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Permutation products, configuration spaces with bounded multiplicity and finite subset spaces

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Contents

Introduction

This thesis is divided into two parts. The first part deals with permutation products and configuration spaces with multiplicity. It establishes many of their fundamental properties and then computes their fundamental group, connectivity and Euler characteristic. The second part studies the space of finite subsets of cardinality at most four associated to a topological space X and proves a number of results about them. In particular we verify a special case of a conjecture of Christopher Tuffley on the connectivity of these spaces. In all of this work, spaces X are assumed to have the homotopy type of a finite CW-complex.

If Γ is a subgroup of the n-th symmetric group $\mathfrak{S}_n$, we define the permutation product $\Gamma\mathrm{P}^n(X)$ to be the quotient of X^n by the permutation action of Γ on coordinates. The prototypical example being of course the n-th symmetric product $\mathrm{SP}^n(X)$ which corresponds to when $\Gamma = \mathfrak{S}_n$. The cyclic product $\mathrm{CP}^n(X)$ corresponds on the other hand to when $\Gamma = \mathbb{Z}_n$ is the cyclic group. To illustrate our constructions and as a first non-trivial example, we give in §2.2 an algebro-geometric characterization of the third cyclic product of a torus $S^1 \times S^1$ (Proposition 2.2.2).

Permutation products have been studied very early on, most notably by [11, 27, 30, 33, 34] and a functorial description of their homology described in [11, 34]. In [33], one of the earliest references on the subject, P.A. Smith showed that the functor $\mathrm{SP}^2(-)$ *abelianizes* fundamental group in the sense that

$$\pi_1(\mathrm{SP}^2(X)) \cong H_1(X;\mathbb{Z})$$

This result is in fact valid for $\pi_1(\mathrm{SP}^n(X))$ for all $n > 1$, and has become folklore with a few available proofs ranging from the simplicial techniques of Dold [11] to the groupoid techniques of Brown-Higgins [8] (see §2.4.1). An explicit constructive and streamlined proof of this result for all n seems however lacking in the literature.

In Chapter 2 of this thesis we extend Smith's result to all permutation products and to all n. More precisely, recall that a permutation subgroup $\Gamma \subset \mathfrak{S}_n$ acting on a finite set $\{1, 2\, \ldots, n\}$ is called transitive if for any pair i, j, there is some permutation in Γ taking i to j. Adapting and expanding Smith's geometric arguments, we show that for transitive subgroups Γ, $\pi_1(\Gamma\mathrm{P}^n X)$ is abelian whenever $\pi_1(X)$ is. Moreover, since any non transitive subgroup of $\mathfrak{S}_n$ is up to conjugation contained in a standard Young subgroup $\mathfrak{S}_k \times \mathfrak{S}_{n-k}$, permutation products of non transitive subgroups can be reduced to transitive ones of smaller size. More explicitly, view $\mathfrak{S}_n$ as the permutation group of $\Omega = \{1, 2, \dots, n\}$. The set Ω breaks into orbits under the action of Γ and we will say that an orbit is non-trivial if it contains at least two elements.

The following is proven in §2.4 and is our first main result.

Theorem 0.0.1 *If $\Gamma \subset \mathfrak{S}_n$, then $\pi_1(\Gamma P^n X) \cong \pi_1(X)^{n-\sum_i k_i} \times H_1(X, \mathbb{Z})^r$, where r is the number of non-trivial transitive orbits under the action of Γ on Ω, and $k_1, k_2, \dots, k_r$ are the sizes of these non-trivial orbits. If Γ is abelian, then $k_i = |\Gamma|$ and $\sum k_i = r|\Gamma|$.*

In proving this theorem we establish and then make use of some useful path lifting criteria for surjective maps (see §1.2).

It is not clear how the higher homotopy groups of $\Gamma\mathrm{P}^n(X)$ compare to the homology of X in general. However and in the case of the symmetric products these groups are the same through a range that we make explicit in §2.4.1.

In Chapter 3 we introduce and study the configuration spaces of bounded multiplicity associated to a topological space X. These have been considered by various authors and in various contexts, but always in limited scope and no systematic study of their algebraic topological properties for general X has been carried out yet.

Define $B^d(X, n)$ and $B_d(X, n)$ to be respectively the subspaces of the symmetric product $\mathrm{SP}^n X$ of all tuples $[x_1, \dots, x_n]$ where no x_i repeats more than d-times (respectively there exists one x_i that repeats at least d-times). For example $B^1(X, n)$ is the configuration space of distinct points and is commonly referred to as the n-th *braid space* of X and it is often written more simply as $B(X, n)$, while $B_2(X, n)$ is known as the *fat diagonal* in $\mathrm{SP}^n X$ (see §3.1). The fundamental group of the configuration space $B^1(X, n)$ is called the *braid group* of X. Braid spaces and braid groups are objects of central importance in all of algebraic topology, geometry and mathematical physics. They were of crucial use in the seventies in the work of Arnold on singularities and were key players in the eighties in the study of gauge theoretic moduli spaces in physics. In the nineties they entered in many works on the topology of holomorphic mapping spaces as shown in work of R.J. Milgram, P. May, G. Segal, F. Cohen, M. Guest and others.

Our starting point is the descending filtration

$$B_1(X, n) = \mathrm{SP}^n X \supset B_2(X, n) \supset \cdots \supset B_n(X, n) = diag(X) \tag{0.0.1}$$

where $diag(X)$ is the diagonal of X having entries of the form $[x, \cdots, x]$. It is not hard to see for example that

$$B_{n-1}(X,n) = X \times X \tag{0.0.2}$$

Note that $\pi_1(B_1(X,n)) = \pi_1(\mathrm{SP}^n X)$ is abelian whereas $\pi_1(B_n(X,n)) = \pi_1(X)$ need not be abelian. We ask the following question: at which stage of the filtration does $\pi_1(B_d(X,n))$ become abelian? We are able to give a complete answer to the question and it turns out that the functor $B_d(-,n)$ abelianizes fundamental group "as soon as it can".

Theorem 0.0.2 *The following statements hold:*
(i) For $n \geq 2$ and $1 \leq d \leq \frac{n}{2}$, $\pi_1(B_d(X,n)) \cong H_1(X;\mathbb{Z})$.
(ii) For all $d \geq 1$, $\pi_1(B_d(X,n))$ is abelian if $\pi_1(X)$ is.
(iii) $B_d(X,n)$ has the connectivity of X.

The most challenging part is (i) and as we show, it is a consequence of the fact that the map

$$\mathrm{SP}^2 X \longrightarrow B_d(X,n)\ , \quad [x,y] \mapsto [x,y,*,\cdots,*]$$

which is defined when $2 \leq d \leq n/2$ induces an isomorphism on fundamental groups. The technical details proceed by giving a description of the fundamental group for the ordered analog via an iterated use of the Van-Kampen theorem, then using our loop-lifting criterion (Corollary 1.2.4) to get commutativity when $1 \leq d \leq \frac{n}{2}$ by "separating" generators into blocks. This method works to show for example that if Γ is a d-transitive subgroup of $\mathfrak{S}_n$, then $\pi_1(B\Gamma_d(X,n))$ is abelian for $n \geq 2$ and $1 \leq d \leq \frac{n}{2}$. Here $B\Gamma_d(X,n)$ is our notation for the quotient under Γ of all ordered tuples of X^n with one entry repeating at least d-times.

In §2.3 we discuss the stabilizer and orbit stratifications induced from permutation actions. We recall that any group Γ acting on a space X stratifies it according to "stabilizer type" and it stratifies the quotient X/Γ according to "orbit type". Indeed, there is a stratum in X for each conjugacy class (H) of a subgroup $H \subset \Gamma$ defined as follows

$$X_{(H)} := \{x \in X \mid stab(x) \sim H\}$$

where $\sim$ means "being conjugate to" and $stab(x) := \{g \in \Gamma \mid gx = x\}$ is the isotropy group of x. Given a finite stratification $\{X_i\}_{i\in\mathcal{I}}$ with the "frontier condition" (see §2.3), we say that $i < j$ if X_i is in the closure of X_j. We define the depth of a stratum X_s to be the maximum length k of a sequence $s = s_0 < s_1 < \cdots < s_k$ in $\mathcal{I}$. The depth of the stratification as a whole is the maximum over the depths of its strata. The depth depends on the action $\Phi : \Gamma \longrightarrow Aut(X)$ and is denoted $d_\Phi(\Gamma, X)$. Given a topological space X and a finite group Γ, the action of Γ on itself embeds Γ into $\mathfrak{S}_{|\Gamma|}$ and the induced action on $X^{|\Gamma|}$, is called the permutation representation action. The main result of §2.3 is to compare $d_\Phi(\Gamma, X)$ to the length l of Γ and we have the following result

Theorem 0.0.3 *If Γ is a finite group, then $0 \leq d_\phi(\Gamma, X) \leq \ell(\Gamma)$ and these equalities are attained when we consider the action of Γ by permutation representation.*

In Chapter 3, we give a computation of the Euler characteristic of the fat diagonal $\chi(B_2(X,n))$ in terms of the Euler characteristic of X. The formula we obtain is pleasantly simple

Theorem 0.0.4 *For X a topological space,*

$$\chi(B_2(X,n)) = \binom{\chi(X)+n-1}{\chi(X)-1} - \frac{1}{n!}\prod_{j=0}^{n-1}(\chi(X)-j)$$

When X is a manifold we show that this can be written as

$$\chi(B_2(X,n)) = \chi(\mathrm{SP}^n X) - \chi(B(X,n))$$

This is intriguing because $B_2(X,n) = \mathrm{SP}^n X - B(X,n)$ and the Euler characteristic of the complement does not behave in general in such a canonical way. However and in joint work [23] we uncover subtle reasons for the additivity of the Euler characteristic for spaces stratified by even dimensional manifolds and use it to give a general formula for $\chi(B_d(X,n))$.

The second part of this thesis studies finite subset spaces and builds on recent work of Kallel-Sjerve [22]. Spaces of finite subsets are also made up of configurations of points but the topology is that of a quotient and not of a complement. More precisely and for X a topological space, $\mathrm{Sub}_n X$ is the space of subsets of X of cardinality at most n. It has the quotient topology from the identification

$$\mathrm{Sub}_n X = X \sqcup X^2 \sqcup \cdots \sqcup X^n/_{\sim}$$

with the identification $(x_1, \cdots, x_r) \sim (y_1, \cdots, y_s)$ if the underlying sets $\{x_1, \cdots, x_r\}$ and $\{y_1, \cdots, y_s\}$ are equal. This space seems to have been first introduced by Borsuk and Ulam in 1931 [6]. Yet, little is known about $\mathrm{Sub}_n X$ for general n and X. In a series of papers [36, 37, 38], C. Tuffley has studied the spaces of finite subsets of the circle, connected graphs, punctured surfaces, and then closed surfaces. In particular,f he computed the homology of these spaces for small n and then conjectured that the connectivity of $\mathrm{Sub}_n X$ is $n + r - 2$ where r is the connectivity of X. In [22], S. Kallel and D. Sjerve verify this conjecture for $n = 3$ and show more generally that $\mathrm{Sub}_n X$ is $(r+1)-$ connnected for all n. In Chapter 4 we establish the following main result

Theorem 0.0.5 *If X is r-connected, $r > 1$, then Sub_4X is $r+2$-connected*

This is done by first improving the bound on the connectivity of Sub_3X based on old calculations of Nakaoka, then using some pushout diagrams describing Sub_4X in terms of Sub_3X, SP^3X and SP^4X (for definition of pushout, see §3.1). As a consequence of the existence of these pushouts, we are able to compute the Euler characteristic χ of Sub_4X in Proposition 4.2.2.

Recall that $X \vee Y$ is the one-point union of two based spaces X and Y. It is a theorem of Tuffley ([38], Theorem 1) that $\mathrm{Sub}_n S^2$ has the rational homology of $S^{2n} \vee S^{2n-2}$ for all n. In the case of $n = 4$, we give a description of the rational homology of $\mathrm{Sub}_n S^d$ for all d. In fact we give more

Theorem 0.0.6
(i) For d odd, $d > 1$, Sub_4S^d has the rational homotopy type of S^{2d+1}.
(ii) For d even, Sub_4S^d has the rational homotopy type of $S^{4d} \vee S^{3d}$.

It is clear that there are many open questions left to be answered. The spaces Sub_nX form an exciting area of investigation. Most recently D. Tanré and Y. Félix have started the study of their rational homotopy type. The configuration spaces $B_d(X, n)$ and $B^d(X, n)$, and their ordered analogs, offer many challenging problems as well. Giving a formula for the Euler characteristic of $B^d(X, n)$ for general X is still an open problem. This characteristic depends strongly on the simplicial structure put on X. In [14], S. Gal has given an expression for $\chi(F(X, n))$ for any simplical X in terms of Euler characteristics of links of cells. It seems reasonable to expect that similar formulae exist for $\chi(F^d(X, n))$ with $d > 1$.

Chapter 4 on Sub_4X has been accepted for publication in the Journal “Topology and its Applications”. The parts on permutation products and configuration spaces with multiplicity is joint work with S. Kallel and will appear in [23].

Chapter 1

Basic Constructions

In this chapter we introduce the spaces and basic constructions needed in the upcoming chapters. We also prove or deduce a number of relevant results scattered in the literature. All spaces unless explicitly mentioned will be path-connected with a chosen basepoint. The symbol $\cong$ will mean *homeomorphic to* or *isomorphic to*, and the symbol $\simeq$ means *homotopy equivalent to*. We will write $\mathbb{P}^n$ the complex n-th projective space which is the quotient of $\mathbb{C}^{n+1} - \{0\}$ by the action of $\mathbb{C}^*$ acting by componentwise multiplication. The unit sphere of $\mathbb{R}^{n+1}$ is denoted by S^n and the unit ball of $\mathbb{R}^n$ is written D^n.

1.1 Permutation products

For X a topological space and Γ a subgroup of the symmetric group $\mathfrak{S}_n$, the permutation product $\Gamma P^n X$ is the quotient of X^n by Γ acting on X^n by permuting coordinates. It inherits the quotient topology from X^n. An element in $\Gamma P^n X$ is called a configuration and will be denoted $[x_1, \cdots, x_n]$.

Note that $\Gamma \mathrm{P}^n(-)$ is a homotopy functor, that is, we have the following result

Lemma 1.1.1 *If $X \simeq Y$ then $\Gamma P^n X \simeq \Gamma P^n Y$*

Proof. If $f : X \longrightarrow Y$ is a homotopy equivalence then the induced map

$$\Gamma\mathrm{P}^n(f) : \Gamma\mathrm{P}^n X \longrightarrow \Gamma\mathrm{P}^n Y,\ [x_1, \cdots, x_n] \mapsto [f(x_1), \cdots, f(x_n)]$$

is a homotopy equivalence. Indeed, let $g : Y \longrightarrow X$ be the homotopy inverse of f, that is, there exists a homotopy $H : f \circ g \simeq 1_Y$ and a homotopy $G : g \circ f \simeq 1_X$. The map defined by $[y_1, \cdots, y_n] \mapsto [H_t(y_1), \cdots, H_t(y_n)]$ for all $0 \leq t \leq 1$ defines a homotopy between $\Gamma\mathrm{P}^n(f) \circ \Gamma\mathrm{P}^n(g)$ and $1_{\Gamma P^n Y}$. Similarly, the map $[x_1, \cdots, x_n] \mapsto [G_t(x_1), \cdots, G_t(x_n)]$ defines a homotopy between $\Gamma\mathrm{P}^n(g) \circ \Gamma\mathrm{P}^n(f)$ and $1_{\Gamma P^n X}$. ∎

The most studied case is the symmetric product $\mathrm{SP}^n X$ corresponding to when $\Gamma = \mathfrak{S}_n$. In the case of $X = \mathbb{C}$ we have the following known result.

Example 1.1.2 The space $\mathrm{SP}^n\mathbb{C}$ is homeomorphic to $\mathbb{C}^n$. The homeomorphism sends a configuration $[x_1, \cdots, x_n]$ to $(a_1, \cdots, a_n)$ where a_i is the i-th elementary symmetric function on $[x_1, \cdots, x_n]$ given by $a_i = \sum_{j_1 < \cdots < j_i} x_{j_1} x_{j_2} \cdots x_{j_i}$.

This result has several corollaries

Corollary 1.1.3 *$SP^n\mathbb{C}^* \cong \mathbb{C}^{n-1} \times \mathbb{C}^*$.*

Proof. By the homeomorphism given in Example 1.1.2, if all the x_i's are different from zero then so is their product a_n. ∎

Corollary 1.1.4 *Let X be a topological surface. Then $SP^n X$ is an n-dimensional complex manifold*

Proof. We exhibit around every configuration of $\mathrm{SP}^n X$ a neighborhood homeomorphic to $\mathbb{C}^n$. Let $\mathbf{x} = [x_1, \cdots, x_n]$ be such a configuration in $\mathrm{SP}^n X$. Decompose $\mathbf{x}$ into blocks of equal entries in the following way.

$$\mathbf{x} = [\underbrace{x_1, \cdots, x_1}_{i_1}, \underbrace{x_2, \cdots, x_2}_{i_2}, \cdots, \underbrace{x_r, \cdots, x_r}_{i_r}]$$

such that $i_1 + \cdots + i_r = n$ and $x_i \neq x_j$ if $i \neq j$. Every x_i of the surface has a neighborhood U_i homeomorphic to $\mathbb{C}$, and the subspace $U = \mathrm{SP}^{i_1} U_1 \times \cdots \times \mathrm{SP}^{i_r} U_r$ is a neighborhood of $\mathbf{x}$ in $\mathrm{SP}^n X$. Since each $\mathrm{SP}^{i_k} U_k$ is homeomorphic to $\mathbb{C}^{i_k}$, U is homeomorphic to $\mathbb{C}^n$. One can show that these give charts of a complex structure on $\mathrm{SP}^n X$ given that the U's are charts of a complex structure on X. ∎

Example 1.1.5 We verify that $\mathrm{SP}^n S^2 \cong \mathbb{P}^n$ the complex n-th projective space. Write the sphere S^2 as $\mathbb{C} \cup \infty$. First, identify the complex projective space $\mathbb{P}^n$ with the space of non zero complex polynomials of degree $\leq n$ modulo scaling by a non zero complex. Indeed, $\mathbb{P}^n = (\mathbb{C}^{n+1} - 0)/_\sim$ where the equivalence relation identifies $(x_1, \cdots, x_{n+1}) \in \mathbb{C}^{n+1} - 0$ with $(\lambda x_1, \cdots, \lambda x_{n+1})$ for any non zero complex λ. Thus the map sending the class of $(x_1, \cdots, x_{n+1})$ to the polynomial $x_{n+1} X^n + \cdots + x_2 X + x_1$ is a homeomorphism. Now the homeomorphism $\mathbb{P}^n \longrightarrow \mathrm{SP}^n S^2$ is given by sending a polynomial $P = a_k X^k + \cdots + a_1 X + a_0$, $k \leq n$, to $[\alpha_1, \cdots, \alpha_k, \infty, \cdots, \infty]$ where $\alpha_1, \cdots, \alpha_k$ are the roots of P.

Remark 1.1.6 The fact that $\mathrm{SP}^n S$ is a manifold, for S a smooth algebraic curve, is an exceptional property that is no longer true for general $\Gamma\mathrm{P}^n S$ as Proposition 2.2.2 ullistrates. In fact and since the argument is local, we can give a simple explanation as to why this is the case: we will argue that $\mathrm{CP}^3(\mathbb{C}) = \mathbb{C}^3/\mathbb{Z}_3$, cannot be homeomorphic to $\mathbb{R}^6$. This is extracted from an argument of Morton ([29], p.2). Decompose $\mathbb{C}^3 = \mathbb{C} \times H$, $\mathbb{C}$ the diagonal line in $\mathbb{C}^3$ and H its orthogonal hyperplane. The action of $\mathbb{Z}_3$ takes place entirely in H. This action restricts to a fixed point free action on the unit sphere $S^3 \subset H$ (since we have factored out the diagonal which consists of the fixed points). It follows that $\mathbb{R}^6/\mathbb{Z}_3$ is homeomorphic to $\mathbb{R}^2 \times cM$ where $M = S^3/\mathbb{Z}_3$ is a three manifold with $H_1(M) = \mathbb{Z}_3$ and cM is the cone on M (with apex at the origin and the cone extending to ∞). This quotient cannot be homeomorphic to $\mathbb{R}^6$ as claimed.

More about manifold structures is in [39].

Remark 1.1.7 Corollary 1.1.3 shows that $\mathrm{SP}^n S^1$ is up to homotopy $\mathbb{C}^*$ and hence S^1. It is not hard to see (using (2.4.2)) that the circle is the only non-contractible CW complex homotopy equivalent to its symmetric products.

The next characterization due to Morton [29] will be important to our work.

Theorem 1.1.8 *[Morton] The multiplication $SP^n S^1 \longrightarrow S^1$, $[x_1, \cdots, x_n] \mapsto x_1 \cdots x_n$ is an $(n-1)$-disc bundle which is orientable if and only if n is odd.*

Indeed we explain here how to describe the fiber as the quotient $\mathbb{R}^{n-1}/G_0$ where G_0 is a semidirect product $\mathbb{Z}^{n-1} \ltimes \mathfrak{S}_n$.

Write S^1 as the quotient of $\mathbb{R}$ by the integer lattice acting by translation, so

$$\mathrm{SP}^n S^1 = (S^1)^n/\mathfrak{S}_n = \mathbb{R}^n/\mathbb{Z}^n/\mathfrak{S}_n \tag{1.1.1}$$

Let $\mathfrak{A}$ be the group of affine motions of $\mathbb{R}^n$. The groups $\mathbb{Z}^n$ and $\mathfrak{S}_n$ are subgroups of $\mathfrak{A}$. Since $\mathbb{Z}^n$ is normalized by $\mathfrak{S}_n$ and $\mathbb{Z}^n \cap \mathfrak{S}_n = 1$, then $G = \mathbb{Z}^n\mathfrak{S}_n$ is a subgroup of $\mathfrak{A}$ which is the internal semidirect product $G = \mathbb{Z}^n \ltimes \mathfrak{S}_n$. Hence

$$\mathrm{SP}^n S^1 = \mathbb{R}^n/\mathbb{Z}^n/\mathfrak{S}_n = \mathbb{R}^n/(\mathbb{Z}^n \ltimes \mathfrak{S}_n) = \mathbb{R}^n/G$$

Now write the vector space $\mathbb{R}^n$ as

$$\mathbb{R}^n = \mathbb{R}^{n-1} \oplus \mathbb{R} \tag{1.1.2}$$

where $\mathbb{R}^{n-1} = \{(x_1, \cdots, x_n) \,|\, x_1 + \cdots + x_n = 0\}$ and $\mathbb{R} = \{(x_1, \cdots, x_n) \,|\, x_1 = \cdots = x_n\}$ its orthogonal complement. With this decomposition, we have that

$$(x_1, \cdots, x_n) = (x_1 - m, \cdots, x_n - m) + (m, \cdots, m)$$

where $m = 1/n(x_1 + \cdots + x_n)$. Write $B_c = (e_1, \cdots, e_n)$ the canonical basis for $\mathbb{R}^n$, $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ where 1 is at the i-th location. To the decomposition (1.1.2), corresponds the basis $B = (v_1, \cdots, v_n)$ where for each $i = 1..n-1$, $v_i = (0, \cdots, 1, 0, \cdots, -1)$ with 1 at the i-th location, -1 at the n-th location and 0 elsewhere, $v_n = (1, \cdots, 1)$. We describe the translation and permutation actions relatively to this basis. First we have that

$$\begin{aligned}
&v_1 = e_1 - e_n\\
&v_2 = e_2 - e_n\\
&\vdots\\
&v_{n-1} = e_{n-1} - e_n\\
&v_n = e_1 + \cdots + e_n
\end{aligned}$$

So the change of basis matrix from B to B_c is

$$P_{B\to B_c} = \begin{pmatrix} 1 & 0 & & 0 & 1 \\ 0 & 1 & & & \vdots \\ \vdots & 0 & \cdots & & \\ 0 & \vdots & & 1 & \\ -1 & -1 & & -1 & 1 \end{pmatrix}$$

For example when $n = 3$,

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}_B = \begin{pmatrix} \alpha_1 + \alpha_3 \\ \alpha_2 + \alpha_3 \\ -\alpha_1 - \alpha_2 + \alpha_3 \end{pmatrix}_{B_c}$$

and we have that

$$1/3 \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{B_c} = \begin{pmatrix} x_1 - m \\ x_2 - m \\ m \end{pmatrix}_B$$

where $m = 1/3(x_1 + x_2 + x_3)$. Now we look at translation and permutation in the new basis. Translation is given by

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}_B \mapsto \begin{pmatrix} \alpha_1 + k_1\alpha_1 \\ \alpha_2 + k_2\alpha_2 \\ \alpha_3 + k_3\alpha_3 \end{pmatrix}_B \quad ; k_i \in \mathbb{Z}$$

The permutation however is more complicated. For example the cyclic permutation $(123) \in \mathfrak{S}_3$ is given as follows

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}_B \mapsto \begin{pmatrix} \alpha_1 + \alpha_3 \\ \alpha_2 + \alpha_3 \\ -\alpha_1 - \alpha_2 + \alpha_3 \end{pmatrix}_{B_c} \overset{(123)}{\mapsto} \begin{pmatrix} -\alpha_1 - \alpha_2 + \alpha_3 \\ \alpha_1 + \alpha_3 \\ \alpha_2 + \alpha_3 \end{pmatrix}_{B_c} \mapsto \begin{pmatrix} -\alpha_1 - \alpha_2 \\ \alpha_1 \\ \alpha_3 \end{pmatrix}_B$$

As we can see the term α_3 is fixed by the permutation. It's a general fact that the last term corresponding to the new decomposition is fixed by any permutation in $\mathfrak{S}_n$. If we denote $\mathbb{Z}^{n-1}$ the group of translations corresponding to the integers $(k_1, \cdots, k_{n-1}, 0)$ then the subgroup $\mathbb{Z}^{n-1}$ is normalized by $\mathfrak{S}_n$ corresponding to the decomposition (1.1.2). This follows immediately from the fact that $\mathfrak{S}_n$ fixes the last coordinate. Now since $\mathbb{Z}^{n-1} \cap \mathfrak{S}_n = \{1\}$, the set $G_0 = \mathbb{Z}^{n-1}\mathfrak{S}_n$ is a subgroup of $\mathfrak{A}$ which is the internal semidirect product that we denote $G_0 = \mathbb{Z}^{n-1} \ltimes \mathfrak{S}_n$. Now if we denote $\mathbb{Z}$ the group of translations corresponding to the integers $(0, \cdots, 0, k_n)$ then it is clear that the subgroup G_0 is normalized by $\mathbb{Z}$ and since $G_0 \cap \mathbb{Z} = \{1\}$, the set $G_0 \ltimes \mathbb{Z}$ is a group isomorphic to G.

