to the eigenvalues $(\lambda_n)_{1 \leq n \leq N}$. We also denote by ψ_n an eigenfunction of A^* corresponding to the eigenvalue $\overline{\lambda_n}$, $1 \leq n \leq N$. It can be shown (see¹⁹) that the family $(\psi_n)_{1 \leq n \leq N}$ can be chosen such that $(\varphi_n)_{1 \leq n \leq N}$ and $(\psi_n)_{1 \leq n \leq N}$ form bi-orthogonal sequences: $(\varphi_n, \psi_m)_X = \delta_{nm}$. It follows then that the projection operator $P^N \in \mathcal{L}(X, X^N)$ can be expressed as

$$P^N z = \sum_{n=1}^N \langle z, \psi_n \rangle \varphi_n, \quad \forall z \in X.$$

With these notation, any solution p of (1) admits the decomposition

$$p(t) = p^N(t) + p_-^N(t) \tag{5}$$

with

$$\begin{cases} p^N(t) := P^N(p(t)) = \sum_{n=1}^N e^{\lambda_n t} p_n^0 \varphi_n, & p_n^0 = \langle p(0), \psi_n \rangle, \\ p_-^N(t) := (\text{Id} - P^N)(p(t)) = e^{tA} p_-^N(0). \end{cases}$$

Throughout the paper, we will use bold letters/symbols to denote vectors and matrices. The above decomposition suggests to introduce the following finite dimensional state variable:

$$\mathbf{p}^N(t) = (p_1^N(t), \dots, p_N^N(t)) := (e^{\lambda_1 t} p_1^0, \dots, e^{\lambda_N t} p_N^0)^T \in \mathbb{C}^N$$

whose dynamics is simply given by

$$\dot{\mathbf{p}}^N(t) = \mathbf{\Lambda}^N \mathbf{p}^N(t), \tag{6}$$

where $\mathbf{\Lambda}^N := \text{diag}(\lambda_1, \dots, \lambda_N)$ (recall that the initial state $\mathbf{p}^N(0)$ is unknown and that the eigenvalues λ_n , $n = 1, \dots, N$, depend on k and are thus also unknown).

¹⁹Barbu and Triggiani, 2004, "Internal stabilization of Navier-Stokes equations with finite-dimensional controllers", p. 1453.

As mentioned above, we shall construct a finite dimensional observer for the M unstable modes of the system, i.e. the M first components of $\mathbf{p}^N(t)$, denoted $\mathbf{p}(t)$:

$$\mathbf{p}(t) := (e^{\lambda_1 t} p_1^0, \dots, e^{\lambda_M t} p_M^0)^T \in \mathbb{C}^M.$$

To do so, system (6) needs to be supplemented by finite dimensional observations, which can be easily obtained from those of the infinite dimensional system as follows. Defining the quantities

$$y_n(t) = \langle y(t), C\varphi_n \rangle_Y = \langle Cp(t), C\varphi_n \rangle_Y, \quad n = 1, \dots, N,$$

and setting

$$\mathbf{y}^N(t) := (y_1(t), \dots, y_N(t))^T \in \mathbb{C}^N,$$

the decomposition (5) immediately shows that

$$\mathbf{y}^N(t) = \mathbf{C}^N \mathbf{p}^N(t) + \mathbf{y}_-^N(t), \quad (7)$$

where the matrix $\mathbf{C}^N := (C_{mn}^N)_{1 \leq n, m \leq N}$ is defined by $C_{mn}^N = \langle C\varphi_n, C\varphi_m \rangle_Y$ and

$$\mathbf{y}_-^N(t) := (\langle Cp_-^N(t), C\varphi_1 \rangle_Y, \dots, \langle Cp_-^N(t), C\varphi_N \rangle_Y) \in \mathbb{C}^N.$$

The family $(C\varphi_n)_{1 \leq n \leq N}$ being linearly independent in X (see²⁰ or²¹), the matrix \mathbf{C}^N is invertible. Consequently, equation (7) equivalently reads

$$\mathbf{q}^N(t) = \mathbf{p}^N(t) + \mathbf{q}_-^N(t), \quad (8)$$

where

$$\mathbf{q}^N(t) = (q_1^N(t), \dots, q_N^N(t))^T := (\mathbf{C}^N)^{-1} \mathbf{y}^N(t)$$

and

$$\mathbf{q}_-^N(t) = (q_{-,1}^N(t), \dots, q_{-,N}^N(t))^T := (\mathbf{C}^N)^{-1} \mathbf{y}_-^N(t).$$

Note that $\mathbf{q}^N(t)$ is an available measure since \mathbf{C}^N and $\mathbf{y}^N(t)$ are known.

Following Kreisselmeier (²²), the proposed finite dimensional observer in \mathbb{C}^M is defined by

$$\hat{\mathbf{p}}(t) = \mathbf{M}(t) \mathbf{E} \theta(t) \quad (9)$$

²⁰Barbu and Triggiani, 2004, "Internal stabilization of Navier-Stokes equations with finite-dimensional controllers".

²¹Ramdani, Tucsnak, and Valein, 2016, "Detectability and state estimation for linear age-structured population diffusion models".

