North-Western European Journal of Mathematics

A subordination principle. Applications

Éric Amar[1]

Received: October 14, 2014/Accepted: December 19, 2014/Online: May 5, 2015

Abstract

This subordination principle states roughly: if a property is true for Hardy spaces in some kind of domains in $\mathbb{C}^n$ then it is also true for the Bergman spaces of the same kind of domains in $\mathbb{C}^{n-1}$.

We give applications of this principle to Bergman-Carleson measures, interpolating sequences for Bergman spaces, A^p-Corona theorem and characterization of the zeros set of Bergman-Nevanlinna class.

These applications give precise results for bounded strictly-pseudo convex domains and bounded convex domains of finite type in $\mathbb{C}^n$.

Keywords: Hardy spaces, Bergman spaces, Carleson measures, Corona problem.

MSC: 32A50, 42B30.

1 Introduction

Let us start with some definitions. *In all the sequel,* domain will mean bounded connected open set in $\mathbb{C}^n$ with smooth $\mathcal{C}^\infty$ boundary defined by a real valued function $r \in \mathcal{C}^\infty(\mathbb{C}^n)$, i.e. $\Omega = \{z \in \mathbb{C}^n \,/\, r(z) < 0\}$, $\forall z \in \partial\Omega$, $\operatorname{grad} r(z) \neq 0$, with the defining function r such that $\forall z \in \Omega$, $-r(z) \simeq \mathrm{d}(z, \Omega^c)$ uniformly on $\bar{\Omega}$. (See the beginning of section 2 on page 33 for the existence of such a function).

Associate to it the "lifted" domain $\tilde{\Omega}$ in $(z,w) \in \mathbb{C}^{n+k}$ with defining function $\tilde{r}(z,w) := r(z) + |w|^2$.

Usually our defining functions will be pluri-sub-harmonic (PSH) or even strictly pluri-sub-harmonic (SPSH) in a neighborhood of $\bar{\Omega}$.

This operation keeps the nature of the domain:

- if Ω is pseudo-convex defined by a r PSH, $\tilde{\Omega}$ is still pseudo-convex defined by $\tilde{r}$ PSH;
- if Ω is strictly pseudo-convex defined by a r SPSH, so is $\tilde{\Omega}$;
- if Ω is convex defined by a function r convex , so is $\tilde{\Omega}$;

[1]Université de Bordeaux, IMB, UMR 5251, F-33400 Talence, France

- if Ω is convex of finite type m, defined by a function r convex, so is $\tilde{\Omega}$.

Moreover we still have $\forall (z,w)\in\tilde{\Omega},\ -(r(z)+|w|^2)\simeq \mathrm{d}((z,w),\tilde{\Omega}^c)$. Let $\mathrm{d}m(z)$ be the Lebesgue measure in $\mathbb{C}^n$ and $\mathrm{d}\sigma(z)$ be the Lebesgue measure on $\partial\Omega$. For $z\in\Omega$, let $\delta(z):=\mathrm{d}(z,\Omega^c)\simeq -r(z)$ be the distance from z to the boundary of Ω. For $k\in\mathbb{N}$, let v_k be the volume of the unit ball in $\mathbb{C}^k$ and set $\forall z\in\Omega, \mathrm{d}m_0(z):=\mathrm{d}m(z)$, $\forall k\geq 1, \forall z\in\Omega$, $\mathrm{d}m_k(z):=(k+1)v_{k+1}(-r(z))^k\,\mathrm{d}m(z)$ a weighted Lebesgue measure in Ω suitable for our needs. Clearly we have that $\mathrm{d}m_k(z)\simeq\delta(z)^k\,\mathrm{d}m(z)$. Let $\mathcal{U}$ be a neighbourhood of $\partial\Omega$ in Ω such that the normal projection π onto $\partial\Omega$ is a smooth well defined application. Define the Bergman, Hardy and Nevanlinna spaces as usual:

Definition 1 – Let f be a holomorphic function in Ω.
We say that $f\in A_k^p(\Omega)$ if $\|f\|_{k,p}^p:=\int_\Omega |f(z)|^p\,\mathrm{d}m_k(z)<\infty$.
We say that $f\in\mathcal{N}_k(\Omega)$ if $\|f\|_{\mathcal{N}_k}=\int_\Omega \ln^+|f(z)|\,\mathrm{d}m_k(z)<\infty$.
We say that $f\in H^p(\Omega)$ if $\|f\|_p^p:=\sup_{\epsilon>0}\int_{\{r(z)=-\epsilon\}}|f(\pi(z))|^p\,\mathrm{d}\sigma(z)<\infty$.
Finally we say that $f\in\mathcal{N}(\Omega)$ if $\|f\|_{\mathcal{N}}=\sup_{\epsilon>0}\int_{\{r(z)=-\epsilon\}}\ln^+|f(\pi(z))|\,\mathrm{d}\sigma(z)<\infty$.

This is meaningful because, for ϵ small enough, the set $\{r(z)=-\epsilon\}$ is a smooth manifold in Ω contained in $\mathcal{U}$. Now we can state our subordination lemma:

Theorem 1 – *(Subordination lemma) Let Ω be a domain in $\mathbb{C}^n$,$\tilde{\Omega}$ its lift in $\mathbb{C}^{n+k}$ and $F(z,w)\in H^p(\tilde{\Omega})$, we have $f(z):=F(z,0)\in A_{k-1}^p(\Omega)$ and $\|f\|_{A_{k-1}^p(\Omega)}\lesssim\|F\|_{H^p(\tilde{\Omega})}$; if $F(z,w)\in\mathcal{N}(\tilde{\Omega})$, then $f(z):=F(z,0)\in\mathcal{N}_{k-1}(\Omega)$ and $\|f\|_{\mathcal{N}_{k-1}(\Omega)}\lesssim\|F\|_{\mathcal{N}(\tilde{\Omega})}$.*

A function f, holomorphic in Ω, is in the Bergman space $A_{k-1}^p(\Omega)$ (resp. in the Nevanlinna Bergman space $\mathcal{N}_{k-1}(\Omega)$) if and only if the function $F(z,w):=f(z)$ is in the Hardy space $H^p(\tilde{\Omega})$ (resp. in the Nevanlinna class $\mathcal{N}(\tilde{\Omega})$) and we have $\|f\|_{A_{k-1}^p}\simeq\|F\|_{H^p(\tilde{\Omega})}$ (resp. $\|f\|_{\mathcal{N}_{k-1}(\Omega)}\simeq\|F\|_{\mathcal{N}(\tilde{\Omega})}$).

In the section 2 on page 33 we prove the subordination lemma as a consequence of a disintegration of Lebesgue measure. In the section 3 on page 39 we introduce the notion of a "good" family of polydiscs $\mathcal{P}$, directly inspired by the work of Catlin (1984) and introduced in É. Amar (2009b) together with a homogeneous hypothesis, (Hg). This notion allows us to define geometric Carleson measure, denoted as $\Lambda(\Omega)$, for Hardy spaces and denoted as $\Lambda_k(\Omega)$, for Bergman spaces and to put it in relation with the Carleson embedding theorem still for these two classes of spaces. In subsection 3.1 on page 41 we apply the subordination lemma to get a Bergman-Carleson embedding theorem from a Hardy-Carleson embedding one. The bounded strictly pseudo-convex domains have Hardy-Carleson embedding property by a result of Hormander[2], hence they have the Bergman-Carleson embedding property by this result. A direct application of it is the following:

[2]Hormander, 1967, "A L^p estimates for (pluri-)subharmonic functions".

Corollary 1 – *A positive Borel measure μ in a strictly pseudo-convex domain Ω in $\mathbb{C}^n$ verifies*

$$\forall p \geq 1, \forall f \in A^p_{k-1}(\Omega), \int_\Omega |f|^p \, \mathrm{d}\mu \lesssim \|f\|_{A^p_{k-1}(\Omega)} \iff \forall a \in \Omega, \mu(P_a(2)) \lesssim \delta(a)^{n+k},$$

where $P_a(2)$ is the polydisc of the good family $\mathcal{P}$ centered at a and of "radius" 2.

This characterization was already proved by Cima and Mercer[3] even for the spaces $A^p_\alpha(\Omega)$ with $\alpha \geq 0$. So, in the case where α is an integer we recover their characterization. We have also a characterization for convex domains of finite type, as shown in subsection 2 on page 33.

Theorem 2 – *Let Ω be a convex domain of finite type in $\mathbb{C}^n$; the measure μ verifies*

$$\exists p > 1, \exists C_p > 0, \forall f \in A^p_{k-1}(\Omega), \int_\Omega |f|^p \, \mathrm{d}\mu \leq C_p^p \|f\|^p_{A^p_{k-1}(\Omega)} \tag{1}$$

iff:

$$\exists C > 0 :: \forall a \in \Omega, \mu(\Omega \cap P_a(2)) \leq C m_{k-1}(\Omega \cap P_a(2)).$$

Hence if μ verifies (1) for a $p > 1$, it verifies (1) for all $q > 1$.

Now let Ω be a domain in $\mathbb{C}^n$. We say that the H^p-Corona theorem is true for Ω if we have: $\forall g_1, \ldots, g_m \in H^\infty(\Omega) :: \forall z \in \Omega, \sum_{j=1}^m \left|g_j(z)\right| \geq \delta > 0$ then $\forall f \in H^p(\Omega)$, $\exists (f_1, \ldots, f_m) \in (H^p(\Omega))^m :: f = \sum_{j=1}^m f_j g_j$. In the same vein, we say that the $A^p_{k-1}(\Omega)$-Corona theorem is true for Ω if we have:

$$\forall g_1, \ldots, g_m \in H^\infty(\Omega) :: \forall z \in \Omega, \sum_{j=1}^m \left|g_j(z)\right| \geq \delta > 0$$

then $\forall f \in A^p_{k-1}(\Omega), \exists (f_1, \ldots, f_m) \in (A^p_{k-1}(\Omega))^m :: f = \sum_{j=1}^m f_j g_j$. In the subsection 5 on page 51, we apply again the subordination principle, because the H^p-Corona theorem is true in these cases, to get:

Corollary 2 – *We have the $A^p_k(\Omega)$-Corona theorem in the following cases:*

- *with $p = 2$ if Ω is a bounded weakly pseudo-convex domain in $\mathbb{C}^n$;*
- *with $1 < p < \infty$ if Ω is a bounded strictly pseudo-convex domain in $\mathbb{C}^n$.*

[3] Cima and Mercer, 1995, "Composition operators between bergman spaces on convex domains in $\mathbb{C}^n$".

In section 4 on page 45 we define and study the interpolating sequences in a domain Ω. We also define the notion of dual bounded sequences in $H^p(\Omega)$ and in $A^p_k(\Omega)$, and applying the subordination principle to the result we proved for $H^p(\Omega)$ interpolating sequences[4], we get the following theorem.

Theorem 3 – *If Ω is a strictly pseudo-convex domain, or a convex domain of finite type in $\mathbb{C}^n$ and if $S\subset\Omega$ is a dual bounded sequence of points in $A^p_k(\Omega)$ then, for any $q<p$, S is $A^p_k(\Omega)$ interpolating with the linear extension property, provided that $p=\infty$ or $p\leq 2$.*

In the unit ball of $\mathbb{C}^n$, we have a better result:

Theorem 4 – *If $\mathbb{B}$ is the unit ball in $\mathbb{C}^n$ and if $S\subset\mathbb{B}$ is a dual bounded sequence of points in $A^p_k(\mathbb{B})$ then, for any $q<p$, S is $A^p_k(\Omega)$ interpolating with the linear extension property.*

Finally in the section 6 on page 52 we study zeros set for Nevanlinna Bergman functions. Let Ω be a domain in $\mathbb{C}^n$ and u a holomorphic function in Ω. Set $X:=\{z\in\Omega :: u(z)=0\}$ the zero set of u and $\Theta_X:=\partial\bar\partial\ln|u|$ its associated $(1,1)$ current of integration.

Definition 2 – A zero set X of a holomorphic function u in the domain Ω is in the Blaschke class, $X\in\mathcal{B}(\Omega)$, if there is a constant $C>0$ such that

$$\forall\beta\in\Lambda^\infty_{n-1,n-1}(\bar\Omega),\ \left|\int_\Omega(-r(z))\Theta_X\wedge\beta\right|\leq C\left\|\beta\right\|_\infty,$$

where $\Lambda^\infty_{n-1,n-1}(\bar\Omega)$ is the space of $(n-1,n-1)$ continuous form in $\bar\Omega$, equipped with the sup norm of the coefficients.

If $u\in\mathcal{N}(\Omega)$ then it is well known[5] that X is in the Blaschke class of Ω. We do the analogue for the Bergman spaces:

Definition 3 – A zero set X of a holomorphic function u in the domain Ω is in the Bergman-Blaschke class, $X\in\mathcal{B}_k(\Omega)$, if there is a constant $C>0$ such that

$$\forall\beta\in\Lambda^\infty_{n-1,n-1}(\bar\Omega),\ \left|\int_\Omega(-r(z))^{k+1}\Theta_X\wedge\beta\right|\leq C\left\|\beta\right\|_\infty,$$

where $\Lambda^\infty_{n-1,n-1}(\bar\Omega)$ is the space of $(n-1,n-1)$ continuous form in $\bar\Omega$, equipped with the sup norm of the coefficients.

