THE GENERALIZED LICHNEROWICZ PROBLEM:
UNIFORMLY QUASIREGULAR MAPPINGS
AND SPACE FORMS

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Abstract. We characterize compact space forms admitting uniformly quasiregular
self mappings. The only compact manifolds on which locally (but not globally)
injective uqr mappings act, turn out to be euclidean space forms. Furthermore,
we show that compact space forms admitting branched uqr maps are precisely the
spherical space forms. We further show that every uqr map of a compact euclidean
space form is a quasiconformal conjugate of a conformal map.

1. Introduction
A quasiregular map \( f \) with a uniform control of the distortion of all its iterates
is called uniformly quasiregular (uqr). Such maps are always conformal with respect
to some measurable riemannian structure. We always suppose uqr mappings to be
non-homeomorphic (otherwise we call the map quasiconformal). We consider such
mappings acting on smooth compact riemannian manifolds \( M \) of dimension at least
three and the question is, what kind of manifolds do admit the action of such a map
and, if so, what kind of uqr mappings do act on a given manifold. It is no restriction
to assume that the uqr maps \( f : M \to M \) are surjective since continuity and openness
of a quasiregular map implies that the image \( fM \) is both compact and open, hence
\( fM = M \).

The first part of the question is a non-injective version of the answer of Ferrand to
a conjecture of Lichnerowicz. She showed that, up to quasiconformal equivalence, the
only compact manifold which admits a non-compact quasiconformal group action is
the euclidean sphere \( S^n \). Now, if there is a uqr map \( f \) of say \( M \) then the semi-group
\( \{ f^n \} \) acts in a non-compact way on \( M \) (since the Julia set of \( f \) is always non-empty).
Therefore, the existence of such a map must be very restrictive for the manifold \( M \).
We make use of the following obstruction for the existence of uqr maps:

**Theorem 1.1.** If \( M^n \) is a smooth compact riemannian manifold and \( f : M^n \to M^n \)
a non-injective uqr mapping, then there exists a non-constant quasiregular mapping
\( g : \mathbb{R}^n \to M^n \).

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Our contribution to the generalized Lichnerowicz problem of determining all compact manifolds which admit non-injective uqr mappings are the following:

**Theorem 1.2.** Let \( f \) be a non-injective uqr map of the compact manifold \( M \) and suppose that \( f \) is locally homeomorphic, i.e. the branch set \( B_f = \emptyset \). Then \( M \) is the quasiconformal image of a euclidean space form.

By a euclidean space form we mean the quotient of \( \mathbb{R}^n \) under a Bieberbach group \( \Gamma \subset \text{Isom}(\mathbb{R}^n) \).

**Theorem 1.3.** If \( M \) is quasiconformally equivalent to a euclidean space form, then \( M \) admits no branched quasiregular (and in particular no branched uqr) mappings.

**Remark 1.4.** The proof we give require only the topological properties of quasiregular mappings. Consequently, the previous result can be generalized as follows:

Suppose \( M \) is homeomorphic to a euclidean space form. Then \( M \) does not admit an orientation preserving, discrete and open map which is branched.

Concerning the space of maps \( UQR(M) \) in the case of the sphere, lens spaces and other spherical manifolds existence is due to [IM1] and [P] and rigidity phenomena have been found in [MM] explaining that the space of uqr mappings of at least three dimensional spheres is very reduced compared with the rational functions of \( S^2 \). For euclidean space forms we can now give a complete description of their uqr mappings:

**Theorem 1.5.** Any non-injective uqr map of a compact euclidean space form \( M \) is the quasiconformal conjugate of a conformal map.

We remark that, in this second result, we no longer suppose that the map has to be locally injective. Theorem 1.5 is a surprising fact because it is false for globally injective mappings. Indeed, it turns out that there are uniformly quasiconformal (even bilipschitz) maps of a (at least)three dimensional torus which cannot be quasiconformally conjugate to a conformal map (see [M1]).

Finally, we can distinguish compact space forms according to the type of uqr maps they support:

**Theorem 1.6.** If \( M \) is a compact space form, then we have the following characterization:

(1) \( M \) admits a branched uqr map if and only if \( M \) is a spherical space form.