²²Kreisselmeier, 1979, "The generation of adaptive law structures for globally convergent adaptive observers";

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with

$$\begin{cases} \dot{\mathbf{M}}(t) = \mathbf{F}\mathbf{M}(t) + [q_1^N(t)\mathbf{Id}, \dots, q_M^N(t)\mathbf{Id}] \\ \mathbf{M}(0) = 0 \\ \dot{\boldsymbol{\theta}}(t) = -\gamma\mathbf{E}^*\mathbf{M}^*(t)(\hat{\mathbf{p}}(t) - \Pi_M\mathbf{q}^N(t)), \\ \boldsymbol{\theta}(0) = 0 \end{cases} \quad (10)$$

where

- $\hat{\mathbf{p}}$ and $\boldsymbol{\theta}$ are vectors in \mathbb{C}^M whose components are respectively \hat{p}_i and θ_i ;
- $\mathbf{M}(t)$ denotes a $M \times M^2$ matrix;
- $\mathbf{E} = (\mathbf{E}_1, \dots, \mathbf{E}_M)^T$ denotes a $M^2 \times M$ matrix, with $\mathbf{E}_n = \text{diag}(0, \dots, 0, 1, 0, \dots, 0)$, the term 1 being at the n^{th} place;
- $\mathbf{F} = \text{diag}(-f_1, \dots, -f_M)$ where f_i are chosen positive;
- \mathbf{Id} denotes the $M \times M$ identity matrix;
- $\Pi_M : \mathbb{C}^N \rightarrow \mathbb{C}^M$ denotes the projection on the M -first components (more precisely, if $\mathbf{z} = (z_1, \dots, z_M, z_{M+1}, \dots, z_N)^T \in \mathbb{C}^N$, then $\Pi_M\mathbf{z} = (z_1, \dots, z_M)^T \in \mathbb{C}^M$);
- γ is a positive real number (gain coefficient);
- the $*$ stands for the conjugate transpose.

Let us emphasize that the adaptive state estimate (9)-(10) is the one proposed by Kreisselmeier (see equation (9) in²³) in the particular case where the initial data of the observer is zero and where the dynamics of $\boldsymbol{\theta}$ is driven by a quadratic error criterion L (see equation (7) in²⁴). However, the available measurement error here is not $\hat{\mathbf{p}}(t) - \mathbf{p}^N(t)$ (since $\mathbf{p}^N(t) = \mathbf{q}^N(t) - \mathbf{q}^N(t)$ is unknown) but only $\hat{\mathbf{p}}(t) - \Pi_M\mathbf{q}^N(t)$, and this generates additional difficulties. In particular, this is why the observer $\hat{\mathbf{p}}(t) \in \mathbb{C}^M$ is computed using the output vector $\Pi_M\mathbf{q}^N(t) = (q_1^N(t), \dots, q_M^N(t))^T \in \mathbb{C}^M$, the latter being obtained from the $N > M$ measurements collected in the vector $\mathbf{y}^N(t) \in \mathbb{C}^N$. As will be seen later, this construction, which might seem unnatural, ensures that the remainder term $\mathbf{q}^N(t)$ decays fast enough to 0 and hence guarantees the convergence of the observer.

Let $\boldsymbol{\theta}^\infty := (\lambda_1 + f_1, \dots, \lambda_M + f_M)^T = (\boldsymbol{\Lambda}^M - \mathbf{F})\mathbf{1} \in \mathbb{C}^M$, where $\mathbf{1}$ denotes the vector of \mathbb{C}^M whose all components are equal to 1 and where $\boldsymbol{\Lambda}^M = \text{diag}(\lambda_1, \dots, \lambda_M)$.

We are now in position to state the two main results of the paper.

²³Kreisselmeier, 1979, "The generation of adaptive law structures for globally convergent adaptive observers".

²⁴Ibid.

Theorem 1 – Let $R > 0$ and $\varepsilon > 0$ be given. Assume that the initial data p_0 satisfies

$$\|p_0\|_X \leq R \quad (11)$$

and

$$|p_n^0| > \varepsilon, \quad \forall n = 1, \dots, M. \quad (12)$$

Assume also that N is chosen such that (3) is satisfied. Finally, let $\hat{\mathbf{p}}(t)$ and $\theta(t)$ be defined by (9) and (10).

Then we can choose the matrix \mathbf{F} such that there exist $\kappa > 0$, $\omega > 0$ satisfying

$$\|\theta(t) - \theta^\infty\|_{\mathbb{C}^M} \leq \kappa e^{-\omega t} \quad \text{and} \quad \|\hat{\mathbf{p}}(t) - \mathbf{p}(t)\|_{\mathbb{C}^M} \leq \kappa e^{-\omega t} \quad (t > 0).$$

Remark 2 – Notice that θ^∞ provides an estimate for the M first eigenvalues of A , and hence of the unknown diffusion coefficient k . In particular, $\theta_1(t)$ converges exponentially to $\theta_1^\infty = \lambda_1 + f_1 = \lambda_1^D - k\lambda_1^D + f_1$.

Theorem 2 – Let $\hat{\mathbf{p}}(t) = (\hat{p}_1(t), \dots, \hat{p}_M(t))^T \in \mathbb{C}^M$ and $\theta(t) = (\theta_1(t), \dots, \theta_M(t))^T \in \mathbb{C}^M$ be defined by (9) and (10) and set

$$\hat{p}(t) = \sum_{n=1}^M \hat{p}_n(t) \varphi_n.$$

Then, under the assumptions of Theorem 1, we can choose the matrix

$$\mathbf{F} = \text{diag}(-f_1, \dots, -f_M), \quad f_i > 0, \quad i = 1, \dots, M,$$

such that there exist $\kappa > 0$ and $\omega > 0$ satisfying

$$\|\hat{p}(t) - p(t)\|_X \leq \kappa e^{-\omega t} \quad (t > 0).$$

Moreover, we have

$$k = \frac{1}{\lambda_1^D} \left(\lambda_1^0 + f_1 - \lim_{t \rightarrow +\infty} \theta_1(t) \right).$$