[4] É. Amar, 2009b, "A weak notion of strict pseudo-convexity. applications and examples".

[5] Skoda, 1976, "Valeurs au bord pour les solutions de l'opérateur et caractérisation des zéros de la classe de Nevanlinna".

If $u \in \mathcal{N}_k(\Omega)$ then X is in the Bergman-Blaschke class of Ω as can be seen again by use of the subordination lemma. Hence exactly as for the Corona theorem we can set the definitions : we say that the *Blaschke characterization is true for* Ω if we have: $X \in \mathcal{B}(\Omega) \Rightarrow \exists u \in \mathcal{N}(\Omega)$ such that $X = \{z \in \Omega :: u(z) = 0\}$. And the same for the Bergman spaces: we say that the *Bergman-Blaschke characterization is true for* Ω if we have: $X \in \mathcal{B}_k(\Omega) \Rightarrow \exists u \in \mathcal{N}_k(\Omega)$ such that $X = \{z \in \Omega :: u(z) = 0\}$. We get, by use of the subordination lemma applied to the corresponding Nevanlinna Hardy results,

Corollary 3 – *The Bergman-Blaschke characterization is true in the following cases:*

- *if* Ω *is a strictly pseudo-convex domain in* $\mathbb{C}^n$ *;*
- *if* Ω *is a convex domain of finite type in* $\mathbb{C}^n$*.*

We stated and proved the subordination lemma for the ball in $\mathbb{C}^n$ in 1978[6], and, since then, we gave seminars and conferences about it in the general situation. As we have seen, Cima and Mercer[7] characterized the Carleson measures for the Bergman spaces in strictly pseudo-convex domains, but the other applications of the subordination principle are new; in particular all the applications to convex domains of finite type. Our treatment has the advantage to be systematic: a result for Hardy space gives automatically the analogous result for Bergman spaces.

2 The subordination lemma

Let $\Omega := \{z \in \mathbb{C}^n :: \rho(z) < 0\}$, $\partial\rho(z) \neq 0$ on $\partial\Omega$ with $\rho \in \mathcal{C}^2(\bar{\Omega})$. Let $\tilde{\Omega} := \{(z,w) \in \mathbb{C}^n \times \mathbb{C} :: \rho(z) + |w|^2 < 0\}$ be the lift of Ω in $\mathbb{C}^{n+1}$. We can always manage to have $\left|\operatorname{grad} \rho(z)\right| = 1$ for $z \in \partial\Omega$ by the well known following lemma.

Lemma 1 – *Let* Ω *be a domain in* $\mathbb{R}^n$*, we can always choose a defining function* s *for* Ω *such that* $\forall z \in \partial\Omega, |\operatorname{grad} s(z)| = 1$.

Proof. Because $\operatorname{grad} r(z) \neq 0$ on $\partial\Omega$, we take any smooth strictly positive extension h of $\frac{1}{|\operatorname{grad} r(z)|}$ in $\bar{\Omega}$; then set $s(z) = h(z)r(z)$. We have that $\operatorname{grad} s(z) = h \operatorname{grad} r(z) + r(z) \operatorname{grad} h(z) = h \operatorname{grad} r$ on $\partial\Omega$, hence $|\operatorname{grad} s| = 1$ on $\partial\Omega$. Of course because $h > 0$, we have that $\Omega = \{z \in \mathbb{R}^n :: s < 0\}$. □

Lemma 2 – *Let* Ω *be a domain in* $\mathbb{R}^n$*, defined by a function* $r \in \mathcal{C}^\infty$*, i.e.*

$$\Omega := \{x \in \mathbb{R}^n :: r(z) < 0\},\ \forall x \in \partial\Omega, \operatorname{grad} r(x) = 1.$$

[6] É. Amar, 1978, "Suites d'interpolation pour les classes de Bergman de la boule et du polydisque de $\mathbb{C}^n$".

[7] Cima and Mercer, 1995, "Composition operators between bergman spaces on convex domains in $\mathbb{C}^n$".

Then the Lebesgue measure σ on $\partial\Omega$ is given by

$$\forall g \in \mathcal{C}(\partial\Omega),\ \int_{\partial\Omega} g\,\mathrm{d}\sigma = \lim_{\eta\to 0}\frac{1}{\eta}\int_{\{-\eta\leq r(x)<0\}} \tilde{g}(x)\,\mathrm{d}m(x),$$

where $\tilde{g}(x)$ is any continuous extension of g near $\partial\Omega$.

Proof. Because $\partial\Omega$ is a codimension one manifold, $\forall x \in \partial\Omega$, $\operatorname{grad} r(x) = 1$ then $\{x \in \mathbb{R}^n :: -\eta \leq r(x) < 0\}$ is "half" a tube of thickness η around $\partial\Omega$, hence we can apply corollary 6.9.12 in Berger and Gostiaux (1988) or the original work by Weyl (1939). □

Lemma 3 – *Let Ω be a domain in $\mathbb{C}^n$. There is a defining function ρ for Ω and $\delta > 0$ such that $\left|\operatorname{grad}\rho(z)\right|^2 - 4\rho(z) \geq \min(4\delta, 1/4)$.*

Proof. Take a defining function ρ such that $\left|\operatorname{grad}\rho\right| = 1$ on $\partial\Omega$. Then the set $K := \{z \in \Omega :: \left|\operatorname{grad}\rho(z)\right| \leq 1/2\}$ is compact in Ω because $\left|\operatorname{grad}\rho(z)\right|$ is continuous. On this set K we have $-\rho(z) \geq \delta > 0$ because $\rho(z) < 0$ in Ω by definition of Ω, hence $\rho(z)$ attains its maximum $-\delta < 0$ on the compact K. Then we have $\forall z \in \bar{\Omega}$, $\left|\operatorname{grad}\rho(z)\right|^2 - 4\rho(z) \geq \min(4\delta, 1/4)$, because

- in $K, -\rho(z) \geq \delta \Rightarrow -4\rho(z) + \left|\operatorname{grad}\rho(z)\right|^2 \geq -4\rho(z) \geq 4\delta$;
- outside $K, \left|\operatorname{grad}\rho(z)\right| > 1/2 \Rightarrow \left|\operatorname{grad}\rho(z)\right|^2 > 1/4 \Rightarrow \left|\operatorname{grad}\rho(z)\right|^2 - 4\rho(z) \geq 1/4$.

Which completes the proof. □

Now back to the lifted domain $\tilde{\Omega}$. The boundary of $\tilde{\Omega}$ is defined by $\rho(z)+|w|^2 = 0$, hence on $\partial\tilde{\Omega}$ we have $|w|^2 = -\rho(z)$.

Lemma 4 – *Let Ω be a domain in $\mathbb{C}^n$. There is a defining function ρ for Ω and $\delta > 0$ such that $\left|\operatorname{grad}(\rho(z) + |w|^2)\right| \geq \min(2\delta, 1/2)$.*

Proof. Let us compute $\operatorname{grad}\left(\rho(z) + |w|^2\right) = \left(\frac{\partial\rho}{\partial x_1}, \frac{\partial\rho}{\partial y_1}, \dots, \frac{\partial\rho}{\partial x_n}, \frac{\partial\rho}{\partial y_n}, 2u, 2v\right)$;
where $z_j = x_j + \mathrm{i}y_j$ and $w = u + \mathrm{i}v$. Hence $\left|\operatorname{grad}(\rho(z) + |w|^2)\right|^2 = \left|\operatorname{grad}\rho(z)\right|^2 + 4|w|^2$. By lemma 3 we get on $\partial\tilde{\Omega}$, replacing δ by δ^2, $\left|\operatorname{grad}(\rho(z) + |w|^2)\right|^2 = \left|\operatorname{grad}\rho(z)\right|^2 - 4\rho(z) \geq \min(4\delta^2, 1/4)$. Taking square root we get the lemma. □

Then we have the main lemma of this section on the disintegration of the Lebesgue measure $\mathrm{d}\tilde{\sigma}$ on $\partial\tilde{\Omega}$:

Lemma 5 – *(Main lemma) For any continuous function g on $\overline{\tilde{\Omega}}$:*

$$\int_{\partial\tilde{\Omega}} g(z,w)\,\mathrm{d}\tilde{\sigma}(z,w) = \int_{\Omega} \sqrt{-\rho(z) + \frac{\left|\operatorname{grad}\rho(z)\right|^2}{4}} \int_{|w|^2=-\rho(z)} g(z,w)\,\mathrm{d}\,|w|\,\mathrm{d}m(z),$$

where $\mathrm{d}\,|w|$ *is the **normalized** Lebesgue measure on the circle* $|w|^2 = -\rho(z)$ *and* $\mathrm{d}m(z)$ *is the Lebesgue measure on* $\mathbb{C}^n$*.*

Proof. we want a defining function whose gradient has norm 1 on the boundary, hence we set $\forall (z,w) \in \partial\tilde{\Omega}$, $h(z,w) := \frac{1}{\left|\operatorname{grad}(\rho(z)+|w|^2)\right|}$, because we have by lemma 4 on page 34 that $\left|\operatorname{grad}(\rho(z)+|w|^2)\right| \geq \min(2\delta, 1/2)$ on $\partial\tilde{\Omega}$; by continuity we have $\left|\operatorname{grad}(\rho(z)+|w|^2)\right| \geq \frac{1}{2}\min(2\delta, 1/2)$ in a neighborhood V of $\partial\tilde{\Omega}$; as in lemma 1 on page 33 we set $\tilde{\rho}(z,w) := \frac{\rho(z)+|w|^2}{\left|\operatorname{grad}(\rho(z)+|w|^2)\right|}$, for $(z,w) \in V$ and we extend it to $\tilde{\Omega}$. Then $\left|\operatorname{grad}\tilde{\rho}(z,w)\right| = 1$ on $\partial\tilde{\Omega}$.

Fix $\eta > 0$ and set $\tilde{\Omega}_\eta := \{(z,w) \in \mathbb{C}^n \times \mathbb{C} :: \tilde{\rho}(z,w) < -\eta\} \subset \tilde{\Omega}$; let η be small enough such that $\tilde{\Omega} \setminus \tilde{\Omega}_\eta \subset V$. the Lebesgue measure on the manifold $\partial\tilde{\Omega}$ can be defined by lemma 2 on page 33 this way: $I := \int_{\partial\tilde{\Omega}} g(z,w)\,\mathrm{d}\sigma(z,w) = \lim_{\eta\to 0} \frac{1}{\eta} \int_{\tilde{\Omega}\setminus\tilde{\Omega}_\eta} g(z,w)\,\mathrm{d}m(z,w)$. Hence, by Fubini, $\int_{\tilde{\Omega}\setminus\tilde{\Omega}_\eta} g(z,w)\,\mathrm{d}m(z,w) = \int_{\Omega} \{\int_{-\eta\leq\tilde{\rho}(z,w)<0} g(z,w)\,\mathrm{d}m(w)\}\,\mathrm{d}m(z)$.

Fix $z \in \Omega$ and let us study

$$-\eta \leq \frac{\rho(z)+|w|^2}{\left|\operatorname{grad}\left(\rho(z)+|w|^2\right)\right|} < 0 \Rightarrow \rho(z)+|w|^2 < 0 \Rightarrow |w|^2 < -\rho(z).$$

Recall that

$$\left|\operatorname{grad}(\rho(z)+|w|^2)\right|^2 = \left|\operatorname{grad}\rho(z)\right|^2 + 4|w|^2 \Rightarrow \left|\operatorname{grad}(\rho(z)+|w|^2)\right| = \sqrt{\left|\operatorname{grad}\rho(z)\right|^2 + 4|w|^2}.$$

The other side of the inequality gives $-\eta\sqrt{\left|\operatorname{grad}\rho(z)\right|^2 + 4|w|^2} \leq \rho(z)+|w|^2 < 0$, hence raising to the square $(\rho(z)+|w|^2)^2 \leq \eta^2(\left|\operatorname{grad}\rho(z)\right|^2 + 4|w|^2)$. Set $a := -\rho(z) \geq 0$, $b := \left|\operatorname{grad}\rho(z)\right|^2 > 0$, $X := |w|^2 \geq 0$, then this inequality becomes

$$(X-a)^2 \leq \eta^2(b+4X) \Rightarrow X^2 - 2(a+2\eta^2)X + a^2 - \eta^2 b \leq 0.$$

This implies that X must be between the 2 roots:

$\Delta^2 := (a+2\eta^2)^2 - (a^2-\eta^2 b) = \eta^2(4a+b+4\eta^2)$; hence the roots are

$$X' := (a+2\eta^2) - \eta\sqrt{4a+b+4\eta^2} \quad ; \quad X'' := (a+2\eta^2) + \eta\sqrt{4a+b+4\eta^2}.$$