This implies that

$$\mathrm{SP}^n S^1 = \mathbb{R}^n/G = \mathbb{R}^n/(G_0 \ltimes \mathbb{Z}) = \mathbb{R}^n/G_0/\mathbb{Z}$$

But

$$\mathbb{R}^n/G_0 = (\mathbb{R}^{n-1} \times \mathbb{R})/G_0 = (\mathbb{R}^{n-1}/G_0) \times \mathbb{R}$$

since G_0 fixes $\mathbb{R}$. Thus

$$\mathrm{SP}^n S^1 = ((\mathbb{R}^{n-1}/G_0) \times \mathbb{R})/\mathbb{Z}$$

Now $\mathbb{Z}$ acts on $\mathbb{R}^{n-1}/G_0$ and acts freely on $\mathbb{R}$ then

$$\mathbb{R}^{n-1}/G_0 \longrightarrow \mathrm{SP}^n S^1 \longrightarrow \mathbb{R}/\mathbb{Z} = S^1$$

is a fiber bundle.

Remark 1.1.9 Following the argument in [29] and viewing the elliptic curve T as $\mathbb{C}/L$ where L is a lattice, we have the following identification. The lattice L^3 acts on $\mathbb{C}^3$

by translations, and $\mathbb{Z}_3$ acts on $\mathbb{C}^3$ by permuting coordinates cyclically. The group G generated by L^3 and $\mathbb{Z}_3$ is the wreath product $L\wr\mathbb{Z}_3$ which we recall is a semi-direct product extension $G = L^3 \ltimes \mathbb{Z}_3$. Since L^3 is normal in G, $\mathbb{C}^3/G = (\mathbb{C}^3/L^3)/\mathbb{Z}_3 = \mathrm{CP}^3(T)$. Now as in Remark 1.1.6, $\mathbb{C}^3$ splits as $\mathbb{C}\times H$ where $\mathbb{C}$ is the diagonal copy, and H is the hyperplane (these describe the irreducible subrepresentations of the permutation representation of $\mathfrak{S}_3$). The action of G preserves this product structure. Let G_0 be the subgroup of G which stabilizes the first factor. Then G_0 is normal in G and $G/G_0 \cong L$. We can write that

$$\mathbb{C}^3/\mathbb{Z}_3 = (\mathbb{C}\times H/G_0)/L = \mathbb{C}\times_L (H/G_0)$$

The projection onto $T = \mathbb{C}/L$ gives the fiber bundle $\mathrm{CP}^3(T)\longrightarrow T$ with fiber H/G_0.

Morton's description has further consequences. For example, the n-th braid space of the circle $B(S^1, n) = \{[x_1, \cdots, x_n] \,|\, x_i \neq x_j\, i \neq j\}$ is a bundle over S^1 with fiber an *open* $(n-1)$-disc. Indeed this comes from observing that the subset of $\mathrm{SP}^n S^1$ corresponding to configurations where one entry repeats at least twice is a sphere bundle over S^1 with fiber the boundary circle of Morton's disk.

In the next example we describe the symmetric product of a wedge of circles $\mathrm{SP}^n(\bigvee^k S^1)$.

Example 1.1.10 We use the fact that $\mathrm{SP}^n(-)$ is a homotopy functor to replace $\mathrm{SP}^n(\bigvee^k S^1)$ by $\mathrm{SP}^n(\mathbb{C}-Q_k)$ where $Q = \{q_1, \ldots, q_k\}$ is a subset of k-distinct points. Given $\zeta = [z_1, \ldots, z_n] \in \mathrm{SP}^n(\mathbb{C})$, we can consider the associated polynomial $P_\zeta := (z-z_1)\ldots(z-z_n)$. We then construct a map

$$\Psi : \mathrm{SP}^n(\mathbb{C}-Q_k)\longrightarrow(\mathbb{C}^*)^k$$

$$\zeta \longmapsto (P_\zeta(q_1), \ldots, P_\zeta(q_k))$$

This is well-defined since the q_i cannot be roots of P_ζ. The fiber of Ψ over a prescribed set of values $s = \{s_1, \ldots, s_k\}$ (in the image) is homeomorphic to all polynomials P of degree n such that $P(q_i) = s_i$. Such polynomials always exist when $n \geq k$ and hence in that range Ψ is surjective. When $n \leq k$, such a polynomial when it exists is unique and Ψ is an embedding. Finally and when $n \geq k$, Ψ is a locally trivial fibration with fiber the complex affine space of dimension $\max(0, n-k)$. Indeed if P_1, P_2 are two polynomials such that $P_1(q_i) = P_2(q_i) = s_i$ for all $1 \leq i \leq k$, then $P_1 = P_2 + q(z)(z-q_1)\ldots(z-q_k)$ where $q(z)$ is an arbitrary polynomial of degree $n-k$.

1.1.1 Braid Spaces

A subspace of $\mathrm{SP}^n X$ that has been extensively studied in the literature is the configuration space or the *braid space* $B(X, n)$ of configurations $[x_1, \cdots, x_n]$ where $x_i \neq x_j$ if $i \neq j$.

This can be defined as the quotient of the ordered configuration space by the action of the symmetric group $B(X,n) = F(X,n)/\mathfrak{S}_n$ where

$$F(X,n) = \{(x_1,\cdots,x_n) \in X^n \mid x_i \neq x_j \text{ if } i \neq j\}$$

Remark 1.1.11 We have that $F(X,n) = X^n - F_2(X,n)$, $n \geq 2$ where $F_2(X,n)$ is the fat diagonal in X^n defined by

$$F_2(X,n) = \{(x_1,\cdots,x_n) \in X^n \mid \text{there exists an entry that repeats at least twice}\}$$

Next is a description of the fundamental group of $F(X,n)$ and $B(X,n)$

Lemma 1.1.12 *For X a manifold of dimension $d > 2$,*

- $\pi_1(F(X,n)) = (\pi_1(X))^n$
- $\pi_1(B(X,n)) = (\pi_1 X)^n \ltimes \mathfrak{S}_n$ *where* $\ltimes$ *denotes a semi direct product and* $(\pi_1 X)^n$ *is a normal subgroup of* $\pi_1(B(X,n))$.

Proof. When X is a manifold of dimension $d > 2$, $F(X,n)$ is the complement of a submanifold of codimension > 2, so $\pi_1(F(X,n)) = \pi_1(X^n) = (\pi_1 X)^n$.

The projection $F(X,n) \longrightarrow B(X,n)$ is a covering of degree $n!$. This induces the following group extension

$$0 \longrightarrow \pi_1(F(X,n)) \longrightarrow \pi_1(B(X,n)) \longrightarrow \mathfrak{S}_n \longrightarrow 0$$

When X is a manifold of dimension $d > 2$, this extension is split. Indeed, any choice of chart $\mathbb{R}^d \subset X$ induces a map of extensions

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\mathbb{R}^{nd}) \arrow[r] \arrow[d] & \pi_1(B(\mathbb{R}^d,n)) \arrow[r, "\gamma"] \arrow[d, "g"] & \mathfrak{S}_n \arrow[r] \arrow[d, "h"] & 1 \\
1 \arrow[r] & \pi_1(X^n) \arrow[r] & \pi_1(B(X,n)) \arrow[r, "\pi"] & \mathfrak{S}_n \arrow[r] & 1
\end{tikzcd}$$

and the map $g\gamma^{-1}h^{-1}$ gives a section to π. ∎

Note that this shows that $\pi_1(B(X,n))$ need not be abelian. In fact and when X is a manifold of dimension greater than 2, $\pi_1(B(X,n)) = \pi_1(X) \wr \mathfrak{S}_n$ is a wreath product. This is folklore and is discussed and used in [17] for example.

Example 1.1.13 We show that $B(\mathbb{R}^n,2) \simeq \mathbb{R}P^{n-1}$. Indeed, consider the projection $\pi : B(\mathbb{R}^n,2) \longrightarrow \mathbb{R}P^{n-1}$ that sends two distinct points of $\mathbb{R}^n$ to the line passing through the origin and parallel to the line defined by the two points. This is a locally trivial fiber bundle. The fiber over a line L is determined as follows. The line L cuts the sphere S^{n-1} in two antipodal points. Now it's easy to see that the set of points that project to L is the upper $(n-1)$-hemisphere. The result follows from contractibility of the fiber.

Recall that the *trefoil* knot is the $(2,3)$-torus knot or equivalently the knot obtained by embedding the circle in $S^1 \times S^1 \hookrightarrow \mathbb{R}^3$ via the map $z \mapsto (z^2, z^3)$. The following result is well-known to the experts but we couldn't find a complete proof of it.

Proposition 1.1.14 *There is a homotopy equivalence $B(\mathbb{R}^2, 3) \simeq S^3 - K$, where K is the trefoil knot.*

PROOF. First of all $B(\mathbb{C}, 3)$ is homotopy equivalent to its subspace of configurations with zero center of mass

$$\hat{B}(\mathbb{C}, 3) = \{[z_1, z_2, z_3] \mid z_1 + z_2 + z_3 = 0, z_i \neq z_j\}$$

This can be seen by either considering the bundle

$$B(\mathbb{C}, 3) \longrightarrow \mathbb{C}\ ,\ [z_1, z_2, z_3] \mapsto z_1 + z_2 + z_3$$

which is trivial since $\mathbb{C}$ is contractible or the retraction

$$r : B(\mathbb{C}, 3) \longrightarrow C_3^0(\mathbb{C})\ ,\ [z_1, z_2, z_3] \mapsto [\frac{2z_1 - z_2 - z_3}{3}, \frac{2z_2 - z_1 - z_3}{3}, \frac{2z_3 - z_1 - z_2}{3}]$$

that induces a deformation retraction of $B(\mathbb{C}, 3)$ onto $\hat{B}(\mathbb{C}, 3)$. On the other hand, the relation $z_1 + z_2 + z_3 = 0$ implies that for any $\alpha \in \mathbb{R}$, $\alpha z_1 + \alpha z_2 + \alpha z_3 = 0$ meaning that for any $[z_1, z_2, z_3]$ with zero center of mass, there exists a unique $\alpha \in \mathbb{R}$ such that $[z_1^\alpha, z_2^\alpha, z_3^\alpha]$, where $z_i^\alpha = \alpha z_i$, is still with zero center of mass in addition to the "normality condition" (NC):

$$|z_1^\alpha z_2^\alpha + z_1^\alpha z_3^\alpha + z_2^\alpha z_3^\alpha|^2 + |z_1^\alpha z_2^\alpha z_3^\alpha|^2 = 1$$

so that

$$\begin{aligned} \hat{B}(\mathbb{C}, 3) &\cong \{(z - z_1)(z - z_2)(z - z_3) \mid z_1 + z_2 + z_3 = 0, z_i \neq z_j,\ \text{NC}\} \\ &= \{z^3 + a_2 z - a_3 \mid a_2 = z_1 z_2 + z_1 z_3 + z_2 z_3, a_3 = z_1 z_2 z_3, z_i \neq z_j, |a_2|^2 + |a_3|^2 = 1\} \end{aligned}$$

This is due to the identity

$$\prod_{j=1}^{n}(z - z_j) = \sum_{j=0}^{n}(-1)^j a_j z^j$$

where the a_i's are the elementary symmetric polynomials on the roots $z_1, \cdots, z_n$. Using the duality between roots and coefficients we have that $\hat{B}(\mathbb{C}, 3)$ is homeomorphic to the space

$$X = \{(a_2, a_3) \in \mathbb{C}^2 \mid a_2 = z_1z_2 + z_1z_3 + z_2z_3, a_3 = z_1z_2z_3, z_i \neq z_j, |a_2|^2 + |a_3|^2 = 1\}$$

which we can write as $S^3 - B$ where

$$B = \{(a_2, a_3) \in \mathbb{C}^2 \mid a_2 = z_1z_2 + z_1z_3 + z_2z_3, a_3 = z_1z_2z_3, |a_2|^2 + |a_3|^2 = 1, z_1 = z_2(\text{say})\}$$

It remains to show that B is the trefoil knot. As a torus knot of type $(2, 3)$, the trefoil knot is given by the image of the map $S^1 \longrightarrow S^1 \times S^1, z \mapsto (z^2, z^3)$. To describe B, we solve the system

$$\begin{cases} z_1 + z_2 + z_3 = 0 \\ vz_1 = z_2 \\ a_2 = z_1z_2 + z_1z_3 + z_2z_3 \\ a_3 = z_1z_2z_3 \end{cases}$$

wich gives that $a_2 = -3z_1^2$ and $a_3 = -2z_1^3$ and

$$B = \{(-3z_1^2, -2z_1^3) \ \mid z_1 \in \mathbb{C}^*\}$$

Since $\mathbb{C}^*$ deformation retracts onto S^1, $B \subset S^3$ is given by the composition of

$$S^1 \longrightarrow B, z_1 \mapsto (-3z_1^2, -2z_1^3)$$

with $B \longrightarrow S^3, (a, b) \mapsto (a/\sqrt(2), b/\sqrt(2))$. ∎

1.2 Path-lifting properties and fundamental group

Let X be a topological space and G a group acting on it. In this section we study the lifting properties of the quotient map $\pi : X \longrightarrow X/G$.

Definition 1.2.1 *A group G is said to act discontinuously on X if*

- $\forall x \in X$, *$stab(x) = \{g \in G \mid gx = x\}$ is finite*

- *$\forall x \in X$, there exists a neighborhood V_x such that $\forall g \notin stab(x)$, $V_x \cap gV_x = \emptyset$.*

The main example of a discontinuous action is given by a finite group acting on a Hausdorff space. This is stated in the following result.

Proposition 1.2.2 *An action of a finite group on a Hausdorff space is discontinuous.*

PROOF. Let $G = \{g_1, \cdots, g_n\}$, $\forall x \in X$, the stabilizer subgroup $stab(x) \subset G$ is also finite. Let $x \in X$, $\pi(x) = [x]$ and $\pi^{-1}[x] = \{g_1 x, \cdots, g_n x\}$. Since X is Hausdorff, we can find pairwise disjoint open neighborhoods N_i of $g_i x$, $i = 1..n$. Let

$$N = \cap\{g_i^{-1}N_i,\ \ i = 1..n\}$$

Then N is an open neighbourhood of x. Now take $g \notin stab(x)$, $gx = g_i x$ for some $i \in \{1, \cdots, n\}$, so $gN \subset N_i$. Hence $N \cap gN = \emptyset$ and the action is discontinuous. ∎

As in every introductory book in homotopy theory, we say a map $\pi : Y \longrightarrow X$ has the path-lifting property if we can lift any path $[0,1] \longrightarrow X$ to a path in Y knowing where it starts in Y.

The following result is the most important feature of discontinuous action regarding lifting properties. The proof is technical (see for example [8], Proposition 1.5)

Proposition 1.2.3 *If the group G acts discontinuously on the Hausdorff space X then the quotient map $\pi : X \longrightarrow X/G$ has the path lifting property. That is, if γ is a path in X/G and x_0 is a point of X such that $\pi(x_0) = \gamma(0)$ then there exists $\tilde{\gamma}$ a path in X such that $\pi \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}(0) = x_0$.*

The following is immediate.

Corollary 1.2.4 *If G is a group which acts discontinuously on a Hausdorff connected space X with a fixed point x_0, then $\pi_1(X, x_0) \longrightarrow \pi_1(X/G, x_0)$ is surjective. In particular $\pi_1(X^n) \longrightarrow \pi_1(\Gamma P^n X)$ is surjective for any $\Gamma \subset \mathfrak{S}_n$.*

We point out that path-lifting properties were discussed way back in [28] where the following is shown

Theorem 1.2.5 *[28] If a compact Lie group acts on a Hausdorff space X, then the quotient map $X \longrightarrow X/G$ has the path-lifting property. Moreover if both X, G are path-connected, then the induced map on fundamental groups is onto.*

The last statement in the above theorem can be seen as a consequence of a more general result.

Lemma 1.2.6 *Let $f : X \longrightarrow Y$ be a quotient map between compact hausdorff and connected spaces with path-connected geometric fibers. Then the map on π_1 is surjective.*

PROOF. Let f be as in the lemma, in particular f is an open map. Then a straightforward argument in general topology shows that the inverse image of a connected subspace must be connected. Now let $Fib(f)$ be the homotopy fiber of f. This as we recall is

$$Fib(f) = \{(x, \gamma) \in X \times PY \mid f(x) = \gamma(1)\}$$

where PY is the space of paths in Y starting at a fixed $y_0 \in Y$. We wish to show that $Fib(f)$ is connected, from which our claim will follow thanks to the exact sequence in homotopy groups for the fibration $Fib(f) \longrightarrow X \longrightarrow Y$. To do that, fix $x_0 \in f^{-1}(y_0)$, and let (x_0, c_{y_0}) be the basepoint of $Fib(f)$, where c_{y_0} is the constant map at y_0. Let (x, γ) be in $Fib(f)$. Since x and x_0 are in $f^{-1}(\gamma)$, and since $f^{-1}(\gamma)$ is path-connected according to our earlier claim, there is a path $\tilde{\gamma} : I \longrightarrow f^{-1}(\gamma)$ with $\tilde{\gamma}(0) = x_0$ and $\tilde{\gamma}(1) = x$. This path can be parameterized so that it is a lifting of γ, i.e. $f(\tilde{\gamma}(t)) = \gamma(t)$. Let then γ_t be the portion of γ between 0 and t, reparameterized to be in $[0, 1]$. Then

$$(\tilde{\gamma}(t), \gamma_t)$$

is a path in $Fib(f)$ from (x, γ) to (x_0, c_{y_0}). Since any element of $Fib(f)$ is connected by a path to (x_0, c_{y_0}), $Fib(f)$ is connected. ∎

Finally we quote a result of Armstrong giving a description of the fundamental group of a quotient by a group action.

Theorem 1.2.7 *Let G be a discontinuous group of homeomorphisms of a path connected, simply connected, locally compact metric space X, and let H be the normal subgroup of G generated by those elements which have fixed points. Then $\pi_1(X/G) \cong G/H$.*

This theorem is stated for simply-connected spaces. However it has adaptations for the non simply-connected case as well. Indeed, if X has a universal cover $\tilde{X}$, then $\pi_1(X)$ acts on $\tilde{X}$ as a group of deck transformations and hence an extension of $\pi_1(X)$ and G acts on $\tilde{X}$. We can then apply the theorem above to this extension acting on the simply connected space $\tilde{X}$.

Chapter 2

Permutation Products

2.1 Introduction

If Γ is a subgroup of the n-th symmetric group $\mathfrak{S}_n$, define the permutation product $\Gamma\mathrm{P}^n(X)$ to be the quotient of X^n by the permutation action of Γ on coordinates. The prototypical example being of course the n-th symmetric product $\mathrm{SP}^n(X)$ which corresponds to when $\Gamma = \mathfrak{S}_n$. The cyclic product $\mathrm{CP}^n(X)$ corresponds on the other hand to when $\Gamma = \mathbb{Z}_n$ is the cyclic group. Note that the homotopy type of the space $\Gamma\mathrm{P}^n(X)$ *does depend* on the way the group Γ embeds in $\mathfrak{S}_n$ (see below).

Elements of $\Gamma\mathrm{P}^n(X)$ are written as equivalence classes $[x_1,\dots,x_n] := \pi(x_1,\dots,x_n)$ where $\pi : X^n \longrightarrow \Gamma\mathrm{P}^n(X)$ is the quotient map. They are also written as abelian products $x_1x_2\cdots x_n$. It will also be convenient to call an element in $\Gamma\mathrm{P}^n(X)$ a *configuration*. When $n = 2$, the only non-trivial permutation product is the symmetric square $\mathrm{SP}^2(X)$. This space is already non-trivial and has been studied early on by Stein but his methods are simplicial and not very helpful for making computations. The case of the circle is illustrative, we discuss it in the next example.

Example 2.1.1 The product map $\mathrm{SP}^2(S^1) \longrightarrow S^1$ is a locally trivial bundle map. The preimage over 1 consists of points of the form $[z, \bar{z}]$, $z \in S^1 \subset \mathbb{C}$ and can be identified with the closed upper semi-circle, thus with the closed interval $I := [0, 1]$. This bundle is non-trivial and $\mathrm{SP}^2(S^1)$ is homeomorphic to the Mobius band.

More generally for the circle there is a well-defined multiplication map

$$m : \Gamma\mathrm{P}^n(S^1) \longrightarrow S^1 \ , \ [x_1,\dots,x_n] \longmapsto x_1x_2\cdots x_n$$

This is a fiber bundle with fiber the quotient of $\mathbb{R}^{n-1}$ by some subgroup Γ_0 of a semi-direct product $\mathbb{Z}^n \ltimes \Gamma$ acting on $\mathbb{R}^n$ by affine motions (Morton). This bundle in the case of $\Gamma = \mathfrak{S}_n$ is described in [29]. In that case the multiplication map $\mathrm{SP}^n(S^1)\longrightarrow S^1$ is an $(n-1)$-disc bundle which is orientable if and only if n is odd. The sphere bundle corresponds to the fat diagonal which corresponds to configurations in $\mathrm{SP}^n X$ where two or more points coincide.

Example 2.1.2 Here we extend Example 2.1.1 to an arbitrary torus. Set $T = S^1 \times S^1$ and let $m : \mathrm{SP}^2 T\longrightarrow T$ be the multiplication map. This is a bundle projection and the fiber at 1 is the quotient $T/\mathfrak{S}_2$, where $\mathfrak{S}_2$ acts by taking $(x, y) \in T$ to $(\bar{x}, \bar{y})$. This action corresponds to the hyperelliptic involution on the torus with 4 fixed points, and so the quotient is a copy of the Riemann sphere. This of course is a special case of the fact that multiplication $\mu : \mathrm{SP}^n(T)\longrightarrow T$ is an analytic fiber bundle with $\mu^{-1}(1) \cong \mathbb{P}^{n-1}$.

Remark 2.1.3 Before getting to more specific examples, we ought to point out why the homotopy type of a permutation product $\Gamma\mathrm{P}^n X$ is dependent on the way Γ embeds in $\mathfrak{S}_n$. Of course if the subgroups are conjugate, then this conclusion is not true and both quotients are homeomorphic. But even for the simplest non-trivial group $\mathbb{Z}_2$, non-conjugate embeddings exists namely the embeddings into $\mathfrak{S}_4$ sending the generator to either (12) or to (13)(24). This in fact provides us with our counterexample : let Γ_1 be the embedded copy of $\mathbb{Z}_2 \subset \mathfrak{S}_4$ generated by (12) and let $\Gamma_2 \cong \mathbb{Z}_2$ be the second copy generated by (13)(24). Then clearly

$$\Gamma_1\mathrm{P}^4 X = \mathrm{SP}^2 X \times X \times X \quad \text{and} \quad \Gamma_2\mathrm{P}^4 X = \mathrm{SP}^2(X \times X)$$

and both spaces are generally non-homotopic (eg. they have different π_1).

Remark 2.1.4 The map $X^n\longrightarrow\Gamma\mathrm{P}^n(X)$ is an example of a *branched covering*; that is a map that is a regular covering over a dense open set. More precisely, let $\phi : M\longrightarrow N$ be a surjective map and $B_\phi \subset N$ a closed subset (the "branch set"). The map $\phi : M\longrightarrow N$ is said to be branched covering, branched over B_ϕ and of degree $d \in \mathbb{Z}^+$ if the restriction

$$\hat{\phi} : M - \phi^{-1}(B_\phi)\longrightarrow N - B_\phi$$

is a connected regular covering of degree d. So away from B_ϕ the map is unbranched. In most situations, B_ϕ is a subcomplex of N. The dimension of B_ϕ in N is the dimension of its largest simplices and one demands that this dimension is at most $\dim N - 2$. Note that if X is simplicial, then the projection $X^n\longrightarrow\Gamma\mathrm{P}^n X$ is branched over a subcomplex.

2.2 Three Fold Cyclic Products

When $n \geq 3$, the cyclic and symmetric products start to differ. We can for example use Morton's description above to deduce the following characterization [39].