Proof. Let us define P^M , X^M , X_-^M , p^M , p_-^M as above, but replacing N by M . Using (5) written with M instead of N , we have

$$\|\hat{p}(t) - p(t)\|_X^2 \leq 2 \|\hat{\mathbf{p}}(t) - \mathbf{p}(t)\|_{\mathbb{C}^M}^2 + 2 \|e^{tA} p_-^M(0)\|_X^2.$$

On the other hand,

$$\|e^{tA} p_-^M(0)\|_X \leq K \|p_-^M(0)\|_X e^{-\alpha_M t},$$

where $K > 0$ and α_M may be any positive constant such that $-\alpha_M > \text{Re } \lambda_{M+1}$. The result follows then from Theorem 1. \square

3 Proof of Theorem 1

We split up the proof of Theorem 1 into three lemmas and two propositions.

Lemma 1 – Let $\theta^\infty = (\Lambda^M - \mathbf{F})\mathbf{1} \in \mathbb{C}^M$. Then $\mathbf{p}(t) \in \mathbb{C}^M$ can be written as

$$\mathbf{p}(t) = \mathbf{M}(t)\mathbf{E}\theta^\infty + e^{t\mathbf{F}}\mathbf{p}(0) - \int_0^t e^{(t-s)\mathbf{F}}(\Lambda^M - \mathbf{F})\Pi_M\mathbf{q}_-^N(s) ds \quad (t > 0). \quad (1)$$

Proof. We set

$$\tilde{\mathbf{p}}(t) := \mathbf{M}(t)\mathbf{E}\theta^\infty + e^{t\mathbf{F}}\mathbf{p}(0) - \int_0^t e^{(t-s)\mathbf{F}}(\Lambda^M - \mathbf{F})\Pi_M\mathbf{q}_-^N(s) ds,$$

and we want to prove that $\tilde{\mathbf{p}}(t) = \mathbf{p}(t)$ for any $t > 0$. We have

$$\frac{d}{dt}\tilde{\mathbf{p}}(t) = \dot{\mathbf{M}}(t)\mathbf{E}\theta^\infty + \mathbf{F}e^{t\mathbf{F}}\mathbf{p}(0) - (\Lambda^M - \mathbf{F})\Pi_M\mathbf{q}_-^N(t) - \int_0^t \mathbf{F}e^{(t-s)\mathbf{F}}(\Lambda^M - \mathbf{F})\Pi_M\mathbf{q}_-^N(s) ds,$$

which implies, using the first equation in (10), that

$$\frac{d}{dt}\tilde{\mathbf{p}}(t) = \mathbf{F}\tilde{\mathbf{p}}(t) + [q_1^N(t)\mathbf{Id}, \dots, q_M^N(t)\mathbf{Id}]\mathbf{E}\theta^\infty - (\Lambda^M - \mathbf{F})\Pi_M\mathbf{q}_-^N(t).$$

Moreover, we can easily verify that

$$[q_1^N(t)\mathbf{Id}, \dots, q_M^N(t)\mathbf{Id}]\mathbf{E}\theta^\infty = \sum_{n=1}^M q_n^N(t)\mathbf{E}_n\theta^\infty = (\Lambda^M - \mathbf{F})\Pi_M\mathbf{q}_-^N(t).$$

Consequently, we have

$$\frac{d}{dt}\tilde{\mathbf{p}}(t) = \mathbf{F}\tilde{\mathbf{p}}(t) + (\Lambda^M - \mathbf{F})\Pi_M(\mathbf{q}^N(t) - \mathbf{q}_-^N(t)),$$

and by (8)

$$\frac{d}{dt}\tilde{\mathbf{p}}(t) = \mathbf{F}\tilde{\mathbf{p}}(t) + (\Lambda^M - \mathbf{F})\Pi_M\mathbf{p}^N(t) = \mathbf{F}(\tilde{\mathbf{p}}(t) - \mathbf{p}(t)) + \Lambda^M\mathbf{p}(t),$$

since $\Pi_M\mathbf{p}^N = \mathbf{p}$ by definition. Using (6), we deduce that

$$\frac{d}{dt}(\tilde{\mathbf{p}}(t) - \mathbf{p}(t)) = \mathbf{F}(\tilde{\mathbf{p}}(t) - \mathbf{p}(t)).$$

Since we have on the other hand that $\tilde{\mathbf{p}}(0) = \mathbf{M}(0)\mathbf{E}\theta^\infty + \mathbf{p}(0) = \mathbf{p}(0)$, we can conclude that $\tilde{\mathbf{p}}(t) = \mathbf{p}(t)$ for any $t > 0$. \square

Remark 3 –

1. Equation (1) provides an explicit formula to compute $\mathbf{p}(t)$ if we knew $\mathbf{p}(0)$, $\boldsymbol{\theta}^\infty$ (and thus λ_n) and $\Pi_M \mathbf{q}_-^N(t)$. This is not the case in the problem studied here.
2. If $\mathbf{q}_-^N = 0$ (i.e. $\mathbf{q}^N = \mathbf{p}^N$) in (1), we recover exactly the adaptive observer proposed by Kreisselmeier²⁵ (see also²⁶).