We already have that $|w|^2 = X < a = -\rho(z)$, hence, setting $c(\eta) := (a+2\eta^2) - \eta\sqrt{4a+b+4\eta^2}$. $-\eta \leq \tilde{\rho}(z,w) < 0 \iff c(\eta) \leq |w|^2 < a$. Now, g being continuous on $\bar{\tilde{\Omega}}$, we get, with $w = r\mathrm{e}^{\mathrm{i}\theta}$ in polar coordinates, $g(z, r\mathrm{e}^{\mathrm{i}\theta}) = g(z, \sqrt{-\rho(z)}\mathrm{e}^{\mathrm{i}\theta}) + o(\eta)$, the $o(\eta)$ being uniform with respect to z, w in V. So let $J := \frac{1}{\eta}\int_{-\eta\leq\tilde{\rho}(z,w)<0} g(z,w)\,\mathrm{d}m(w)$ we have $J = \frac{1}{\eta}\int_{c(\eta)\leq|w|^2<a} g(z,w)\,\mathrm{d}m(w)$; computing with polar coordinates,

$$J = \frac{1}{\eta}\int_{\sqrt{c(\eta)}}^{\sqrt{a}}\int_0^{2\pi}(g(z, \sqrt{-\rho(z)}\mathrm{e}^{\mathrm{i}\theta}) + o(\eta))\frac{\mathrm{d}\theta}{2\pi} r\,dr,$$

hence

$$J = \int_0^{2\pi}(g(z, \sqrt{-\rho(z)}\mathrm{e}^{\mathrm{i}\theta}) + o(\eta))\frac{\mathrm{d}\theta}{2\pi} \times \frac{1}{\eta}\int_{\sqrt{c(\eta)}}^{\sqrt{a}} r\,dr,$$

but

$$\frac{1}{\eta}\int_{\sqrt{c(\eta)}}^{\sqrt{a}} r\,dr = \frac{1}{2\eta}(a - c(\eta)) = \frac{1}{2\eta}(a - ((a+2\eta^2) - \eta\sqrt{4a+b+4\eta^2})) = \sqrt{a + \frac{b}{4} + \eta^2} - \eta,$$

so we get

$$J = (\sqrt{a + \frac{b}{4} + \eta^2} - \eta)\int_0^{2\pi}(g(z, \sqrt{-\rho(z)}\mathrm{e}^{\mathrm{i}\theta}) + o(\eta))\frac{\mathrm{d}\theta}{2\pi}.$$

Hence, letting $\eta \to 0$, we get $J \to \sqrt{a + \frac{b}{4}}\int_0^{2\pi} g(z, \sqrt{-\rho(z)}\mathrm{e}^{\mathrm{i}\theta})\frac{\mathrm{d}\theta}{2\pi}$. Putting it in I

$$I = \int_\Omega \sqrt{a + \frac{b}{4}}\left\{\int_0^{2\pi} g(z, \sqrt{-\rho(z)}\mathrm{e}^{\mathrm{i}\theta})\frac{\mathrm{d}\theta}{2\pi}\right\}\mathrm{d}m(z),$$

i.e.

$$I = \int_\Omega \sqrt{-\rho(z) + \frac{\left|\operatorname{grad}\rho(z)\right|^2}{4}}\int_{|w|^2=-\rho(z)} g(z,w)\,\mathrm{d}|w|\,\mathrm{d}m(z),$$

with $\mathrm{d}|w|$ the normalized Lebesgue measure on the circle $\{|w|^2 = -\rho(z)\}$. □

Corollary 4 – *Setting* $h(z) := \sqrt{-\rho(z) + \frac{|\operatorname{grad}\rho(z)|^2}{4}}$, *we have that* $\exists \alpha > 0, \beta > 0$ *such that*

- $\forall z \in \bar{\Omega}, \alpha \leq h(z) \leq \beta;$
- $\forall g \in \mathcal{C}(\partial\tilde{\Omega}), \int_{\partial\tilde{\Omega}} g(z,w)\,\mathrm{d}\sigma(z,w) = \int_\Omega h(z)\int_{|w|^2=-\rho(z)} g(z,w)\,\mathrm{d}|w|\,\mathrm{d}m(z).$

- $\forall f \in \mathcal{C}(\partial\tilde{\Omega}), \int_{\partial\tilde{\Omega}} f(z,w)\frac{1}{h(z)}\,d\sigma(z,w) = \int_{\Omega}\int_{|w|^2=-\rho(z)} f(z,w)\,d|w|\,dm(z).$

Proof. We have $\alpha = \min(\delta, 1/16)$ by lemma 3 on page 34 and $\beta = \|h\|_\infty < \infty$ because h is continuous on $\bar{\Omega}$ and $\bar{\Omega}$ is compact. The second point is the main lemma. So it remains to prove the last assertion and for it we set $g(z,w) := \frac{f(z,w)}{h(z)} \in \mathcal{C}(\partial\tilde{\Omega})$ and we apply the main lemma. □

Remark 1 – In the case of the unit ball $\mathbb{B}$ in $\mathbb{C}^n$ we get, with $\rho(z) = |z|^2 - 1$ as defining function, that $-\rho(z) + \frac{|\mathrm{grad}\,\rho(z)|^2}{4} = 1$, hence we have a disintegration of the Lebesgue measure on $\partial\tilde{\mathbb{B}}$ without weight.

Now we can prove our subordination theorem 1 on page 30 stated in the introduction. We copy from É. Amar[8], and adapt from the ball to this general case. We shall prove it with several steps.

Proposition 1 – *Let Ω be a domain in $\mathbb{C}^n$ and $\tilde{\Omega}$ its lift in $\mathbb{C}^{n+1}$. There are constants $\alpha > 0, \beta > 0$ depending only on Ω such that if $F \in H^p(\tilde{\Omega})$ then $F(z,0) \in A^p(\Omega)$ and $\|F(\cdot,0)\|_{A^p(\Omega)} \leq \frac{1}{\alpha}\|F\|_{H^p(\tilde{\Omega})}$. Conversely if $f \in A^p(\Omega)$, for $(z,w) \in \tilde{\Omega}$ set $F(z,w) := f(z)$, then we have $\|F\|_{H^p(\tilde{\Omega})} \leq \beta\|f\|_{A^p(\Omega)}$.*

Proof. If $F(z,w) \in H^p(\tilde{\Omega})$ we have $\|F\|_p^p := \sup_{\epsilon>0}\int_{\{\tilde{r}(z,w)=-\epsilon\}} |F(z,w)|^p\,d\tilde{\sigma}(z,w) < \infty$. Fix $\epsilon > 0$ and set $\tilde{\Omega} = \tilde{\Omega}_\epsilon := \{(z,w) \in \mathbb{C}^{n+1} :: \tilde{r}(z,w) < \epsilon\}$ to apply what precedes. By corollary 4 on page 36 the Lebesgue measure on $\partial\tilde{\Omega}$ is

$$\forall g \in \mathcal{C}(\overline{\tilde{\Omega}}), \int_{\partial\tilde{\Omega}} g(z,w)\,d\tilde{\sigma}(z,w) = \int_{\Omega} h(z)\int_{|w|^2=-\rho(z)} g(z,w)\,d|w|\,dm(z),$$

with $\forall z \in \bar{\Omega}, 0 < \alpha \leq h(z) \leq \beta < \infty$. So

$$\int_{\Omega} h(z)\left\{\int_{|w|^2=-\rho(z)} |F(z,w)|^p\,d|w|\right\}dm(z) =: \|F\|_p^p < \infty.$$

But $F(z,w)$ is holomorphic in w for z fixed, hence $|F(z,w)|^p$ is sub harmonic in w which implies $\int_{|w|^2=-\rho(z)} |F(z,w)|^p\,d|w| \geq |F(z,0)|^p$.

Hence $\int_{\Omega} h(z)|F(z,0)^p|\,dm(z) \leq \|F\|_p^p < \infty$, which implies, because $h(z)$ is bounded below and above in $\bar{\Omega}$, that $\int_{\Omega} |F(z,0)|^p\,dm(z) \leq \frac{1}{\alpha}\|F\|_p^p < \infty$.

Now apply this for $\tilde{\Omega}_\epsilon$ instead of $\tilde{\Omega}$; we have that $F(z,w)$ is continuous up to $\partial\tilde{\Omega}_\epsilon$ because $\epsilon > 0$. So $\int_{\partial\tilde{\Omega}_\epsilon} |F(z,w)|^p\,d\tilde{\sigma}(z,w) \geq \alpha\int_{\Omega_\epsilon} |F(z,0)|^p\,dm(z)$. Hence by

[8] É. Amar, 1978, "Suites d'interpolation pour les classes de Bergman de la boule et du polydisque de $\mathbb{C}^n$".

Fatou's lemma with $\epsilon \to 0$, $\alpha \|F(\cdot,0)\|_{A^p(\Omega)} \leq \|F\|^p_{H^p(\tilde{\Omega})}$. So we have the first part of the lemma.

Conversely if $f \in A^p(\Omega)$, setting $F(z,w) := f(z)$ and reversing the previous computations, using equalities this time,

$$\int_{\partial\tilde{\Omega}} |F|^p \, d\tilde{\sigma} = \int_{\Omega} h(z) \int_{|w|^2=-\rho(z)} |F(z,w)|^p \, d|w| \, dm(z) = \int_{\Omega} h(z)|f(z)|^p \, dm(z),$$

because $\int_{|w|^2=-\rho(z)} d|w| = 1$. Hence $\int_{\partial\tilde{\Omega}} |F|^p \, d\tilde{\sigma} \leq \beta \int_{\Omega} |f(z)|^p \, dm(z) = \beta \|f\|^p_{A^p(\Omega)}$. □

The only thing we used was that $|F(z,w)|^p$ is sub harmonic in w for z fixed. This being also true for $F(z,w) \in \mathcal{N}(\tilde{\Omega})$, the very same proof gives:

Proposition 2 – *Let Ω be a domain in $\mathbb{C}^n$ and $\tilde{\Omega}$ its lift in $\mathbb{C}^{n+1}$. There are constants $\alpha > 0, \beta > 0$ depending only on Ω such that if $F \in \mathcal{N}(\tilde{\Omega})$, then $F(z,0) \in \mathcal{N}_0(\Omega)$ and $\|F(\cdot,0)\|_{\mathcal{N}_0(\Omega)} \leq \frac{1}{\alpha}\|F\|_{\mathcal{N}(\tilde{\Omega})}$. Conversely if $f \in \mathcal{N}_0(\Omega)$, for $(z,w) \in \tilde{\Omega}$ set $F(z,w) := f(z)$, then we have $\|F\|_{\mathcal{N}(\tilde{\Omega})} \leq \beta \|f\|_{\mathcal{N}_0(\Omega)}$.*

Now if we start with a function $F(z,w) \in A^p(\tilde{\Omega})$ what happens? We have:

Proposition 3 – *Let Ω be a domain in $\mathbb{C}^n$ and $\tilde{\Omega}$ its lift in $\mathbb{C}^{n+1}$. If $F \in A^p(\tilde{\Omega})$, then $F(z,0) \in A^p_1(\Omega)$ and $\|F(\cdot,0)\|_{A^p_1(\Omega)} \leq \frac{1}{\pi}\|F\|_{A^p(\tilde{\Omega})}$. Conversely if $f \in A^p_1(\Omega)$, for $(z,w) \in \tilde{\Omega}$ set $F(z,w) := f(z)$, then we have $\|F\|_{A^p(\tilde{\Omega})} \leq \pi \|f\|_{A^p_1(\Omega)}$.*

Proof. By Fubini we have $\int_{\tilde{\Omega}} |F(z,w)|^p \, dm(z,w) = \int_{\Omega} \int_{|w|^2<-r(z)} |F(z,w)|^p \, dm(w) \, dm(z)$. But again $|F(z,w)|^p$ is sub harmonic in w for z fixed hence

$$|F(z,0)|^p \leq \frac{1}{\pi(-r(z))} \int_{|w|^2<-r(z)} |F(z,w)|^p \, dm(w),$$

because the area of the disc $\{|w|^2 < -r(z)\}$ is $\pi(-r(z))$. So

$$\pi \int_{\Omega} |F(z,0)|^p \, (-r(z)) \, dm(z) \leq \int_{\tilde{\Omega}} |F(z,w)|^p \, dm(z,w)$$

hence $\|F(\cdot,0)\|_{A^p_1(\Omega)} \leq \frac{1}{\pi}\|F\|_{A^p(\tilde{\Omega})}$.

Conversely if $F(z,w) = f(z) \in A^p_1(\Omega)$,

$$\begin{aligned} \int_{\tilde{\Omega}} |F(z,w)|^p \, dm(z,w) &= \int_{\Omega} |f(z)|^p \int_{|w|^2<-r(z)} dm(w) \, dm(z) \\ &= \int_{\Omega} |f(z)|^p \, \pi(-r(z)) \, dm(z) \end{aligned}$$

hence $\|F\|_{A^p(\tilde{\Omega})} \leq \pi \|f\|_{A^p_1(\Omega)}$. □

We have the same results with the same proofs replacing Bergman classes by Nevanlinna ones.