Lemma 2.2.1 *There is a homeomorphism $CP^3(S^1) = S^1 \times S^2$.*

Proof. The key point is to observe that the quotient map $\mathrm{CP}^3(S^1) \longrightarrow \mathrm{SP}^3(S^1)$ has two sections given as follows. Note first of all that the clockwise (or counterclockwise) order of any three points on the circle is not changed by a cyclic $\mathbb{Z}_3$-permutation. This means if we write the points $\{x_1, x_2, x_3\}$ in clockwise order, we get a well-defined element in $\mathrm{CP}^3(S^1)$ and if we write them in counterclockwise we get another. This gives the two sections. When two of the points among x_1, x_2 and x_3 are equal, the sections coincide. Each section produces a copy of $\mathrm{SP}^3(S^1)$ in $\mathrm{CP}^3(S^1)$. Since any configuration in $\mathrm{CP}^3(S^1)$ is either clockwise or counterclockwise, we have the decomposition

$$\mathrm{CP}^3(S^1) = \mathrm{SP}^3(S^1) \cup \mathrm{SP}^3(S^1)$$

The union is over configurations of the form $[x, x, y]$. But $\mathrm{SP}^3(S^1)$ is a (trivial) bundle over S^1 with fiber the two disk D^2, and the configurations with repeated points consists of its sphere bundle $\partial D \times S^1 = S^1 \times S^1$. This means that $\mathrm{CP}^3(S^1)$ is a union of two disk bundles joined along their common circle bundle. In other words $\mathrm{CP}^3(S^1)$ is the *fiberwise* pushout

$$S^2 = D^2 \cup_{S^1} D^2 \longrightarrow \mathrm{CP}^3(S^1) \longrightarrow S^1$$

and this is a trivial bundle obtained from glueing two trivial disk bundles. ∎

It is slightly harder to determine $\mathrm{CP}^3(T)$ when $T = S^1 \times S^1$ is the torus. There is a branched degree two covering $\pi : \mathrm{CP}^3 T \longrightarrow \mathrm{SP}^3 T$ but there are no obvious sections this time. Since T is an abelian topological group, we have again a multiplication map $\mu : \mathrm{SP}^3(T) \longrightarrow T$. This map, known in other circles as the *Abel-Jacobi* map, is a bundle map with fiber $\mathbb{P}^2$.

The following result characterizes $\mathrm{CP}^3(T)$ completely and must be compared to ([5], corollary 3.4). We write the group structure on T additively.

Proposition 2.2.2 *Addition $\mu_c : CP^3(T) \longrightarrow T$, $[x_1, x_2, x_3] \mapsto x_1 + x_2 + x_3$, is a bundle projection with fiber a simply-connected algebraic surface with 9 cusps ramified over $\mathbb{P}^2$ along the dual curve of a plane cubic.*

Proof. The map μ_c makes CP^3T into a bundle over the elliptic curve T and we write E the preimage of $0 \in T$ (as an additive abelian group). This preimage is a complex algebraic surface with complex structure depending on the T we pick. Pick T a plane elliptic curve; i.e. $T \subset \mathbb{C}^2 \subset \mathbb{P}^2$. The fiber of μ_c over $0 \in T$ consists of the triple of points $[x, y, z] \in \mathrm{CP}^3(T)$ with $x + y + z = 0$. We identify this fiber with the quotient $E := (T \times T)/\mathbb{Z}_3$ where $\mathbb{Z}_3$ acts via its generator τ of order three as follows

$$\tau : (x, y) \mapsto (-x - y, x)$$

We can write the diagram

$$\begin{array}{ccccc} \mathbb{T}/\mathbb{Z}_3 & \longrightarrow & \mathrm{CP}^3(T) & \xrightarrow{\mu_c} & T \\ \downarrow{\scriptstyle 2:1} & & \downarrow{\scriptstyle \pi} & & {\scriptstyle =}\downarrow \\ \mathbb{T}/\mathfrak{S}_3 & \longrightarrow & \mathrm{SP}^3(T) & \xrightarrow{\mu_s} & T \end{array}$$

where $\mathbb{T} := T \times T$. The symmetric group acts on $\mathbb{T}$ via its generators τ of order 3 and $J : (x, y) \mapsto (y, x)$ of order 2. Of course $\mu_c^{-1}(0) \to \mu_s^{-1}(0)$ is a branched degree two cover. The branching locus is a copy of the elliptic curve $C := \{(x, x, -2x)\}$ under this identification. But now $\mu_s^{-1}(0) = \mathbb{T}/\mathfrak{S}_3$ can be identified with $\mathbb{P}^2$ as follows: since $T \subset \mathbb{C}^3$ is a plane curve, then to any triple $(x, y, -x - y)$ corresponds a unique line in $\mathbb{C}^3$ passing through x, y (in fact and by definition of the group law on T, this line meets the curve again at $x + y$). Under this identification, the branching curve C corresponds to the space of tangents to that curve in $\mathbb{P}^2$ and this is by definition the dual curve which we write $C^* \in (\mathbb{P}^2)^*$. But the dual curve to a smooth plane cubic C has 9 singular points (cusps) which correspond under the duality to the 9 flexes of C (i.e the inflection points of C or points of order 3 in the group law for C). Note that this dual curve is of course of genus 1 and hence must have degree 6 (i.e. a sextic) by the degree-genus formula. Since $E \longrightarrow \mathbb{P}^2$ ramifies over the projective curve C^*, then E has 9 singular points which map to the cusps in C^*. Note that E is simply connected because of the short exact sequence

$$0 = \pi_2(T) \longrightarrow \pi_1(E) \longrightarrow \pi_1(\mathrm{CP}^3T) \xrightarrow{\mu_{c*}} \pi_1(T) \longrightarrow 0$$

associated to the bundle μ_c. Indeed $\mu_{c*} : \pi_1(\mathrm{CP}^3T) \longrightarrow \pi_1(T)$ must be an isomorphism between two copies of $\mathbb{Z} \oplus \mathbb{Z}$, this being a consequence of Theorem 0.0.1 and the fact that $\mathrm{CP}^3T \longrightarrow T$ has a section. ∎

The homology of $\mathrm{CP}^3(T)$ is listed in the following table. This computation is a special case of more elaborate recent calculations in [2] where the homology of $\mathrm{CP}^p(T)$ is obtained for any prime p and any product of circles T.

Lemma 2.2.3 *The following table lists the non-trivial homology groups of* $CP^3(S^1 \times S^1)$

H_0	H_1	H_2	H_3	H_4	H_5	H_6
$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^5$	$\mathbb{Z}^8$	$\mathbb{Z}^5$	$\mathbb{Z}^2 \oplus \mathbb{Z}_3$	$\mathbb{Z}$

PROOF. Set $T = S^1 \times S^1$. Note that $\mathrm{CP}^3(T)$ has the homology of a closed manifold whenever the characteristic of the field of coefficients is prime to 3 (this follows from a well-known general fact that applies to global quotient orbifolds under the action of a finite group). This homology consists of the invariant classes in $H_*(T)^{\otimes 3}$ under the induced permutation action. If $e_1, e_2 \in H_1(T)$ are the generators, then the torsion free part of $H_2(\mathrm{CP}^3T)$ is additively generated by

$$e_i \otimes e_j \otimes 1 + 1 \otimes e_i \otimes e_j + e_j \otimes 1 \otimes e_i$$

for $i, j \in \{1, 2\}$, and by the class $[T] \otimes 1 \otimes 1 + 1 \otimes [T] \otimes 1 + 1 \otimes 1 \otimes [T]$. The total is 5 classes as asserted. Similarly for obtaining the rank of the other torsion free groups. Since $H_*(T)$ is torsion-free, it's equally easy to see that $H_*(\mathrm{CP}^3T)$ has no p-torsion if p is prime to 3. The only real issue then is to compute $H_*(\mathrm{CP}^3T, \mathbb{Z}_3)$. This is done in [2] and we only state their result.

One idea is to think of $\mathrm{CP}^3(T)$ as a quotient of $(\mathbb{R}^2/\mathbb{Z}^2)^3$ by an action of $\mathbb{Z}_3$ preserving the lattice. This action is the sum of two regular representations $\mathbb{Z}_3 \longrightarrow GL_3(\mathbb{R})$ since any such representation corresponds to cyclically permuting the basis vectors. In the notation of [2], this action is of "type $(0, 2, 0)$" and so we can apply Theorem 1 of that paper to deduce the 3-torsion in $H_*(\mathrm{CP}^3T; \mathbb{Z}) \cong \mathbb{Z}^{\alpha_k} \oplus (\mathbb{Z}_3)^{\beta_k}$. The coefficient β_k is obtained by writing the formal power series in x

$$T_L(x) = \frac{x}{1 - x^2}\left[x^2(1+x)^2 - x^2 + 1 - (1 + \alpha x)(1 + x^3)^2\right] = p(x) + \alpha q(x) = x^5 + \alpha q(x)$$

and reading the coefficients of $p(x)$ which in this case is x^5, thus $\beta_5 = 1$. There is therefore only one non-trivial torsion group in dimension 5. ∎

2.3 Orbit stratifications

The category of stratified orbifolds is a suitable category in which to consider permutation products of manifolds. Following Beshears [4], we define for a G-space X, a stratum for each conjugacy class (H) of a subgroup $H \subset G$ as follows

$$X_{(H)} := \{x \in X \mid stab(x) \sim H\} = \bigcup_{K \sim H} X_K$$

where $\sim$ means "being conjugate to", $stab(x) := \{g \in G \mid gx = x\}$ is the isotropy group of x and X_K is the fixed point set of K. For example $X_{(G)}$ consists of the fixed points of the action. As H ranges over all subgroups of G we get a stratification of X called the *stabilizer stratification.* One checks that $X_{(H)}$ is a G-invariant subspace since $\mathrm{stab}(gx) = g\, \mathrm{stab}(x)g^{-1}$, and that the corresponding quotients $X_{(H)}/G$ stratify X/G making up the so-called *orbit type stratification*

Example 2.3.1 Strata corresponding to proper subgroups can be empty. For example if G is a non-trivial group acting trivially on a space X; i.e. $gx = x$ for all $g \in G$, $x \in X$, then $X_{(H)} = \emptyset$ for all proper subgroups $H \subset G$. In this case there is only one non empty stratum $X_{(G)} = X$.

Example 2.3.2 The stabilizer stratification of $\mathbb{Z}_2$ acting on S^n by reflection with respect to the plane of the equator has two strata : the two open hemispheres form a stratum and the equator forms another. The stratification associated to $\mathbb{Z}_2$ acting on S^n via the antipodal map consists of only one stratum $X_{(1)}$, 1 being the identity in G. Generally $X_{(1)}$ is the subset of points of X on which G acts freely, so that the action of G on X is free if and only if $X_{(1)} = X$.

Some Properties: We assume that G acts discontinuously on X (see definition 1.2.1).

- With G acting on X, H a subgroup of G, the quotient map $\pi : X \longrightarrow X/G$ restricts to a *regular covering* projection $\pi_H : X_{(H)} \longrightarrow X_{(H)}/G$, see [4].
- If $X_K \subset \overline{X}_H$, then $H \subset K$. To see this let $x \in X_K$. Since x is in the closure of X_H, there is a sequence of points $x_i \in X_H$ converging to x. But for any $h \in H$, $hx_i = x_i$ and so hx_i converges to $hx = x$, and $H \subset \mathrm{stab}(x) = K$. The reciprocal statement is not always true. Let $\mathfrak{S}_4$ act on X^4 by permuting coordinates, $K = \mathbb{Z}_2$ and $H = \mathbb{Z}_4 \subset \mathfrak{S}_4$. Then $X^4_{(\mathbb{Z}_4)}$ is empty but $X^4_{(\mathbb{Z}_2)}$ is not (see Corollary 2.3.4).

For a stratified space X one defines the relation $\leq$ on the index set $\mathcal{I}$ of a stratification by setting $i \leq j$ if and only if $X_i \subset \overline{X}_j$; the closure of X_j in X. A partition $\{X_i\}_{i \in \mathcal{I}}$ satisfies the "frontier condition" if and only if $\leq$ is a partial ordering. That is if for every $i, j \in \mathcal{I}$

$$X_i \cap \overline{X}_j \neq \emptyset \quad \text{implies} \quad X_i \subseteq \overline{X}_j$$

We write $i < j$ if $i \neq j$. For example a locally finite simplicial complex is such a stratified space with strata the open simplices. Similarly if N is a submanifold of M, then N and $M - N$ form a stratification of M satisfying the frontier condition.

It is not hard to come up with group actions such that both the orbit and stabilizer stratifications do not satisfy the frontier condition. For example X is a fork and G is $\mathbb{Z}_2$ acting by spinning the fork around its axis an angle π. In the case of permutation products however, the situation is well behaved.

Example 2.3.3 (Permutation Products) In the case of permutation products, the point to make is that the stabilizer of a point $(x_1, \ldots, x_n) \in X^n$ only depends on the multiplicities of factors and not on their location. Let $\sigma \in \Gamma \subset \mathfrak{S}_n$ be a permutation. It has a *cycle structure*; that is can be expressed as a product of pairwise disjoint cycles

$$\sigma = (i_1^1 \ldots i_{r_1}^1)(i_1^2 \ldots i_{r_2}^2) \cdots (i_1^k \ldots i_{r_k}^k) \tag{2.3.1}$$

Fixed points of $\sigma \in \Gamma$ with such a cycle structure consist of all n-tuples having equal entries at $i_1^1, ..., i_{r_1}^1$, then equal entries at $i_1^2, \ldots, i_{r_2}^2$, etc.

We refer to "young" any subgroup $H \subset \mathfrak{S}_n$ of the form $\mathfrak{S}_{d_1} \times \ldots \times \mathfrak{S}_{d_k}$, with $\sum d_i = n$. Any two embeddings of H in $\mathfrak{S}_n$ are conjugate so for our purposes we don't need to specify the embedding. The stratum $X^n_{(\mathfrak{S}_{d_1} \times \ldots \times \mathfrak{S}_{d_k})}$ corresponds to all n-tuples of points which after permutation can be brought to the form

$$(\underbrace{x_1, \ldots, x_1}_{d_1}, \underbrace{x_2, \ldots, x_2}_{d_2}, \cdots, \underbrace{x_k, \ldots, x_k}_{d_k}) \tag{2.3.2}$$

with $x_i \neq x_j$ for $i \neq j$ and $\sum d_i = n$. Now any permutation σ is a product of cycles as in (2.3.1) (its *cycle structure*), and the fixed points of such a product acting on X^n are, up to permutation, elements of the form (2.3.2). From this description, it is easy to see that X^n_K is trivial if K is not a Young subgroup. Moreover it is known that each cycle structure corresponds to a young subgroup of $\mathfrak{S}_n$. This implies in particular that for various choices of Young subgroups, the subspaces $X^n_{(young)}$ give rise to a stratification of X^n, which is $\mathfrak{S}_n$-equivariant. We refer to the subspace $X^n_{(young)}$ as a "Young stratum". The closure of the Young stratum $X^n_{(\mathfrak{S}_{d_1} \times \ldots \times \mathfrak{S}_{d_k})}$ is made up of all young strata obtained by merging $\mathfrak{S}_{d_1} \times \ldots \times \mathfrak{S}_{d_k}$ into Young subgroups with smaller factors, so for example $X^n_{(\mathfrak{S}_{d_1+d_2} \times \mathfrak{S}_{d_3} \times \ldots \times \mathfrak{S}_{d_k})}$ is in the closure.

The following corollary is a consequence of the previous discussion.

Corollary 2.3.4 *If $\Gamma \subset \mathfrak{S}_n$ acts on X^n by permutation, then both the stabilizer and orbit stratifications satisfy the frontier condition.*

2.3.1 Depth

We assume here that all groups are finite. Given a finite stratification $\{X_i\}_{i\in\mathcal{I}}$ of a space X satisfying the frontier condition, we define the *depth* of a stratum X_s to be the maximal length k of a sequence $s = s_0 < s_1 < \cdots < s_k$ in $\mathcal{I}$. The depth of a (finite) stratification as a whole is the maximum over the depths of its strata.

It is easy to see that the depth of the stabilizer and orbit type stratifications of a finite group Γ acting on X coincide. This is a consequence of the fact that the map $\pi : X \longrightarrow X/\Gamma$ must carry distinct strata to distinct strata.

Definition 2.3.5 *An action of Γ on a space X corresponds to a homomorphism $\phi : \Gamma \longrightarrow Aut(X)$. Denote by $d_\phi(\Gamma, X)$ the depth of the stabilizer stratification associated to this action. As is clear, this depth depends on both ϕ and X.*

For given Γ, the various depths $d_\phi(\Gamma, X)$ can be compared to the length of Γ. Recall that the length $\ell(\Gamma)$ of a finite group Γ is the length of the longest chain of subgroups in Γ

$$0 = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_\ell = \Gamma$$

Theorem 2.3.6 *If Γ is a finite group, then $0 \le d_\phi(\Gamma, X) \le \ell(\Gamma)$ and these equalities are sharp.*

Proof. To see that $d_\phi(\Gamma, X)$ can be 0, simply choose X to be a point. More generally and as we pointed out, if Γ acts on a space X then in the stabilizer stratification, if $X_{(K)} \subset \overline{X}_{(H)}$ then $H \subset K^\sigma$ for some conjugate subgroup $K^\sigma \sim K$. It follows that any chain of strata in the stratification must correspond to a chain of subgroups and hence the depth of the stratification is at most $\ell(\Gamma)$. This establishes the right inequality. To see that this bound is sharp, consider the permutation action of Γ on the set of its elements Ω. This action is free and gives an embedding $\psi : \Gamma \hookrightarrow \mathfrak{S}_n$, $n = |\Gamma|$ (this is the so-called permutation representation of Γ). From now on we identify Γ with $\psi(\Gamma)$ and $H \subset \Gamma$ with $\psi(H)$. We then get an induced action on X^n by permuting coordinates and this we represent via

$$\phi : \Gamma \longrightarrow Aut(X^n) \quad , \quad n = |\Gamma|$$

Suppose X has at least n-distinct elements $x_1, \ldots, x_n$. We claim that $d_\phi(\Gamma, X^n) = \ell(\Gamma)$. To that end, it is enough to check that any subgroup $H \subset \Gamma$ gives rise to a non-empty stratum $X^n_{(H)}$. This is our main claim which we now verify.

We use some terminology: the *support* of a permutation σ on $\Omega = \{1, 2, \ldots, |\Gamma|\}$ is the complement of those elements of Ω fixed by σ. For example the support of $(12)(34)$ is

$\{1,2,3,4\}$. We write $\sup(\sigma)$ such a support. If Γ is embedded into $\mathfrak{S}_{|\Gamma|}$ via its permutation representation, then the support of $1 \neq \sigma \in \Gamma$ is all of $\{1, \dots, |\Gamma|\}$, and its cycle structure (2.3.1) is made out of cycles of lengths at least two.

As already pointed out, fixed points of $\sigma \in \Gamma$ having cycle structure as in (2.3.1) consist of all n-tuples having equal entries at $i_1^1, ..., i_{r_1}^1$, then equal entries at $i_1^2, \dots, i_{r_2}^2$, etc. If we choose the entries corresponding to different cycles to be distinct and to take values in $\{x_1, \dots, x_n\}$, then the corresponding fixed point of (2.3.1) is an n-tuple that decomposes into k-blocs, one bloc for each cycle. For instance (124)(356) has as fixed point $(x_1, x_1, x_2, x_1, x_2, x_2)$ consisting of a bloc of x_1's and a bloc of x_2's. We say here that $\{1,2,4\}$ is the support of x_1 and $\{3,5,6\}$ is the support of x_2. Generally if $\zeta \in X^n$ and x is a factor in ζ that appears in positions $i_1, \dots, i_r$, then we write $\sup(x) = \{i_1, \dots, i_r\}$.

Given $H \subset \Gamma$, we construct a remarkable element ζ in its fixed point set as follows. For $i \in \Omega = \{1, 2, \dots, |\Gamma|\}$, we write $\mathrm{orb}(i)$ the subset of all integers $j \in \Omega$ such that $h(i) = j$, for some $h \in H$. Starting with $1 \in \Omega$, we fill every j-th entry of ζ by x_1 if $j \in \mathrm{orb}(1)$. Now we look at the smallest integer m not in $\mathrm{orb}(1)$. We fill in the j-th entry of ζ by x_2 if $j \in \mathrm{orb}(m)$. Etc. We obtain an element ζ consisting of blocs of x_i's with the following key properties: (i) each $h \in H$ acts on each bloc and doesn't permute elements from different blocs, and (ii) this action is transitive on elements of the bloc. We claim that such a fixed point ζ has stabilizer subgroup H; i.e. no bigger group stabilizes it.

Choose an element $\zeta \in X^n$ as above and let $g \in \mathrm{Stab}(\zeta)$. We wish to show that $g \in H$ necessarily. Decompose ζ into blocs of x_i's. Then g leaves the bloc of x_1 invariant. Choose τ a cycle appearing in the cycle structure of g leaving invariant $\sup(x_1)$, and assume $\tau(i) = j$ for some i, j in that support. Since H acts transitively on that bloc, there is $h \in \psi(H)$ such that $h(i) = j$. This means that at the level of the group Γ, there are generators $g_i, g_j \in \Gamma$ such that $gg_i = g_j$ and $hg_i = g_j$, which means that $h = g$ and $g \in H$.

In conclusion, $\zeta \in X_H$ and the stratum $X_{(H)}$ is non empty. This last fact being true for every $H \subset \Gamma$, we have one stratum for every subgroup of Γ and the equality $d_\phi(\Gamma, X^n) = \ell(\Gamma)$ holds. ∎

Remark 2.3.7 The proof above highlights some special features of permutation products. Let $\phi : \Gamma \hookrightarrow \mathfrak{S}_n$. Then the depth of the orbit stratification of Γ acting on X^n by permutations via ϕ doesn't depend on the space X as long as X contains at least n-distinct elements.

Example 2.3.8 Let X be a space with at least n distinct points. Then the strata of the orbit type stratification of $\mathrm{SP}^n(X)$ are in one to one correspondence with integer partitions of the form

$$P = [p_1, p_2, \ldots, p_k] \quad , \quad p_i \leq p_{i+1} \; , \; \sum p_i = n$$

and thus the depth of this stratification is $n-1$. To the partition $[p_1, p_2, \ldots, p_k]$ corresponds the stratum of points of the form

$$[x_1, \ldots, x_1, x_2 \ldots, x_2, \cdots, x_k, \cdots x_k]$$

where $x_i \neq x_j$ if $i \neq j$, and each x_i repeats p_i-times. The generic (open and dense) stratum is the configuration space of distinct points $B(X,n) = F(X,n)/\mathfrak{S}_n = X^n_{(1)}/\mathfrak{S}_n$ and this corresponds to the partition $[1, \ldots, 1]$. The diagonal in $\mathrm{SP}^n X$ corresponds to the partition $[n]$. The poset of strata here is the set of partitions partially ordered according to $P \leq Q$ if Q is a refinement of P, that is if Q is obtained by further partitioning the parts of P.

Example 2.3.9 Let X be a space with at least n distinct points. The depth of $\mathbb{Z}_n$ acting on X^n is the length of $\mathbb{Z}_n$. This is because the inclusion $\mathbb{Z}_n \hookrightarrow \mathfrak{S}_n$, sending the generator to the n-cycle $(12 \cdots n)$ corresponds to the permutation representation of $\mathbb{Z}_n$. Note that if $p|n$, then $X^n_{(\mathbb{Z}_p)}$ consists up to cyclic permutation of all points of the form

$$(x_1, x_2, \ldots, x_{n/p}, x_1, x_2, \ldots, x_{n/p}, \ldots, x_1, x_2, \ldots, x_{n/p})$$

with $x_i \neq x_j$ if $i \neq j$, each x_i repeating p times.

2.3.2 The Diagonal Stratum

The orbit stratification is "Whitney stratified" if M is a smooth manifold (see [31], paragraph 4.2). In particular the orbit stratification of $\Gamma\mathrm{P}^n(M)$ for M a smooth manifold, is Whitney regular. As an illustrative example, let us analyze the "neighborhood" of the diagonal in $\mathrm{SP}^2 X$.

Lemma 2.3.10 *Assume X is a closed smooth manifold. A neighborhood deformation retract V of the diagonal X in SP^2X is homeomorphic to the fiberwise cone on the projectivized tangent bundle of X. The section corresponding to the cone points is the inclusion of X into V.*

Proof. Write the diagonal copy of X in $\mathrm{SP}^2 X$ again as X and consider its sphere bundle SX. We claim that when we projectivize SX and do the fiberwise cone construction on it, we obtain a neighborhood V of X, and that X is sitting in V as the image of the section sending x to the vertex of the cone over x. To verify this claim, choose a metric on X with injectivity radius greater than 2 and let $DX \subset TX$ be the unit disc bundle of the tangent bundle of X. We will write *exp* the exponential map $DX \longrightarrow X$ and exp_x its restriction to $D_xX := T_xX \cap DX$, for $x \in X$. The image $U \subset X \times X$ of DX via $(x,v) \mapsto (exp_x(v), exp_x(-v))$ is a tubular neighbourhood of the submanifold $\Delta X \subset X \times X$. This is of course saying that the normal bundle of the diagonal embedding of X in $X \times X$ is isomorphic to the tangent bundle of X. Mapping down via the quotient map $\pi : X^2 \longrightarrow \mathrm{SP}^2X$, we identify V with $\pi(U)$, and $\pi(\Delta X)$ with X. Since $[exp_x(v), exp_x(-v)] = [exp_x(-v), exp_x(v)] \in \mathrm{SP}^2X$, it follows that V is up to homeomorphism the quotient

$$V = \{(x,v) \in DX\}/\sim \quad , \quad (x,v) \sim (x,-v)$$

In other words V is obtained by quotienting fiberwise the disc bundle of X by $D^n/v \sim -v$ and this is a cone on $\mathbb{R}P^{n-1}$. ∎

Remark 2.3.11 We recall that the *homotopy pushout* of two maps $f : A \longrightarrow B$ and $g : A \longrightarrow C$ is the double mapping cylinder of these two maps. This is generally represented by a square with A at the top left corner and the resulting pushout at the bottom right corner. Define $B(X,2) = \mathrm{SP}^2X - X$, with X sitting diagonally in SP^2X. This is the configuration of distinct points (see §3.1). If X a closed smooth manifold, then a reformulation of Lemma 2.3.10 is the existence of the homotopy pushout

$$\begin{tikzcd} \mathbb{T}X \arrow[r, "f"] \arrow[d, "g"] & B(X,2) \arrow[d] \\ X \arrow[r] & \mathrm{SP}^2X \end{tikzcd}$$

where $g : \mathbb{T}X \longrightarrow X$ is the projectivized tangent bundle and $f(x,v) = [exp_x(v), exp_x(-v)]$.

