SOME REMARKS ON SYMMETRIC PRODUCTS OF CURVES

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Abstract. Symmetric products of curves are important spaces for both geometers and topologists, and increasingly useful objects for physicists. We summarize below some of their basic homotopy theoretic properties and derive a handful of known and less well-known results about them.

We combine in a slightly leisurely way both geometry and topology to describe some useful properties of symmetric products of algebraic curves. We record a straightforward derivation of Clifford's theorem from a calculation of MacDonald, point to an equally simple characterization of hyperelliptic curves and discuss the embeddability (both in the continuous and holomorphic categories) of the unique spherical generator in dimension two in the homology of these spaces. A homotopy retract statement about the Abel-Jacobi map is also proven.

1. Cohomology Structure and Clifford's Theorem

Given a complex algebraic curve C and $n \geq 1$, the n-th symmetric product of C is the quotient $C^{(n)} = C^n/\Sigma_n$, where Σ_n is the symmetric group acting on C^n by permuting coordinates[1]. Elements in $C^{(n)}$ are referred to as *effective divisors* on C. A point $D \in C^{(n)}$ is said to have degree n and we can write it as a formal linear combination $\sum n_i x_i$ where $x_i \neq x_j \in C$ for $i \neq j$, and n_i are positive integers with $\sum n_i = n$.

1.1. MacDonald and Clifford's theorems. It it assumed well-known that $C^{(n)}$ is a complex (smooth) algebraic variety for all positive n. The n-th Abel-Jacobi map is an algebraic map

$$\mu_n : C^{(n)} \longrightarrow J(C)$$

where $J(C)$ is the "Jacobian" of C (a complex torus of dimension the genus of C). It is additive in the sense that the following commutes

$$\begin{array}{ccc} C^{(r)} \times C^{(s)} & \longrightarrow & C^{(r+s)} \\ \big\downarrow{\scriptstyle \mu_r \times \mu_s} & & \big\downarrow{\scriptstyle \mu_{r+s}} \\ J(C) \times J(C) & \longrightarrow & J(C) \end{array}$$

where the bottom map is addition in the abelian torus $J(C)$, and the top map is concatenation of points (which is also an abelian pairing). If C is an elliptic curve for example, then $J(C) \cong C$. The inverse preimages of μ_n are complex projective spaces $\mathbb{P}^m$, where m is related to the dimension of some complete linear series on C(cf. [1]). The dimension of the preimages $\mu_n^{-1}(x), x \in J(C)$ is an upper semi-continuous function of x. When $n \geq 2g$, the dimension is constant for given n and equals $n - g$ (this is a direct consequence of Riemann Roch for curves). In fact, Mattuck shows that μ_n is a projectivized analytic bundle projection with fiber $\mathbb{P}^{n-g}$. The fiber being Kahler

[1] In the algebraic topology literature, this is often written $\mathrm{SP}^n(C)$.

and the coefficients being simple (as can be checked), it follows by a theorem of Blanchard that this fibration is homologically trivial and

$$H_*(C^{(n)};\mathbb{Z}) \cong H_*(\mathbb{P}^{n-g}) \otimes H_*(S^1)^{\otimes 2g} \quad , \quad n \geq 2g$$

here of course we use the fact that $J(C) \simeq (S^1)^{2g}$. In fact we will show that $J(C)$ is a homotopy retract of $C^{(n)}$ for $n \geq 2g$ (section 1.3). The situation is less clear for $2 \leq n < 2g$ since the fibers can jump up in dimension. This jump is however well controlled (see corollary 1.3).

The following main theorem of MacDonald describes the cohomology of $C^{(n)}$ for all n. As it turns out, these spaces have interesting cup products.

Theorem 1.1. *(I.G. Macdonald) Let $e_i^* \in H^1(C)$ be the one dimensional generators, $1 \leq i \leq 2g$ and $b^* \in H^2(C)$ the orientation class. Then the cohomology ring of $C^{(n)}$ over the integers $\mathbf{Z}$ is generated by the e_i^* and b^* subject to the following relations:*
(i) The e_i^*'s anti-commute with each other and commute with b^*;
(ii) If $i_1, \ldots, i_a, j_1, \ldots, j_b, k_1, \ldots, k_c$ are distinct integers from 1 to g inclusive, then

$$e_{2i_1-1}^* \cdots e_{2i_a-1}^* e_{2j_1}^* \cdots e_{2j_b}^* (e_{2k_1-1}^* e_{2k_1}^* - b^*) \cdots (e_{2k_c-1}^* e_{2k_c}^* - b^*)(b^*)^q = 0$$

provided that $a + b + 2c + q = n + 1$.
If $n < 2g$ all the relations above are consequences of those for which $q = 0, 1$, and if $n > 2g - 2$ all the relations are consequences of the single relation

$$(b^*)^{n-2g+1} \prod_{i=1}^{g} (e_{2i-1}^* e_{2i}^* - b^*) = 0$$

In other words $H^*(C^{(n)})$ is the quotient of $H^*(J;\mathbb{Z})[b^*] = E(e_1^*, \ldots, e_{2g}^*) \otimes \mathbb{Z}[b^*]$ by the above relations, where E stands for an exterior algebra. In the first part of this note we describe the geometry behind MacDonald's theorem and give some simple manipulations and applications of the homology of symmetric products. For example a rather pleasant application is the derivation, in purely topological terms, of a pivotal theorem in the geometry of algebraic curves due to Clifford.