Proposition 4 – *Let Ω be a domain in $\mathbb{C}^n$ and $\tilde{\Omega}$ its lift in $\mathbb{C}^{n+1}$. If $F \in \mathcal{N}_0(\tilde{\Omega})$, then $F(z,0) \in \mathcal{N}_1(\Omega)$ and $\|F(\cdot,0)\|_{\mathcal{N}_1(\Omega)} \le \frac{1}{\pi}\|F\|_{\mathcal{N}_0(\tilde{\Omega})}$. Conversely if $f \in \mathcal{N}_1(\Omega)$, for $(z,w) \in \tilde{\Omega}$ set $F(z,w) := f(z)$, then we have $\|F\|_{\mathcal{N}_0(\tilde{\Omega})} \le \pi\|f\|_{\mathcal{N}_1(\Omega)}$.*

Proof (of the subordination lemma). We prove the subordination lemma for a one level lift. To get it for k levels lift, we just proceed by induction remarking that $\widetilde{(\tilde{\Omega}_{k-1})} = \tilde{\Omega}_k$. Let Ω be a domain in $\mathbb{C}^n$ and set $\tilde{\Omega}_k$ its k steps lift. Let $F(z,w_1,\dots,w_k) \in H^p(\tilde{\Omega}_k)$ then by the one level lift, proposition 1 on page 37 we have

$$F(z,w_1,\dots,w_{k-1},0) \in A^p(\tilde{\Omega}_{k-1}), \|F(\cdot,0)\|_{A^p(\tilde{\Omega}_{k-1})} \le \frac{1}{\alpha}\|F\|_{H^p(\tilde{\Omega}_k)}.$$

Now set $F_1(z,w_1,\dots,w_{k-1}) := F(z,w_1,\dots,w_{k-1},0) \in A^p(\tilde{\Omega}_{k-1})$ and apply proposition 3 on page 38, we get

$$F_1(z,w_1,\dots,w_{k-2},0) \in A_1^p(\tilde{\Omega}_{k-2}), \|F_1(\cdot,0)\|_{A_1^p(\tilde{\Omega}_{k-2})} \le \frac{1}{\pi}\|F_1\|_{A^p(\tilde{\Omega}_{k-1})} \le \frac{1}{\alpha\pi}\|F\|_{H^p(\tilde{\Omega}_k)}.$$

And so on. The converse is done the same way as for the Nevanlinna classes. □

Exactly the same induction gives the easy corollary:

Corollary 5 – *Let Ω be a domain in $\mathbb{C}^n$, $\tilde{\Omega}$ its lift in $\mathbb{C}^{n+k}$ and $F(z,w) \in A_l^p(\tilde{\Omega})$, we have $f(z) := F(z,0) \in A_{k+l}^p(\Omega)$ and $\|f\|_{A_{k+l}^p(\Omega)} \lesssim \|F\|_{A_l^p(\tilde{\Omega})}$; if $F(z,w) \in \mathcal{N}_l(\tilde{\Omega})$, then $f(z) := F(z,0) \in \mathcal{N}_{k+l}(\Omega)$ and $\|f\|_{\mathcal{N}_{k+l}(\Omega)} \lesssim \|F\|_{\mathcal{N}_l(\tilde{\Omega})}$.*

A function f, holomorphic in Ω, is in the Bergman space $A_{k+l}^p(\Omega)$ (resp. in the Nevanlinna Bergman space $\mathcal{N}_{k+l}(\Omega)$) if and only if the function $F(z,w) := f(z)$ is in the Bergman space $A_l^p(\tilde{\Omega})$ (resp. in the Nevanlinna class $\mathcal{N}_l(\tilde{\Omega})$) and we have $\|f\|_{A_{k+l}^p} \simeq \|F\|_{A_l^p(\tilde{\Omega})}$ (resp. $\|f\|_{\mathcal{N}_{k+l}(\Omega)} \simeq \|F\|_{\mathcal{N}_l(\tilde{\Omega})}$).

3 Geometric Carleson measures and p-Carleson measures

In order to define precisely the geometric Carleson measures, we need the notion of a "good" family of polydiscs, directly inspired by the work of Catlin (1984) and introduced in É. Amar (2009b).

Let $\mathcal{U}$ be a neighbourhood of $\partial\Omega$ in Ω such that the normal projection π onto $\partial\Omega$ is a smooth well defined application. Let $\alpha \in \partial\Omega$ and let $b(\alpha) = (L_1, L_2, \dots, L_n)$ be an orthonormal basis of $\mathbb{C}^n$ such that $(L_2,\dots,L_n)$ is a basis of the tangent complex space $T_{\alpha}^{\mathbb{C}}$ of $\partial\Omega$ at α; hence L_1 is the complex normal at α to $\partial\Omega$. Let $m(\alpha) = (m_1, m_2, \dots, m_n) \in \mathbb{N}^n$ be a multi-index at α with $m_1 = 1, \forall j \geq 2, m_j \geq 2$. For $a \in \mathcal{U}$ and $t > 0$ set $\alpha = \pi(a)$ and $P_a(t) := \prod_{j=1}^{n} tD_j$, the polydisc such that tD_j is the disc centered at a, parallel to $L_j \in b(\alpha)$, with radius $t\left|r(a)\right|^{1/m_j}$ (recall that we have $|r(a)| \simeq \delta(a)$). Set $b(a) := b(\pi(a)), m(a) := m(\pi(a))$, for $a \in \mathcal{U}$. This way we have a family of polydiscs $\mathcal{P} := \{P_a(t)\}_{a \in \mathcal{U}}$ defined by the family of basis $\{b(a)\}_{a \in \mathcal{U}}$, the family of multi-indices $\{m(a)\}_{a \in \mathcal{U}}$ and the number t. Notice that the polydisc $P_a(2)$ always overflows the domain Ω. It will be useful to extend this family to the whole of Ω. In order to do so let $(z_1, \dots, z_n)$ be the canonical coordinates system in $\mathbb{C}^n$ and for $a \in \Omega \setminus \mathcal{U}$, let $P_a(t)$ be the polydisc of center a, of sides parallel to the axis and radius $t\delta(a)$ in the z_1 direction and $t\delta(a)^{1/2}$ in the other directions. So the points $a \in \Omega \setminus \mathcal{U}$ have automatically a "minimal" multi-index $m(a) = (1, 2, \dots, 2)$. Now we can set:

Definition 4 – We say that $\mathcal{P}$ is a "good family" of polydiscs for Ω if the $m_j(a)$ are uniformly bounded on Ω and if it exists $\delta_0 > 0$ such that all the polydiscs $P_a(\delta_0)$ of $\mathcal{P}$ are contained in Ω. In this case we call $m(a)$ the multi-type at a of the family $\mathcal{P}$.

We notice that, for a good family $\mathcal{P}$, by definition the multi-type is always finite. Moreover there is no regularity assumptions on the way that the basis $b(a)$ varies with respect to $a \in \Omega$. We can see easily that there are always good families of polydiscs in a domain Ω in $\mathbb{C}^n$: for a point $a \in \Omega$, take any orthonormal basis $b(a) = (L_1,\ L_2, \dots, L_n)$, with L_1 the complex normal direction, and the "minimal" multitype $m(a) = (1, 2, \dots, 2)$. Then, because the level sets $\partial\Omega_a$ are uniformly of class $\mathcal{C}^2$ and compact, we have the existence of a uniform $\delta_0 > 0$ such that the family $\mathcal{P}$ is a good one. As seen in É. Amar (2009b), in the strictly pseudo-convex domains, this family with "minimal" multi-type is the right one. We can give the definitions relative to Carleson measures.

Definition 5 – A positive borelian measure μ on Ω is a geometric Carleson measure, $\mu \in \Lambda(\Omega)$, if

$$\exists C = C_\mu > 0 :: \forall a \in \Omega,\ \mu(\Omega \cap P_a(2)) \leq C\sigma(\partial\Omega \cap P_a(2)).$$

Definition 6 – A positive borelian measure μ on Ω is a p-Carleson measure in Ω if

$$\exists C > 0 :: \forall f \in H^p(\Omega),\ \int_{\Omega} |f(z)|^p \, \mathrm{d}\mu(z) \leq C^p \|f\|_{H^p(\Omega)}^p.$$

And analogously for the Bergman spaces:

Definition 7 – A positive borelian measure μ on Ω is a k-geometric Bergman-Carleson measure, $\mu \in \Lambda_k(\Omega)$, if

$$\exists C = C_\mu > 0 :: \forall a \in \Omega,\ \mu(\Omega \cap P_a(2)) \leq C m_{k-1}(\Omega \cap P_a(2)).$$

Notice the gap $k \to k-1$.

Definition 8 – A positive borelian measure μ is (p,k)-Bergman-Carleson measure in Ω if

$$\exists C>0 :: \forall f \in A^p_{k-1}(\Omega),\ \int_\Omega |f(z)|^p \, \mathrm{d}\mu(z) \le C^p \|f\|^p_{A^p_{k-1}(\Omega)}.$$

Definition 9 – We shall say that the domain Ω has the p-Carleson embedding property, p-CEP, if

$$\forall \mu \in \Lambda(\Omega),\ \exists C = C_\mu > 0 :: \forall f \in H^p(\Omega),\ \int_\Omega |f|^p \, \mathrm{d}\mu \le C \|f\|^p_{H^p(\Omega)}.$$

And the same for the Bergman spaces:

Definition 10 – We shall say that the domain Ω has the (p,k)-Bergman-Carleson embedding property, (p,k)-BCEP, if

$$\forall \mu \in \Lambda_k(\Omega),\ \exists C = C_{\mu,p} > 0 :: \forall f \in A^p_{k-1}(\Omega),\ \int_\Omega |f|^p \, \mathrm{d}\mu \le C \|f\|^p_{A^p_{k-1}(\Omega)}.$$

3.1 The subordination lemma applied to Carleson measures

We shall fix $k \in \mathbb{N}$ and lift the measure on the domain $\tilde{\Omega} := \{\tilde{r}(z,w) := r(z) + |w|^2 < 0\}$, with $w = (w_1, \dots, w_k) \in \mathbb{C}^k$. We already know how to lift a function, the lifted measure $\tilde{\mu}$ of a measure μ is just $\tilde{\mu} := \mu \otimes \delta$, with δ the delta Dirac measure of the origin in $\mathbb{C}^k$. We shall need a lemma linking Bergman and Hardy geometric Carleson measures. Let Ω be a domain in $\mathbb{C}^n$, $\tilde{\Omega}$ be its lift in $\mathbb{C}^{n+k}$, and suppose that $\tilde{\Omega}$ is equipped with a good family of polydiscs $\tilde{\mathcal{P}}$, we have the definition:

Definition 11 – We shall say that the good family of polydiscs $\tilde{\mathcal{P}}$ on the domain $\tilde{\Omega}$ is "homogeneous" if

$$\begin{gathered}\exists t>0, \exists C>0 :: \forall a \in \tilde{\Omega}, \Omega \cap \tilde{P}_a(2) \neq \emptyset, \\ \forall b \in \Omega \cap \tilde{P}_a(2), \tilde{P}_b(t) \supset \tilde{P}_a(2) \text{ and } \tilde{\sigma}(\partial\tilde{\Omega} \cap \tilde{P}_b(t)) \le C \tilde{\sigma}(\partial\tilde{\Omega} \cap \tilde{P}_a(2)),\end{gathered} \tag{Hg}$$

where $\Omega = \tilde{\Omega} \cap \{w = 0\} \subset \tilde{\Omega}$.

Naturally the domain Ω is equipped with the family $\mathcal{P}$ induced by $\tilde{\mathcal{P}}$ the following way $\forall a \in \Omega, P_a(u) := \tilde{P}_{(a,0)}(u) \cap \{w = 0\}$, which is easily seen to be a good family for Ω. As examples we have the strictly pseudo-convex domains and the convex domains of finite type, because both are domains of homogeneous type in the sense of Coifman and Weiss[9].

[9] Coifman and Weiss, 1971, *Analyse harmonique non commutative sur certains espaces homogènes*.