(2) \( M \) admits a non-injective, locally injective uqr map if and only if \( M \) is a euclidean space form.

(3) \( M \) admits no non-injective uqr map if and only if \( M \) is a hyperbolic space form.

We suggest that the correct counterpart for the Lichnerowicz conjecture for uqr maps could be that if a compact riemannian manifold supports a non-injective uqr map, then it must be a finite product \( M_1 \times \cdots \times M_k \) of riemannian manifolds \( M_i \) which are quasiconformal images of either euclidean or spherical space forms. It remains an interesting open question whether for example a space like \( S^2 \times S^2 \# S^2 \times S^2 \) supports
uqr maps. This manifold was very recently proven to be elliptic [Ri].

2. Proofs

Proof of Theorem 1.1:
Suppose \( h \) is a non-injective uqr mapping acting on a compact riemannian manifold \( M \). We may assume that the Julia set is not empty and contains a fixed point \( x_0 \). Introduce coordinates \((U_{x_0}, h)\), where \( U_{x_0} \) is an open set in \( M \) containing \( x_0 \) and \( h : U_{x_0} \to \mathbb{B}^n \) is a 2-bilipschitz map. We get a non-normal family \( \mathcal{F} = \{ \hat{f}_k \} \) of mappings \( \hat{f}_k = f_k \circ h^{-1} : \mathbb{B}^n \to M \). Hence there exists positive numbers \( \rho_j \downarrow 0 \) and points \( x_j \to 0, x_j \in \mathbb{B}^n \) and mappings \( \hat{f}_j \in \mathcal{F} \) so that \( \hat{f}_j(x_j + \rho_j x) \to \psi(x) \) locally uniformly on \( \mathbb{R}^n \), where \( \psi : \mathbb{R}^n \to M \) is a non-constant quasiregular map. See [IM2, Theorems 19.7.3, 19.9.3].

Recall that the conical set of a uqr map is, using the notation of the above proof, the set of those points \( x_0 \) in the Julia set for which the ratio \( |x_j - x_0|/\rho_j \) stays bounded with a uniform constant when \( j \to \infty \).

Proof of Theorem 1.2:
Suppose \( f : M \to M \) is a non-injective uqr map with empty branch set. Since \( M \) is compact we may find a point \( x_0 \) in the Julia set \( J_f \) of \( f \). It is now possible, as in the proof of Theorem 1.1 to construct a family \( \mathcal{F} = \{ \hat{f} \} \) of mappings \( \hat{f} : \mathbb{B}^n \to M \) and find positive numbers \( \rho_j \downarrow 0 \) and points \( x_j \to 0, x_j \in \mathbb{B}^n \) and mappings \( \hat{f}_j \in \mathcal{F} \) so that \( \hat{f}_j(x_j + \rho_j x) \to \psi(x) \) locally uniformly in \( \mathbb{R}^n \), where \( \psi : \mathbb{R}^n \to M \) is a non-constant quasiregular map. Now clearly the limit map \( \psi \) is also a local homeomorphism. Due to the Picard-type theorem of [HR] \( \psi \) omits at most \( q < \infty \) points (number \( q \) depending only on dimension \( n \) and the dilatation of the map \( \psi \)). Denote by \( \tilde{\psi} : \mathbb{R}^n \to \tilde{M} \) the lift of \( \psi \) under a locally isometric covering map \( \pi : M \to \tilde{M} \), where \( \tilde{M} \) is the covering space of \( M \). If the fundamental group of \( M \) is infinite the mapping \( \psi \) is necessarily a surjection, since otherwise the lifted map \( \tilde{\psi} \) would omit infinitely many points (corresponding every omitted value of \( \psi \)) and this is again impossible due to [HR]. The covering space \( \tilde{M} \) is hence first of all a homeomorphic image of the euclidean space \( \mathbb{R}^n \). This homeomorphism is also necessarily quasiconformal since on the underlying compact manifold all smooth riemannian metrics are quasiconformally equivalent.