Example 2.3.12 When S^n is the diagonal in $\mathrm{SP}^2(S^n)$, V a small deformation retract of it and U the complement of V, then as we know

$$U \simeq B^1(S^n, 2) = F(S^n, 2)/\mathfrak{S}_2 = \mathbb{R}P^n$$

since $F(S^n,2)$ is $\mathfrak{S}_2$-equivariantly homotopic to S^n via the map $(x,y) \mapsto \frac{x-y}{|x-y|}$. In fact we can exhibit an explicit deformation retraction of U onto the antidiagonal $\bar{\Delta} = \{[x,-x] \in \mathrm{SP}^2X)\}$ which is homeomorphic to $\mathbb{R}P^n$. First observe that if $[x,y] \in U$, then $-y$ and x are not antipodal and can be joined by a shortest geodesic. Similarly $-x$ and y can be joined by a shortest geodesic. These geodesics are paths γ_1 from $-y$ to x and γ_2 from $-x$ to y. They give in turn a path $\gamma_{x,y}$ from

$$\left[\frac{x-y}{|x-y|}, \frac{y-x}{|y-x|}\right]$$

to $[x,y]$ by defining $\gamma_{x,y}(t) = [\gamma_1(t)-\gamma_2(t), \gamma_2(t)-\gamma_1(t)]$ (normalized vectors). This path is well-defined since γ_1 and γ_2 don't cross ; i.e. $\gamma_1(t) \neq \gamma_2(t), \forall t$ (in fact they are disjoint paths). It is now easy to see that

$$F_t([x,y]) = \gamma_{x,y}(t)$$

is a well-defined deformation retraction of $\mathrm{SP}^2 S^n - \Delta$ onto $\mathbb{R}P^n$.

Corollary 2.3.13 *(Nakaoka)* $H_*(SP^2(S^{2n});\mathbb{Z}) = \begin{cases} \mathbb{Z} & , k = 0, 2n, 4n \\ \mathbb{Z}_2 & , k \text{ even } 2n < k < 4n \end{cases}$.

2.4 The fundamental group of a transitive group quotient

This section proves that if Γ is a transitive subgroup of $\mathfrak{S}_n$ acting by permutations on X^n, then $\pi_1(\Gamma\mathrm{P}^n X) \cong H_1(X;\mathbb{Z})$. We then use this to prove Theorem 0.0.1.

The starting point is the projection $\pi : X^n \longrightarrow \Gamma\mathrm{P}^n X$, which is a branched covering over the orbit stratified quotient. We will proceed in three steps and show that : (i) π_* is surjective on fundamental groups for any $\Gamma \subset \mathfrak{S}_n$, (ii) $\pi_1(\Gamma\mathrm{P}^n X)$ is abelian if Γ is transitive and (iii) $\pi_1(\Gamma\mathrm{P}^n X) = H_1(X,\mathbb{Z})$.

Step (i) is saying the following. Let $\mathbf{x} = [x_1, \ldots, x_n]$, $\mathbf{y} = [y_1, \ldots, y_n]$ be two points in $\Gamma\mathrm{P}^n(X)$, and $\gamma(t)$ a path between them. Then there are paths $\gamma_1, \ldots, \gamma_n$ in X such that

$$\gamma(t) = [\gamma_1(t), \ldots, \gamma_n(t)]$$

with $\gamma_i(0) = x_i$ and $\gamma_i(1) = y_{\sigma(i)}$ for some permutation $\sigma \in \Gamma$. This claim is very intuitive and a justification for it was given in §1.2. If we choose as our basepoints the diagonal element $(x_0, \ldots, x_0) \in X^n$ and its image $[x_0, \ldots, x_0] \in \Gamma\mathrm{P}^n X$, then we deduce immediately that

$$\pi_* : \pi_1(X^n) \twoheadrightarrow \pi_1(\Gamma\mathrm{P}^n X) \tag{2.4.1}$$

is surjective since a loop in $\Gamma\mathrm{P}^n X$ is a path based at $[x_0, \ldots, x_0]$ and any of its lifts has to be again a loop based at $(x_0, \ldots, x_0)$. For a more general statement of this kind, see

Corollary 1.2.4. In particular $\pi_1(\Gamma\mathrm{P}^n X)$ is abelian whenever $\pi_1(X)$ is.

We next check step (ii) that $\pi_1(\Gamma\mathrm{P}^n X)$ is abelian if Γ acts transitively. Write $\eta_j, 1 \leq j \leq k$ the j-th generator of $\pi_1(X)$, and write a presentation of the product as

$$\pi_1(X)^n =< \eta_{11}, \ldots, \eta_{1k}, \cdots, \eta_{n1}, \ldots, \eta_{nk} \mid \text{relations} >$$

where η_{ij} is the j-th generator of the i-th copy of $\pi_1(X)$ in $\pi_1(X)^n$.

Let α and β be two homotopy classes in $\pi_1(\Gamma\mathrm{P}^n(X))$. We wish to show that they commute. By our lifting criterion (2.4.1) there exist $\tilde{\alpha}$ and $\tilde{\beta}$ in $\pi_1(X^n) = \pi_1(X)^n$ that project respectively to α and β. Write these two classes additively as

$$\tilde{\alpha} = \sum_{ij} a_{ij}\eta_{ij} \quad , \quad \tilde{\beta} = \sum_{ij} b_{ij}\eta_{ij}$$

Observe that the transitivity of the action garantees that

$$\pi_*(\eta_{i\ell}) = \pi_*(\eta_{j\ell}) \quad , \quad 1 \leq \forall i, j \leq n$$

We can then define

$$\tilde{\alpha}_1 = \sum_{ij} a_{ij}\eta_{1j} \quad , \quad \tilde{\beta}_2 = \sum_{ij} b_{ij}\eta_{2j}$$

so that

$$\pi_*(\tilde{\alpha}_1) = \pi_*(\tilde{\alpha}) = \alpha \quad , \quad \pi_*(\tilde{\beta}_2) = \pi_*(\tilde{\beta}) = \beta$$

The point of this manipulation is that now $\tilde{\alpha}_1$ and $\tilde{\beta}_2$ commute in $\pi_1(X^n)$ since they are made out of generators which live in different factors of $\pi_1(X)^n$. Since these classes commute so do their images α and β in $\pi_1(\Gamma\mathrm{P}^n(X))$. This proves that $\pi_1(\Gamma\mathrm{P}^n X)$ is abelian.

Note that in fact we have proven a little more

Corollary 2.4.1 *The composite* $\tau : \pi_1(X) \hookrightarrow \pi_1(X)^n \longrightarrow \pi_1(\Gamma P^n X) = H_1(\Gamma P^n X)$ *is surjective.*

Proof. This is restating what we previously constructed. Let's refer to the first factor of $\pi_1(X)^n$ by $\pi_1(X)$ which is generated by the $\eta_{i1}, 1 \leq i \leq k$. Then for $\alpha \in \pi_1(\Gamma\mathrm{P}^n X)$, choose $\tilde{\alpha}$ a preimage in $\pi_1(X)^n$; $\tilde{\alpha} = \sum a_{ij}\eta_{ij}$. But now as we pointed out $\tilde{\alpha}_1 := \sum a_{ij}\eta_{i1}$ has the property that $\pi_*(\tilde{\alpha}_1) = \alpha$. ∎

Finally we verify (iii) that $\pi_1(\Gamma\mathrm{P}^n X) = H_1(\Gamma\mathrm{P}^n X)$ coincides with $H_1(X;\mathbb{Z})$ if $n \geq 2$. The composite τ in Corollary 2.4.1 factors necessarily through the abelianization π_* :

$H_1(X;\mathbb{Z})\longrightarrow H_1(\Gamma\mathrm{P}^nX)$ and this map as we indicated is surjective. It then remains to show it is also injective. This can be done through standard arguments in the next lemma.

Lemma 2.4.2 *Let $i : X \hookrightarrow \Gamma P^nX$ be the basepoint inclusion $x \mapsto [x, *, \dots, *]$, $n \geq 2$. Then for all $k \geq 1$, $i_* : H_k(X;\mathbb{Z})\longrightarrow H_k(\Gamma P^nX)$ is a monomorphism.*

Proof. We show that on cohomology, the induced map $i^* : H^k(\Gamma\mathrm{P}^nX)\longrightarrow H^k(X)$ is surjective. Let $\alpha \in H^k(X;\mathbb{Z})$. Then α is represented by the (homotopy class) of a map $f : X\longrightarrow K(\mathbb{Z}, k)$, where $K(\mathbb{Z}, k)$ is the Eilenberg-MacLane space; that is $\alpha = f^*(\iota)$ where $\iota \in H^k(K(\mathbb{Z}, k))$ is a generator. A model for $K(\mathbb{Z}, k)$ is $\mathrm{SP}^\infty(S^k)$ and this is an abelian topological monoid (which we write multiplicatively). We can then extend f to a map $\hat{f} : \Gamma\mathrm{P}^nX\longrightarrow K(\mathbb{Z}, k)$ sending $[x_1, \dots, x_n] \longmapsto f(x_1)\cdots f(x_n)$. The diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & K(\mathbb{Z},k) \\ \Gamma\mathrm{P}^nX \arrow[ur, "\hat{f}"'] & \end{tikzcd}$$

defines a class $\hat{\alpha} := \hat{f}^*(\iota)$ such that $i^*(\hat{\alpha}) = \alpha$. Since i^* is surjective, then i_* is injective. ∎

For non-transitive subgroups Γ, $\pi_1(\Gamma\mathrm{P}^nX)$ is not necessarily abelian. For example if $\Gamma = \mathfrak{S}_2$ acts on X^3 by permuting the first two coordinates, then $\Gamma\mathrm{P}^3X = \mathrm{SP}^2X \times X$ cannot be abelian unless $\pi_1(X)$ is. Observe as well that the quotient of $X^4 = X^2 \times X^2$ by $\Gamma = \mathfrak{S}_2 \times \mathfrak{S}_2$ acting in the obvious way, has abelian fundamental group eventhough Γ is not transitive.

Proof. (of Theorem 0.0.1) By the remark made in the introduction, any non-transitive subgroup $\Gamma \subset \mathfrak{S}_n$ is up to conjugation contained in $\mathfrak{S}_j \times \mathfrak{S}_{n-j}$ as a product of two subgroups $\Gamma_1 \times \Gamma_2$, and so iteratively Γ is up to conjugation a subgroup $\Gamma_1 \times \cdots \times \Gamma_s$ of $\mathfrak{S}_{r_1} \times \cdots \times \mathfrak{S}_{r_s}$, $\sum r_i = n$, with each Γ_i acting transitively on X^{r_i} (here X^{r_i} is viewed as a subset of X^n defined by a choice of r_i-factors. This choice is determined by the conjugation). If $r_1, \dots, r_k > 1$ describe the non-trivial orbits, while $r_{k+1} = \cdots = r_s = 1$, then each quotient $\Gamma_iP^{r_i}$ has fundamental group $H_1(X, Z)$ since Γ_i acts transitively on X^{r_i} when $i \leq k$. The rest is immediate. Finally and if Γ is abelian, it is known that the length of any non-trivial transitive orbit is $|\Gamma|$ from which the last claim follows (indeed it is known that an abelian transitive permutation group acting on Ω must be regular; that is the stabilizer of any $x \in \Omega$ must be the trivial subgroup). ∎

Remark 2.4.3 The study of the transitive actions of a group is equivalent to the study of the actions of the group on the sets of rights cosets of its various subgroups. Indeed if

$H \subset G$ is any subgroup, then G acts transitively on the right cosets of H in G. Conversely if G acts transitively on Ω, fix a point $x \in \Omega$ and let $H = G_x$ be the stabilizer of α in G. For an arbitrary point $y \in \Omega$, let $\pi(y) = \{g \in G \mid gx = y\}$. Then $\pi(y)$ is a right coset of H in G and π is a bijection from Ω onto the set $\Lambda = \{gH, g \in G\}$ of all right cosets of H [18].

2.4.1 The Symmetric Product Case

In the special case $\Gamma = \mathfrak{S}_n$, there are various arguments leading to $\pi_1(\mathrm{SP}^n X) \cong H_1(X;\mathbb{Z})$. We give two very short proofs that rely on the following general results of Dold, Thom and Steenrod :

- (i) (Dold and Thom) As an abelian topological monoid and for connected CW complexes X;
$$\mathrm{SP}^\infty(X) \simeq \prod_{i\geq 0} K(\tilde{H}_i(X), i) \tag{2.4.2}$$
- (ii) (Steenrod, Nakaoka) $H_*(\mathrm{SP}^n X)$ embeds in $H_*(\mathrm{SP}^{n+1}X)$ and hence in $H_*(\mathrm{SP}^\infty X)$.
- (iii) (Dold) If $H_*(X) \cong H_*(Y)$ for $* \leq N$, then $H_*(\Gamma\mathrm{P}^n X) \cong H_*(\Gamma\mathrm{P}^n Y)$ for $* \leq N$, for all $n \geq 1$ and all permutation subgroups $\Gamma \subset \mathfrak{S}_n$.

We now show that $\pi_1(\mathrm{SP}^n X) \cong H_1(X;\mathbb{Z})$:

Proof 1: This uses (i), (ii) and the fact that $\pi_1(\mathrm{SP}^n X)$ is abelian for $n \geq 2$; that is that $\pi_1(\mathrm{SP}^n X) \cong H_1(\mathrm{SP}^n X;\mathbb{Z})$. But there is a series of monomorphisms according to (ii)

$$H_1(X) \hookrightarrow H_1(\mathrm{SP}^n X) \hookrightarrow H_1(\mathrm{SP}^\infty X) = \pi_1(\mathrm{SP}^\infty X) = H_1(X)$$

which shows that $H_1(\mathrm{SP}^n X) \cong H_1(X;\mathbb{Z})$ and hence the claim.

Proof 2: This only uses (iii) and the elementary cell decomposition of [21]. According to Dold's result, $H_1(\mathrm{SP}^n X) = H_1(\mathrm{SP}^n X^{(2)})$ where $X^{(2)}$ is the two skeleton of X. Write $Y = X^{(2)}$. Then Y has the homotopy type of a two dimensional based CW-complex $\bigvee^r S^1 \cup D_1 \cup \cdots D_s$, where the D_i's are two dimensional cells. In [21], the authors show via elementary constructions that for any such complex $H_1(\mathrm{SP}^n Y;\mathbb{Z}) = H_1(Y;\mathbb{Z})$. But $H_1(Y;\mathbb{Z}) = H_1(X;\mathbb{Z})$ and the claim follows.

Remark 2.4.4 Since the symmetric products have the beautiful property of converting π_1 into H_1, we can ask if more generally and for which n, $\pi_i(\mathrm{SP}^n X) \cong H_i(X;\mathbb{Z})$. This is the case if $n = \infty$ of course. Dold and Puppe show using simplicial techniques that

$$\pi_i(\mathrm{SP}^n X) \cong H_i(X;\mathbb{Z}) \quad , \quad i < r + 2n - 2 \ \text{ and } \ r > 1, n > 1 \tag{2.4.3}$$

where r is the connectivity of X. A different proof can be obtained by noticing that $\mathrm{SP}^\infty X/\mathrm{SP}^{n-1}X$ has at least the connectivity of the reduced symmetric product $\overline{\mathrm{SP}}^n X = \mathrm{SP}^n X/\mathrm{SP}^{n-1}X$ which is known to be $(r+2n-2)$-connected. Next is an example where (2.4.3) is no longer true for small values of n. The following lemma we borrow from [20].

Lemma 2.4.5 *Let C be a closed topological surface of genus $g = 2$. Then there is a Laurent polynomial description*

$$\pi_2(SP^2C) = \mathbb{Z}[t_i, t_i^{-1}],\ 1 \leq i \leq 4$$

Proof. The curve C being of genus two, it is necessarily hyperelliptic (after choosing some complex structure of course) and so it is obtained from its Jacobian (a complex 2-torus) by blowing up a single point. The universal cover of the torus is $\mathbb{C}^2$ and $J(C)$ is the quotient by a lattice $\mathcal{L}$ whose vertices are in one-to-one correspondence with $\mathbb{Z}^4$. It follows that the universal cover $\tilde{X}$ of SP^2C is a blowup of $\mathbb{C}^2$ at each and everyone of these vertices. Each exceptional fiber of this blowup being a copy of $\mathbb{P}^1$, it contributes a generator to $H_2(\tilde{X};\mathbb{Z})$. But then

$$H_2(\tilde{X}) = \pi_2(\tilde{X}) = \pi_2(\mathrm{SP}^2C)$$

and so $\pi_2(\mathrm{SP}^2C)$ is infinitely generated indeed. Choose a fundamental domain for the lattice made up of four essential edges mapping to the fundamental group generators of $J(C)$, which we call $t_1,\dots,t_4$ and which also correspond to the generators of $\pi_1(\mathrm{SP}^2C) = \mathbb{Z}^4$ since the Abel-Jacobi map is an isomorphism on π_1. Moving from generator to generator of $H_2(\tilde{X})$ through the lattice corresponds in SP^2C to multiplication by a word in the t_i's or their inverses. ∎

We can also compare the higher homotopy groups of SP^nX with the homology of X. Next we re-derive a known result of Dold and Puppe.

Theorem 2.4.6 *Let X be r connected, $r \geq 1$, then for $n > 1$*

$$\pi_i(SP^nX) \cong \tilde{H}_i(X;\mathbb{Z})$$

provided that $0 < i < r+2n-2$.

Proof. This theorem can be deduced from a result of Kallel on the connectivity of the reduced symmetric products $\mathrm{SP}^nX/SP^{n-1}X$. Indeed, in ([19], Theorem 1.3), it is asserted that $\mathrm{SP}^{n+1}X/\mathrm{SP}^n X$ is $(r+2n)$-connected. Then we use a relative form of Hurewicz theorem (see Hatcher's book [16], section 4.2) to deduce that

$$\pi_i(SP^{n+1}X, SP^n X) = 0 \text{ for } i \leq r + 2n$$

which implies that $\pi_i(\mathrm{SP}^n X) = \pi_i(\mathrm{SP}^\infty X)$ for $i \leq r + 2n - 1$ ∎

Chapter 3

Configuration Spaces with Bounded Multiplicity

3.1 Preliminaries

There are naturally defined $\mathfrak{S}_n$-invariant subspaces in X^n (for $n \geq 2$ a fixed integer) which we list below. Throughout $* \in X$ denotes the basepoint.

- $F^d(X,n) = \{(x_1,\ldots,x_n) \in X^n \mid \text{each } x_i \text{ has multiplicity} \leq d\}$.
- The d-th fat diagonal is

$$F_d(X,n) = \{(x_1,\ldots,x_n) \in X^n \mid \text{at least one entry with multiplicity} \geq d\}$$

- We have $F^d(X,n) = X^n - F_{d+1}(X,n)$.
- The symmetric group $\mathfrak{S}_n$ acts by permuting coordinates and hence we can define the unordered analogs of the above constructions

$$B_d(X,n) = F_d(X,n)/\mathfrak{S}_n \ \text{ and } \ B^d(X,n) = F^d(X,n)/\mathfrak{S}_n$$

  We have $B^1(X,n) = B(X,n)$, $B_n(X,n) = X$ while $B_2(X,n)$ is the image of the fat diagonal in $\mathrm{SP}^n X$.

Note that $B^d(X,n)$ is not a functorial construction since the constant map $X \longrightarrow X$, $x \mapsto *$, doesn't induce a self-morphism of $B^d(X,n)$. On the other hand the fat diagonal $B_d(X,n)$ is not only functorial but is an invariant of homotopy type so that $X \simeq Y$ implies $B_d(X,n) \simeq B_d(Y,n)$.

Remark 3.1.1 $F_d(X,n)$ is a union of closure of strata corresponding to the stabilizer stratification of $\mathfrak{S}_n$ acting on X^n by permutation, while $B_d(X,n)$ coincides with a union of closure of strata of its orbit type stratification.

When X is a simplicial complex (i.e. a triangulated space), $\Gamma\mathrm{P}^n X$ has the structure of a CW complex. This cell structure for the case of symmetric products is discussed in [22]. Below is a more elaborate statement.

Proposition 3.1.2 *For simplicial X, $\Gamma P^n X$ is a cellular complex such that the closure of strata of its associated orbit stratification are subcomplexes.*

Proof. The argument is similar to [22] and hence we keep it short. The point is the existence of a $\mathfrak{S}_n$-equivariant (hence Γ-equivariant) simplicial structure on X^n, induced from one on X, such that the fat diagonal is a CW-subcomplex. Because this simplicial structure is equivariant, it induces a *cellular* (not necessarily simplicial) structure on the quotient $\Gamma\mathrm{P}^n X$. Write $\pi : X^n \to \Gamma\mathrm{P}^n X$ the projection. The skeleta of $\Gamma\mathrm{P}^n X$ are determined by the way points come together. This means that the inverse image under π of a skeleton is a simplicial subcomplex of the fat diagonal and hence the skeleton is a CW-subcomplex of $\Gamma\mathrm{P}^n X$ ∎

We will need the following standard definition and representation.

Definition 3.1.3 *A (strict) pushout of two cofibrations $f : A \longrightarrow B$ and $g : A \longrightarrow C$ along a closed subset A is defined as the identification space*

$$B \sqcup C/_{\sim}, \quad f(a) \sim g(a), \quad \forall a \in A$$

This is often written $B \cup_A C$.

We will represent a strict pushout $B \cup_A C$ by a commuting square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \alpha} \\ C & \xrightarrow{\beta} & X \end{array}$$

where at least one of the maps f and g is a cofibration, and where α and β are the projections.

3.2 Configurations with multiplicity bounded below

The subspaces $B_d(X,n)$ are closed subspaces of $\mathrm{SP}^n X$ and form as we observed a decreasing filtration

$$B_1(X,n) = \mathrm{SP}^n(X) \supset B_2(X,n) \supset \cdots \supset B_n(X,n) = X$$

For example there is a homeomorphism

$$B_d(X,n) = X \times \mathrm{SP}^{n-d}(X) \qquad \text{when } d > [n/2] \text{ and } n \geq 3$$

This is true because there can be only one entry x with multiplicity at least d and thus can be singled out. In particular $B_{n-1}(X,n) = X \times X$. The above example shows for instance that $\pi_1(B_d(X,n))$ can be non-abelian if $\pi_1(X)$ is not.

Remark 3.2.1 We will be heavily using the Van-Kampen theorem in this section and so we state it here in the form we need and which applies to finitely generated groups. Let $G =< g_1,\dots,g_n \mid r_G >$ where r_G stands for a generating set of relations, $H =< h_1,\dots,h_m \mid r_H >$ and $K =< k_1,\dots,k_t \mid r_K >$, where K is viewed as a subgroup of both H and G. Then the pushout $H *_K G = H \leftarrow K \rightarrow G$ has presentation

$$< g_1,\dots,g_n,h_1,\dots,h_m \mid r_G, r_H, f(k_i)g(k_i)^{-1} >$$

where $f : K \to H$ and $g : K \to G$ are the morphisms of the pushout. Note that the relations r_K do not matter. Note also that if f is surjective, then the "universal" map $G \longrightarrow H *_K G$ is surjective as well.

Lemma 3.2.2 *Suppose $[n/3] < d \leq [n/2]$. Then there is a pushout diagram*

$$\begin{tikzcd}
(X\times X)\times SP^{n-2d}(X) \arrow[r, "f"] \arrow[d, "\pi\times 1"] & X\times SP^{n-d}(X) \arrow[d, "\beta"] \\
SP^2(X)\times SP^{n-2d}(X) \arrow[r, "\alpha"] & B_d(X,n)
\end{tikzcd} \tag{3.2.1}$$

where $f(x,y,z) = (x, y^d z)$, $\beta(a,b) = a^d b$ and $\alpha(xy,z) = (x^d y^d z)$. In particular $\pi_1(B_d(X,n))$ is abelian in the range $n/3 < d \leq n/2$.

Proof. The proof of the pushout statement is an easy inspection; the point being that when $[n/3] < d \leq [n/2]$, there are at most two points having multiplicity at least d and when both occur they are indistinguishable. We will show that $\pi_1(B_d(X,n))$ is abelian when $n/3 < d < n/2$ by showing that f_* is surjective at the level of π_1 (see remark 3.2.1). Consider the composition

$$\alpha : X \times * \times \mathrm{SP}^{n-2d}(X) \hookrightarrow X \times X \times \mathrm{SP}^{n-2d}(X) \xrightarrow{f} X \times \mathrm{SP}^{n-d}(X)$$

sending $(x, *, z) \mapsto (x, z)$ where here we view $\mathrm{SP}^{n-2d}(X)$ as a subset of $\mathrm{SP}^{n-d}(X)$. If this composite is surjective on π_1 then so is f. But α is surjective on π_1 because the basepoint inclusion $\mathrm{SP}^{n-2d}(X) \hookrightarrow \mathrm{SP}^{n-d}(X)$ is surjective on π_1 according to Corollary 2.4.1 and the fact that $X \subset \mathrm{SP}^{n-2d}X$ if $d < n/2$.

Finally and when $d = n/2$ we need a slightly different argument using Van-Kampen again but this is immediate and $\pi_1(B_d(X, 2d))$ must be abelian as well. ∎

In all cases we will establish the second statement of Lemma 3.2.2 in greater generality in Lemma 3.2.7.

Remark 3.2.3 Notice that the diagonal map $\Delta : X \longrightarrow \mathrm{SP}^d(X), x \mapsto x^d$ doesn't generally induce an epimorphism on the level of π_1. For example, consider the inclusion $S^1 \hookrightarrow \mathrm{SP}^2 S^1$. This embeds the circle $B_2(S^1, 2)$ into the Mobius band where $B_2(S^1, 2)$ is a 0-sphere bundle on the circle S^1 hence the map Δ induces the multiplication by 2, $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$ which is not surjetive.