Lemma 6 – *Let* $(\Omega,\tilde{\Omega})$ *be as above and suppose that* $\tilde{\Omega}$ *is equipped with a good family of polydiscs* $\tilde{\mathcal{P}}$ *which verifies the hypothesis* (Hg)*. The measure* μ *is a* k*-geometric Bergman-Carleson measure in* Ω *iff the measure* $\tilde{\mu}$ *is a geometric Carleson measure in* $\tilde{\Omega}$*.*

Proof. Suppose that μ is a k-geometric Bergman-Carleson measure in Ω, we want to show: $\exists C>0 :: \forall (a,b)\in\tilde{\Omega}, \tilde{\mu}(\tilde{\Omega}\cap\tilde{P}_{(a,b)}(2))\leq C\tilde{\sigma}(\partial\tilde{\Omega}\cap\tilde{P}_{(a,b)}(2))$, with $\tilde{P}_c$ the polydisc of center $c=(a,b)\in\tilde{\Omega}$ of the family $\tilde{\mathcal{P}}$. Let us see first the case where $b=0$, i.e. $(a,b)=(a,0)\in\Omega\subset\tilde{\Omega}$. Then, by definition of $\tilde{\mu}$, we have $\tilde{\mu}(\tilde{\Omega}\cap\tilde{P}_{(a,0)}(2))=\mu(\Omega\cap P_a(2))$. On the other hand, we have, exactly as in the proof of the subordination lemma, $\tilde{\sigma}(\partial\tilde{\Omega}\cap\tilde{P}_{(a,0)}(2))\simeq\int_{\Omega\cap P_a(2)}kv_k(-r(z))^{k-1}\,\mathrm{d}m(z)=m_{k-1}(\Omega\cap P_a(2))$. But if μ is a k-geometric Bergman-Carleson measure in Ω, we have $\exists C>0 :: \forall a\in\Omega, \mu(\Omega\cap P_a(2))\leq Cm_{k-1}(\Omega\cap P_a(2))$, so $\tilde{\mu}(\tilde{\Omega}\cap\tilde{P}_{(a,0)}(2))=\mu(\Omega\cap P_a(2))\leq Cm_{k-1}(\Omega\cap P_a(2))\simeq C\tilde{\sigma}(\partial\tilde{\Omega}\cap\tilde{P}_{(a,0)}(2))$. Now take a general $\tilde{P}_{(a,b)}(2)$. In order for $\tilde{\mu}(\tilde{\Omega}\cap\tilde{P}_{(a,b)}(2))$ to be non zero, we must have $\tilde{P}_{(a,b)}(2)\cap\{w=0\}\neq\emptyset\Rightarrow\exists(c,0)\in\tilde{P}_{(a,b)}(2)\cap\{w=0\}$. By the (Hg) hypothesis, this means that we have $\tilde{P}_{(c,0)}(t)\supset\tilde{P}_{(a,b)}(2)$ with the uniform control $\tilde{\sigma}(\partial\tilde{\Omega}\cap\tilde{P}_{(c,0)}(t))\lesssim\tilde{\sigma}(\partial\tilde{\Omega}\cap\tilde{P}_{(a,b)}(2))$. We apply the above inequality $\tilde{\mu}(\tilde{\Omega}\cap\tilde{P}_{(a,b)}(2))\leq\tilde{\mu}(\tilde{\Omega}\cap\tilde{P}_{(c,0)}(t))\leq Cm_k(\Omega\cap P_c(t))=C\tilde{\sigma}(\partial\tilde{\Omega}\cap\tilde{P}_{(c,0)}(t))\lesssim\tilde{\sigma}(\partial\tilde{\Omega}\cap\tilde{P}_{(a,b)}(2))$, hence $\tilde{\mu}$ is a geometric Carleson measure on $\tilde{\Omega}$.

Conversely suppose that $\tilde{\mu}$ is a geometric Carleson measure on $\tilde{\Omega}$, this means $\forall(a,b)\in\tilde{\Omega}$, $\tilde{\mu}(\tilde{\Omega}\cap\tilde{P}_{(a,b)}(2))\leq C\tilde{\sigma}(\partial\tilde{\Omega}\cap\tilde{P}_{(a,b)}(2))$, hence, in particular for $b=0$, $\forall a\in\Omega$, $\tilde{\mu}(\tilde{\Omega}\cap P_{(a,0)}(2))\leq C\tilde{\sigma}(\tilde{\Omega}\cap P_{(a,0)}(2))$, but then, by definition of $\tilde{\mu}$ and with the previous computation of $\tilde{\sigma}(\tilde{\Omega}\cap P_{(a,0)}(2))$, we get $\forall a\in\Omega, \mu(\Omega\cap P_a(2))\leq Cm_{k-1}(\Omega\cap P_a(2))$, hence the measure μ is a k-geometric Bergman-Carleson measure in Ω. □

Now we shall use the subordination lemma to get a Bergman-Carleson embedding theorem from a Hardy-Carleson embedding one.

Theorem 5 – *Let* $(\Omega,\tilde{\Omega})$ *be as usual and suppose that* $\tilde{\Omega}$ *is equipped with a good family of polydiscs* $\tilde{\mathcal{P}}$ *which verifies the hypothesis* (Hg)*. If the lifted domain* $\tilde{\Omega}$ *has the* p*-CEP then* Ω *has the* (p,k)*-BCEP.*

Proof. Suppose the positive measure μ is a k-geometric Bergman-Carleson measure; by the previous lemma, we have that the lifted measure $\tilde{\mu}$ is a geometric Carleson measure in $\tilde{\Omega}$. By the p-CEP we have $\forall F\in H^p(\tilde{\Omega}), \int_{\tilde{\Omega}}|F|^p\,\mathrm{d}\tilde{\mu}\leq C_\mu^p\|F\|_{H^p(\tilde{\Omega})}^p$. Choose $f(z)\in A_{k-1}^p(\Omega)$ and set $\forall(z,w)\in\tilde{\Omega}, F(z,w)=f(z)$. By the subordination lemma we have $\|f\|_{A_{k-1}^p(\Omega)}\simeq\|F\|_{H^p(\tilde{\Omega})}$, and by definition of $\tilde{\mu}$, we have

$$\int_\Omega|f|^p\,\mathrm{d}\mu=\int_{\tilde{\Omega}}|F|^p\,\mathrm{d}\tilde{\mu}\leq C_\mu^p\|F\|_{H^p(\tilde{\Omega})}^p\lesssim\|f\|_{A_{k-1}^p(\Omega)},$$

hence μ is a (k,p)-Bergman-Carleson measure in Ω. □

Theorem 6 – *Let $(\Omega, \tilde{\Omega})$ be as usual and suppose that $\tilde{\Omega}$ is equipped with a good family of polydiscs $\tilde{\mathcal{P}}$ which verifies the hypothesis (Hg). If p-Carleson implies geometric Carleson in $\tilde{\Omega}$, then (p,k)-Bergman-Carleson implies geometric k-Bergman-Carleson in Ω.*

Proof. If the positive measure μ is (p,k)-Bergman-Carleson in Ω then $\tilde{\mu}$ is a p-Carleson measure in $\tilde{\Omega}$ by lemma 6 on page 42 hence a geometric Carleson measure in $\tilde{\Omega}$ by the assumption of the theorem. Then applying lemma 6 on page 42 we get that μ is a k-geometric Carleson measure in Ω hence the theorem. □

Remark 2 – The definition of geometric Carleson measures depends on the chosen good family of polydiscs on the domain; the theorem asserts the equivalence of properties between a domain Ω and its lift $\tilde{\Omega}$. The fact that a lifted domain $\tilde{\Omega}$ equipped with a good family of polydiscs $\tilde{\mathcal{P}}$ has the Carleson embedding property has to be proved directly but if it has the p-CEP then Ω equipped with the induced family $\mathcal{P}$ has the (p,k)-BCEP without any further proof.

3.2 Application to strictly pseudo-convex domains.

Corollary 6 – *Let Ω be a strictly pseudo-convex domain equipped with its minimal good family of polydiscs, then Ω has the (p,k) Bergman Carleson embedding property.*

Proof. The domain Ω equipped with its minimal good family has the p-CEP by Hormander[10], hence we can apply theorem 6. □

This corollary gives a characterization of the (p,k)-Bergman-Carleson measures of the strictly pseudo-convex domains. Let Ω be a strictly pseudo-convex domain and $\tilde{\Omega}$ its lift in $\mathbb{C}^{n+k}$. Let $\tilde{\mathcal{P}}$ be its minimal good family of polydiscs in $\tilde{\Omega}$; one can see easily that the induced family of polydiscs $\mathcal{P}$ on Ω is again the minimal good family of polydiscs. Recall that $\forall a \in \Omega, \delta(a) = \mathrm{d}(a, \partial\Omega)$; we have this characterization:

Corollary 7 – *A positive Borel measure μ in a strictly pseudo-convex domain in $\mathbb{C}^n$ is a (p,k)-Bergman-Carleson measure iff:*

$$\forall a \in \Omega,\ \mu(P_a(2)) \lesssim \delta(a)^{n+k}.$$

This means that it is a characterization of the measures such that

$$\forall p \geq 1,\ \forall f \in A^p_{k-1}(\Omega),\ \int_\Omega |f|^p \,\mathrm{d}\mu \lesssim \|f\|_{A^p_{k-1}(\Omega)}.$$

In particular this characterization is independent of $p \geq 1$.

[10] Hormander, 1967, "A L^p estimates for (pluri-)subharmonic functions".

Proof. Let $\tilde{\Omega}$ be the lift of Ω in $\mathbb{C}^{n+k}$ and $\tilde{\mu}$ be the lift of μ on $\tilde{\Omega}$. Suppose that μ is a (p,k)-Bergman Carleson measure in Ω, then $\tilde{\mu}$ is a p-Carleson measure in $\tilde{\Omega}$ by lemma 6 on page 42 then by a theorem of Hormander[11] the p-Carleson measures are precisely the geometric ones in $\tilde{\Omega}$, hence we have $\forall \tilde{a} \in \tilde{\Omega}, \tilde{\mu}(\tilde{\Omega} \cap \tilde{P}_{\tilde{a}}(2)) \lesssim \tilde{\sigma}(\partial\tilde{\Omega} \cap \tilde{P}_{\tilde{a}}(2))$.

Now let $a \in \Omega, \tilde{a} := (a,0) \in \tilde{\Omega}$ then a classical computation gives $\tilde{\sigma}(\partial\tilde{\Omega} \cap \tilde{P}_{\tilde{a}}(2)) \lesssim \tilde{\delta}(\tilde{a})^{n+k} = \delta(a)^{n+k}$. By the definition of $\tilde{\mu}$ we have $\delta(a)^{n+k} \gtrsim \tilde{\mu}(\tilde{\Omega} \cap \tilde{P}_{\tilde{a}}(2)) = \mu(\tilde{P}_{\tilde{a}}(2) \cap \Omega) = \mu(P_a(2) \cap \Omega)$, so $\forall a \in \Omega, \mu(P_a(2) \cap \Omega) \lesssim \delta(a)^{n+k}$.

Now suppose that $\forall a \in \Omega, \mu(P_a(2) \cap \Omega) \lesssim \delta(a)^{n+k}$ then we have, by the definition of $\tilde{\mu}$, with $\tilde{a} := (a,0) \in \tilde{\Omega}$, $\tilde{\mu}(\tilde{\Omega} \cap \tilde{P}_{\tilde{a}}(2)) \leq \tilde{\delta}(a)^{n+k} \simeq \tilde{\sigma}(\partial\tilde{\Omega} \cap \tilde{P}_{\tilde{a}}(2))$. Doing exactly as in the proof of lemma 6 on page 42 we have the same inequality with a bigger constant for all $\tilde{a} \in \tilde{\Omega}$, hence $\tilde{\mu}$ is a geometric Carleson measure in $\tilde{\Omega}$. So by Hormander[12], $\tilde{\mu}$ is a p-Carleson measure in $\tilde{\Omega}$ hence we have the embedding $\forall F \in H^p(\tilde{\Omega}), \int_{\tilde{\Omega}} |F|^p \, d\tilde{\mu} \lesssim \|F\|_{H^p(\tilde{\Omega})}$.

Now we take $f \in A^p_{k-1}(\Omega)$ and we set $\forall (z,w) \in \tilde{\Omega}, F(z,w) := f(z)$ by the subordination lemma we have $\|F\|_{H^p(\tilde{\Omega})} \simeq \|f\|_{A^p_{k-1}(\Omega)}$ and $\int_{\tilde{\Omega}} |F|^p \, d\tilde{\mu} = \int_\Omega |f|^p \, dm_{k-1} \lesssim \|F\|^p_{H^p(\tilde{\Omega})} \simeq \|f\|^p_{A^p_{k-1}(\Omega)}$. □

Cima and Mercer[13] characterized the Carleson measures for the spaces $A^p_\alpha(\Omega)$ for Ω strictly pseudo-convex, and with $\alpha \geq 0$. In the case where α is an integer we recover their characterization, because one has easily, when Ω is a strictly pseudo-convex domain, that $P_a(2) \cap \Omega \simeq W(\pi(a), \delta(a))$ where $W(\zeta, h)$ is the classical Carleson window in Ω.

Remark 3 – In the case of the unit ball Ω of $\mathbb{C}^n$, $\tilde{\Omega} \subset \mathbb{C}^{n+1}$ N. Varopoulos indicated me an alternative proof for the fact that $F(z,w) \in H^p(\tilde{\Omega}) \Rightarrow F(z,0) \in A_p(\Omega)$: the Lebesgue measure on $\{w = 0\} \cap \tilde{\Omega}$ is easily seen to be a geometric Carleson measure in $\tilde{\Omega}$, hence by the Carleson-Hörmander embedding theorem[14] we have

$$\int_\Omega |F(z,0)|^p \, dm(z) \leq C \|F\|_{H^p(\tilde{\Omega})},$$

and the assertion. Of course this is still valid in codimension $k \geq 1$, with the weighted Lebesgue measure on Ω, and for strictly pseudo-convex domains because the Carleson-Hörmander embedding theorem is still valid there. But this is just one direction of the lemma, it works only if there is a Carleson embedding theorem and this proof is much less elementary than the previous one. In fact we can reverse things and say that one part of the subordination lemma asserts that the weighted Lebesgue measure on Ω is always a Carleson measure in $\tilde{\Omega}$, Ω strictly pseudo-convex or not.

[11] Hormander, 1967, "A L^p estimates for (pluri-)subharmonic functions".

[12] Ibid.

[13] Cima and Mercer, 1995, "Composition operators between bergman spaces on convex domains in $\mathbb{C}^n$".

3.3 Application to convex domains of finite type in $\mathbb{C}^n$

In É. Amar (2009b) we prove a Carleson embedding theorem for the convex domains of finite type in $\mathbb{C}^n$.