Lemma 2 – Assume that N is chosen according to condition (3) and that p_0 satisfies (11)-(12). For f_n large enough ($n = 1, \dots, M$), there exist positive constants m_0, m_1, m_2 independent of p_0 such that, for any $t > t_0 := m_0/\text{Re } \lambda_M$, the two following inequalities hold:

$$\left| (\mathbf{E}^* \mathbf{M}(t)^* \mathbf{M}(t) \mathbf{E})_{nn} \right|^{1/2} \geq m_1 (\text{Re}(\lambda_M) t - m_0), \quad n = 1, \dots, M, \quad (2)$$

$$\|\mathbf{M}(t) \mathbf{E}\| \leq m_2 e^{\lambda_1 t}. \quad (3)$$

Proof. We first compute explicitly the matrix $\mathbf{E}^* \mathbf{M}^*(t) \mathbf{M}(t) \mathbf{E}$. Writing

$$\mathbf{M}(t) = [\mathbf{M}_1(t), \dots, \mathbf{M}_M(t)],$$

where the matrices \mathbf{M}_n are $M \times M$ matrices, we have $\mathbf{M}(t) \mathbf{E} = \sum_{n=1}^M \mathbf{M}_n(t) \mathbf{E}_n$. Moreover the matrices $\mathbf{M}_n(t)$ satisfy the following differential equations

$$\dot{\mathbf{M}}_n(t) = \mathbf{F} \mathbf{M}_n(t) + q_n^N(t) \mathbf{Id}, \quad n = 1, \dots, M.$$

As $\mathbf{M}(0) = 0$, we have by (8)

$$\mathbf{M}_n(t) = \int_0^t q_n^N(s) e^{(t-s)\mathbf{F}} \mathbf{d}s = \int_0^t p_n^N(s) e^{(t-s)\mathbf{F}} \mathbf{d}s + \int_0^t q_{-,n}^N(s) e^{(t-s)\mathbf{F}} \mathbf{d}s.$$

Now, recalling that $p_n^N(t) = p_n^0 e^{\lambda_n t}$ are the components of $\mathbf{p}^N(t)$, we have

$$\begin{aligned} \int_0^t p_n^N(s) e^{(t-s)\mathbf{F}} \mathbf{d}s &= \int_0^t p_n^0 e^{\lambda_n s} e^{(t-s)\mathbf{F}} \mathbf{d}s \\ &= p_n^0 \text{diag} \left(\frac{e^{\lambda_n t} - e^{-f_1 t}}{\lambda_n + f_1}, \dots, \frac{e^{\lambda_n t} - e^{-f_M t}}{\lambda_n + f_M} \right). \end{aligned}$$

²⁵Kreisselmeier, 1979, “The generation of adaptive law structures for globally convergent adaptive observers”.

²⁶Kreisselmeier, 1977, “Adaptive observers with exponential rate of convergence”.

3. Proof of Theorem 1

Therefore

$$\mathbf{M}_n(t) = p_n^0 \operatorname{diag} \left(\frac{e^{\lambda_n t} - e^{-f_1 t}}{\lambda_n + f_1}, \dots, \frac{e^{\lambda_n t} - e^{-f_M t}}{\lambda_n + f_M} \right) + \mathbf{M}_n^-(t)$$

where

$$\mathbf{M}_n^-(t) = \operatorname{diag} \left(\int_0^t e^{-(t-s)f_1} q_{-,n}^N(s) ds, \dots, \int_0^t e^{-(t-s)f_M} q_{-,n}^N(s) ds \right).$$

We deduce that

$$\mathbf{M}_n(t) \mathbf{E}_n = \left(p_n^0 \frac{e^{\lambda_n t} - e^{-f_n t}}{\lambda_n + f_n} + \int_0^t e^{-(t-s)f_n} q_{-,n}^N(s) ds \right) \mathbf{E}_n. \quad (4)$$

As $\mathbf{E}_n^* = \mathbf{E}_n$ and $\mathbf{E}_n \mathbf{E}_m = \delta_{nm} \mathbf{E}_n$ (δ_{mn} denotes the Kronecker symbol), we obtain

$$\mathbf{E}^* \mathbf{M}^*(t) \mathbf{M}(t) \mathbf{E} = \sum_{n=1}^M \left(p_n^0 \frac{e^{\lambda_n t} - e^{-f_n t}}{\lambda_n + f_n} + \int_0^t e^{-(t-s)f_n} q_{-,n}^N(s) ds \right)^2 \mathbf{E}_n.$$

We shall seek for an upper-bound for the diagonal terms of $\mathbf{E}^* \mathbf{M}^*(t) \mathbf{M}(t) \mathbf{E}$. Denoting by ρ_n the real part of λ_n , we have

$$|(\mathbf{E}^* \mathbf{M}^*(t) \mathbf{M}(t) \mathbf{E})_{nn}|^{1/2} \geq \frac{|p_n^0|}{|\lambda_n + f_n|} (e^{\rho_n t} - e^{-f_n t}) - \int_0^t e^{-(t-s)f_n} |q_{-,n}^N(s)| ds.$$

It follows from the fact that the semigroup e^{tA} generated on X by A is compact for $t \geq a^*$ and the definition of p_-^N that there exists $\kappa > 0$ such that

$$\|p_-^N(t)\|_X \leq \kappa \|p_-^N(0)\|_X e^{-\alpha t} \quad (t > 0),$$

where α satisfies (4).