We say a space X is r-connected, $r \geq 0$, if the homotopy groups $\pi_i(X)$ vanish for $i \leq r$.

Lemma 3.2.4 *If X is an r-connected simplicial complex, then $B_d(X, n)$ is also r-connected.*

Proof. The proof of ([21], Corollary 3.5) applies verbatim to this situation. It uses the CW decomposition on $B_d(X, n)$ as a subcomplex of $\mathrm{SP}^n X$ (Proposition 3.1.2). ∎

The next result is Theorem 0.0.2, part (ii), of the introduction.

Lemma 3.2.5 *If $\pi_1(X)$ is abelian, then so is $\pi_1(B_d(X, n))$, $d \geq 1$.*

Proof. We have defined $F_d(X, n) \subset X^n$ at the beginning of this section and defined $B_d(X, n)$ as its quotient via the $\mathfrak{S}_n$-permutation action

$$\pi : F_d(X, n) \longrightarrow B_d(X, n) \tag{3.2.2}$$

The thin diagonal is in $F_d(X, n)$ and hence according to Corollary 1.2.4, the map π is surjective on π_1. Rewrite

$$F_d(X, n) = \bigcup_{i_1 < i_2 < \cdots < i_d} F_{i_1, \ldots, i_d}(X, n)$$

where

$$F_{i_1,\dots,i_d}(X,n) = \{(x_1,\dots,x_d) \in X^n \mid x_{i_1} = \cdots = x_{i_d}\}$$

Write $G := \pi_1(X)$ and $\pi_1(X^n) \cong G^n$. The following facts are easy to check :

1. $\pi_1(F_{i_1,\dots,i_d}(X,n)) = \{(g_1,\dots,g_n) \in G^n \mid g_{i_1} = \cdots = g_{i_d}\}$, and this is isomorphic to $G \times G^{n-d}$.

2. The permutation action of $\mathfrak{S}_n$ on X^n induces a permutation action on $F_d(X,n)$ and

$$\sigma(F_{i_1,\dots,i_d}(X,n)) = F_{\sigma(i_1),\dots,\sigma(i_d)}(X,n) \quad \text{for } \sigma \in \mathfrak{S}_n$$

3. The restriction of π_* to $\pi_1(F_{1,\dots,d}(X,n)) \longrightarrow \pi_1(B_d(X,n))$ is surjective.

Fact (1) and (2) follow because the diagonal map $\Delta : X \to X \times X$ induces again the diagonal embedding at the level of fundamental groups. Fact (3) is a consequence of the Van-Kampen theorem since $F_d(X,n)$ is an iterated pushout and hence according to Remark 3.2.1 $\pi_1(F_d(X,n))$ is generated by the generators of the various $\pi_1(F_{i_1,\dots,i_d}(X,n))$ subject to relations there and to additional van-kampen relations. The action of $\mathfrak{S}_n$ can then take any generator of $\pi_1(F_d(X,n))$ and send it to a generator of $F_{1,2,\dots,d}(X,n)$. Since π_* is surjective then so is $\pi_1(F_{1,2,\dots,d}(X,n)) \longrightarrow \pi_1(B_d(X,n))$. But since $\pi_1(F_{1,2,\dots,d}(X,n)) \cong G \times G^{n-d}$ is abelian, then so is $\pi_1(B_d(X,n))$. ∎

Remark 3.2.6 The arguments above indicate that $\pi_1(F_d(X,n))$ doesn't in general need to be abelian even if $\pi_1(X)$ is. For example, let $X = S^1$, $n = 3, d = 2$ and write

$$F_2(S^1,3) = F_{12} \cup F_{13} \cup F_{23}$$

where $F_{12} = \{(x,x,y) \in X^3\}$, $F_{13} = \{(x,y,x) \in X^3\}$ and $F_{23} = \{(y,x,x) \in X^3\}$. Each of these subspaces is homeomorphic to $X \times X$. An iterated use of the Van-Kampen theorem shows that

$$\pi_1(F_2(S^1,3)) \cong ((\mathbb{Z} \times \mathbb{Z}) * (\mathbb{Z} \times \mathbb{Z}) * (\mathbb{Z} \times \mathbb{Z}))/N$$

where N is the normalizer of the subgroup generated by $g_{11}g_{12}g_{22}^{-1}g_{21}^{-1}$ and $g_{11}g_{12}g_{32}^{-1}g_{31}^{-1}$, where g_{i1}, g_{i2} are the two generators of the i-th copy of $\mathbb{Z} \times \mathbb{Z}$, $1 \leq i \leq 3$.

Next is the main calculation of this section

Lemma 3.2.7 *For $n \geq 2$ and $1 \leq d \leq \frac{n}{2}$, $\pi_1(B_d(X,n))$ is always abelian.*

Proof. As in the proof of the commutativity of $\pi_1(\Gamma\mathrm{P}^n X)$ in §2.4, the idea here is the following. Given two elements in $\pi_1(B_d(X,n))$, we lift them to $\pi_1(F_d(X,n))$ via the surjection π_* in (3.2.2), then using the action of $\mathfrak{S}_n$ we separate them into elements that commute. Write

$$\pi_1(X) = G = \langle g_i, i \in J \mid \text{relations}\rangle \quad , \text{ with } J \text{ some finite indexing set}$$

For $I = \{i_1, \ldots, i_d\}$, $\pi_1(F_{i_1,\cdots,i_d}(X,n)) \subset G^n$ has generators

$$g_i^I \ , \ g_{a,j}^I \quad , \quad i,j \in J, a \notin I$$

where $g_i^I = (e,\cdots,g_i,\cdots,g_i,\cdots,e)$ is the element of G^n with g_i in the $i_1, i_2, \cdots, i_d$ entries and the identity e in the remaining entries, while $g_{a,j}^I = (e,\cdots,g_j,e,\cdots,e)$ is the element of G^n with g_j in the a^{th} entry and the identity e in the remaining entries. Under the isomorphism $\pi_1(F_{i_1,\cdots,i_d}(X,n)) \cong G \times G^{n-d}$, the g_i^I generate the first factor G and the $g_{a,j}^I$ generate the second factor G^{n-d}. For example $\pi_1(F_{1,2}(X,3))$ has generators

$$g_i^{12} \ , \ g_{3,j}^{12} \ , \ i,j \in J$$

Since $F_d(X,n) = \bigcup F_I(X,n)$, with I a sequence of the form $1 \le i_1 < \cdots < i_d \le n$, an iterated use of the Van-Kampen theorem shows that $\pi_1(F_d(X,n))$ is generated by the g_i^I and $g_{a,j}^I$ with I varying over all such sequences, $a \notin I$ and $i,j \in J$, subject to various relations. Among these relations are those coming from $\pi_1(F_I(X,n))$ which are:

- g_i^I and $g_{a,j}^I$, $a \notin I$, commute since they come from different factors of G^n.
- Similarly $g_{a,r}^I$ and $g_{b,s}^I$ commute if $a \neq b$.

Write $I_d = \{1,2,\ldots,d\}$ and $I^d = \{n-d+1, n-d+2, \ldots, n\}$. These are disjoint subsets since $n \ge 2d$. Consider the pushout diagram

$$\begin{tikzcd}
X \times X \times X^{n-2d} \arrow[r, "f"] \arrow[d, "g"] & F_{I_d}(X,n) \arrow[d] \\
F_{I^d}(X,n) \arrow[r] & F_{I^d}(X,n) \cup F_{I_d}(X,n)
\end{tikzcd}$$

where the image of $(x,y,(z_1,\ldots,z_{n-2d}))$ under both inclusions f and g is

$$(x,x,\ldots,x,z_1,\ldots,z_{n-2d},y,y,\ldots,y)$$

If we write $\pi_1(X \times X \times X^{n-2d})$ as $G \times G \times G^{n-2d}$, then the generators $g_{1,i}$ coming from the first copy of G map to

$$f_*(g_{1,i}) = g_i^{I_d} \quad , \quad g_*(g_{1,i}) = g_{1,i}^{I^d} + g_{2,i}^{I^d} + \cdots + g_{d,i}^{I^d}$$

By the Van-Kampen theorem, this implies that in $\pi_1(F_d(X,n))$, there are generators of the same name $g_i^{I_d}$ and $g_{a,i}^{I^d}$ with the relation for given $i \in J$

$$g_i^{I_d} = g_{1,i}^{I^d} + g_{2,i}^{I^d} + \cdots + g_{d,i}^{I^d} \tag{3.2.3}$$

Let now α, β be two elements in $\pi_1(B_d(X,n))$ and write $\tilde{\alpha}, \tilde{\beta}$ in $\pi_1(F_d(X,n))$ such that $\pi_*(\tilde{\alpha}) = \alpha$ and $\pi_*(\tilde{\beta}) = \beta$. In terms of generators we can express $\tilde{\alpha}$ as a finite sum

$$\tilde{\alpha} = \sum n_i^I g_i^I + \sum m_j^{I'} g_{a,j}^{I'} \quad , \quad n^I, m^{I'} \in \mathbb{Z}$$

for some choice of I, I', a, i, j. We know that under the $\mathfrak{S}_n$ quotient,

$$\pi_*(g_i^I) = \pi_*(g_i^{\sigma(I)}) \quad , \quad \pi_*(g_{a,j}^I) = \pi_*(g_{\sigma(a),j}^{\sigma(I)}) \ , \ a \notin I$$

Another better choice of a lift would then be

$$\tilde{\alpha}_1 = \sum n_i g_i^{I^d} + \sum m_j g_{1,j}^{I^d}$$

and this lives in $\pi_1(F_{I^d}(X,n))$. We replace $g_i^{I^d}$ by $g_i^{I_d}$ and this in turn can be replaced by $g_{1,i}^{I^d} + \cdots + g_{d,i}^{I^d}$ according to (3.2.3). Similarly we can replace each $g_{a,i}^{I^d}$ by $g_{1,i}^{I^d}$ since one is mapped into the other by the transposition $(1,a)$. At the end and for $n \geq 2d$ there is a choice of lift

$$\tilde{\alpha}_2 = \sum t_k g_{1,k}^{I^d} \quad t_k \in \mathbb{Z} \tag{3.2.4}$$

such that $\pi_*(\tilde{\alpha}_2) = \alpha$. Similarly there is a choice of a lift for β of the form

$$\tilde{\beta}_2 = \sum \ell_k g_{2,k}^{I^d} \quad \ell_k \in \mathbb{Z} \tag{3.2.5}$$

This choice is possible since $n \geq 2$. The expressions in (3.2.4) and (3.2.5) commute in $\pi_1(F_{I^d}(X,n)$ and hence α commutes with β in $\pi_1(B_d(X,n))$. We have shown that $\pi_1(B_d(X,n))$ is abelian. ∎

Corollary 3.2.8 *The map $SP^2X \longrightarrow B_d(X,n)$, $[x,y] \mapsto [x,y,*,\cdots,*]$ induces an isomorphism in fundamental groups for $2 \leq d \leq n/2$.*

Proof. The previous proof shows that if $X \hookrightarrow F_{I_d}(X,n)$ is the inclusion of the first factor, then the composite $X \longrightarrow F_{I_d}(X,n) \longrightarrow B_d(X,n)$ is surjective on π_1. In particular this implies that the composite $X \hookrightarrow \mathrm{SP}^2 X \longrightarrow B_d(X,n)$ is surjective on π_1 and thus on H_1. It remains to see that $H_1 X \longrightarrow H_1(B_d(X,n))$ is injective but this is true since the composite $X \hookrightarrow B_d(X,n) \hookrightarrow \mathrm{SP}^n X$ induces a homology monomorphism. ∎

A subgroup of $\mathfrak{S}_n$ is "d-transitive" if it takes any *ordered* d points in $\{1, \dots n\}$ to any other ordered d points. The following result generalizes the main theorem of this section and its proof is an adaptation of the proof of that theorem as well (thus it is left to the reader). Define

$$B\Gamma_d(X,n) := F_d(X,n)/\Gamma$$

This is well-defined since Γ acts on $F_d(X,n)$.

Theorem 3.2.9 *If Γ is d-transitive and $1 \leq d \leq n/2$, then $\pi_1(B\Gamma_d(X,n))$ is abelian.*

3.3 Euler characteristic of configuration spaces with multiplicity bounded below

We first notice that the Euler characteristic of the braid space $F(X,n)$ does not depend in general on the Euler of characteristic of X. We give a counter example. Let X and Y be two graphs as in the figure 3.1

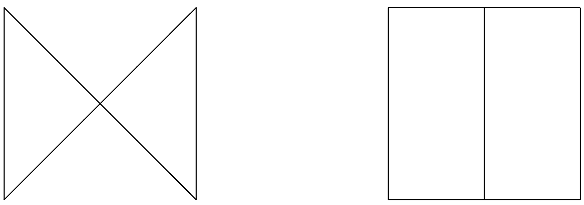

Figure 3.1: Two graphs with the same homotopy type but their corresponding braid spaces have different Euler characteristics

Both X and Y are homotopy equivalent to a wedge of two circles. The graph X has 5 vertices and 6 edges while the graph Y has 6 vertices and 7 edges and so $\chi(X) = \chi(Y) = -1$. Now by applying Barnett and Farber's formula (see [3], Corollary 1.2), we get $\chi(F(X,2)) = -6$ while $\chi(F(Y,2)) = -4$.

Of course the case of X a manifold could be misleading since it's already known that for X a manifold $\chi(F(X,n))$ depends only on $\chi(X)$. We recall this briefly. It begins with a classical theorem of Fadell and Neuwirth, here $Q_k \subset X$ denotes a subset with k points.

Theorem 3.3.1 (Fadell-Neuwirth) *For X a connected manifold, the projection onto any choice of q coordinates $F(X,n) \longrightarrow F(X,q)$ is a fiber bundle with fiber $F(X - Q_q, n - q)$.*

In particular, the projection

$$F(X - Q_1, n-1) \longrightarrow F(X,n) \longrightarrow X$$

is a fibration. Hence

$$\chi(F(X,n)) = \chi(F(X - Q_1, n-1))\chi(X)$$

In the same way

$$F(X - Q_2, n-2) \longrightarrow F(X - Q_1, n-1) \longrightarrow X - Q_1$$

is a fibration, hence

$$\chi(F(X - Q_1, n)) = \chi(F(X - Q_2, n-2))\chi(X - Q_1)$$

By induction,

$$\chi(F(X,n)) = \chi(X - Q_{n-1})\chi(X - Q_{n-2}) \cdots \chi(X)$$

Now since

$$\chi(X - Q_j) = \chi(X) - j(-1)^d = \chi(X) + j(-1)^{d-1}$$

we get the following desired formula

$$\chi(F(X,n)) = \prod_{j=0}^{n-1} (\chi(X) + j(-1)^{d-1})$$

Now we look at spaces where entries are allowed to collide. We can verify that

Theorem 3.3.2 *Let X be of the homotopy type of a finite simplicial complex and assume that either $[n/3] < d \leq [n/2]$ or that $d = 2$ and n arbitrary, then*

$$\begin{aligned} \chi\left(B_d(X,n)\right) &= \binom{\chi(X) + n - 1}{\chi(X) - 1} \\ &- \sum_{\sum_{i=1}^{d-1} i\alpha_i = n} \frac{1}{\alpha_1! \cdots \alpha_{d-1}!} \chi(X)(\chi(X) - 1) \cdots (\chi(X) - (\sum \alpha_i - 1)) \end{aligned}$$

where the α_i are non-negative integers. In particular $\chi(B_d(X,n)) = 0$ if $\chi(X) = 0$ (eg. a closed odd dimensional manifold).

Remark 3.3.3 We must observe that the above formula holds for all d and n and a proof will appear in [23]. This proof uses more elaborate techniques necessary to handle the more general situation. For d between $[n/3]$ and $[n/2]$ this is verified using the pushout (3.2.1). The proof of the case $d = 2$ and n arbitrary is given in the next subsection.

3.3.1 Euler characteristic of the fat diagonal

We first look at the Euler characteristic of the ordered fat diagonal $F_2(X,n)$, then we deduce that of its image under the action of the symmetric group. For simplicity, χ denotes $\chi(X)$.

To derive the Euler characteristic of $F_2(X,n)$ we start by writing the formula for lower cases of n. For $n = 3$,

$$\chi(F_2(X,3)) = \binom{3}{2}\chi^2 - \binom{\binom{3}{2}}{2}\chi + \binom{\binom{3}{2}}{3}\chi$$

This simplifies to

$$\chi(F_2(X,3)) = 3\chi^2 - 2\chi$$

Or

$$\chi(F_2(X,3)) = \chi^3 - \chi(\chi-1)(\chi-2)$$

Similarly we can see for $n = 4$

$$\chi(F_2(X,4)) = \binom{4}{2}\chi^3 - \binom{\binom{4}{2}}{2}\chi^2 + [\binom{\binom{4}{2}}{3} - \binom{4}{3}]\chi + \binom{4}{3}\chi^2 - \binom{\binom{4}{2}}{4}\chi + \binom{\binom{4}{2}}{5}\chi - \binom{\binom{4}{2}}{6}\chi$$

This simplifies to

$$\chi(F_2(X,4)) = 6\chi^3 - 11\chi^2 + 6\chi$$

Or

$$\chi(F_2(X,4)) = \chi^4 - \chi(\chi-1)(\chi-2)(\chi-3)$$

Now we write the general formula $\chi(F_2(X,n))$.

Proposition 3.3.4 $\chi(F_2(X,n)) = \chi^n - \chi(\chi-1)\cdots(\chi-(n-1))$.

Proof. We proceed by induction. The claim is true for $n = 3$ as we pointed out. Suppose the claim is true for $n \geq 3$ and write

$$F_2(X, n+1) = (F_2(X,n) \times X) \cup \bigcup_{1 \leq i \leq n} F_i$$

where

$$F_i = \{(x_1, \cdots, x_n, y) \mid x_i = y\}$$

Of course $F_i \cong X^n$. Now

- $F_{i_1} \cap \cdots \cap F_{i_k} = X^{n-k+1}$, and there are $\binom{n}{k}$ such intersections.
- $(F_2(X,n) \times X) \cap F_i \cong F_2(X,n)$, for each i.
- $F_{i_1} \cap \cdots \cap F_{i_k} \subset F_2(X,n) \times X$ for $k \geq 2$.

Write $e_n = \chi(F_2(X,n))$. The additive formula for the Euler characteristic (alternating sum over k-fold intersections) gives

$$\begin{aligned} e_{n+1} &= e_n\chi + n\chi^n \\ &\quad - \left[ne_n + \binom{n}{2}\chi^{n-1}\right] \\ &\quad + \left[\binom{n}{2}\chi^{n-1} + \binom{n}{3}\chi^{n-2}\right] \\ &\quad \vdots \\ &\quad \pm \left[\binom{n}{n-1}\chi^2 + \binom{n}{n}\chi\right] \\ &\quad \mp\chi \\ &= e_n\chi + n\chi^n + ne_n \\ &= (\chi - n)(\chi^n - \chi(\chi-1)\cdots(\chi-(n-1)) + n\chi^n \\ &= \chi^{n+1} - \chi(\chi-1)\cdots(\chi-n) \end{aligned}$$

∎

To give the Euler characteristic of $B_2(X,n)$ we use the following result

Theorem 3.3.5 (Zagier) *Let G be a group acting on X, then*

$$\chi(X/G) = \sum_{[g] \in G_*} (1/|Z_g|)\chi(X^g)$$

where G_* is the set of all conjugacy classes in G, Z_g is the centralizer of g in G and $X^g = \{x \in X \mid gx = x\}$.

Now since $B_2(X,n) = F_2(X,n)/\mathfrak{S}_n$,

$$\chi(B_2(X,n)) = \sum_{\sigma\in\mathfrak{S}_{n*}} \frac{1}{Z_\sigma} B_2(X,n)^\sigma$$

we split this sum by singling out the case $\sigma = id$ this gives that

$$\chi(B_2(X,n)) = \frac{1}{|Z_{id}|}\chi(F_2(X,n)^{id}) + \sum_{\substack{\sigma\in\mathfrak{S}_{n*}\\ \sigma\neq id}} \frac{1}{Z_\sigma}\chi(B_2(X,n)^\sigma)$$

Note that $F_2(X,n)^\sigma \subset (X^n)^\sigma$. In addition, if $\sigma \neq id$, a configuration is fixed by σ only if at least two of its entries are equal, that is, it is a configuration in $F_2(X,n)$, this implies that for $\sigma \neq id$, $F_2(X,n)^\sigma = (X^n)^\sigma$. Thus

$$\chi(B_2(X,n)) = \frac{1}{|Z_{id}|}\chi(F_2(X,n)^{id}) + \sum_{\sigma\in\mathfrak{S}_{n*}} \frac{1}{Z_\sigma}(X^n)^\sigma - \frac{1}{|Z_{id}|}\chi((X^n)^{id})$$

Since $Z_{id} = \mathfrak{S}_n$, $F_2(X,n)^{id} = F_2(X,n)$ and $(X^n)^{id} = X^n$ we get

$$\chi(B_2(X,n)) = \frac{1}{n!}\chi(F_2(X,n)) + \chi(\mathrm{SP}^n X) - \frac{1}{n!}\chi(X^n)$$

Which when replacing $\chi(F_2(X,n))$ by its value can be rewritten as follows

$$\chi(B_2(X,n)) = \chi(\mathrm{SP}^n X) - \frac{1}{n!}\chi(\chi-1)\cdots(\chi-(n-1))$$

Remark 3.3.6 Note that since the projection

$$F(X,n) \longrightarrow B(X,n)$$

is a $n!$ covering then $\chi(B(X,n)) = \frac{1}{n!}\chi(F(X,n))$. When X is a manifold of odd degree, it's known that $\chi(X) = 0$ and when X is a manifold of even degree then $\chi(F(X,n)) = \chi(\chi-1)\cdots(\chi-(n-1))$ and in this case $\chi(B_2(X,n)) = \chi(\mathrm{SP}^n X) - \chi(B(X,n))$. This is intriguing since as we observed earlier $B_2(X,n) = \mathrm{SP}^n X - B(X,n)$ and the euler characteristic does not in general behave in such a canonical way.

Remark 3.3.7 When X is a manifold of dimension d, we can use duality to determine the Euler characteristic of $B_2(X,n)$. We proceed as follows starting as usual by the ordered case. Since $F(X,n) = X^n - F_2(X,n)$, where $F_2(X,n)$ is seen as the singular locus, we have by duality that

$$H_{nd-i}F(X,n) = H^i(X^n, F_2(X,n)) = \tilde{H}^i(X^n/F_2(X,n))$$

Using betti numbers

$$b_{nd-i}F(X,n) = b_i(X^n/F_2(X,n))\ ,\ i \geq 1 \text{ and } b_{nd}F(X,n) + 1 = b_0(X^n/F_2(X,n))$$

Thus

$$\chi(X^n/F_2(X,n)) = (-1)^{nd}\chi(F(X,n)) + 1$$

On the other hand

$$\chi(X^n/F_2(X,n)) = \chi(X^n \cup_{F_2(X,n)} CF_2(X,n))$$

where $CF_2(X,n)$ is the cone of $F_2(X,n)$. Since $CF_2(X,n)$ is contractible,

$$\chi(X^n/F_2(X,n)) = \chi(X^n) - \chi(F_2(X,n)) + 1$$

We combine the formulas to get

$$\chi(F_2(X,n)) = \chi^n - (-1)^{nd}\chi(F(X,n)) = \chi^n - (-1)^{nd}\prod_{i=0}^{n-1}(\chi + i(-1)^{d-1})$$

Now we use the same formula of Zagier to deduce the Euler characteristic of $B_2(X,n)$.

3.4 Homology of the fat diagonal of the d-sphere

Recall that the fat diagonal $B_2(X,n)$ is the subspace of SP^nX having one entry that repeats at least twice. In this section we compute the homology of $B_2(S^d,4)$ and describe the manifold structure of $B_2(S^d,n)$.

We use the following pushout (see Lemma 4.2.1)

$$\begin{array}{ccc} S^d \times S^d & \xrightarrow{h} & S^d \times \mathrm{SP}^2S^d \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \delta} \\ \mathrm{SP}^2S^d & \xrightarrow{\lambda} & B_2(S^d,4) \end{array} \tag{3.4.1}$$

where $h(x,y) = (x,[y,y])$, $\delta(x,[y,z]) = [x,x,y,z]$, $\lambda[x,y] = [x,x,y,y]$ and π is the quotient map by the $\mathfrak{S}_2$-permutation action.