Theorem 7 – *Let Ω be a convex domain of finite type in $\mathbb{C}^n$; if the measure μ is a geometric Carleson measure we have*

$$\forall p > 1,\ \exists C_p > 0,\ \forall f \in H^p(\Omega),\ \int_\Omega |f|^p \, d\mu \leq C_p^p \|f\|_{H^p}^p.$$

Conversely if the positive measure μ is p-Carleson for a $p \in [1,\infty[$, then it is a geometric Carleson measure, hence it is q-Carleson for any $q \in]1,\infty[$.

We already know that if Ω is a convex domain of finite type, so is $\tilde{\Omega}$ with the same type. Moreover the hypothesis (Hg) is true for these domains equipped with a (slightly modified) McNeal family of polydiscs, so we can apply what precedes in this case to get from the Carleson embedding theorem the Bergman-Carleson embedding one.

Theorem 8 – *Let Ω be a convex domain of finite type in $\mathbb{C}^n$; if the measure μ is a k-geometric Bergman-Carleson measure, i.e.*

$$\exists C > 0 :: \forall a \in \Omega, \mu(\Omega \cap P_a(2)) \leq C m_{k-1}(\Omega \cap P_a(2)),$$

we have

$$\forall p > 1, \exists C_p > 0, \forall f \in A_{k-1}^p(\Omega), \int_\Omega |f|^p \, d\mu \leq C_p^p \|f\|_{A_{k-1}^p(\Omega)}^p.$$

Conversely if the positive measure μ is (p,k)-Bergman-Carleson for a $p \in [1,\infty[$, then it is a k-geometric Bergman-Carleson measure, hence it is (q,k)-Bergman-Carleson for any $q \in]1,\infty[$.

4 Interpolating sequences for Bergman spaces

4.1 On Bergman and Szegö projections

Let Ω be a domain in $\mathbb{C}^n$, recall the definition of its Szegö projection: this is the orthogonal projection P from $L^2(\partial\Omega)$ onto $H^2(\Omega)$; we shall note its kernel by $S(z,\zeta)$, i.e. $\forall f \in L^2(\partial\Omega), Pf(z) = \int_{\partial\Omega} S(z,\zeta) f(\zeta) \, d\sigma(\zeta)$. The same way, recall the definition of the Bergman projection : this is the orthogonal projection P_k from $L^2(\Omega, dm_k)$

[14] Hormander, 1967, "A L^p estimates for (pluri-)subharmonic functions".

onto $A^2_k(\Omega)$, the holomorphic functions on Ω still in $L^2(\Omega, dm_k)$. We shall note its kernel by $B_k(z,\zeta)$ i.e. $\forall f \in L^2(\Omega, dm_k), P_k f(z) = \int_\Omega B_k(z,\zeta)f(\zeta)\,dm_k(\zeta)$. Let $\tilde{\Omega}$ be the lifted domain of Ω in $\mathbb{C}^{n+k}$; we shall use the notation $\forall z \in \Omega, \tilde{z} := (z,0) \in \tilde{\Omega}$.

Corollary 8 – *For any $a \in \Omega$, the Bergman kernel $B_{k-1}(z,a)$ and the Szegö kernel $\tilde{S}((z,w),\tilde{a})$ for the lifted domain $\tilde{\Omega}$, verify,*

$$\forall a \in \Omega, \forall z \in \Omega, B_{k-1}(z,a) = \tilde{S}(\tilde{z},\tilde{a}).$$

Moreover we have

$$\forall a \in \Omega, \|B_{k-1}(\cdot,a)\|_{A^p_{k-1}(\Omega)} \simeq \left\|\tilde{S}(\cdot,\tilde{a})\right\|_{H^p(\tilde{\Omega})}.$$

Proof. Let $f \in A(\Omega)$ be a holomorphic function in Ω, continuous up to $\partial\Omega$. Let $\forall (z,w) \in \tilde{\Omega}, F(z,w) := f(z)$. We have $\int_\Omega f(z)\bar{B}_{k-1}(z,a)\,dm_{k-1}(z) = f(a) = F(a,0) = \int_{\partial\tilde{\Omega}} F(z,w)\bar{\tilde{S}}((z,w),\tilde{a})\,d\sigma(z,w)$, by the reproducing property of these kernels. But F does not depend on w and $\bar{\tilde{S}}((z,w),\tilde{a})$ is anti-holomorphic in w for z fixed in Ω, so

$$\frac{1}{\eta}\int_{\{w\in\mathbb{C}^k::-\eta-r(z)\leq|w|^2<-r(z)\}} \bar{\tilde{S}}((z,w),\tilde{a})\,dm(w) \to \overline{\tilde{S}((z,0),\tilde{a})}v_k k(-r(z))^{k-1},$$

by the proof of the subordination lemma, hence

$$\begin{aligned}\int_\Omega f(z)\bar{B}_{k-1}(z,a)\,dm_{k-1}(z) &= \int_\Omega f(z)\bar{\tilde{S}}((z,0),\tilde{a})v_k k(-r(z))^{k-1}\,dm(z)\\ &= \int_\Omega f(z)\bar{\tilde{S}}((z,0),\tilde{a})\,dm_{k-1}(z).\end{aligned}$$

So we have

$$\forall f \in A(\Omega), \int_\Omega f(z)(\bar{\tilde{S}}((z,0),\tilde{a}) - \bar{B}_{k-1}(z,a))\,dm_{k-1}(z) = 0,$$

hence $\tilde{S}((z,0),\tilde{a}) - B_{k-1}(z,a) \perp A(\Omega)$ in $A^2_{k-1}(\Omega)$. But $\tilde{S}((z,0),\tilde{a}) - B_k(z,a)$ is holomorphic in z, hence $\forall z \in \Omega, \tilde{S}((z,0),\tilde{a}) = B_{k-1}(z,a)$. The second part is a direct application of the first part in the subordination theorem 1 on page 30. □

4.2 Interpolating sequences

For $a \in \Omega$, let $k_a(z) := S(z,a)$ denotes the Szegö kernel of Ω at the point a. It is also the reproducing kernel for $H^2(\Omega)$, i.e. $\forall a \in \Omega, \forall f \in H^2(\Omega), f(a) = \int_{\partial\Omega} f(z)\bar{k}_a(z)\,d\sigma(z) = \langle f, k_a\rangle$. Set $\|k_a\|_p := \|k_a\|_{H^2(\Omega)}$ and:

Definition 12 – We say that the sequence Λ of points in Ω is $H^p(\Omega)$ interpolating if

$$\forall \lambda \in \ell^p(\Lambda), \exists f \in H^p(\Omega) :: \forall a \in \Lambda, f(a) = \lambda_a \|k_a\|_{p'},$$

with p' the conjugate exponent for $p, \frac{1}{p} + \frac{1}{p'} = 1$.

We say that Λ has the linear extension property if Λ is $H^p(\Omega)$ interpolating and if moreover there is a bounded linear operator $E\ \ell^p(\Lambda) \to H^p(\Omega)$ making the interpolation, i.e. $\forall \lambda \in \ell^p(\Lambda), E(\lambda) \in H^p(\Omega), \forall a \in \Lambda, E(\lambda)(a) = \lambda_a \|k_a\|_{p'}$.

A weaker notion is the dual boundedness:

Definition 13 – We shall say that the sequence Λ of points in Ω is dual bounded in $H^p(\Omega)$ if there is a bounded sequence of elements in $H^p(\Omega), \{\rho_a\}_{a\in\Lambda} \subset H^p(\Omega)$ which dualizes the associated sequence of reproducing kernels, i.e.

$$\exists C > 0 :: \forall a \in \Lambda, \left\|\rho_a\right\|_p \leq C, \forall a, c \in \Lambda, \langle \rho_a, k_c \rangle = \delta_{a,c} \|k_c\|_{p'}.$$

If Λ is $H^p(\Omega)$ interpolating then it is dual bounded in $H^p(\Omega)$: just interpolate the elements of the basic sequence in $\ell^p(\Lambda)$. The converse is the crux of the characterization by Carleson[15] of $H^\infty(\mathbb{D})$ interpolating sequences and the same by Shapiro and Shields[16] for $H^p(\mathbb{D})$ interpolating sequences in $\mathbb{D}$. We do the same for the Bergman spaces. For $k \in \mathbb{N}$ and $a \in \Omega$, let $b_{k,a}(z) := B_k(z,a)$ denotes the Bergman kernel of Ω at the point a. It is also the reproducing kernel for $A^2_k(\Omega)$, i.e. $\forall a \in \Omega, \forall f \in A^2_k(\Omega), f(a) = \int_\Omega f(z)\bar{b}_{k,a}(z)\, \mathrm{d}m_k(z) = \langle f,\ b_{k,a}\rangle$. Now we set $\left\|b_{k,a}\right\|_p := \left\|b_{k,\ a}\right\|_{A^p_k(\Omega)}$ and:

Definition 14 – We say that the sequence Λ of points in Ω is $A^p_k(\Omega)$ interpolating if

$$\forall \lambda \in \ell^p(\Lambda), \exists f \in A^p_k(\Omega) :: \forall a \in \Lambda, f(a) = \lambda_a \left\|b_{k,a}\right\|_{p'},$$

with p' the conjugate exponent for $p, \frac{1}{p} + \frac{1}{p'} = 1$.

We say that Λ has the linear extension property if Λ is $A^p_k(\Omega)$ interpolating and if moreover there is a bounded linear operator $E\ \ell^p(\Lambda) \to A^p_k(\Omega)$ making the interpolation.

Definition 15 – We shall say that the sequence Λ of points in Ω is dual bounded in $A^p_k(\Omega)$ if there is a bounded sequence of elements in $A^p_k(\Omega), \{\rho_a\}_{a\in\Lambda} \subset A^p_k(\Omega)$ which dualizes the associated sequence of reproducing kernels, i.e.

$$\exists C > 0 :: \forall a \in \Lambda, \left\|\rho_a\right\|_p \leq C, \forall a, c \in \Lambda, \langle \rho_a, b_{k,c} \rangle = \delta_{a,c} \left\|b_{k,a}\right\|_{p'}.$$

[15] Carleson, 1958, *An interpolation problem for bounded analytic functions*.

[16] Shapiro and Shields, 1961, "On some interpolation problems for analytic functions".

Again if Λ is $A^p_k(\Omega)$ interpolating then it is dual bounded in $A^p_k(\Omega)$: just interpolate the elements of the basic sequence in $\ell^p(\Lambda)$.

4.3 Case of the unit disc $\mathbb{D}$ in $\mathbb{C}$

In that case the interpolating sequences for $H^\infty(\mathbb{D})$ where characterized by Carleson[17] and for $H^p(\mathbb{D})$ by Shapiro and Shields[18]. The interpolating sequences for the Bergman spaces $A^p_k(\mathbb{D})$ were characterized by Seip[19].

In these cases it appears that dual boundedness implies interpolation. For Hardy spaces dual boundedness is easily seen to be equivalent to the Carleson condition and for Bergman spaces, it is proved by Schuster and Seip[20].

4.4 General case

We shall apply the subordination lemma to interpolating sequences in general domains Ω. Let $\tilde{\Omega}$ be the lifted domain in $\mathbb{C}^{n+k}$ associated to Ω. Let $\tilde{\Lambda}$ be the sequence Λ viewed in $\tilde{\Omega}$, $\tilde{\Lambda} := \Lambda \subset \Omega \subset \tilde{\Omega}$. Let us denote by $k_{\tilde{a}}(z,w) := S((z,w),\tilde{a})$ the Szegö kernel of $\tilde{\Omega}$, for $\tilde{a} = (a,0)$.

Theorem 9 – *Let Ω be a domain in $\mathbb{C}^n$ and $\tilde{\Omega}$ its lift to $\mathbb{C}^{n+k}$. If $\Lambda \subset \Omega$ is a sequence of points in Ω, let $\tilde{\Lambda}$ be the sequence Λ viewed in $\tilde{\Omega}, \tilde{\Lambda} := \Lambda \subset \Omega \subset \tilde{\Omega}$. We have:*

1. *Λ is dual bounded in $A^p_{k-1}(\Omega)$ iff $\tilde{\Lambda}$ is dual bounded in $H^p(\tilde{\Omega})$.*
2. *Λ is $A^p_{k-1}(\Omega)$ interpolating iff $\tilde{\Lambda}$ is $H^p(\tilde{\Omega})$ interpolating.*
3. *Λ has the linear extension property in $A^p_{k-1}(\Omega)$ iff $\tilde{\Lambda}$ has the linear extension property in $H^p(\tilde{\Omega})$.*

Proof.