From the definition of \mathbf{y}_-^N , we infer that there exists a positive constant L' such that

$$\|\mathbf{y}_-^N(t)\|_{\mathbb{C}^N} \leq L' \|p_-^N(0)\|_X e^{-\alpha t} \quad (t > 0).$$

As $\mathbf{q}_-^N(t) := (\mathbf{C}^N)^{-1} \mathbf{y}_-^N(t)$, we can write an analogous inequality for its components $q_{-,n}^N$: there exists a positive constant L such that

$$|q_{-,n}^N(t)| \leq L \|p_-^N(0)\|_X e^{-\alpha t}, \quad \text{for } n = 1, \dots, M. \quad (5)$$

Thus, if the f_n 's are chosen greater than α , we can write

$$|(\mathbf{E}^* \mathbf{M}^*(t) \mathbf{M}(t) \mathbf{E})_{nn}|^{1/2} \geq K_n (e^{\rho_n t} - e^{-f_n t}) - L_n (e^{-\alpha t} - e^{-f_n t})$$

where

$$K_n = \frac{|p_n^0|}{|\lambda_n + f_n|}, \quad L_n = \frac{L\|p_-^N(0)\|_X}{f_n - \alpha}.$$

Now, if $L_n \leq K_n$, we have

$$\begin{aligned} K_n(e^{\rho_n t} - e^{-f_n t}) - L_n(e^{-\alpha t} - e^{-f_n t}) &\geq K_n(e^{\rho_n t} - e^{-\alpha t}) + (L_n - K_n)e^{-f_n t} \\ &\geq K_n(e^{\rho_n t} - 1) + L_n - K_n \\ &\geq K_n \left(\rho_n t - \frac{K_n - L_n}{K_n} \right), \end{aligned}$$

and if $L_n > K_n$, we have

$$\begin{aligned} K_n(e^{\rho_n t} - e^{-f_n t}) - L_n(e^{-\alpha t} - e^{-f_n t}) &\geq K_n e^{\rho_n t} - L_n e^{-\alpha t} \\ &\geq K_n(\rho_n t + 1) - L_n \\ &= K_n \left(\rho_n t - \frac{L_n - K_n}{K_n} \right). \end{aligned}$$

Hence, in both cases we have

$$|(\mathbf{E}^* \mathbf{M}^*(t) \mathbf{M}(t) \mathbf{E})_{nn}|^{1/2} \geq K_n(\rho_n t - K'_n), \quad n = 1, \dots, M$$

with $K'_n = |L_n - K_n| K_n^{-1}$.

Due to assumptions (11)-(12), there exist constants $\ell, \ell' > 0$ independent of p_0 such that for all $n = 1, \dots, M$:

$$\ell' \varepsilon \leq K_n \leq \ell R, \quad L_n \leq \ell R, \quad K'_n \leq \ell \frac{R}{\varepsilon}.$$

Hence, $K_n(\rho_n t - K'_n) \geq K_n(\rho_M t - m_0)$ where $m_0 := \ell R / \varepsilon$. Letting $m_1 := \ell' \varepsilon$ and $t_0 = m_0 / \rho_M$, we have

$$|(\mathbf{E}^* \mathbf{M}^*(t) \mathbf{M}(t) \mathbf{E})_{nn}|^{1/2} \geq m_1(\rho_M t - m_0), \quad \forall t > t_0.$$

We now prove the second point of the lemma. From (4) and (5), we have

$$|(\mathbf{M}(t) \mathbf{E})_{nn}| \leq K_n |e^{\lambda_n t} - e^{-f_n t}| + L_n (e^{-\alpha t} - e^{-f_n t}) \leq (2K_n + L_n) e^{\rho_n t},$$

where, for the last inequality, we have used $-f_n \leq -\alpha \leq \rho_n$, $n = 1, \dots, M$. Hence, since $\mathbf{M}(t) \mathbf{E}$ is a diagonal matrix,

$$\|\mathbf{E}^* \mathbf{M}^*(t)\| = \|\mathbf{M}(t) \mathbf{E}\| = \max_{n=1, \dots, M} |(\mathbf{M}(t) \mathbf{E})_{nn}| \leq m_2 e^{\lambda_1 t}$$

where $m_2 = 3\ell R$. Notice that λ_1 is real and is equal to the largest value taken by the numbers ρ_n ($n = 1, \dots, M$). \square

3. Proof of Theorem 1

Proposition 1 – Assume that N is chosen according to condition (3) and that p_0 satisfies (11)-(12). For f_n large enough ($n = 1, \dots, M$), the function V defined by

$$V(t) = \|\boldsymbol{\theta}(t) - \boldsymbol{\theta}^\infty\|_{\mathbb{C}^M}^2 \quad (t > 0)$$

satisfy for all $t > t_0 = m_0/\text{Re } \lambda_M$:

$$\dot{V}(t) \leq -2\gamma m_1^2 (\text{Re}(\lambda_M)t - m_0)^2 V(t) + 2\gamma m_6 e^{(\lambda_1 - \alpha)t} V(t)^{1/2}, \quad (6)$$

where m_6 is a positive constant independent of p_0 and where α satisfies (4).