The Mayer-Vietoris sequence associated to this pushout

$$\cdots \longrightarrow H_k(S^d\times S^d) \xrightarrow{h_*\oplus\pi_*} H_k(S^d\times \mathrm{SP}^2S^d)\oplus H_k(\mathrm{SP}^2S^d) \xrightarrow{\delta_*-\lambda_*} H_k(B_2(S^d,4)) \longrightarrow H_{k-1}(S^d\times S^d) \tag{3.4.2}$$

We will need the following result due to Steenrod (for a proof see Hatcher, Example 4k.2)

Lemma 3.4.1 *The base point inclusion $S^d \longrightarrow SP^2S^d$ is a cofibration with cofibre homeomorphic to $\Sigma^d\mathbb{R}P^{d-1}$ and so*

$$H_i(SP^2S^d) = H_iS^d \oplus \tilde{H}_i(\Sigma^{d+1}\mathbb{R}P^{d-1})$$

In particular,

$$H_i\mathrm{SP}^2S^d = \begin{cases} \mathbb{Z}, & i = d \\ \mathbb{Z}, & i = 2d \text{ and } d \text{ even} \\ 0, & \text{otherwise} \end{cases} \tag{3.4.3}$$

Indeed, the long exact sequence associated to the pair (SP^2S^d, S^d) is given by

$$\cdots H_{i+1}\Sigma^{d+1}\mathbb{R}P^{d-1} \longrightarrow H_iS^d \longrightarrow H_i\mathrm{SP}^2S^d \longrightarrow H_i\Sigma^{d+1}\mathbb{R}P^{d-1} \longrightarrow H_{i-1}S^d \cdots$$

and $H_i\Sigma^j\mathbb{R}P^k = H_{i-j}\mathbb{R}P^k$. Moreover, $H_k\mathbb{R}P^k = \mathbb{Z}$ if k is odd and 0 if k is even. Now the homology of $B_2(S^d,4)$ is given as follows

Proposition 3.4.2 *If d is even*

$$\tilde{H}_*(B_2(S^d,4);\mathbb{Z}) = \begin{cases} \mathbb{Z}, & * = 3d \\ \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}_2, & * = 2d \\ \mathbb{Z}, & * = d \end{cases}$$

If d is odd

$$\tilde{H}_*(B_2(S^d,4);\mathbb{Z}) = \begin{cases} \mathbb{Z}_2, & * = 2d \\ \mathbb{Z}, & * = d \end{cases}$$

Proof. In dimension $3d$, the short exact sequence reduces to

$$0 \longrightarrow H_dS^d \otimes H_{2d}\mathrm{SP}^2S^d \longrightarrow H_{3d}B_2(S^d,4) \longrightarrow 0$$

so

$$H_{3d}B_2(S^d,4) = H_dS^d \otimes H_{2d}\mathrm{SP}^2S^d$$

Hence by (3.4.3)

$$H_{3d}B_2(S^d,4) = \begin{cases} \mathbb{Z}, & d \text{ even} \\ 0, & d \text{ odd} \end{cases}$$

In dimension $2d$, we have that

$$H_dS^d \otimes H_dS^d \xrightarrow{h_*\oplus\pi_*} H_dS^d \otimes H_d\mathrm{SP}^2S^d \oplus H_0S^d \otimes H_{2d}\mathrm{SP}^2S^d \oplus H_{2d}\mathrm{SP}^2S^d \longrightarrow H_{2d}B_2(S^d,4) \longrightarrow 0$$

Let u be the fundamental class of S^d.

If d is even, then $d-1$ is odd and $H_{2d}(\mathrm{SP}^2S^d) = H_{2d}(\Sigma^{d+1}\mathbb{R}P^{d-1}) = H_{d-1}(\mathbb{R}P^{d-1}) = \mathbb{Z}\{u_2\}$. The homomorphism h_* sends

$$u \otimes u \longmapsto 2(u \otimes u)$$

The homomorphism π_* sends

$$u \otimes u \longmapsto 2u_2$$

where u_2 is the top homology class of SP^2S^d. Hence

$$H_{2d}B_2(S^d,4) = coker\left(\begin{array}{c} \mathbb{Z} \xrightarrow{h_*\oplus\pi_*} \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \\ 1 \mapsto (2,0,2) \end{array}\right) = \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}_2$$

If d is odd, the short exact sequence reduces to

$$0 \longrightarrow H_dS^d \otimes H_dS^d \xrightarrow{h_*} H_dS^d \otimes H_d\mathrm{SP}^2S^d \longrightarrow H_{2d}B_2(S^d,4) \longrightarrow 0$$

where $h_*(u\otimes u) = 2(u \otimes u)$ and then

$$H_{2d}B_2(S^d,4) = coker\left(\begin{array}{c} \mathbb{Z} \longrightarrow \mathbb{Z} \\ 1 \mapsto 2 \end{array}\right) = \mathbb{Z}_2$$

In dimension d, the group $H_d(B_d(S^d,4))$ is the cokernel of the homomorphism

$$\mathbb{Z}\{1\otimes u\} \oplus \mathbb{Z}\{u \otimes 1\} \xrightarrow{h_*\oplus\pi_*} \mathbb{Z}\{1\otimes u\} \oplus \mathbb{Z}\{u\otimes 1\} \oplus \mathbb{Z}\{u\}$$

The homomorphism h_* sends

$$1 \otimes u \longmapsto 2(1 \otimes u), \quad u \otimes 1 \longmapsto u \otimes 1$$

The homomorphism π_* sends

$$1 \otimes u \longmapsto u, \quad u \otimes 1 \longmapsto u$$

Hence

$$H_d(B_d(S^d, 4)) = coker \left(\begin{array}{c} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \\ (1,0) \longmapsto (1,0,1) \quad , \quad (0,1) \longmapsto (0,2,1) \end{array} \right)$$

The Smith normal form of $\begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ so $H_d(B_d(S^d, 4)) = \mathbb{Z}$. ∎

In particular,

$$\tilde{H}_*(B_2(S^2, 4); \mathbb{Z}) = \begin{cases} \mathbb{Z}, & * = 6 \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2, & * = 4 \\ \mathbb{Z}, & * = 2 \end{cases} \tag{3.4.4}$$

Remark 3.4.3 The bottom class must be generated by the inclusion $S^2 \hookrightarrow B_2(S^2, 4)$ sending $x \longmapsto (x, *, *, *)$. This is because the composite $S^2 \hookrightarrow B_2(S^2, 4) \hookrightarrow \mathrm{SP}^4 S^2$ sends generator to generator.

Corollary 3.4.4 *The space $B_2(S^d, 4)$ cannot be homotopic to a closed manifold.*

PROOF. We will check that $B_2(S^d, 4)$ is not a manifold by showing that it doesn't satisfy Poincaré duality. Consider as above the pushout diagram (3.4.1).

This strict pushout says that if $B_2(S^d, 4)$ were a manifold, it would be of (real) dimension $3d$. If $B_2(S^d, 4)$ were a manifold of dimension $3d$, then the homology calculations of Proposition 3.4.2 would violate Poincaré duality in dimension $2d$ and d and we get a contradiction. ∎

3.5 Configurations with multiplicity bounded above

The space $B^d(X, n)$ is the subspace of $\mathrm{SP}^n X$ of configurations $[x_1, \cdots, x_n]$ such that an entry can't repeat more than d times. Note that the configuration space $B(X, n)$ is just

$B^1(X,n)$. These spaces fit into filtration

$$B^1(X,n) \subset B^2(X,n) \subset \cdots \subset B^n(X,n) = \mathrm{SP}^n X \tag{3.5.1}$$

where $\pi_1(B^1(X,n))$ is not abelian but $\pi_1(B^n(X,n)) = \pi_1(\mathrm{SP}^n X)$ is. Clearly $B^d(X,n) = \mathrm{SP}^n(X)$ if $d \geq n$, while

$$B^{n-1}(X,n) = \mathrm{SP}^n(X) - diag(X)$$

where $diag(X)$ is the image of the thin diagonal in $\mathrm{SP}^n X$. Actually we have a more general relation

$$B^d(X,n) = \mathrm{SP}^n X - B_{d+1}(X,n)$$

Next is an example.

Example 3.5.1

$$B^d(\mathbb{C},n) = \mathrm{SP}^n\mathbb{C} - B_{d+1}(\mathbb{C},n)$$

where $\mathrm{SP}^n\mathbb{C} \cong \mathbb{C}^n$. In particular,

$$B^{n-1}(\mathbb{C},n) = \mathrm{SP}^n\mathbb{C} - B_n(\mathbb{C},n) = \mathbb{C}^n - V(n)$$

where $V(n)$ is the subspace of $\mathbb{C}^n$ called the "Veronese" whose elements are of the form $(x, x^2, \cdots, x^n)$.

In the case of the affine plane and of orientable surfaces we have the following results.

Proposition 3.5.2 *Suppose S is a closed topological surface. Then $\pi_1(B^d(S,n)) = H_1(X;\mathbb{Z})$.*

Proof. As is known (see [32], proposition 5.6.1), if A is a closed analytic subspace of N of real codimension 2, then the induced map

$$\pi_1(N - A) \longrightarrow \pi_1(N) \tag{3.5.2}$$

is surjective and it is an isomorphism if the codimension is greater than 2. This happens because there is "enough space" for a loop to avoid A after performing a homotopy (locally supported) if necessary. When X is a compact topological surface, then $\mathrm{SP}^n(X)$ is a closed manifold of dimension $2n$ and $B_{d+1}(X,n)$ is the union of submanifolds of dimension $2(n-d) = 2n-2d$. This means that $B^d(S,n) = \mathrm{SP}^n S - B_{d+1}(S,n)$ is the complement of a finite union of submanifolds of codimension $2d > 2$ and hence $\pi_1(B^d(X,n)) = \pi_1(\mathrm{SP}^n X) = H_1(X;\mathbb{Z})$. ∎

Chapter 4

Finite Subset Spaces

4.1 Introduction

We discuss topological features of the space of subsets of cardinality at most four. Recall that these spaces are defined as follows. For X a topological space, $\mathrm{Sub}_n X$ is the space of subsets $\{x_1, \cdots, x_d\} \subset X$, $d \leq n$. It has the quotient topology from the identification

$$\mathrm{Sub}_n X = X \sqcup X^2 \sqcup \cdots \sqcup X^n /_{\sim}$$

with $(x_1, \cdots, x_r) \sim (y_1, \cdots, y_s)$ if the underlying sets $\{x_1, \cdots, x_r\}$ and $\{y_1, \cdots, y_s\}$ are equal.

Example 4.1.1 It's easy to see that $\mathrm{Sub}_2 X \cong \mathrm{SP}^2 X$. The homeomorphism

$$\mathrm{SP}^2 X \longrightarrow \mathrm{Sub}_2 X$$

maps $[x, y]$ to $\{x, y\}$ if $x \neq y$ and maps $[x, x]$ to $\{x\}$. In particular, $\mathrm{Sub}_2 S^1 \simeq S^1$.

Example 4.1.2 For $n \geq 3$ both spaces start to differ since $[x, y, y]$ and $[x, x, y]$ represent different configurations in $\mathrm{SP}^3 X$ unless $x = y$, but they yield the same element $\{x, y\}$ in $\mathrm{Sub}_3 X$. When $X = S^1$ for example, $\mathrm{SP}^3 S^1 = S^1 \times D^2$ by the Theorem of Morton cited earlier, while $\mathrm{Sub}_3 S^1$ has the following cute description due to Bott

Theorem 4.1.3 (Bott) *There is a homeomorphism $Sub_3 S^1 \cong S^3$*

Bott used a "cut and paste" argument and a fundamental group calculation to prove his result. Alternative proofs can be found in Tuffley [36] using the classification of simply connected Seifert fibrations, or in [22] based on the Poincaré conjecture for Seifert fibrations also.

In what follows we will focus on $\mathrm{Sub}_4 X$. Our main tools will be diagrams of pushouts describing $\mathrm{Sub}_4 X$ in terms of $\mathrm{Sub}_3 X$, $\mathrm{SP}^3 X$ and $\mathrm{SP}^4 X$. This allows us to give an explicit formula for the Euler characteristic $\chi(\mathrm{Sub}_4 X)$ in terms of that of X. Indeed, we show (Proposition 4.2.2) that

$$\chi(\mathrm{Sub}_4 X) = \frac{1}{24}(\chi(X)^4 - 2\chi(X)^3 + 11\chi(X)^2 + 14\chi(X))$$

These pushout diagrams enable us to compute the rational homology of $\mathrm{Sub}_4 S^d$ (see §4.4).

4.2 Basic Constructions and Euler Characteristic

Throughout X will be a CW complex with finitely many cells in every dimension. We will represent a strict pushout $X = B \cup_A C$ by a commuting square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \alpha} \\ C & \xrightarrow{\beta} & X \end{array} \tag{4.2.1}$$

The Euler characteristic χ is additive with respect to such pushouts; i.e. $\chi(X) = \chi(B) + \chi(C) - \chi(A)$.

Recall that the fat diagonal $F_2(X, n)$ is the subspace of X^n of $n-$tuples with 2 equal entries. Its image in $\mathrm{SP}^n X$ is denoted $B_2(X, n)$ and is also called the fat diagonal. For example $B_2(X, 4)$, the fat diagonal in $\mathrm{SP}^4 X$, is the set of configurations of the form $[x, x, y, z] \in \mathrm{SP}^4 X$ and there is a pushout

$$\begin{array}{ccc} B_2(X,4) & \xrightarrow{f} & \mathrm{SP}^4 X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \alpha} \\ \mathrm{Sub}_3 X & \xrightarrow{\beta} & \mathrm{Sub}_4 X \end{array} \tag{4.2.2}$$

where f and β are inclusions, $g[x, x, y, z] = \{x, y, z\}$ and α is the quotient map. We have also the following pushout

$$\begin{array}{ccc} X \times X & \xrightarrow{l} & \mathrm{SP}^3 X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \mu} \\ \mathrm{SP}^2 X & \xrightarrow{\tau} & \mathrm{Sub}_3 X \end{array} \tag{4.2.3}$$

where $l(x, y) = [x, x, y]$, $\tau([x, y]) = \{x, y\}$, μ and π are quotient maps. The following description of $B_2(X, 4)$ is equally easy to establish.

Lemma 4.2.1 *The following diagram*

$$\begin{tikzcd} X \times X \arrow[r, "h"] \arrow[d, "\pi"] & X \times SP^2X \arrow[d, "\delta"] \\ SP^2X \arrow[r, "\lambda"] & B_2(X,4) \end{tikzcd} \tag{4.2.4}$$

is a pushout, where $h(x,y) = (x,[y,y])$, $\delta(x,[y,z]) = [x,x,y,z]$, $\lambda([x,y]) = [x,x,y,y]$ *and* π *is the quotient map by the* $\mathfrak{S}_2$*-permutation action.*

It is convenient to put the diagrams together to see the whole picture in the following way

$$\begin{tikzcd} X \times X \arrow[r, "h"] \arrow[d, "\pi"] & X \times \mathrm{SP}^2X \arrow[d, "\delta"] & \\ \mathrm{SP}^2X \arrow[r, "\lambda"] & B_2(X,4) \arrow[r, "f"] \arrow[d, "g"] & \mathrm{SP}^4X \arrow[d, "\alpha"] \\ \mathrm{SP}^3X \arrow[r, "\mu"] & \mathrm{Sub}_3X \arrow[r, "\beta"] & \mathrm{Sub}_4X \\ X \times X \arrow[r, "\pi"] \arrow[u, "l"] & \mathrm{SP}^2X \arrow[u, "\tau"] & \end{tikzcd} \tag{4.2.5}$$

Associated to these pushouts are Mayer-Vietoris sequences, in short MVS, which we will be using throughout this chapter. As a consequence we can already derive the following

Proposition 4.2.2 *Given* X *a topological space with Euler characteristic* $\chi(X)$*, the Euler characteristics of* Sub_3X *and* Sub_4X *are given as follows*

$$\begin{aligned} \chi(Sub_3X) &= \frac{1}{6}(\chi(X)^3 + 5\chi(X)) \\ \chi(Sub_4X) &= \frac{1}{24}(\chi(X)^4 - 2\chi(X)^3 + 11\chi(X)^2 + 14\chi(X)) \end{aligned}$$

Proof. To prove the first equality, we use the pushout (4.2.3) and the additivity formula for the Euler characteristic to obtain the equation

$$\chi(\mathrm{Sub}_3X) = \chi(\mathrm{SP}^2X) + \chi(\mathrm{SP}^3X) - \chi(X)^2 \tag{4.2.6}$$

Now we refer to Macdonald formula [26]:

$$\sum_{n=0}^{\infty} \chi(\mathrm{SP}^n X) q^n = \frac{1}{(1-q)^{\chi(X)}} \tag{4.2.7}$$

Upon expanding we see that

$$\chi(\mathrm{SP}^n X) = \begin{pmatrix} \chi(X) + n - 1 \\ \chi(X) - 1 \end{pmatrix} \tag{4.2.8}$$

Combining this with (4.2.6) yields the computation for $\chi(\mathrm{Sub}_3 X)$. To derive the second equation, we use the diagram of pushouts (4.2.5). Indeed, we have that

$$\chi(\mathrm{Sub}_4 X) = \chi(\mathrm{SP}^4 X) + \chi(\mathrm{Sub}_3 X) - \chi(B_2(X,4)) \tag{4.2.9}$$

and

$$\chi(B_2(X,4)) = \chi(\mathrm{SP}^2 X) + \chi(X)\chi(\mathrm{SP}^2 X) - \chi(X)^2 \tag{4.2.10}$$

hence

$$\chi(\mathrm{Sub}_4 X) = \chi(\mathrm{SP}^4 X) + \chi(\mathrm{SP}^3 X) - \chi(X)\chi(\mathrm{SP}^2 X) \tag{4.2.11}$$

The Euler characteristic of all the terms on the right are known and the formula follows. ∎

Example 4.2.3 When $X = S_g$ is a closed orientable surface of genus g, Tuffley ([38]) shows that

$$\chi(\mathrm{Sub}_3 S_g) = 1/3(-4g^3 + 12g^2 - 17g + 9)$$

a fact that is obviously in agreement with Proposition 4.2.2 after replacing $\chi(X)$ by $2-2g$.

4.2.1 Homology of Symmetric Products

For any based connected CW complex X, it is a standard result of Steenrod (see [12]) that we have a splitting

$$\begin{aligned} H_*(\mathrm{SP}^n X;\mathbb{Z}) &\cong H_*(\mathrm{SP}^{n-1} X;\mathbb{Z}) \oplus H_*(\mathrm{SP}^n X, \mathrm{SP}^{n-1} X;\mathbb{Z}) \\ &\cong H_*(X;\mathbb{Z}) \oplus \tilde{H}_*(\overline{\mathrm{SP}}^2 X;\mathbb{Z}) \oplus \cdots \oplus \tilde{H}_*(\overline{\mathrm{SP}}^n X;\mathbb{Z}) \end{aligned} \tag{4.2.12}$$

where $\overline{\mathrm{SP}}^n X := \mathrm{SP}^n X/\mathrm{SP}^{n-1} X$ is the reduced symmetric product. The homology of $\mathrm{SP}^n X$ embeds in the homology of the infinite symmetric product $\mathrm{SP}^\infty X$ obtained as the direct limit under the basepoint inclusions $\mathrm{SP}^n X \hookrightarrow \mathrm{SP}^{n+1} X$. The infinite symmetric product is an abelian topological monoid and by work of Dold and Thom

$$\mathrm{SP}^{\infty}X \simeq \prod_{i\geq 0} K(\tilde{H}_i(X;\mathbb{Z}), i)$$

Determining the direct summands corresponding to $H_*(\mathrm{SP}^n X)$ in the homology of this product is work of [27].

Example 4.2.4 In the case of a sphere S^d, $\mathrm{SP}^{\infty}S^d \simeq K(\mathbb{Z}, d)$. It is known that for d odd, $K(\mathbb{Z}, d)$ has the rational homology of S^d . When d is even, we have that

$$H^*(K(\mathbb{Z}, d); \mathbb{Q}) \cong \mathbb{Q}[u] \quad , \quad \deg u = d$$

The rational cohomology of $\mathrm{SP}^n S^d$ in the even case is a truncated algebra

$$H^*(\mathrm{SP}^n S^d; \mathbb{Q}) \cong \mathbb{Q}[u]/(u^{n+1})$$

In homology, $H_*(\mathrm{SP}^n S^d; \mathbb{Q})$ will be generated by classes u_r dual to the u^r. We have the following properties which we will use freely:

- $\pi_*(u_1^{\otimes r}) = r! u_r$, where $u_1 = [S^d]$ is the top homology class of S^d and $\pi : (S^d)^r \longrightarrow \mathrm{SP}^r S^d$ is the quotient map.
- The coproduct on the u_r is given by

$$u_r \longmapsto \sum_{i=0}^{r} u_i \otimes u_{r-i} \tag{4.2.13}$$

- The concatenation pairing $\mathrm{SP}^r S^d \times \mathrm{SP}^s S^d \longrightarrow \mathrm{SP}^{r+s} S^d$ sends $u_r \otimes u_s \longmapsto \binom{r+s}{s} u_{r+s}$.

Example 4.2.5 In the case $d = 2$, $\mathrm{SP}^n S^2 \cong \mathbb{P}^n$ is the complex projective space. Integrally, $H_*(\mathrm{SP}^n S^2) = H_*(\mathbb{P}^n)$ has generators

$$\gamma_i, \ 1 \leq i \leq n, \ \deg(\gamma_i) = 2i \tag{4.2.14}$$

Geometrically, $\gamma_i = [\mathbb{P}^i]$ where $\mathbb{P}^i \subset \mathbb{P}^n$ is any linearly embedded projective subspace.

A fact we will use very often in this chapter is the following

Lemma 4.2.6 *The power map $X \longrightarrow SP^n X, x \longrightarrow x^n$ induces in homology a map which is multiplication by n on the primitives.*

Proof. Write the power map as a composite

$$X \xrightarrow{\Delta} X^n \xrightarrow{\pi} \mathrm{SP}^n X,\ x \longmapsto (x,\dots,x) \longmapsto x^n \tag{4.2.15}$$

On primitive classes, Δ_* factors through $H_*(X)^{\oplus n} \hookrightarrow H_*(X^n)$, and the image of $H_*(X)^{\oplus n}$ under π_* is in the direct summand $H_*(X) \hookrightarrow H_*(\mathrm{SP}^n X)$. Since the coproduct on a primitive class a is $\sum_{i=0}^{n-1} 1^{\otimes i} \otimes a \otimes 1^{\otimes n-i-1}$, and each such term maps to $a \in H_*(X) \subset H_*(\mathrm{SP}^n X)$, the claim follows. ∎

We next recall that by a fundamental result of Dold, if $H_i(X;\mathbb{Z}) \cong H_i(Y;\mathbb{Z})$ for $i \le N$ then $H_i(\mathrm{SP}^n X;\mathbb{Z}) \cong H_i(\mathrm{SP}^n Y;\mathbb{Z})$ for $i \le N$ for each non negative integer n. This result is no longer valid for the functor $\mathrm{Sub}_n(-)$ as is illustrated next. To construct our counterexample, we use $\mathbb{P}^2$ and $S^2 \vee S^4$ which constitute one of the simplest examples of spaces having the same homology but distinct cohomology rings.

Lemma 4.2.7 *$H_6(Sub_3(S^2 \vee S^4);\mathbb{Q}) \cong \mathbb{Q} \oplus \mathbb{Q}$ and $H_6(Sub_3\mathbb{P}^2;\mathbb{Q}) \cong \mathbb{Q}$.*

Proof. We will be using the pushout (4.2.3) and its MVS

$$H_k(X \times X) \xrightarrow{l_* \oplus \pi_*} H_k(\mathrm{SP}^3 X) \oplus H_k(\mathrm{SP}^2 X) \longrightarrow H_k(\mathrm{Sub}_3 X) \tag{4.2.16}$$

$$\xrightarrow{\partial} H_{k-1}(X \times X) \xrightarrow{l_* \oplus \pi_*} H_{k-1}(\mathrm{SP}^3 X) \oplus H_{k-1}(\mathrm{SP}^2 X)$$

To compute $H_6(\mathrm{Sub}_3(S^2 \vee S^4);\mathbb{Q})$, we need to determine $H_*(\mathrm{SP}^3(S^2 \vee S^4);\mathbb{Q})$ and $H_*(\mathrm{SP}^2(S^2 \vee S^4);\mathbb{Q})$. As we mentioned, these are direct summands in

$$\begin{aligned} H_*(\mathrm{SP}^\infty(S^2 \vee S^4)) &\cong H_*(\mathrm{SP}^\infty S^2 \times \mathrm{SP}^\infty S^4) \\ &\cong H_*(K(\mathbb{Z},2) \times K(\mathbb{Z},4)) \\ &\cong \mathbb{Q}\{u_1,u_2,\cdots\} \otimes \mathbb{Q}\{v_1,v_2,\cdots\} \quad , \quad \deg u_i = 2i, \deg v_i = 4i \end{aligned}$$

In our notation $\mathbb{Q}\{x_1,x_2,\dots\}$ means the vector space over $\mathbb{Q}$ with basis generators the x_i. Hence $H_*(\mathrm{SP}^3(S^2 \vee S^4);\mathbb{Q})$ is generated by all classes of the form $u_r \otimes v_s, r+s \le 3$ (or simply $u_r v_s$), where as before u_r are dual to u^r and similarly v_s are dual to v^s.

Since this homology is concentrated in even degrees, we deduce from (4.2.16) that $H_6(\mathrm{Sub}_3(S^2 \vee S^4)) = coker(l_* \oplus \pi_*)$ where

$$l_* \oplus \pi_* : H_6((S^2 \vee S^4) \times (S^2 \vee S^4)) \longrightarrow H_6(\mathrm{SP}^3(S^2 \vee S^4)) \oplus H_6(\mathrm{SP}^2(S^2 \vee S^4))$$

$$\mathbb{Q}\{u_1 \otimes v_1, v_1 \otimes u_1\} \longrightarrow \mathbb{Q}\{u_3, u_1 v_1\} \oplus \mathbb{Q}\{u_1 v_1\}$$

To understand l_*, we rewrite the map $l : X\times X\longrightarrow \mathrm{SP}^3X$, $(x,y)\longmapsto [x,x,y]$ in the following way

$$l \quad : \quad X\times X\xrightarrow{\Delta\times 1}X\times X\times X\xrightarrow{\pi\times 1}\mathrm{SP}^2X\times X\longrightarrow \mathrm{SP}^3X \qquad (4.2.17)$$
$$(x,y)\longmapsto (x,x,y)\longmapsto (x^2,y)\longmapsto x^2y$$

So the effect of $l_*\oplus\pi_*$ is given as follows:

$$l_*\oplus\pi_* : u_1\otimes v_1\longmapsto 2u_1v_1\oplus u_1v_1 \quad , \quad v_1\otimes u_1\longmapsto 2u_1v_1\oplus u_1v_1$$

If we write $\mathbb{Q}\{u_1\otimes v_1, v_1\otimes u_1\}\cong\mathbb{Q}\oplus\mathbb{Q}$ and $\mathbb{Q}\{u_3,u_1v_1\}\oplus\mathbb{Q}\{u_1v_1\}\cong\mathbb{Q}\oplus\mathbb{Q}\oplus\mathbb{Q}$, then the map above takes the form

$$\mathbb{Q}\oplus\mathbb{Q}\longrightarrow\mathbb{Q}\oplus\mathbb{Q}\oplus\mathbb{Q} \quad , \quad (1,0)\longmapsto (0,2,1) \text{ and } (0,1)\longmapsto (0,2,1)$$

and has cokernel $H_6(\mathrm{Sub}_3(S^2\vee S^4))=\mathbb{Q}\oplus\mathbb{Q}$.