1. Suppose that Λ is dual bounded $A^p_{k-1}(\Omega)$ and let $\{\rho_a\}_{a\in\Lambda} \subset A^p_{k-1}(\Omega)$ be the dual sequence to the sequence $\{b_{k-1,a}\}_{a\in\Lambda}$; extend it to $\tilde{\Omega}$: $\forall a \in \Lambda$, $\Gamma_a(z,w) := \rho_a(z)$, then the subordination lemma gives us that $\|\Gamma_a\|_{H^p(\tilde{\Omega})} \simeq \left\|\rho_a\right\|_{A^p_{k-1}(\Omega)}$ and we have, using corollary 8 on page 46,

$$\forall a,c \in \Lambda,\ \langle \Gamma_a, k_{\tilde{c}}\rangle = \langle \Gamma_a, S((\cdot,0),\ \tilde{c})\rangle = \langle \rho_a, B(\cdot,\ c)\rangle = \langle \rho_a, b_{k-1,c}\rangle = \delta_{ab}\left\|b_{k-1,c}\right\|_{p'}$$

[17] Carleson, 1958, *An interpolation problem for bounded analytic functions.*
[18] Shapiro and Shields, 1961, “On some interpolation problems for analytic functions”.
[19] Seip, 1993, “Beurling type density theorems in the unit disk”.
[20] Schuster and Seip, 1998, “A Carleson type condition for interpolation in Bergman spaces”.

because Λ is dual bounded in $A^p_{k-1}(\Omega)$. Then we have, by corollary 8 on page 46,

$$\forall \tilde{c} = (c,0), c \in \Omega, \left\|b_{k-1,c}\right\|_{A^p_{k-1}(\Omega)} \simeq \|k_{\tilde{c}}\|_{H^p(\Omega)}, \tag{2}$$

hence $\forall a, c \in \Lambda, \langle \Gamma_a, k_{\tilde{c}} \rangle = \delta_{ac} \left\|b_{k-1,c}\right\|_{p'} \simeq \delta_{ac} \|k_{\tilde{c}}\|_{p'}$ hence $\tilde{\Lambda}$ is dual bounded in $H^p(\tilde{\Omega})$. Because we used only equivalences in this proof, it works also for the converse, hence if $\tilde{\Lambda}$ is dual bounded in $H^p(\tilde{\Omega})$ then Λ is dual bounded in $A^p_{k-1}(\Omega)$.

2. Suppose that $\tilde{\Lambda}$ is interpolating in $H^p(\tilde{\Omega})$. We want to show that Λ is $A^p_{k-1}(\Omega)$ interpolating, so let $\mu = \{\mu_a\}_{a\in\Lambda} \in \ell^p(\Lambda)$ the sequence to be interpolated. Set $\lambda = \{\lambda_{\tilde{a}}\}_{a\in\Lambda}$ with $\forall \tilde{a} \in \tilde{\Lambda}, \lambda_{\tilde{a}} := \mu_a \times \frac{\|b_{k-1,\,a}\|_{A^{p'}_{k-1}(\Omega)}}{\|k_{\tilde{a}}\|_{H^{p'}(\tilde{\Omega})}}$; then $\lambda \in \ell^p(\tilde{\Lambda}), \|\lambda\|_p \simeq \left\|\mu\right\|_p$ by equation (2). Let $F \in H^p(\tilde{\Omega})$ be the function making the interpolation of the sequence λ, which exists because $\tilde{\Lambda}$ is $H^p(\tilde{\Omega})$ interpolating. It means that $F(\tilde{a}) = \lambda_a \|k_{\tilde{a}}\|_{H^{p'}(\tilde{\Omega})} = \mu_a \left\|b_{k-1,a}\right\|_{A^{p'}_{k-1}(\Omega)}$. Set $\forall z \in \Omega, f(z) := F(z,0)$ then we have $\forall a \in \Lambda, f(a) = F(a,0) = F(\tilde{a}) = \mu_a \left\|b_{k-1,a}\right\|_{A^{p'}_{k-1}(\Omega)}$. Hence Λ is $A^p_{k-1}(\Omega)$ interpolating. Again the converse is straightforward because we use only equivalences.

3. Suppose that $\tilde{\Lambda}$ has the bounded extension linear property, i.e. there is a linear operator $\tilde{E}\colon \ell^p(\tilde{\Lambda}) \to H^p(\tilde{\Omega})$ such that $F(z,w) := \tilde{E}(\lambda)(z,w)$, $F \in H^p(\tilde{\Omega})$, $\forall a \in \Lambda$, $F(\tilde{a}) = \mu_a \|k_{\tilde{a}}\|_{H^{p'}(\tilde{\Omega})}$, $\|F\|_{H^p(\tilde{\Omega})} \lesssim \left\|\mu\right\|_p$. With the same notations λ and μ as above, set $f(z) := F(z,0) = \tilde{E}(\mu)(z,0) =: E(\lambda)(z)$, then clearly E is linear in λ and then still using the subordination lemma we have $\|f\|_{A^p_{k-1}(\Omega)} \lesssim \|F\|_{H^p(\tilde{\Omega})} \lesssim \left\|\mu\right\|_p \simeq \|\lambda\|_p$ and $\forall a \in \Lambda,\ f(a) = \mu_{\tilde{a}} \|k_{\tilde{a}}\|_{H^{p'}(\tilde{\Omega})} = \lambda_a \left\|b_{k-1,a}\right\|_{A^{p'}_{k-1}(\Omega)}$. Hence $\lambda \to E(\lambda)$ is bounded from $\ell^p(\Lambda)$ in $A^p_{k-1}(\Omega)$ and Λ is $A^p_{k-1}(\Omega)$ interpolating with the linear extension. Again the converse is straightforward. □

4.5 Application to strictly pseudo-convex domains

In É. Amar (2008) we proved a general theorem on interpolating sequences in the spectrum of a uniform algebra. In the case of strictly pseudo-convex domains, it says that:

Theorem 10 – *If Ω is a strictly pseudo-convex domain in $\mathbb{C}^n$ and if $\Lambda \subset \Omega$ is a dual bounded sequence of points in $H^p(\Omega)$, then, for any $q < p, \Lambda$ is $H^q(\Omega)$ interpolating with the linear extension property, provided that $p = \infty$ or $p \leq 2$.*

We have, as a consequence of the subordination lemma the following theorem:

Theorem 11 – *Let Ω be a strictly pseudo-convex domain in $\mathbb{C}^n$ and $\Lambda \subset \Omega$ be a dual bounded sequence of points in $A^p_k(\Omega)$, then, for any $q < p, \Lambda$ is $A^p_k(\Omega)$ interpolating with the linear extension property, provided that $p = \infty$ or $p \leq 2$.*

Proof. Let $\tilde{\Omega}$ be the lift of Ω in $\mathbb{C}^{n+k+1}$ and $\tilde{\Lambda} \subset \tilde{\Omega}$ the sequence Λ viewed in $\tilde{\Omega}$. We apply theorem 9 on page 48 *(i)* to have that $\tilde{\Lambda}$ is dual bounded in $H^p(\tilde{\Omega})$ because Λ is dual bounded in $A^p_k(\Omega)$. Now we apply theorem 10 on page 49 to get that $\tilde{\Lambda}$ is $H^q(\tilde{\Omega})$ interpolating with $q < p$, and has the bounded linear extension property, provided that $p = \infty$ or $p \leq 2$. Then again theorem 9 on page 48 *(iii)* to get the same for Λ in $A^p_k(\Omega)$. □

We have a better result for the unit ball in $\mathbb{C}^n$. In É. Amar (2009a) we proved:

Theorem 12 – *If Λ is a dual bounded sequence in the unit ball $\mathbb{B}$ of $\mathbb{C}^n$ for the Hardy space $H^p(\mathbb{B})$, then for any $q < p$, Λ is $H^q(\mathbb{B})$ interpolating with the bounded linear extension property.*

So copying the proof of theorem 11, just replacing theorem 10 on page 49 by theorem 12 we get:

Theorem 13 – *Let Λ be a dual bounded sequence in the unit ball $\mathbb{B}$ of $\mathbb{C}^n$ for the Bergman space $A^p_k(\mathbb{B})$, then for any $q < p$, S is $A^q_k(\mathbb{B})$ interpolating with the bounded linear extension property.*

Remark 4 – If we apply this theorem in the unit disc $\mathbb{D}$ of $\mathbb{C}$ we get that if Λ is a dual bounded sequence in $A^p_k(\mathbb{D})$ then it is interpolating in $A^q_k(\mathbb{D})$ for any $q < p$. In this particular case, one variable, the Schuster-Seip theorem[21] says that we have the interpolation up to $q = p$.

4.6 Application to convex domains of finite type

To apply the general theorem on interpolating sequences in the spectrum of a uniform algebra to the case of convex domains of finite type in $\mathbb{C}^n$, we need to have a precise knowledge of the good family of polydiscs associated to the domain and in É. Amar (2009b), we proved:

Theorem 14 – *If Ω is a convex domain of finite type in $\mathbb{C}^n$ and if $\Lambda \subset \Omega$ is a dual bounded sequence of points in $H^p(\Omega)$, then, for any $q < p$, Λ is $H^q(\Omega)$ interpolating with the linear extension property, provided that $p = \infty$ or $p \leq 2$.*

Then, again, copying the proof of theorem 11, just replacing theorem 10 on page 49 by theorem 14 we get:

[21] Schuster and Seip, 1998, "A Carleson type condition for interpolation in Bergman spaces".

Theorem 15 – *If Ω is a convex domain of finite type in $\mathbb{C}^n$ and if $\Lambda \subset \Omega$ is a dual bounded sequence of points in $A^p_{k-1}(\Omega)$ then, for any $q < p$, Λ is $A^p_{k-1}(\Omega)$ interpolating with the linear extension property, provided that $p = \infty$ or $p \leq 2$.*

Remark 5 – We applied the subordination principle since 1978[22] essentially in this case. For instance in D. Amar and É. Amar (1978) we used it to show that the interpolating sequences for $H^p(\mathbb{B})$, with $\mathbb{B}$ the unit ball in $\mathbb{C}^n$, $n \geq 2$, are different for different values of p, opposite to the one variable case of $H^p(\mathbb{D})$.

5 The H^p-Corona theorem for Bergman spaces

Let Ω be a domain in $\mathbb{C}^n$. We say that the H^p-Corona theorem is true for Ω if we have: $\forall g_1,\dots,g_k \in H^\infty(\Omega) :: \forall z \in \Omega,\ \sum_{j=1}^m \left|g_j(z)\right| \geq \delta > 0$ then $\forall f \in H^p(\Omega)$, $\exists (f_1,\dots,f_m) \in (H^p(\Omega))^m :: f = \sum_{j=1}^m f_j g_j$. In the same vein, we say that the $A^p_{k-1}(\Omega)$-Corona theorem is true for Ω if we have:

$$\forall g_1,\dots,g_m \in H^\infty(\Omega) :: \forall z \in \Omega, \sum_{j=1}^m \left|g_j(z)\right| \geq \delta > 0 \tag{3}$$

then $\forall f \in A^p_{k-1}(\Omega)$, $\exists (f_1,\dots,f_m) \in (A^p_{k-1}(\Omega))^m :: f = \sum_{j=1}^m f_j g_j$. Then we have:

Theorem 16 – *Suppose that the H^p-Corona is true for the domain $\tilde{\Omega}$, then the $A^p_{k-1}(\Omega)$-Corona theorem is also true for Ω.*

Proof. Let $\tilde{\Omega}$ be the lifted domain; then set $\forall j = 1,\dots,m$, $g_j \in H^\infty(\Omega)$, $f \in H^p(\Omega)$, $G_j(z,w) := g_j(z)$, $F(z,w) := f(z)$. Clearly the G_j are in $H^\infty(\tilde{\Omega})$ and by the subordination lemma, $F \in H^p(\tilde{\Omega})$. Moreover, if the condition of equation (3) is true, we have $\forall (z,w) \in \tilde{\Omega}$, $\sum_{j=1}^m \left|G_j(z,w)\right| \geq \delta$ with the same δ. So we can apply the hypothesis: $\exists (F_1,\dots,F_m) \in (H^p(\tilde{\Omega}))^m :: F = \sum_{j=1}^m F_j G_j$.

Now set $f_j(z) = F_j(z,0)$ then applying again the subordination lemma, we have

$$f(z) = F(z,0) = \sum_{j=1}^m F_j(z,0)G_j(z,0) = \sum_{j=1}^m f_j(z)g_j(z). \qquad \square$$

[22] D. Amar and É. Amar, 1978, "Sur les suites d'interpolation en plusieurs variables";
É. Amar, 1978, "Suites d'interpolation pour les classes de Bergman de la boule et du polydisque de $\mathbb{C}^n$".

5.1 Application to pseudo-convex domains

Corollary 9 – *We have the $A^p_{k-1}(\Omega)$-Corona theorem in the following cases:*

- *with $p = 2$ if Ω is a bounded weakly pseudo-convex domain in $\mathbb{C}^n$;*
- *with $1 < p < \infty$ if Ω is a bounded strictly pseudo-convex domain in $\mathbb{C}^n$.*

The first case because Andersson[23] (with a preprint in 1990) proved the H^2-Corona theorem for Ω bounded weakly pseudo-convex domain in $\mathbb{C}^n$; the last one for two generators because we proved[24] the H^p-Corona theorem for two generators in the ball; for any number of generators because Andersson and Carlsson[25] proved the H^p-Corona theorem in this case.

6 Zeros set of the Nevanlinna-Bergman class

Let Ω be a domain in $\mathbb{C}^n$, u a holomorphic function in Ω. Set $X := \{z \in \Omega :: u(z) = 0\}$ the zero set of u and $\Theta_X := \partial\bar{\partial}\ln|u|$ its associated $(1,1)$ current of integration.