Proof. Let us compute \dot{V} , the derivative of V along the trajectories of system (10). Using the expression of \mathbf{q}^N given by (8), we have

$$\begin{aligned} \dot{V}(t) &= 2(\boldsymbol{\theta}(t) - \boldsymbol{\theta}^\infty)^* \dot{\boldsymbol{\theta}}(t) \\ &= -2\gamma(\boldsymbol{\theta}(t) - \boldsymbol{\theta}^\infty)^* \mathbf{E}^* \mathbf{M}^*(t)(\hat{\mathbf{p}}(t) - \Pi_M \mathbf{q}^N(t)) \\ &= -2\gamma(\boldsymbol{\theta}(t) - \boldsymbol{\theta}^\infty)^* \mathbf{E}^* \mathbf{M}^*(t)(\hat{\mathbf{p}}(t) - \mathbf{p}(t) - \Pi_M \mathbf{q}_-^N(t)) \\ &= -2\gamma(\boldsymbol{\theta}(t) - \boldsymbol{\theta}^\infty)^* \mathbf{E}^* \mathbf{M}^*(t)(\mathbf{M}(t)\mathbf{E}(\boldsymbol{\theta}(t) - \boldsymbol{\theta}^\infty) \\ &\quad - e^{t\mathbf{F}} \mathbf{p}(0) + \int_0^t e^{(t-s)\mathbf{F}}(\boldsymbol{\Lambda}^M - \mathbf{F})\Pi_M \mathbf{q}_-^N(s)ds - \Pi_M \mathbf{q}_-^N(t)), \end{aligned} \quad (7)$$

using, for the last formula, (9) and (1). Substituting (2) in (7), we obtain

$$\begin{aligned} \dot{V}(t) &\leq -2\gamma m_1^2 (\text{Re}(\lambda_M)t - m_0)^2 V(t) \\ &\quad + 2\gamma \|\mathbf{E}^* \mathbf{M}^*(t)\| \|\boldsymbol{\theta}(t) - \boldsymbol{\theta}^\infty\| \left\{ \|e^{t\mathbf{F}}\| \|\mathbf{p}(0)\| \right. \\ &\quad \left. + \left\| \int_0^t e^{(t-s)\mathbf{F}}(\boldsymbol{\Lambda}^M - \mathbf{F})\Pi_M \mathbf{q}_-^N(s)ds \right\| + \|\Pi_M \mathbf{q}_-^N(t)\| \right\} \end{aligned} \quad (8)$$

for every $t > t_0$ (the norms are taken in \mathbb{C}^M). We need an upperbound for the second term in the hand-right side of this inequality. From the definition of \mathbf{q}_-^N and from inequalities (5), there exists a positive constant m_3 independent of p_0 such that

$$\|\Pi_M \mathbf{q}_-^N(t)\|_{\mathbb{C}^M} \leq m_3 e^{-\alpha t} \quad (t > 0). \quad (9)$$

We deduce from this inequality and from $f_n > \alpha$ that we have

$$\left\| \int_0^t e^{(t-s)\mathbf{F}}(\boldsymbol{\Lambda}^M - \mathbf{F})\Pi_M \mathbf{q}_-^N(s)ds \right\|_{\mathbb{C}^M} \leq m_3 \frac{\lambda_1 + f_\infty}{f_0 - \alpha} (e^{-\alpha t} - e^{-f_\infty t}) \leq m_4 e^{-\alpha t} \quad (10)$$

where $f_0 = \min_{n=1, \dots, M} f_n$, $f_\infty = \max_{n=1, \dots, M} f_n$ and $m_4 = m_3(\lambda_1 + f_\infty)(f_0 - \alpha)^{-1}$. We deduce from inequalities (9), (10) and from $f_n > \alpha$ that

$$\|e^{t\mathbf{F}}\|_{\mathbb{C}^M} \|\mathbf{p}(0)\|_{\mathbb{C}^M} + \left\| \int_0^t e^{(t-s)\mathbf{F}}(\boldsymbol{\Lambda}^M - \mathbf{F})\Pi_M \mathbf{q}_-^N(s)ds \right\|_{\mathbb{C}^M} + \|\Pi_M \mathbf{q}_-^N(t)\|_{\mathbb{C}^M} \leq m_5 e^{-\alpha t},$$

(11)

where $m_5 = R + m_4 + m_3$. Substituting inequalities (3) and (11) in (8), we obtain

$$\dot{V}(t) \leq -2\gamma m_1^2 (\rho_M t - m_0)^2 V(t) + 2\gamma m_6 e^{(\lambda_1 - \alpha)t} V(t)^{1/2}$$

with $m_6 = m_2 m_5$. \square

Lemma 3 – Assume that N is chosen according to condition (3) and that p_0 satisfies (11)-(12). The function

$$W(t) := V(t)^{1/2} = \|\theta(t) - \theta^\infty\|_{\mathbb{C}^M}$$

is continuous and right differentiable on \mathbb{R}_+ . Moreover, denoting by \dot{W}_r the right derivative of W , \dot{W}_r satisfies the following inequality for every $t > t_0 = m_0/\text{Re } \lambda_M$:

$$\dot{W}_r(t) \leq -\gamma m_1^2 (\text{Re } (\lambda_M) t - m_0)^2 W(t) + \gamma m_6 e^{(\lambda_1 - \alpha)t}, \quad (12)$$

where α satisfies (4)

Proof. Notice first that as $t \mapsto \langle Cp(t), C\varphi \rangle_Y$ is a C^1 function (for every $\varphi \in X$), \mathbf{y}^N and \mathbf{q}^N are also C^1 function (by definition), and so θ is a C^2 function due to (10). Thus W is continuous.

Moreover if t_1 is such that $V(t_1) \neq 0$, then W is differentiable at $t = t_1$ and so is right differentiable. In this case we have

$$\dot{W}_r(t_1) = \frac{\dot{V}(t_1)}{2V(t_1)^{1/2}}.$$

By (6), (12) holds.