For the finite fundamental group case \( \tilde{\psi} : \mathbb{R}^n \to \tilde{M} \setminus \{ p_1, \ldots, p_{q'} \} \) is a covering mapping to a simply connected space and \( q' \) is a finite number. Hence \( \tilde{M} \setminus \{ p_1, \ldots, p_{q'} \} \) is a quasiconformal (homeomorphic) image of the euclidean space and only the \( n \)-sphere can occur as a covering space. This forces also \( f \) to be homeomorphism since \( f \circ \pi : \mathbb{S}^n \to M \) is also a covering map whose degree must agree with that of the
mapping $\pi$. Since we were dealing with non-injective uqr maps this case cannot occur.

We now show that $M$ is a euclidean space form. Denote by $F : \mathbb{R}^n \to \mathbb{R}^n$ the lift of $f$ to the universal covering space so that the condition $\pi \circ F = f \circ \pi$ holds. Now $F$ is also necessarily a local homeomorphism and hence a quasiconformal homeomorphism as a mapping between simply connected spaces. We can now extend $F$ to a quasiconformal mapping acting on $\mathbb{R}^n$ and fixing infinity. If the family $\{F^k\}^\infty_{k=1}$ were finite, it would follow that also $f^k = \text{Id}_M$ holds for some integer $k$ forcing $f$ to be a homeomorphism. Hence the family $\{F^k\}^\infty_{k=1}$ must be infinite and $F$ is either parabolic or loxodromic. In either case the family $\{F^k\}^\infty_{k=1}$ is not normal in a neighborhood of any point in $\mathbb{R}^n$ (see [GM]). The Julia set $\mathcal{J}_F$ is hence the whole space, forcing also $f$ to be chaotic. Next we show that all the points in $M$ are conical as well. Cover $M$ with finitely many coordinates $(U_i, \varphi_i), i = 1, \ldots, k$, where $\varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{R}^n$ is a 2-bilipschitz homeomorphism. Fix any point $x_0 \in M = \mathcal{J}_f$. Let $U_{i_0}$ be a coordinate neighborhood containing point $x_0$ and $\Omega \subset U_{i_0}$ a neighborhood of $x_0$. Now $f^m(x_0) \in U_{i_m}$ for some $i_m \in \{1, \ldots, k\}$ and $\varphi_{i_m}(\Omega') \subset \mathbb{R}^n$ is an open set, where $\Omega' = f^m(\Omega) \cap U_{i_m}$. It is no restriction to assume that $\varphi_{i_0}(x_0) = 0$ and $\varphi_{i_m}(f^m(x_0)) = 0$. Now the mapping $\hat{f}_m = \varphi_{i_m} \circ f^m \circ \varphi_{i_0}^{-1} : \varphi_{i_0}(\Omega) \to \varphi_{i_m}(\Omega')$ is a locally injective quasiregular map. There exists a positive number $r_m$ and an inverse branch $\hat{F}_m : B^n(r_m) \to \varphi_{i_0}(\Omega)$ of $\hat{f}_m$. Moreover, it follows from the local lipschitz behavior of uqr maps [HMM, Lemma 4.1] near fixed points that there exists a positive radius $r$ such that the inverse branch is defined in a ball $B^n(r)$ independently of $m$. Hence the family $\{\hat{F}_m\}_m$ is normal and any convergent subsequence has a constant limit. This forces $x_0$ to be a conical point. The measurable conformal structure preserved by the uqr mapping $f$ is continuous in measure at a conical point. We can hence apply the rigidity theorem for uqr maps [MM, Theorem 6.1] to deduce that $f$ is actually a Lattès type map. Especially $F$ is then an expanding similarity. This forces $M$ to be a euclidean space form.

**Proof of Theorem 1.3:**

Suppose $M = \mathbb{R}^n / \Gamma$ is a compact euclidean space form and $f : M \to M$ a non-constant quasiregular mapping. First of all, we may suppose that $\Gamma$ consists of translations only and, because of compactness of $M$, that $\Gamma$ is isomorphic to $\mathbb{Z}^n$. This is because of Bieberbach’s theorem (see [W]): the translation subgroup $\mathcal{T}$ of $\Gamma$ is a normal subgroup of finite index and $\mathbb{R}^n / \mathcal{T}$ is a normal riemannian covering space of $M$. Therefore it suffices to consider $\mathbb{R}^n / \mathcal{T}$ together with a lift of $f$ to this covering of $M$ instead of the couple $M, f$.