Definition 16 – An analytic set $X := u^{-1}(0), u \in \mathcal{H}(\Omega)$, in the domain Ω is in the Blaschke class, $X \in \mathcal{B}(\Omega)$, if there is a constant $C > 0$ such that

$$\forall \beta \in \Lambda^\infty_{n-1,n-1}(\bar{\Omega}), \left|\int_\Omega (-r(z))\Theta_X \wedge \beta\right| \leq C \left\|\beta\right\|_\infty$$

where $\Lambda^\infty_{n-1,n-1}(\bar{\Omega})$ is the space of $(n-1, n-1)$ continuous form in $\bar{\Omega}$, equipped with the sup norm of the coefficients.

If $u \in \mathcal{N}(\Omega)$ then it is well known[26] that X is in the Blaschke class of Ω. We do the analogue for the Bergman spaces:

Definition 17 – An analytic set $X := u^{-1}(0), u \in \mathcal{H}(\Omega)$, in the domain Ω is in the Bergman-Blaschke class, $X \in \mathcal{B}_{k-1}(\Omega)$, if there is a constant $C > 0$ such that

$$\forall \beta \in \Lambda^\infty_{n-1,n-1}(\bar{\Omega}), \left|\int_\Omega (-r(z))^{k+1}\Theta_X \wedge \beta\right| \leq C\left\|\beta\right\|_\infty,$$

where $\Lambda^\infty_{n-1,n-1}(\bar{\Omega})$ is the space of $(n-1, n-1)$ continuous form in $\bar{\Omega}$, equipped with the sup norm of the coefficients.

[23] Andersson, 1994, "The H^2 corona problem and $\bar{\partial}_b$ in weakly pseudoconvex domains."

[24] É. Amar, 1991, "On the corona problem"; with É. Amar, 1980, *Généralisation d'un théorème de Wolff à la boule de C^n*, already in 1980.

[25] Andersson and Carlsson, 1994, "Wolff-type estimates for $\bar{\partial}_b$ and the H^p-corona problem in strictly pseudo-convex domains"; see also É. Amar and Menini, 2000, "Universal divisors in Hardy spaces".

[26] Skoda, 1976, "Valeurs au bord pour les solutions de l'opérateur et caractérisation des zéros de la classe de Nevanlinna".

If $u \in \mathcal{N}_{k-1}(\Omega)$ then X is in the Bergman-Blaschke class of Ω, for instance again by use the subordination lemma from the case $\mathcal{N}(\tilde{\Omega})$. Hence exactly as for the Corona theorem we can set the definitions: we say that the *Blaschke characterization is true for* Ω if we have: $X \in \mathcal{B}(\Omega) \Rightarrow \exists u \in \mathcal{N}(\Omega)$ such that $X = \{z \in \Omega :: u(z) = 0\}$. And the same for the Bergman spaces: we say that the *Bergman-Blaschke characterization is true for* Ω if we have: $X \in \mathcal{B}_k(\Omega) \Rightarrow \exists u \in \mathcal{N}_k(\Omega)$ such that $X = \{z \in \Omega :: u(z) = 0\}$.

Theorem 17 – *Suppose that the Blaschke characterization is true for the lifted domain* $\tilde{\Omega}$, *then the Bergman-Blaschke characterization is also true for* Ω.

Proof. Let $\tilde{\Omega}$ be the lifted domain in $\mathbb{C}^{n+k}$ of Ω; then set $X = u^{-1}(0)$, Θ_X its associated current and suppose that $X \in \mathcal{B}_k(\Omega)$. This means that

$$\forall \beta \in \Lambda^{\infty}_{n-1,n-1}(\bar{\Omega}), \left| \int_{\Omega} (-r(z))^{k+1} \Theta_X \wedge \beta \right| \leq C \left\| \beta \right\|_{\infty}.$$

Let $\forall w \in \mathbb{C}^k, U(z,w) := u(z), \tilde{X} := U^{-1}(0) \cap \tilde{\Omega} \subset \tilde{\Omega}, \tilde{\Theta}_{\tilde{X}} = \partial\bar{\partial}\ln|U|$; we shall show that $\tilde{X} \in \mathcal{B}(\tilde{\Omega})$. We have that $\tilde{\Theta}_{\tilde{X}}$ does not depend on w, hence, $\forall \tilde{\beta} \in \Lambda^{\infty}_{n+k-1,n+k-1}(\overline{\tilde{\Omega}})$,

$$A := \int_{\tilde{\Omega}} (-\tilde{r}(z,w)) \tilde{\Theta}_{\tilde{X}} \wedge \tilde{\beta} = \int_{\Omega} \Theta_X(z) \wedge \int_{|w|^2 < -r(z)} -(r(z) + |w|^2) \tilde{\beta}(z,w).$$

Because Θ_X is a $(1,1)$ current depending only on z, this means that in the integral in w we have only the terms containing $dw_1 \wedge d\bar{w}_1 \wedge \cdots \wedge dw_k \wedge d\bar{w}_k$, the other terms being 0 against Θ_X. So this integral in w gives a $(n-1, n-1)$ form in z. Now set $\beta_1(z) := \int_{|w|^2 < -r(z)} (1 + \frac{|w|^2}{-r(z)}) \tilde{\beta}(z,w)$, we have $A = \int_{\Omega} \Theta_X(z) \wedge (-r(z)) \beta_1(z)$ and, because $1 + \frac{|w|^2}{-r(z)} < 2$ in $\{|w|^2 < -r(z)\}$, we have $\left|\beta_1(z)\right| \leq 2 \left\|\tilde{\beta}\right\|_{\infty} \int_{|w|^2 < -r(z)} dm_k(w) \leq 2v_k \left\|\tilde{\beta}\right\|_{\infty} (-r(z))^k$, because we get the volume in $\mathbb{C}^k$ of the ball centered in 0 and of radius $\sqrt{-r(z)}$. Set $\beta_2(z) := (-r(z))^{-k} \beta_1(z)$, we have $\left\|\beta_2\right\|_{\infty} \leq 2v_k \left\|\tilde{\beta}\right\|_{\infty}$ and $A = \int_{\Omega} \Theta_X(z) \wedge (-r(z)) \beta_1(z) = \int_{\Omega} \Theta_X(z) \wedge (-r(z))^{k+1} \beta_2(z)$. We can apply the hypothesis $X \in \mathcal{B}_{k-1}(\Omega)$ to the integral A: $|A| \leq \left\|\beta_2\right\|_{\infty} \lesssim 2 \left\|\tilde{\beta}\right\|_{\infty}$, hence $\tilde{X} \in \mathcal{B}(\tilde{\Omega})$. Now we apply the hypothesis of the theorem, $\exists V \in \mathcal{N}(\tilde{\Omega}) :: \tilde{X} = V^{-1}(0)$, and clearly $X = V^{-1}(0) \cap \{w = 0\}$, because if $z \in X$ then $\forall w :: |w|^2 < -r(z)$, $(z,w) \in \tilde{X}$. Hence we set $v(z) := V(z,0) \in \mathcal{N}_{k-1}(\Omega)$, by the subordination lemma, and we are done. □

6.1 Application to pseudo-convex domains

Corollary 10 – *The Bergman-Blaschke characterization is true in the following cases:*

- *if* Ω *is a strictly pseudo-convex domain in* $\mathbb{C}^n$;
- *if* Ω *is a convex domain of finite type in* $\mathbb{C}^n$.

Proof. The first case is true by the famous theorem proved by Henkin[27] and Skoda[28] which says that the Blaschke characterization is true for strictly bounded pseudo-convex domain in $\mathbb{C}^n$.

The second one because the Blaschke characterization is true for convex domain of finite strict type by a theorem of Bruna, Charpentier, and Dupain[29] generalized to all convex domains of finite type by Cumenge[30] and Diederich and Mazzilli[31].□

References

Amar, D. and É. Amar (1978). "Sur les suites d'interpolation en plusieurs variables". *Pacific J. Math* **15**, pp. 15–20 (cit. on p. 51).

Amar, É. (1978). "Suites d'interpolation pour les classes de Bergman de la boule et du polydisque de $\mathbb{C}^n$". *Canad. J. Math* **30**, pp. 711–737 (cit. on pp. 33, 37, 51).

Amar, É. (1980). *Généralisation d'un théorème de Wolff à la boule de C^n*. Technical Report 42. Université d'Orléans et Université d'Orsay (cit. on p. 52).

Amar, É. (1991). "On the corona problem". *J. Geom. Anal* **1** (4), pp. 291–305 (cit. on p. 52).

Amar, É. (2008). "On linear extension for interpolating sequences". *Studia Mathematica* **186** (3), pp. 251–265 (cit. on p. 49).

Amar, É. (2009a). "A Carleson type condition for interpolating sequences in the Hardy spaces of the ball of C^n". *Publicacions Matematiques* **53**, pp. 481–488 (cit. on p. 50).

Amar, É. (2009b). "A weak notion of strict pseudo-convexity. applications and examples". *arXiv:0906.1956v2 [math.CV]* (cit. on pp. 30, 32, 39, 40, 45, 50).

Amar, É. and C. Menini (2000). "Universal divisors in Hardy spaces". *Studia Mathematica* **143** (1), pp. 1–21 (cit. on p. 52).

Andersson, M. (1994). "The H^2 corona problem and $\bar{\partial}_b$ in weakly pseudoconvex domains." *Trans. Amer. Math. Soc* **342**, pp. 241–255 (cit. on p. 52).

Andersson, M. and H. Carlsson (1994). "Wolff-type estimates for $\bar{\partial}_b$ and the H^p-corona problem in strictly pseudo-convex domains". *Ark. Mat.* **32**, pp. 255–276 (cit. on p. 52).

Berger, M. and B. Gostiaux (1988). *Differential geometry*. **115**. Graduate texts in Mathematics. Springer Verlag (cit. on p. 34).

[27] Henkin, 1975, "Solutions with estimates of the H. Lewy and Poincaré-Lelong equations".

[28] Skoda, 1976, "Valeurs au bord pour les solutions de l'opérateur et caractérisation des zéros de la classe de Nevanlinna".

[29] Bruna, Charpentier, and Dupain, 1998, "Zero varieties for the Nevanlinna class in convex domains of finite type in $\mathbb{C}^n$."

[30] Cumenge, 2001, "Zero sets of functions in the Nevanlinna and Nevanlinna-Drjbachian classes in convex domains of finite type".

[31] Diederich and Mazzilli, 2001, "Zero varieties for the nevanlinna class on all convex domains of finite type".

Bruna, J., P. Charpentier, and Y. Dupain (1998). "Zero varieties for the Nevanlinna class in convex domains of finite type in $\mathbb{C}^n$." *Annals of Math* **147**, pp. 391–415 (cit. on p. 54).

Carleson, L. (1958). *An interpolation problem for bounded analytic functions*. Amer. J. Math (cit. on pp. 47, 48).

Catlin, D. (1984). "Boundary invariants of pseucoconvex domains". *Ann. of Math* **12**, pp. 529–586 (cit. on pp. 30, 39).

Cima, J. A. and P. R. Mercer (1995). "Composition operators between bergman spaces on convex domains in $\mathbb{C}^n$". *J. Operator Theory* **33**, pp. 363–369 (cit. on pp. 31, 33, 44).

Coifman, R. and G. Weiss (1971). *Analyse harmonique non commutative sur certains espaces homogènes*. **242**. Lecture Notes in mathematics. Springer-Verlag (cit. on p. 41).

Cumenge, A. (2001). "Zero sets of functions in the Nevanlinna and Nevanlinna-Drjbachian classes in convex domains of finite type". *Pacific J. Math* **199**, pp. 79–92 (cit. on p. 54).

Diederich, K. and E. Mazzilli (2001). "Zero varieties for the nevanlinna class on all convex domains of finite type". *Nagoya Math. J* **163**, pp. 215–227 (cit. on p. 54).

Henkin, G. (1975). "Solutions with estimates of the H. Lewy and Poincaré-Lelong equations". *Soviet Math. Dokl* **16**, pp. 1310–1314 (cit. on p. 54).

Hormander, L. (1967). "A L^p estimates for (pluri-)subharmonic functions". *Math. Scand* **20**, pp. 65–78 (cit. on pp. 30, 43–45).

Schuster, A. P. and K. Seip (1998). "A Carleson type condition for interpolation in Bergman spaces". *Journal für die reine und angewandte Mathematik* **497**, pp. 223–233 (cit. on pp. 48, 50).

Seip, K. (1993). "Beurling type density theorems in the unit disk". *Inventiones mathmaticæ* **113**, pp. 21–39 (cit. on p. 48).

Shapiro, H. and A. L. Shields (1961). "On some interpolation problems for analytic functions". *Amer. J. Math* **83**, pp. 513–522 (cit. on pp. 47, 48).

Skoda, H. (1976). "Valeurs au bord pour les solutions de l'opérateur et caractérisation des zéros de la classe de Nevanlinna". *Bull. Soc. Math. France* **104**, pp. 225–299 (cit. on pp. 32, 52, 54).

Weyl, H. (1939). "On the volume of tubes". *American J. Math* **61** (2), pp. 461–472 (cit. on p. 34).

Contents