If $V(t_1) = 0$, from formula (7), we obtain that $\dot{V}(t_1) = 0$. So we can write

$$V(t) = \frac{\beta}{2}(t - t_1)^2 + o((t - t_1)^2)$$

with $\beta = \ddot{V}(t_1) \geq 0$ because $V(t) \geq 0$ near t_1 . From this equality, and taking $h > 0$, we have

$$\frac{W(t_1 + h) - W(t_1)}{h} = \frac{\sqrt{\beta h^2/2 + o(h^2)}}{h} = \sqrt{\beta/2 + o(1)},$$

which proves that W is right differentiable at $t = t_1$ and that $\dot{W}_r(t_1) = \sqrt{\beta/2}$.

Notice that if $\beta > 0$, then $V(t) > 0$ for t in some interval $(t_1, t_1 + \tau)$ (here $\tau > 0$), so in this case we have

$$\frac{\dot{V}(t_1 + h)}{2V(t_1 + h)^{1/2}} = \frac{\beta h + o(h)}{2\sqrt{\beta h^2/2 + o(h^2)}} = \frac{\beta + o(1)}{2\sqrt{\beta/2 + o(1)}}$$

3. Proof of Theorem 1

which proves that

$$\lim_{\substack{t \rightarrow t_1 \\ t > t_1}} \frac{\dot{V}(t)}{2V(t)^{1/2}} = \dot{W}_r(t_1).$$

Thus, in the case where $V(t_1) = 0$ and $\beta > 0$, inequality (12) can be obtained by dividing inequality (6) by $V(t)^{1/2}$ and by taking the limit as t tends to t_1^+ . Finally, if $V(t_1) = 0$ and $\beta = 0$, we have $\dot{W}_r(t_1) = W(t_1) = 0$ and the inequality of the lemma is obvious. \square

Proposition 2 – Assume that N is chosen according to condition (3) and that p_0 satisfies (11)-(12). For f_n large enough ($n = 1, \dots, M$), there exist $m_7 > 0$ and $t_1 \geq t_0 = m_0/\text{Re } \lambda_M$ independent of p_0 such that, for any $t > t_1$, we have

$$\|\theta(t) - \theta^\infty\|_{\mathbb{C}^M} \leq m_7 e^{(\lambda_1 - \alpha)(t+t_0)/2}, \quad (13)$$

where α satisfies (4).

Proof. Let us set $\rho_M = \text{Re}(\lambda_M)$ and consider the function \mathcal{W} defined as

$$\mathcal{W}(t) = \chi(t)W(t),$$

where W is defined in Lemma 3 and (recall that $m_0 = \rho_M t_0$),

$$\chi(t) = \exp\left(\frac{\gamma m_1^2}{3\rho_M}(\rho_M t - m_0)^3\right) = \exp\left(\frac{\gamma m_1^2 \rho_M^2}{3}(t - t_0)^3\right).$$

Then, \mathcal{W} is right differentiable on \mathbb{R}_+ and for every $t > t_0$, we get from (12):

$$\dot{\mathcal{W}}_r(t) = \left\{ \dot{W}_r(t) + \gamma m_1^2 (\rho_M t - m_0)^2 W(t) \right\} \chi(t) \leq \gamma m_6 e^{(\lambda_1 - \alpha)t} \chi(t).$$

Applying the mean value inequality, we deduce that

$$\mathcal{W}(t) - \mathcal{W}(t_0) \leq \gamma m_6 \int_{t_0}^t e^{(\lambda_1 - \alpha)s} \chi(s) ds$$

and hence, since $W(t_0) = \mathcal{W}(t_0)$

$$W(t) \leq \frac{W(t_0)}{\chi(t)} + \gamma m_6 \frac{1}{\chi(t)} \int_{t_0}^t e^{(\lambda_1 - \alpha)s} \chi(s) ds. \quad (14)$$

The first term in the right-hand side of this inequality tends to zero as t tends to infinity. We will see that the same is true for the second term. To this end we divide

the integral appearing in the second term into two parts. As $\lambda_1 < \alpha$ we first have

$$\begin{aligned}
\int_{t_0}^{(t+t_0)/2} e^{(\lambda_1-\alpha)s} \chi(s) ds &\leq \int_{t_0}^{(t+t_0)/2} \chi(s) ds \\
&= \int_0^{\frac{\rho_M^3}{8}(t-t_0)^3} \frac{\exp\left(\frac{\gamma m_1^2}{3\rho_M} \sigma\right)}{3\rho_M \sigma^{2/3}} d\sigma \\
&\leq \exp\left(\frac{\gamma m_1^2 \rho_M^2}{24}(t-t_0)^3\right) \int_0^{\frac{\rho_M^3}{8}(t-t_0)^3} \frac{d\sigma}{3\rho_M \sigma^{2/3}} \\
&= \exp\left(\frac{\gamma m_1^2 \rho_M^2}{24}(t-t_0)^3\right) \frac{t-t_0}{2}. \tag{15}
\end{aligned}$$

On the other hand

$$\int_{(t+t_0)/2}^t e^{(\lambda_1-\alpha)s} \chi(s) ds \leq \chi(t) \int_{(t+t_0)/2}^t e^{(\lambda_1-\alpha)s} ds \leq \frac{\chi(t)}{\alpha-\lambda_1} e^{(\lambda_1-\alpha)(t+t_0)/2}. \tag{16}$$

Substituting inequalities (15) and (16) into (14), we obtain

$$W(t) \leq \frac{W(t_0)}{\chi(t)} + \gamma m_6 \left\{ \exp\left(-\frac{7\gamma m_1^2 \rho_M^2}{24}(t-t_0)^3\right) \frac{t-t_0}{2} + \frac{1}{\alpha-\lambda_1} e^{(\lambda_1-\alpha)(t+t_0)/2} \right\}.$$

Therefore there exist $t_1 \geq t_0$ and a constant $m_7 > 0$ such that if $t > t_1$ we have

$$\|\theta(t) - \theta^\infty\|_{\mathbb{C}^M} = W(t) \leq m_7 e^{(\lambda_1-\alpha)(t+t_0)/2}. \quad \square$$

We are now in position to prove Theorem 1.