Denote by $F : \mathbb{R}^n \to \mathbb{R}^n$ a quasiregular lift of $f$ and let $D$ be a fundamental domain of $M$ containing the origin. We can further assume that $F(0) \in D$. Then there exist a group homomorphism $A : \Gamma \to \Gamma$ so that for any $x \in \mathbb{R}^n$ and $\gamma(x) = x + v$, $\gamma \in \Gamma$,
the following holds:
\[(2.1) \quad F(x + v) = F(x) + A(v).\]

Since \(\Gamma\) is a translation group isomorphic to \(\mathbb{Z}^n\), the above homomorphism is given by a linear map \(A : \mathbb{R}^n \to \mathbb{R}^n\). Furthermore, \(A\) is invertible because \(A\Gamma\) is again isomorphic to \(\mathbb{Z}^n\).

We now show that \(F\) has an invertible derivative at \(\infty\) which concludes the proof. Indeed, we then have \(i(\infty, F) = 1\). On the other hand, this local index does agree with the degree of the map \(F\) since the only preimage of \(\infty\) is infinity itself. In other words, \(F\) is a homeomorphism.

Let \(i(x) = \frac{x}{\|x\|^2}\) be the chart at infinity, denote \(y = i(x)\),
\[G = i \circ F \circ i \quad \text{and} \quad A^* = i \circ A \circ i.\]

Clearly \(G(0) = 0\). For \(x\) close to the origin,
\[G(x) = \frac{F(y)}{\|F(y)\|^2} = \frac{F(y_0) + A(y - y_0)}{\|F(y_0) + A(y - y_0)\|^2} = \frac{A(y) + (F(y_0) - A(y_0))}{\|A(y) + (F(y_0) - A(y_0))\|^2}\]
where \(y_0 \in D\) such that there is \(\gamma \in \Gamma\) with \(\gamma(y_0) = y_0 + v = y\). Clearly \(B(y_0) = F(y_0) - A(y_0)\) is bounded on \(D\). Therefore,
\[\frac{\|A(y)\|^2}{\|A(y) + B(y_0)\|^2} \leq \frac{1}{1 - \left(\frac{\|B(y_0)\|}{\|A(y)\|}\right)^2} \leq 1 + C \frac{1}{\|A(y)\|^2} \leq 1 + C'\|x\|^2.\]

Similarly
\[\frac{\|A(y)\|^2}{\|A(y) + B(y_0)\|^2} \geq 1 - C''\|x\|^2.\]

This leads to
\[G(x) = \frac{A(y)\|A(y)\|^2}{\|A(y)\|^2 \|A(y) + B(y_0)\|^2} + \frac{B(y_0)}{\|A(y) + B(y_0)\|^2} = A^*(x) \left(1 + O(\|x\|^2)\right) + O(\|x\|^2) = A^*(x) + O(\|x\|^2),\]
meaning that \(DG(0) = A^*\).

**Proof of Theorem 1.5:** Theorem 1.3 gives the local injectivity of the map. We note that alternatively this can be seen shortly by using more dynamical arguments as follows. Let \(M = \mathbb{R}^n/\Gamma\) be a compact euclidean space form and \(f \in UQR(M)\). Consider \(\pi : \mathbb{R}^n \to M\) the natural projection and take \(F \in UQR(\mathbb{R}^n)\) a lift of \(f\). We proceed by contradiction and suppose that \(F\) is not injective.