Similarly, $H_*(\mathrm{SP}^3\mathbb{P}^2)\cong H_*(\mathrm{SP}^3(S^2\vee S^4))$ by the result of Dold. The effect of the squaring map on this homology is however different. Indeed taking $X=\mathbb{P}^2$ and $n=2$ in (4.2.15), we obtain

$$v_1\longmapsto v_1\otimes 1+u_1\otimes u_1+1\otimes v_1\longmapsto 2v_1+2u_2$$

The map $l_*\oplus\pi_*$ in this case sends

$$u_1\otimes v_1\longmapsto 2u_1v_1\oplus u_1v_1 \quad , \quad v_1\otimes u_1\longmapsto 6u_3\oplus 2u_1v_1\oplus u_1v_1$$

As before this translates into the map $\mathbb{Q}^2\longrightarrow\mathbb{Q}^3$ sending $(1,0)\longmapsto (0,2,1)$ and $(0,1)\longmapsto (6,2,1)$. The cokernel is a copy of $\mathbb{Q}$ as asserted. ∎

Next we re-derive Tuffley's calculation of $H_*(\mathrm{Sub}_3S^2;\mathbb{Z})$. This computation will be useful in §4.5.

Lemma 4.2.8

$$\tilde{H}_*(Sub_3S^2;\mathbb{Z})=\begin{cases}\mathbb{Z} & \textit{if } *=6\\ \mathbb{Z}\oplus\mathbb{Z}_2 & \textit{if } *=4\end{cases} \qquad (4.2.18)$$

Proof. Consider the pushout (4.2.3). There are only two non trivial terms in the MVS associated to this pushout. First,

$$0\longrightarrow H_6(\mathrm{SP}^3S^2)=H_6(\mathbb{P}^3)\xrightarrow{\mu_*}H_6(\mathrm{Sub}_3S^2)\longrightarrow 0$$

and $H_6(\mathrm{Sub}_3 S^2)$ is then a copy of $\mathbb{Z}$. On the other hand, in dimension 4 we have that $H_4(\mathrm{Sub}_3 S^2;\mathbb{Z}) = coker(l_* \oplus \pi_*)$ where

$$l_* \oplus \pi_* : \mathbb{Z}\{\gamma_1 \otimes \gamma_1\} \longrightarrow \mathbb{Z}\{\gamma_2\} \oplus \mathbb{Z}\{\gamma_2\}$$

and the generators γ_i are as defined in (4.2.14). Since the induced map from $\pi : S^2 \times S^2 \longrightarrow \mathrm{SP}^2 S^2$ sends $\gamma_1 \otimes \gamma_1$ to $2\gamma_2$ and the induced map from the squaring map $S^2 \longrightarrow \mathrm{SP}^2 S^2$ sends γ_1 to $2\gamma_1$, it follows by (4.2.17) that $l_* \oplus \pi_*(\gamma_1 \otimes \gamma_1) = 4\gamma_2 \oplus 2\gamma_2$ hence the cokernel is

$$H_4(\mathrm{Sub}_3 S^2) = \langle\, \mu_*(\gamma_2), \tau_*(\gamma_2) \mid 4\mu_*(\gamma_2) = 2\tau_*(\gamma_2)\,\rangle$$

which we identify with $\mathbb{Z} \oplus \mathbb{Z}_2$. ∎

4.3 Connectivity

We say a space X is r-connected, $r \geq 0$, if the homotopy groups $\pi_i(X)$ vanish for $i \leq r$. The case X path connected corresponds to $r = 0$ and simply connected to $r = 1$. In [35], Tuffley has shown that $\mathrm{Sub}_n X$ is $(n-2)$-connected and later conjectured that this connectivity should be $n-2+r$. In [22], the authors show that $\mathrm{Sub}_n X$ is $(r+1)$-connected if $r > 1$ hence verifying the conjecture in the case $n = 3$. In this section we prove that Tuffley's conjecture holds for $n = 4$ as well.

The starting point is the following useful calculation of Nakaoka ([30], Corollary 4.7).

Lemma 4.3.1 *Let $i : X \longrightarrow SP^n X;\ x \longmapsto [x, *, \cdots, *]$ be the base point inclusion. If X is r-connected, $r > 1$, then the induced homomorphism $i_* : H_k(X;\mathbb{Z}) \longrightarrow H_k(SP^n X;\mathbb{Z})$ is an isomorphism for $k = r+1$ and $k = r+2$.*

When $n = 4$, we have also the following result.

Lemma 4.3.2 *If X is r-connected, $r > 1$, then $H_{r+1}(B_2(X,4);\mathbb{Z}) = H_{r+1}(X;\mathbb{Z})$. In particular, the induced homomorphism $f_* : H_{r+1}(B_2(X,4);\mathbb{Z}) \longrightarrow H_{r+1}(SP^4 X;\mathbb{Z})$ is an isomorphism, where $f : B_2(X,4) \longrightarrow SP^4 X$ is the inclusion.*

Proof. First notice that since X is r-connected, $H_r(X \times X) = 0$ hence the first non trivial terms of the MVS associated to the pushout (4.2.4) are given as follows

$$\cdots \longrightarrow H_{r+1}(X \times X) \xrightarrow{h_* \oplus \pi_*} H_{r+1}(X \times \mathrm{SP}^2 X) \oplus H_{r+1}(\mathrm{SP}^2 X) \xrightarrow{\delta_* - \lambda_*} H_{r+1}(B_2(X,4)) \longrightarrow 0$$

so $H_{r+1}(B_2(X,4)) = coker(h_* \oplus \pi_*)$. It is then enough to show that $coker(h_* \oplus \pi_*)$ is isomorphic to $H_{r+1}(X)$. Since X is r-connected, $H_{r+1}(X \times X) = H_{r+1}(X) \oplus H_{r+1}(X)$.

Moreover, $H_{r+1}(\mathrm{SP}^2X) = H_{r+1}(X)$ by Lemma 4.3.1. Also because X is r-connected, all classes a in $H_{r+1}(X)$ are primitive and we have that

$$h_* \oplus \pi_* : \ a \otimes 1 \longmapsto a \otimes 1 \oplus a \ \text{ and } \ 1 \otimes a \longmapsto 2(1 \otimes a) \oplus a$$

Denote $G := H_{r+1}(X)$ then we can write

$$h_* \oplus \pi_* : G \oplus G \longrightarrow G \oplus G \oplus G \ ; \ (g_1, g_2) \longmapsto (g_1, 2g_2, g_1 + g_2)$$

To show that $coker(h_* \oplus \pi_*) = (G \oplus G \oplus G)/(Im(h_* \oplus \pi_*))$ is isomorphic to G, consider the homomorphism

$$\phi : G \oplus G \oplus G \longrightarrow G \ ; \ (g_1, g_2, g_3) \longmapsto 2g_1 + g_2 - 2g_3$$

This is obviously surjective with $ker(\phi) = Im(h_* \oplus \pi_*)$ and the claim follows. ∎

The following algebraic fact will be used throughout. Given a pushout as in (4.2.1) with Mayer-Vietoris long exact sequence

$$\cdots H_i(A) \xrightarrow{f_* \oplus g_*} H_i(B) \oplus H_i(C) \xrightarrow{\alpha_* - \beta_*} H_i(X) \xrightarrow{\partial} H_{i-1}(A) \xrightarrow{f_* \oplus g_*} H_{i-1}(B) \oplus H_{i-1}(C) \cdots$$

then the following is immediate

Lemma 4.3.3 *If $H_i(A) \xrightarrow{f_* \oplus g_*} H_i(B) \oplus H_i(C)$ is surjective and $H_{i-1}(A) \xrightarrow{f_* \oplus g_*} H_{i-1}(B) \oplus H_{i-1}(C)$ is injective then $H_i(X) = 0$*

This is useful in obtaining the following main result.

Lemma 4.3.4 *If X is r-connected, $r > 1$, then Sub_3X is $(r+2)$-connected.*

Proof. Since Sub_3X is $r+1$-connected, then by the Hurewicz theorem ([16], p.366) $H_{r+2}(\mathrm{Sub}_3X)$ is isomorphic to $\pi_{r+2}(\mathrm{Sub}_3X)$ and so it is enough to show that $H_{r+2}(\mathrm{Sub}_3X) = 0$. Consider the pushout (4.2.3) and its associated MVS as in (4.2.16). We will show that

$$H_{r+2}(X \times X) \xrightarrow{l_* \oplus \pi_*} H_{r+2}(\mathrm{SP}^3X) \oplus H_{r+2}(\mathrm{SP}^2X)$$

is surjective and that

$$H_{r+1}(X \times X) \xrightarrow{l_* \oplus \pi_*} H_{r+1}(\mathrm{SP}^3X) \oplus H_{r+1}(\mathrm{SP}^2X)$$

is injective from which we conclude that $H_{r+2}(\mathrm{Sub}_3 X) = 0$ using Lemma 4.3.3. In fact we show that both maps are isomorphisms. By Lemma 4.3.1 we have that

$$H_{r+i}(\mathrm{SP}^3 X) = H_{r+i}(\mathrm{SP}^2 X) = H_{r+i}(X) \quad , \quad i = 1, 2$$

In degree $r+1$, $H_{r+1}(X \times X) = H_{r+1}(X) \oplus H_{r+1}(X)$ since X is r connected and we can rewrite $l_* \oplus \pi_*$ as the map

$$l_* \oplus \pi_* : B \oplus B \longrightarrow B \oplus B \quad , \quad B := H_{r+1}(X)$$

Since all classes in $H_{r+1}(X)$ are primitive, then by Lemma 4.2.6 the squaring map is multiplication by 2 in homology, so by (4.2.17), $l_*(a,b) = 2a + b$ as a map $B \oplus B \longrightarrow B$. On the other hand, $\pi_* : B \oplus B \longrightarrow B$ is the abelian sum so that

$$l_* \oplus \pi_* : (a, b) \longmapsto (2a + b, a + b)$$

which is an isomorphism. The exact same argument applies for H_{r+2} because it also consists of primitive classes since $r > 1$. ∎

Theorem 4.3.5 *If X is r-connected, $r > 1$, then $Sub_4 X$ is $(r+2)$-connected.*

Proof. As in Lemma 4.3.4, it is enough to show that $H_{r+2}(\mathrm{Sub}_4 X) = 0$. Consider the pushout (4.2.2). The associated MVS is

$$\longrightarrow H_{r+2}(B_2(X,4)) \xrightarrow{f_* \oplus g_*} H_{r+2}(\mathrm{SP}^4 X) \oplus H_{r+2}(\mathrm{Sub}_3 X)$$

$$\xrightarrow{\alpha_* - \beta_*} H_{r+2}(\mathrm{Sub}_4 X) \xrightarrow{\partial} H_{r+1}(B_2(X,4)) \xrightarrow{f_* \oplus g_*} H_{r+1}(\mathrm{SP}^4 X) \oplus H_{r+1}(\mathrm{Sub}_3 X)$$

By Lemma 4.3.4, $H_{r+2}(\mathrm{Sub}_3 X) = H_{r+1}(\mathrm{Sub}_3 X) = 0$, so to deduce that $H_{r+2}(\mathrm{Sub}_4 X) = 0$, it is enough by Lemma 4.3.3 to show that
i) the map $H_{r+2}(B_2(X,4)) \xrightarrow{f_*} H_{r+2}(\mathrm{SP}^4 X)$ is surjective
ii) the map $H_{r+1}(B_2(X,4)) \xrightarrow{f_*} H_{r+1}(\mathrm{SP}^4 X)$ is injective.
Now i) follows from Lemma 4.3.1 stating that the composite map

$$H_{r+2}(X) \xrightarrow{i_*} H_{r+2}(B_2(X,4)) \xrightarrow{f_*} H_{r+2}(\mathrm{SP}^4 X)$$

is an isomorphism. The statement ii) follows directly from Lemma 4.3.2. ∎

4.4 Sub$_4$ for Spheres

In this section we describe the rational homology of $\mathrm{Sub}_4 S^d$ using Mayer-Vietoris sequences associated to the pushouts depicted in (4.2.5). We need to know the homology of $B_2(S^d, 4)$, $\mathrm{Sub}_3 S^d$ and $\mathrm{SP}^4 S^d$. The rational homology of $\mathrm{SP}^k S^d$ was described in §4.2.1 and takes the form:

$$H_*(\mathrm{SP}^k S^d; \mathbb{Q}) \cong \begin{cases} H_*(S^d; \mathbb{Q}) & \text{, if } d \text{ is odd} \\ H_*(S^d \vee S^{2d} \vee \cdots \vee S^{kd}; \mathbb{Q}) & \text{, if } d \text{ is even} \end{cases}$$

4.4.1 Rational homotopy type of Sub$_4 S^d$, d odd, $d > 1$

Lemma 4.4.1 *For d odd, $\mathrm{Sub}_3 S^d$ has the rational homotopy type of the sphere S^{2d+1}.*

Proof. The MVS associated to the pushout (4.2.3) and as written up in (4.2.16), has only two non-trivial terms. First

$$0 \longrightarrow H_{2d+1}(\mathrm{Sub}_3 S^d) \longrightarrow H_{2d}(S^d \times S^d) \longrightarrow 0$$

and thus $H_{2d+1}(\mathrm{Sub}_3 S^d) \cong \mathbb{Q}$. The second term is

$$0 \longrightarrow H_{d+1}(\mathrm{Sub}_3 S^d) \longrightarrow H_d(S^d \times S^d) \xrightarrow{l_* \oplus \pi_*} H_d(\mathrm{SP}^3 S^d) \oplus H_d(\mathrm{SP}^2 S^d) \longrightarrow H_d(\mathrm{Sub}_3 S^d) \longrightarrow 0$$

The map

$$l_* \oplus \pi_* : \mathbb{Q}\{u_1 \otimes 1\} \oplus \mathbb{Q}\{1 \otimes u_1\} \longrightarrow \mathbb{Q}\{u_1\} \oplus \mathbb{Q}\{u_1\}$$

where $u_1 = [S^d]$ is the top homology class of S^d, sends

$$u_1 \otimes 1 \longmapsto 2u_1 \oplus u_1 \quad \text{and} \quad 1 \otimes u_1 \longmapsto u_1 \oplus u_1$$

This is clearly an isomorphism between two copies of $\mathbb{Q} \oplus \mathbb{Q}$. The injectivity of this map implies that $H_{d+1}(\mathrm{Sub}_3 S^d) = 0$ while surjectivity implies that $H_d(\mathrm{Sub}_3 S^d) = 0$. Since all other reduced groups are trivial, $\mathrm{Sub}_3 S^d$ has the rational homology of the sphere S^{2d+1}. Now the claim that $\mathrm{Sub}_3 S^d$ has the rational homotopy type of S^{2d+1} follows from a standard result stating that if a simply connected space has the rational homology of a sphere, then it has the rational homotopy type of that sphere. This result is a direct consequence of a rational Hurewicz theorem ([24], Corollary 2.5) and the rational Whitehead theorem ([13], Theorem 8.6). ∎

Lemma 4.4.2 *For d odd, $B_2(S^d, 4)$ has the rational homotopy type of S^d.*

PROOF. Consider the pushout (4.2.4). For d odd, the squaring map $S^d \longrightarrow \mathrm{SP}^2 S^d$ is multiplication by 2 in rational homology so h induces isomorphism in rational homology and by the rational Whitehead theorem, h is a rational homotopy equivalence since $S^d \times S^d$ and $S^d \times \mathrm{SP}^2 S^d$ are simply connected, d being strictly greater than 1. The claim that the map λ is a rational homotopy equivalence, follows from the following standard fact. Given a pushout as in (4.2.1) of simply connected spaces, if f is a rational homotopy equivalence, then so is β. ∎

Corollary 4.4.3 *For d odd, the inclusion $\beta : Sub_3 S^d \longrightarrow Sub_4 S^d$ is a rational homotopy equivalence.*

PROOF. There are two non trivial parts in the MVS associated to the pushout (4.2.2). The first part gives immediately that

$$\beta_* : H_{2d+1}(\mathrm{Sub}_3 S^d; \mathbb{Q}) \longrightarrow H_{2d+1}(\mathrm{Sub}_4 S^d; \mathbb{Q})$$

is an isomorphism. The second part gives that $H_d(\mathrm{Sub}_4 S^d; \mathbb{Q}) = 0$ since

$$f_* : H_d(B_2(S^d, 4); \mathbb{Q}) \longrightarrow H_d(\mathrm{SP}^4 S^d; \mathbb{Q})$$

is an isomorphism. It follows that β is a rational homology equivalence. Since $\mathrm{Sub}_3 S^d$ and $\mathrm{Sub}_4 S^d$ are simply connected, it follows by rational Whitehead theorem that β is a rational homotopy equivalence. ∎

4.4.2 Rational homotopy type of Sub$_4 S^d$, d even

In ([38], Theorem 1) Tuffley has proven that $\mathrm{Sub}_n S^2$ has the rational homology of $S^{2n} \vee S^{2n-2}$. In this section and for $n = 3$ and $n = 4$, we generalize this result to all even spheres.

We start by determining the rational homology of $\mathrm{Sub}_3 S^d$.

Lemma 4.4.4 *For d even, the space $Sub_3 S^d$ has the rational homology of $S^{3d} \vee S^{2d}$.*

PROOF. There are only three non trivial terms in the MVS associated to the pushout (4.2.3). The first one is

$$0 \longrightarrow H_{3d}(\mathrm{SP}^3 S^d) \xrightarrow{\mu_*} H_{3d}(\mathrm{Sub}_3 S^d) \longrightarrow 0$$

so that $H_{3d}(\mathrm{Sub}_3 S^d) = \mathbb{Q}\{\mu_*(u_3)\}$. The second term is

$$0 \longrightarrow H_{2d+1}(\mathrm{Sub}_3 S^d) \longrightarrow H_{2d}(S^d \times S^d) \xrightarrow{l_* \oplus \pi_*} H_{2d}(\mathrm{SP}^3 S^d) \oplus H_{2d}(\mathrm{SP}^2 S^d) \longrightarrow H_{2d}(\mathrm{Sub}_3 S^d) \longrightarrow 0$$

where

$$l_* \oplus \pi_* : \mathbb{Q}\{u_1 \otimes u_1\} \longrightarrow \mathbb{Q}\{u_2\} \oplus \mathbb{Q}\{u_2\}$$

sending

$$u_1 \otimes u_1 \longmapsto 4u_2 \oplus 2u_2$$

with cokernel $\mathbb{Q}$. The last non trivial term is

$$0 \longrightarrow H_{d+1}(\mathrm{Sub}_3 S^d) \longrightarrow H_d(S^d \times S^d) \xrightarrow{l_* \oplus \pi_*} H_d(\mathrm{SP}^3 S^d) \oplus H_d(\mathrm{SP}^2 S^d) \longrightarrow H_d(\mathrm{Sub}_3 S^d) \longrightarrow 0$$

where

$$l_* \oplus \pi_* : \mathbb{Q}\{u_1 \otimes 1\} \oplus \mathbb{Q}\{1 \otimes u_1\} \longrightarrow \mathbb{Q}\{u_1\} \oplus \mathbb{Q}\{u_1\}$$

sending

$$u_1 \otimes 1 \longmapsto 2u_1 \oplus u_1 \ \text{ and } \ 1 \otimes u_1 \longmapsto u_1 \oplus u_1$$

and this is an isomorphism. The injectivity of this map implies that $H_{d+1}(\mathrm{Sub}_3 S^d) = 0$ while surjectivity implies that $H_d(\mathrm{Sub}_3 S^d) = 0$. ∎

When $d = 2$ this is just Tuffley's computations on the sphere S^2. Next we determine the rational homology of $B_2(S^d, 4)$.

Lemma 4.4.5 *For d even, $B_2(S^d, 4)$ has the rational homology of $S^{3d} \vee S^{2d} \vee S^{2d} \vee S^d$.*

Proof. Use the MVS associated to the pushout (4.2.4). The first non trivial term is

$$0 \longrightarrow H_{3d}(S^d \times \mathrm{SP}^2 S^d) \xrightarrow{\delta_*} H_{3d}(B_2(S^d, 4)) \longrightarrow 0$$

which gives that $H_{3d}(B_2(S^d, 4)) = \mathbb{Q}\{\delta_*(u_1 \otimes u_2)\}$. The second non trivial term is

$$0 \longrightarrow H_{2d+1}(B_2(S^d, 4)) \longrightarrow H_{2d}(S^d \times S^d) \xrightarrow{h_* \oplus \pi_*}$$

$$H_{2d}(S^d \times \mathrm{SP}^2 S^d) \oplus H_{2d}(\mathrm{SP}^2 S^d) \longrightarrow H_{2d}(B_2(S^d, 4)) \longrightarrow 0$$

where

$$h_* \oplus \pi_* : \mathbb{Q}\{u_1 \otimes u_1\} \longrightarrow \mathbb{Q}\{u_1 \otimes u_1\} \oplus \mathbb{Q}\{1 \otimes u_2\} \oplus \mathbb{Q}\{u_2\}$$

sending

$$u_1 \otimes u_1 \longmapsto 2(u_1 \otimes u_1) \oplus 2u_2$$

This map is injective and has cokernel $\mathbb{Q} \oplus \mathbb{Q}$. The last non trivial term is

$$0 \longrightarrow H_{d+1}(B_2(S^d, 4)) \longrightarrow H_d(S^d \times S^d) \xrightarrow{h_* \oplus \pi_*}$$

$$H_d(S^d \times \mathrm{SP}^2 S^d) \oplus H_d(\mathrm{SP}^2 S^d) \longrightarrow H_d(B_2(S^d, 4)) \longrightarrow 0$$

where

$$h_* \oplus \pi_* : \mathbb{Q}\{u_1 \otimes 1\} \oplus \mathbb{Q}\{1 \otimes u_1\} \longrightarrow \mathbb{Q}\{u_1 \otimes 1\} \oplus \mathbb{Q}\{1 \otimes u_1\} \oplus \mathbb{Q}\{u_1\}$$

sending

$$u_1 \otimes 1 \longmapsto (u_1 \otimes 1) \oplus u_1 \ \text{ and } \ 1 \otimes u_1 \longmapsto 2(1 \otimes u_1) \oplus u_1$$

so this is a map

$$\mathbb{Q} \oplus \mathbb{Q} \longrightarrow \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}$$

sending

$$(1, 0) \longmapsto (1, 0, 1) \ \text{ and } \ (0, 1) \longmapsto (0, 2, 1)$$

which is injective and has cokernel isomorphic to $\mathbb{Q}$. ∎

Now we prove the main result of this subsection

Theorem 4.4.6 *For d even, $Sub_4 S^d$ has the rational homotopy type of $S^{4d} \vee S^{3d}$*

Proof. As a first step, we show that $\mathrm{Sub}_4 S^d$ has the rational homology of $S^{4d} \vee S^{3d}$. To that end we will use the MVS associated to the pushouts in diagram (4.2.5). The first non trivial term in the MVS associated to the middle pushout is

$$0 \longrightarrow H_{4d}(\mathrm{SP}^4 S^d) \longrightarrow H_{4d}(\mathrm{Sub}_4 S^d) \longrightarrow 0$$

which gives that $H_{4d}(\mathrm{Sub}_4 S^d) = \mathbb{Q}$. The second non trivial term is

$$0 \longrightarrow H_{3d+1}(\mathrm{Sub}_4 S^d) \longrightarrow H_{3d}(B_2(S^d, 4)) \xrightarrow{f_* \oplus g_*} H_{3d}(\mathrm{SP}^4 S^d) \oplus H_{3d}(\mathrm{Sub}_3 S^d) \longrightarrow H_{3d}(\mathrm{Sub}_4 S^d) \longrightarrow 0$$

Now recall that

$$H_{3d}(B_2(S^d, 4)) = \mathbb{Q}\{\delta_*(u_1 \otimes u_2)\}$$

so that

$$f_* \oplus g_* : \mathbb{Q}\{\delta_*(u_1 \otimes u_2)\} \longrightarrow \mathbb{Q}\{u_3\} \oplus \mathbb{Q}\{\mu_*(u_3)\}$$

The map

$$f \circ \delta : S^d \times \mathrm{SP}^2 S^d \longrightarrow \mathrm{SP}^4 S^d \ , \ (x, [y, z]) \longmapsto [x, x, y, z]$$

is the same as

$$f \circ \delta : \quad S^d \times \mathrm{SP}^2 S^d \longrightarrow S^d \times S^d \times \mathrm{SP}^2 S^d \longrightarrow \mathrm{SP}^2 S^d \times \mathrm{SP}^2 S^d \longrightarrow \mathrm{SP}^4 S^d \qquad (4.4.1)$$

$$(x,[y,z]) \longmapsto (x,x,[y,z]) \longmapsto ([x,x],[y,z]) \longmapsto [x,x,y,z]$$

Its effect in homology is given by $(f\circ\delta)_*(u_1\otimes u_2) = 6u_3$. Similarly,

$$g\circ\delta : S^d \times \mathrm{SP}^2 S^d \longrightarrow \mathrm{Sub}_3 S^d \ , \ (x,[y,z]) \longmapsto \{x,y,z\}$$

is the same as

$$g\circ\delta : S^d \times \mathrm{SP}^2 S^d \longrightarrow \mathrm{SP}^3 S^d \xrightarrow{\mu} \mathrm{Sub}_3 S^d \ , \ (x,[y,z]) \longmapsto [x,y,z] \longmapsto \{x,y,z\} \quad (4.4.2)$$

which in homology sends $u_1 \otimes u_2$ to $3\mu_*(u_3)$. In conclusion,

$$f_*\oplus g_* : \mathbb{Q}\{\delta_*(u_1\otimes u_2)\} \longrightarrow \mathbb{Q}\{u_3\}\oplus\mathbb{Q}\{\mu_*(u_3)\} \ , \ \delta_*(u_1\otimes u_2) \longmapsto 6u_3 \oplus 3\mu_*(u_3)$$

which is injective and has cokernel isomorphic to $\mathbb{Q}$.