Proof (Proof of Theorem 1). Proposition 2 obviously implies that $\theta(t)$ tends exponentially to θ^∞ as $t \rightarrow \infty$ since $\lambda_1 < \alpha$.

Concerning the convergence of $\hat{\mathbf{p}} - \mathbf{p}$, from (9) and (1), we obtain (taking the norms in \mathbb{C}^M)

$$\begin{aligned}
\|\hat{\mathbf{p}}(t) - \mathbf{p}(t)\| &\leq \|\mathbf{M}(t)\mathbf{E}\| \|\theta(t) - \theta^\infty\| + \|e^{t\mathbf{F}}\| \|\mathbf{p}(0)\| \\
&\quad + \left\| \int_0^t e^{(t-s)\mathbf{F}} (\mathbf{\Lambda}^M - \mathbf{F}) \Pi_M \mathbf{q}_-^N(s) ds \right\|.
\end{aligned}$$

Gathering (3), (13) and (11), we have, for every $t > t_1$,

$$\|\hat{\mathbf{p}}(t) - \mathbf{p}(t)\|_{\mathbb{C}^M} \leq m_2 m_7 e^{(\lambda_1-\alpha)t_0/2} e^{(3\lambda_1-\alpha)t/2} + m_5 e^{-\alpha t},$$

which implies that $\hat{\mathbf{p}}(t) - \mathbf{p}(t)$ tends exponentially to zero due to (4), and this ends the proof. \square

References

- Barbu, V. and R. Triggiani (2004). “Internal stabilization of Navier-Stokes equations with finite-dimensional controllers”. *Indiana Univ. Math. J.* **53** (5), pp. 1443–1494 (cit. on pp. 44–46).
- Chan, W. L. and B. Z. Guo (1989). “On the semigroups of age-size dependent population dynamics with spatial diffusion”. *Manuscripta Math.* **66** (2), pp. 161–181 (cit. on p. 42).
- Di Blasio, G. and A. Lorenzi (2011). “Direct and inverse problems in age-structured population diffusion”. *Discrete Contin. Dyn. Syst. Ser. S* **4** (3), pp. 539–563 (cit. on p. 41).
- Di Blasio, G. and A. Lorenzi (2013). “An identification problem in age-dependent population diffusion”. *Numer. Funct. Anal. Optim.* **34** (1), pp. 36–73 (cit. on p. 41).
- Engl, H. W., W. Rundell, and O. Scherzer (1994). “A regularization scheme for an inverse problem in age-structured populations”. *J. Math. Anal. Appl.* **182** (3), pp. 658–679 (cit. on p. 40).
- Gyllenberg, M., A. Osipov, and L. Päivärinta (2002). “The inverse problem of linear age-structured population dynamics”. *J. Evol. Equ.* **2** (2), pp. 223–239 (cit. on p. 40).
- Kato, T. (1995). *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin (cit. on p. 44).
- Kreisselmeier, G. (1977). “Adaptive observers with exponential rate of convergence”. *IEEE Trans. Automatic Control* **AC-22** (1), pp. 2–8. issn: 0018-9286 (cit. on pp. 41, 46, 50).
- Kreisselmeier, G. (1979). “The generation of adaptive law structures for globally convergent adaptive observers”. *IEEE Trans. Automatic Control* **AC-24** (3), pp. 510–513 (cit. on pp. 41, 46, 47, 50).
- Perasso, A. and U. Razafison (2016). “Identifiability problem for recovering the mortality rate in an age-structured population dynamics model”. *Inverse Probl. Sci. Eng.* **24** (4), pp. 711–728 (cit. on p. 40).
- Perthame, B. and J. P. Zubelli (2007). “On the inverse problem for a size-structured population model”. *Inverse Problems* **23** (3), pp. 1037–1052 (cit. on p. 40).
- Pilant, M. and W. Rundell (1991). “Determining a coefficient in a first-order hyperbolic equation”. *SIAM J. Appl. Math.* **51** (2), pp. 494–506 (cit. on p. 40).
- Ramdani, K., M. Tucsnak, and J. Valein (2016). “Detectability and state estimation for linear age-structured population diffusion models”. *ESAIM: M2AN* **50** (6), pp. 1731–1761 (cit. on pp. 41, 42, 44, 46).
- Rundell, W. (1993). “Determining the death rate for an age-structured population from census data”. *SIAM J. Appl. Math.* **53** (6), pp. 1731–1746 (cit. on p. 40).
- Traore, O. (2007). “Approximate controllability and application to data assimilation problem for a linear population dynamics model”. *IAENG Int. J. Appl. Math.* **37** (1), Paper 1, 12 (cit. on p. 41).

Traore, O. (2010). “Null controllability and application to data assimilation problem for a linear model of population dynamics”. *Ann. Math. Blaise Pascal* **17** (2), pp. 375–399 (cit. on p. 41).

Zabczyk, J. (1975/76). “Remarks on the algebraic Riccati equation in Hilbert space”. *Appl. Math. Optim.* **2** (3), pp. 251–258 (cit. on p. 43).

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