From the proof of Theorem 1.2 the map \(F\) extends at infinity to a UQR map of the sphere \(\mathbb{R}^n\) and \(\infty\) is a completely invariant fixed point which is super-attracting
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since we supposed $F$ is not quasiconformal; indeed since $\infty$ is completely invariant, the local index of $F$ at that point is equal to the degree of $F$ and hence $\infty$ is in the branch set of the extended map. It then follows that the Fatou set of $F$ contains an open neighborhood of $\infty$, the basin of attraction of $\infty$. See [HMM] for details. From standard estimates near $\infty$ we have, after replacing $F$ by a power of itself if necessary, the following: there is $P > 1$ and $R > 0$ such that

$$
\|F(x)\| \geq \|x\|^P \quad \text{provided} \quad \|x\| > R.
$$

Now, for $\gamma \in \Gamma$ there is $\gamma' \in \Gamma$ such that $F \circ \gamma^N = (\gamma')^N \circ F$. There are $U, V \in O(n)$ and $v, w \in \mathbb{R}^n$ such that $\gamma^N(x) = U^N x + Nv$ and $(\gamma')^N(x) = V^N x + NW$. We take $\gamma$ parabolic such that $\|\gamma^N(0)\| > R$ for all $N$ large enough. Then:

$$
\|F \circ \gamma^N(0)\| \geq \|\gamma^N(0)\|^P = (N\|v\|)^P
$$

and thus

$$(N\|v\|)^P \leq \|(\gamma')^N(F(0))\| \leq \|F(0)\| + N\|w\|.$$ 

This is impossible if $N$ has been chosen big enough. The map $F$ is in fact quasiconformal.

To conclude the proof we will look for a quasiconformal map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$, a Bieberbach group $\Gamma'$ and a conformal (loxodromic) map $G : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$
\Phi :< F, \Gamma > \to < G, \Gamma' >
$$

meaning that $\Phi$ conjugates the maps $F, G$ and the groups $\Gamma, \Gamma'$. It suffices then to project $\Phi$ to a quasiconformal map $\varphi : M \to M' = \mathbb{R}^n/\Gamma'$ because this new map conjugates $f$ and $g : M' \to M'$, the projection of $G$.

In order to do this we take a conformal structure $\mu$ on $M$ for which $f$ is conformal. Lift this structure to $\mathbb{R}^n$ to a structure for which the group $< F, \Gamma >$ is conformal and use again a renormalization to find $\Phi$.

**Proof of Theorem 1.6:**

The existence of a branched uqr map for an arbitrary spherical space form $M^n$ is proved in [P]. There one starts with an arbitrary branched covering map $f : M^n \to S^n$ followed by the covering projection $g : S^n \to M^n$. The mapping $g \circ f : M^n \to M^n$ can now be modified to an uqr mapping by the so called trapping method. The branch set does not change under the construction.

For a hyperbolic space form there are no non-injective uqr maps at all. This follows from Theorem 1.1, since hyperbolicity implies exponential volume growth for the fundamental group. Fundamental groups of manifolds quasiregularly covered by $\mathbb{R}^n$ have at most polynomial growth of order $n$ [VSC].

The non-existence of a branched uqr map in the euclidean space form case is Theorem 1.3 above. The existence of a non-injective uqr map can be shown by constructing
Lattès type maps. This can be seen by using the following characterization [R, Theorem 13, p. 373]. Suppose $\Gamma$ is the fundamental group of a compact euclidean space form $M$. The group $\Gamma$ consists of the translation subgroup and the rotation subgroup. One can introduce coordinates $\{e_i\}_{i=1}^n$ of the covering space $\mathbb{R}^n$ in such a way that the elements of the translation part are of the form $x \mapsto x + \sum_{i=1}^n m_i e_i$, $m_i \in \mathbb{Z}$ and the general element is of the form

$$x \mapsto R(x) + \sum_{i=1}^n \frac{n_i}{g} e_i,$$

where the matrix of $R$ representing the rotational part, when presented in this fixed base, contains integer entries only, $n_i \in \mathbb{Z}$ and $g$ is the order of the rotational part. The Lattès type map $f : M \to M$ now rises as a solution of the equation

$$(f \circ \pi)(x) = (\pi \circ A)(x),$$

where $\pi : \mathbb{R}^n \to M$ is the conformal covering projection. For the dilation map $A : \mathbb{R}^n \to \mathbb{R}^n$ one can choose $A : x \mapsto 2gx$, which preserves the lattice identifications as needed. See [M2] for details on Lattès type maps.

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