The second non trivial term in the MVS is

$$0 \longrightarrow H_{2d+1}(\mathrm{Sub}_4 S^d) \longrightarrow H_{2d}(B_2(S^d,4)) \xrightarrow{f_*\oplus g_*} H_{2d}(\mathrm{SP}^4 S^d) \oplus H_{2d}(\mathrm{Sub}_3 S^d) \longrightarrow H_{2d}(\mathrm{Sub}_4 S^d) \longrightarrow 0$$

where

$$f_*\oplus g_* : \mathbb{Q}\{\delta_*(1\otimes u_2)\}\oplus\mathbb{Q}\{\lambda_*(u_2)\} \longrightarrow \mathbb{Q}\{u_2\}\oplus\mathbb{Q}\{\mu_*(u_2)\}$$

such that

$$f_*\circ\delta_*(1\otimes u_2) = u_2 \ \text{ and } \ g_*\circ\delta_*(1\otimes u_2) = \mu_*(u_2)$$

On the other hand, the map

$$f\circ\lambda : \mathrm{SP}^2 S^d \longrightarrow \mathrm{SP}^4 S^d \ , \ [x,y] \longmapsto [x,x,y,y]$$

is the same as

$$f\circ\lambda : \mathrm{SP}^2 S^d \longrightarrow \mathrm{SP}^2 S^d \times \mathrm{SP}^2 S^d \longrightarrow \mathrm{SP}^4 S^d \ , \ [x,y] \longmapsto ([x,y],[x,y]) \longmapsto [x,x,y,y] \quad (4.4.3)$$

which in homology sends u_2 to $4u_2$ since by (4.2.13) the coproduct on $u_2 \in H_{2d}(\mathrm{SP}^2 S^d)$ is $u_2\otimes 1 + u_1\otimes u_1 + 1\otimes u_2$.

Consider the pushout (4.2.5) again. We have that

$$g_*\circ\lambda_*(u_2) = \tau_*(u_2)$$

On the other hand, the relation $\tau_* \circ \pi_*(u_1 \otimes u_1) = \mu_* \circ l_*(u_1 \otimes u_1)$ gives that

$$2\tau_*(u_2) = 4\mu_*(u_2)$$

Hence rationally

$$g_* \circ \lambda_*(u_2) = 2\mu_*(u_2)$$

Since $\delta \circ h = \lambda \circ \pi$, we have that

$$g_* \circ \lambda_*(u_2) = g_* \circ \delta_*(u_1 \otimes u_1) = 2\mu_*(u_2) \tag{4.4.4}$$

To sum up, the homomorphism

$$f_* \oplus g_* : \mathbb{Q}\{\delta_*(1 \otimes u_2)\} \oplus \mathbb{Q}\{\lambda_*(u_2)\} \longrightarrow \mathbb{Q}\{u_2\} \oplus \mathbb{Q}\{\mu_*(u_2)\}$$

sends

$$\delta_*(1 \otimes u_2) \longmapsto u_2 \oplus \mu_*(u_2) \ \text{ and } \ \lambda_*(u_2) \longmapsto 4u_2 \oplus 2\mu_*(u_2)$$

which is equivalent to the map

$$\mathbb{Q} \oplus \mathbb{Q} \longrightarrow \mathbb{Q} \oplus \mathbb{Q}\ , \quad (1,0) \longmapsto (1,1) \ \text{ and } \ (0,1) \longmapsto (4,2)$$

which is an isomorphism.

The last non trivial term in the MVS is

$$0 \longrightarrow H_{d+1}(\mathrm{Sub}_4 S^d) \longrightarrow H_d(B_2(S^d,4)) \xrightarrow{f_* \oplus g_*} H_d(\mathrm{SP}^4 S^d) \oplus H_d(\mathrm{Sub}_3 S^d) \longrightarrow H_d(\mathrm{Sub}_4 S^d) \longrightarrow 0$$

Since $H_d(\mathrm{Sub}_3 S^d) = 0$ and $f_* : H_d(B_2(S^d,4)) \longrightarrow H_d(\mathrm{SP}^4 S^d)$ is an isomorphism (Lemma 4.3.2), then $H_d(\mathrm{Sub}_4 S^d) = 0$.

To show that $\mathrm{Sub}_4 S^d$ and $S^{4d} \vee S^{3d}$ have the same rational homotopy type, we need to exhibit a weak homotopy equivalence $(S^{4d} \vee S^{3d})_{\mathbb{Q}} \longrightarrow (\mathrm{Sub}_4 S^d)_{\mathbb{Q}}$ between their corresponding rationalizations.

We construct $(S^{3d})_{\mathbb{Q}} \longrightarrow (\mathrm{Sub}_4 S^d)_{\mathbb{Q}}$ from a map $\zeta : S^{3d} \longrightarrow (\mathrm{Sub}_4 S^d)_{\mathbb{Q}}$ whose existence is a direct consequence of the rational Hurewicz theorem since

$$\pi_{3d}((\mathrm{Sub}_4 S^d)_{\mathbb{Q}}) = \pi_{3d}(\mathrm{Sub}_4 S^d) \otimes \mathbb{Q} = H_{3d}(\mathrm{Sub}_4 S^d; \mathbb{Q}) = \mathbb{Q}$$

However we can be more explicit and show that ζ is induced from the composite

$$f : (S^d)^3 \longrightarrow \mathrm{Sub}_3 S^d \hookrightarrow \mathrm{Sub}_4 S^d \longrightarrow (\mathrm{Sub}_4 S^d)_{\mathbb{Q}}$$

where the first map is the projection $(x, y, z) \longmapsto \{x, y, z\}$. Indeed, by rational Hurewicz theorem

$$\pi_k((\mathrm{Sub}_4 S^d)_{\mathbb{Q}}) = \pi_k(\mathrm{Sub}_4 S^d) \otimes \mathbb{Q} = \tilde{H}_k(\mathrm{Sub}_4 S^d; \mathbb{Q}) = 0, \text{ for } k = 0 \mathinner{..} 2d$$

so f factors through the quotient of $(S^d)^3$ by its $2d-$skeleton, but the $2d-$skeleton of $(S^d)^3$ is precisely the fat wedge consisting of triples $(x_1, x_2, x_3) \in (S^d)^3$ where at least one x_i is the base point of S^d. Now this quotient is the smash product $S^d \wedge S^d \wedge S^d = S^{3d}$ and we get a homotopy commutative diagram

$$\begin{tikzcd}
(S^d)^3 \arrow[r] \arrow[d] & \mathrm{Sub}_3 S^d \arrow[r, hook] & \mathrm{Sub}_4 S^d \arrow[r] & (\mathrm{Sub}_4 S^d)_{\mathbb{Q}} \\
S^{3d} \arrow[urrr] & & &
\end{tikzcd}$$

It follows by previous homology computations that the constructed map $S^{3d} \longrightarrow (\mathrm{Sub}_4 S^d)_{\mathbb{Q}}$ induces an isomorphism on $H_{3d}(-; \mathbb{Q})$.

Now we treat the case of dimension $4d$. The space $\mathrm{Sub}_4 S^d$ is $d+1$ connected so by rational Hurewicz theorem ([24], Theorem 1.1), the map

$$\pi_i(\mathrm{Sub}_4 S^d) \otimes \mathbb{Q} \longrightarrow H_i(\mathrm{Sub}_4 S^d; \mathbb{Q})$$

is an isomorphism for $i = 1 \mathinner{..} 2d+2$. By previous homology computations, this means that $\pi_i(\mathrm{Sub}_4 S^d) \otimes \mathbb{Q} = 0$ for $i = 1 \mathinner{..} 2d+2$. By rational Hurewicz theorem again we have that

$$\pi_i(\mathrm{Sub}_4 S^d) \otimes \mathbb{Q} \longrightarrow H_i(\mathrm{Sub}_4 S^d; \mathbb{Q})$$

is an isomorphism for $i = 1 \mathinner{..} 4d+4$. In particular, in dimension $4d$ the Hurewicz map

$$\pi_{4d}(\mathrm{Sub}_4 S^d) \otimes \mathbb{Q} \longrightarrow H_{4d}(\mathrm{Sub}_4 S^d; \mathbb{Q})$$

is an isomorphism. It follows that

$$\pi_{4d}((\mathrm{Sub}_4 S^d)_{\mathbb{Q}}) = \pi_{4d}(\mathrm{Sub}_4 S^d) \otimes \mathbb{Q} = H_{4d}(\mathrm{Sub}_4 S^d; \mathbb{Q}) = \mathbb{Q}$$

which means that the class in dimension $4d$ of $(\mathrm{Sub}_4 S^d)_{\mathbb{Q}}$ is spherical.

In conclusion, there is a map $\theta : S^{4d} \vee S^{3d} \longrightarrow (\mathrm{Sub}_4 S^d)_{\mathbb{Q}}$ that induces isomorphism on rational homology. By rational Whitehead theorem, θ is a rational homotopy equivalence, hence the induced rationalization map $\theta_{\mathbb{Q}} : (S^{4d} \vee S^{3d})_{\mathbb{Q}} \longrightarrow ((\mathrm{Sub}_4 S^d)_{\mathbb{Q}})_{\mathbb{Q}}$ is a weak homotopy equivalence. Now the claim follows from the fact that the rationalization of

$(\mathrm{Sub}_4 S^d)_{\mathbb{Q}}$ is $(\mathrm{Sub}_4 S^d)_{\mathbb{Q}}$ itself. ∎

Remark 4.4.7 Note that a map $S^{4d} \longrightarrow \mathrm{Sub}_4 S^d$ that induces an isomorphism on rational H_{4d} cannot factor through the projection $\mathrm{SP}^4 S^d$, since for example when $d = 2$, $\mathrm{SP}^4 S^2 = \mathbb{P}^4$ and there are no essential map from S^8 to $\mathbb{P}^4$.

Next we give an integral homology calculation hence re-deriving a calculation of $H_*(\mathrm{Sub}_4 S^2; \mathbb{Z})$ by Tuffley [38] which will be useful in §4.5. We use a MVS associated to the pushouts (4.2.5) to do this computation . This is different from Tuffley's method who uses cellular decomposition.

Lemma 4.4.8

$$\tilde{H}_*(Sub_4 S^2; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \textit{if } * = 8 \\ \mathbb{Z} \oplus \mathbb{Z}_3 & \textit{if } * = 6 \\ \mathbb{Z}_2 & \textit{if } * = 4 \end{cases} \tag{4.4.5}$$

Proof. In dimension 8, $H_8(\mathrm{Sub}_4 S^2)$ is obviously isomorphic to $H_8(\mathrm{SP}^4 S^2) \cong H_8(\mathbb{P}^4) \cong \mathbb{Z}$.

To determine $H_6(\mathrm{Sub}_4 S^2)$ we analyse the map

$$f_* \oplus g_* : H_6(B_2(S^2, 4)) \longrightarrow H_6(\mathrm{SP}^4 S^2) \oplus H_6(\mathrm{Sub}_3 S^2)$$

which is a map $\mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$. The group $\mathbb{Z} \cong H_6(B_2(S^2, 4))$ is generated by the top class in $S^2 \times \mathrm{SP}^2 S^2$, so we have to understand the effect of the maps $f \circ \delta$ and $g \circ \delta$ on H_6 in light of (4.4.1) and (4.4.2) respectively. The first map is multiplication by 6 because it is the image of $(\gamma_1 \otimes 1 + 1 \otimes \gamma_1) \otimes \gamma_2$ which is $2\binom{3}{1}\gamma_3 = 6\gamma_3$. The second map is multiplication by 3, using the fact that on H_6 the map $\mathrm{SP}^3 S^2 \longrightarrow \mathrm{Sub}_3 S^2$ is an isomorphism. This then means that $f_* \oplus g_*$ is injective and that

$$H_6(\mathrm{Sub}_4 S^2; \mathbb{Z}) = coker \begin{pmatrix} \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \\ 1 \longmapsto (6, 3) \end{pmatrix}$$

and this is $\mathbb{Z} \oplus \mathbb{Z}_3$.

Now we determine $H_4(\mathrm{Sub}_4 S^2; \mathbb{Z})$ using the MVS associated to the pushout (4.2.2). To do this, we need to determine $H_4(B_2(S^2, 4); \mathbb{Z})$ and $H_4(\mathrm{Sub}_3 S^2; \mathbb{Z})$. The latter is determined in Lemma 4.2.8. The group $H_4(B_2(S^2, 4); \mathbb{Z}) = coker(h_* \oplus \pi_*)$, where

$$\mathbb{Z}\{\gamma_1 \otimes \gamma_1\} \xrightarrow{h_* \oplus \pi_*} \mathbb{Z}\{\gamma_1 \otimes \gamma_1\} \oplus \mathbb{Z}\{1 \otimes \gamma_2\} \oplus \mathbb{Z}\{\gamma_2\}$$

sending

$$\gamma_1 \otimes \gamma_1 \longmapsto 2(\gamma_1 \otimes \gamma_1) \oplus 2\gamma_2$$

so the cokernel

$$H_4(B_2(S^2,4)) = \langle \delta_*(\gamma_1 \otimes \gamma_1),\ \delta_*(1 \otimes \gamma_2),\ \lambda_*(\gamma_2) \mid 2\delta_*(\gamma_1 \otimes \gamma_1) = 2\lambda_*(\gamma_2) \rangle$$

which we identify with $\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}_2$ by rearranging generators and relations in the following way

$$H_4(B_2(S^2,4)) = \langle\, \delta_*(\gamma_1 \otimes \gamma_1),\ \delta_*(1 \otimes \gamma_2),\ \delta_*(\gamma_1 \otimes \gamma_1) - \lambda_*(\gamma_2) \mid 2[\delta_*(\gamma_1 \otimes \gamma_1) - \lambda_*(\gamma_2)] = 0 \,\rangle$$

Now $H_4(\mathrm{Sub}_4 S^2; \mathbb{Z}) = coker(f_* \oplus g_*)$, where

$$f_* \oplus g_* : \mathbb{Z}\{\delta_*(\gamma_1 \otimes \gamma_1)\} \oplus \mathbb{Z}\{\delta_*(1 \otimes \gamma_2)\} \oplus \mathbb{Z}_2\{\delta_*(\gamma_1 \otimes \gamma_1) - \lambda_*(\gamma_2)\}$$

$$\longrightarrow \mathbb{Z}\{\gamma_2\} \oplus \mathbb{Z}\{\mu_*(\gamma_2)\} \oplus \mathbb{Z}_2\{2\mu_*(\gamma_2) - \tau_*(\gamma_2)\}$$

By (4.4.1) we have that

$$f_* \circ \delta_*(\gamma_1 \otimes \gamma_1) = 4\gamma_2 \ \text{ and } \ f_* \circ \delta_*(1 \otimes \gamma_2) = \gamma_2$$

Similarly, by (4.4.2) we have that

$$g_* \circ \delta_*(\gamma_1 \otimes \gamma_1) = 2\mu_*(\gamma_2) \ \text{ and } \ g_* \circ \delta_*(1 \otimes \gamma_2) = \mu_*(\gamma_2)$$

By (4.4.3), $f_* \circ \lambda_*(\gamma_2) = 4\gamma_2$. On the other hand we have that $g_* \circ \lambda_*(\gamma_2) = \tau_*(\gamma_2)$.

In conclusion, $H_4(\mathrm{Sub}_4 S^2; \mathbb{Z}) = coker(f_* \oplus g_*)$ where

$$f_* \oplus g_* : \mathbb{Z}\{\delta_*(\gamma_1 \otimes \gamma_1)\} \oplus \mathbb{Z}\{\delta_*(1 \otimes \gamma_2)\} \oplus \mathbb{Z}_2\{\delta_*(\gamma_1 \otimes \gamma_1) - \lambda_*(\gamma_2)\}$$

$$\longrightarrow \mathbb{Z}\{\gamma_2\} \oplus \mathbb{Z}\{\mu_*(\gamma_2)\} \oplus \mathbb{Z}_2\{2\mu_*(\gamma_2) - \tau_*(\gamma_2)\}$$

sending

$$\begin{aligned} &\delta_*(\gamma_1 \otimes \gamma_1) \longmapsto 4\gamma_2 \oplus 2\mu_*(\gamma_2), \\ &\delta_*(1 \otimes \gamma_2) \longmapsto \gamma_2 \oplus \mu_*(\gamma_2), \\ &\delta_*(\gamma_1 \otimes \gamma_1) - \lambda_*(\gamma_2) \longmapsto 2\mu_*(\gamma_2) - \tau_*(\gamma_2) \end{aligned}$$

and then the cokernel $H_4(\mathrm{Sub}_4 S^2)$ is the group generated by

$$\alpha_*(\gamma_2), \ \ \beta_* \circ \mu_*(\gamma_2), \ \ 2\beta_* \circ \mu_*(\gamma_2) - \beta_* \circ \tau_*(\gamma_2)$$

modulo the following relations

$$\begin{aligned}
&4\alpha_*(\gamma_2) = 2\beta_* \circ \mu_*(\gamma_2),\\
&\alpha_*(\gamma_2) = \beta_* \circ \mu_*(\gamma_2),\\
&2\beta_* \circ \mu_*(\gamma_2) - \beta_* \circ \tau_*(\gamma_2) = 0,\\
&2[2\beta_* \circ \mu_*(\gamma_2) - \beta_* \circ \tau_*(\gamma_2)] = 0
\end{aligned}$$

which we identify with $\mathbb{Z}_2$ by writing $H_4(\mathrm{Sub}_4 S^2) = \langle\, \alpha_*(\gamma_2) \mid 2\alpha_*(\gamma_2) = 0\,\rangle$ ∎

4.5 The Concatenation Pairing

By taking union of points we obtain pairings

$$\mu_{r,s} : \mathrm{Sub}_r X \times \mathrm{Sub}_s X \longrightarrow \mathrm{Sub}_{r+s} X \;\;,\; r+s \leq 4$$

which we refer to as the concatenation pairings. We wish to investigate the effect in homology of these pairings for spheres.

In the case X is an odd sphere, this pairing is necessarily trivial. Next we study the case of X is the 2-sphere. Here is a first observation.

Lemma 4.5.1 *Consider*

$$\mu_{2,2} : Sub_2 S^2 \times Sub_2 S^2 \longrightarrow Sub_4 S^2$$

and write $H_4(Sub_2 S^2;\mathbb{Z}) = \mathbb{Z}\{a\}$ *and* $H_8(Sub_4 S^2;\mathbb{Z}) = \mathbb{Z}\{z\}$. *Then*

$$(\mu_{2,2})_*(a \otimes a) = 6z$$

Proof. Since $\mathrm{Sub}_2 S^2 = \mathrm{SP}^2 S^2 = \mathbb{P}^2$, then $H_8(\mathrm{Sub}_2 S^2 \times \mathrm{Sub}_2 S^2) = H_4(\mathrm{Sub}_2 S^2) \otimes H_4(\mathrm{Sub}_2 S^2)$. Hence the induced map in homology at dimension 8 is

$$(\mu_{2,2})_* : H_4(\mathrm{Sub}_2 S^2) \otimes H_4(\mathrm{Sub}_2 S^2) \longrightarrow H_8(\mathrm{Sub}_4 S^2)$$

Observe that the following diagram commutes

$$\begin{tikzcd}
\mathrm{SP}^2 S^2 \times \mathrm{SP}^2 S^2 \arrow[r] \arrow[d] & \mathrm{SP}^4 S^2 \arrow[d] \\
\mathrm{Sub}_2 S^2 \times \mathrm{Sub}_2 S^2 \arrow[r] & \mathrm{Sub}_4 S^2
\end{tikzcd}$$

where $\mathrm{Sub}_2 S^2 \cong \mathrm{SP}^2 S^2$ and $H_8(\mathrm{Sub}_4 S^2) \cong H_8(\mathrm{SP}^4 S^2)$, hence $(\mu_{2,2})_*$ is the same as the top pairing $H_4(\mathbb{P}^2) \otimes H_4(\mathbb{P}^2) \longrightarrow H_8(\mathbb{P}^4)$ which is a pairing of divided power algebra. The generator $a = \gamma_2$ and so $(\gamma_2 \otimes \gamma_2) \longmapsto \binom{4}{2}\gamma_4 = 6\gamma_4 = 6z$. ∎

Consider the diagram of pushouts as in (4.2.5) in the special case of the 2-sphere S^2. We want to analyse the map induced by α

$$\tilde{H}_*(\mathrm{SP}^4 S^2; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } * = 8 \\ \mathbb{Z} & \text{if } * = 6 \\ \mathbb{Z} & \text{if } * = 4 \\ \mathbb{Z} & \text{if } * = 2 \end{cases} \xrightarrow{H(\alpha)} \tilde{H}_*(\mathrm{Sub}_4 S^2; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } * = 8 \\ \mathbb{Z} \oplus \mathbb{Z}_3 & \text{if } * = 6 \\ \mathbb{Z}_2 & \text{if } * = 4 \end{cases} \tag{4.5.1}$$

As well as the map

$$\tilde{H}_*(\mathrm{Sub}_3 S^2; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } * = 6 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & \text{if } * = 4 \end{cases} \xrightarrow{H(\beta)} \tilde{H}_*(\mathrm{Sub}_4 S^2; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } * = 8 \\ \mathbb{Z} \oplus \mathbb{Z}_3 & \text{if } * = 6 \\ \mathbb{Z}_2 & \text{if } * = 4 \end{cases} \tag{4.5.2}$$

induced from the inclusion $\beta : \mathrm{Sub}_3 S^2 \hookrightarrow \mathrm{Sub}_4 S^2$.

We write $H_i(\alpha)$ the map $H_i(\mathrm{SP}^4 S^2) \longrightarrow H_i(\mathrm{Sub}_4 S^2)$. Similarly for $H_i(\beta)$.

Because of connectivity, $H_2(\alpha) = 0$. Because the homological dimension for $B_2(S^2, 4)$ and $\mathrm{Sub}_3 S^2$ is 6, $H_8(\alpha)$ is an isomorphism. It remains to determine $H_6(\alpha)$, $H_4(\alpha), H_6(\beta)$ and $H_4(\beta)$. We get the following result

Proposition 4.5.2

$$H_4(\alpha) : \mathbb{Z}\{a\} \longrightarrow \mathbb{Z}_2\{t\}; \;\; a \longmapsto t$$

$$H_4(\beta) : \mathbb{Z}\{b\} \oplus \mathbb{Z}_2\{c\} \longrightarrow \mathbb{Z}_2\{t\}; \;\; b \longmapsto t \;\;, \;\; c \longmapsto 0$$

And

$$H_6(\alpha) : \mathbb{Z}\{a\} \longrightarrow \mathbb{Z}_2\{t\} \oplus \mathbb{Z}_3\{s\}; \;\; a \longmapsto t$$

$$H_6(\beta) : \mathbb{Z}\{b\} \longrightarrow \mathbb{Z}\{t\} \oplus \mathbb{Z}_3\{s\}; \;\; b \longmapsto 2t - s$$

Proof. To find out about the map $H_6(\mathrm{SP}^4 S^2) = \mathbb{Z} \longrightarrow H_6(\mathrm{Sub}_4 S^2) = \mathbb{Z} \oplus \mathbb{Z}_3$, we identify $H_6(\mathrm{Sub}_4 S^2; \mathbb{Z})$ with $coker(f_* \oplus g_*)$ (see the pushout (4.2.2)) and we analyse the quotient map

$$H_6(\mathrm{SP}^4 S^2) \oplus H_6(\mathrm{Sub}_3 S^2) \xrightarrow{q} H_6(\mathrm{Sub}_4 S^2)$$

$$\mathbb{Z} \oplus \mathbb{Z} \quad \longrightarrow \quad \mathbb{Z} \oplus \mathbb{Z}_3$$

We need to label the generators on the left hand of q, say a and b. We will write the images of a and b in the cokernel $H_6(\mathrm{Sub}_4 S^2)$ as $\bar{a}$ and $\bar{b}$ respectively. Then $H_6(\mathrm{Sub}_4 S^2)$ is the abelian group generated by $\bar{a}$ and $\bar{b}$ subject to the relation

$$6\bar{a} = 3\bar{b}$$

which we identify with $\mathbb{Z} \oplus \mathbb{Z}_3$ by choosing new generators $(t, s) := (\bar{a}, 2\bar{a} - \bar{b})$. This means that the map

$$q : \mathbb{Z}\{a\} \oplus \mathbb{Z}\{b\} \longrightarrow \mathbb{Z}\{t\} \oplus \mathbb{Z}_3\{s\}$$

sends $q(a) = t$ and $q(b) = 2t - s$. Hence

$$H_6(\alpha) : \mathbb{Z}\{a\} \longrightarrow \mathbb{Z}\{t\} \oplus \mathbb{Z}_3\{s\};\ a \longmapsto t$$

and

$$H_6(\beta) : \mathbb{Z}\{b\} \longrightarrow \mathbb{Z}\{t\} \oplus \mathbb{Z}_3\{s\}; \quad b \longmapsto 2t - s$$

In light of computations made in the proof of Lemma 4.4.8, we determine $H_4(\alpha)$ and $H_4(\beta)$ in exactly the same way. ∎

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