Proposition 1.2. *Let C be a closed oriented topological surface of genus $g > 0$, and $\mathbb{P}^m \longrightarrow C^{(n)}$ a map that is non-zero on second homology groups. Id*
(i) $n < 2g$, then necessarily $m \leq \left[\frac{n}{2}\right]$,
(ii) and if $n \geq 2g$, then $m \leq n - g$.

Proof: Suppose $m = [\frac{n}{2}] + 1$ and $n < 2g$ and suppose there is a continuous $h : \mathbb{P}^m \longrightarrow C^{(n)}$ with $h^*(b^*) \neq 0$ (the induced map in cohomology). Then necessarily $h^*(b^*)^m \neq 0$ by the ring structure of projective space. On the other hand, $h^*(e_i^*) = 0$ and hence

$$h^*(b^{*m}) = h^*(\prod_{k=1}^{m} (b^* - e_{2k-1}^* e_{2k}^*)) = 0$$

using the relation of MacDonalds theorem 1.1. This contradicts the choice of m. One uses a similar argument for (ii). ∎

Corollary 1.3. *(Clifford) Suppose $0 \leq n \leq 2g - 1$, and pick $x \in J(C)$. Write $\mu_n^{-1}(x) = \mathbb{P}^m$ for some m. Then necessarily $m \leq \frac{n}{2}$.*

Remark 1.4. It is well-known that when C is hyperelliptic, $C^{(2)}$ is the blowup at a point of $W_2 = \mu(C^{(2)}) \subset J(C)$. The exceptional fiber is $\mathbb{P}^1 \subset C^{(2)}$ and hence for all m one has an essential map $(\mathbb{P}^1)^{(m)} = \mathbb{P}^m \longrightarrow C^{(2m)}$. The bound for Clifford's theorem is clearly sharp.

Remark 1.5. The class $\theta := \sum_1^g e_{2i-1}^*e_{2i}^* \in H^2(J;\mathbb{Z})$ is particularly important since it is Poincaré dual to the class of the divisor $W_{g-1} := \mu[C^{(g-1)}]$. This (or a translate of it) is known as the Θ-divisor ([1], chap I, §4).

1.2. Homology. Alternatively, C being a genus g closed topological surface, one can express the homology of $C^{(n)}$ as a "truncated Pontryagin ring" (as in [7] for instance). To be more explicit, write

$$\tilde{H}_*(C;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}\{b\}, & \text{when } * = 2 \text{ with generator } b \\ \mathbb{Z}^{2g}\{e_1,\dots,e_{2g}\}, & \text{when } * = 1 \end{cases}$$

The generators e_i are spherical classes corresponding to the inclusion of the one skeleton $\bigvee_1^{2g} S^1$ in C. It will be often convenient to appeal to the cofibration sequence describing C

$$\bigvee^{2g} S^1 \xrightarrow{\ i\ } C \xrightarrow{\ p\ } S^2 \tag{1}$$

We choose the e_i's so that representative cycles of e_{2i-1} and e_{2i} intersect transversally at a point (e_i and e_j are disjoint otherwise). These generators have (canonical) duals b^* and e_i^* (b^* is the pull-back of the generator under the quotient map $C \longrightarrow S^2$). This as we know implies that $H^*(C;\mathbb{Z})$ is generated by $b^*, e_1^*, \dots, e_{2g}^*$ with the relation $b^* = e_{2i-1}^*e_{2i}^*$ (and graded commutativity).

The following, of which we give a short proof, is a well-known property of symmetric products. Choose once and for all a basepoint $x_0 \in C$.

Lemma 1.6. *The basepoint inclusion $i_n : C^{(n)} \longrightarrow C^{(n+1)}$, $D \mapsto D + x_0$ induces a homology monomorphism.*

PROOF: A class $\alpha \in H^i(C^{(n)})$ is determined by a map $f : C^{(n)} \longrightarrow K(\mathbb{Z}, i)$ into an Eilenberg-MacLane space. Composition gives a map $g : C \hookrightarrow C^{(n)} \longrightarrow K(\mathbb{Z}, i)$. Since $K(\mathbb{Z}, i)$ can be chosen to be an abelian monoid[2], we get a map $f \times g : C \times C^{(n)} \longrightarrow K(\mathbb{Z}, i)$ and since this monoid is moreover commutative, we further get a map $\beta : C^{(n+1)} \longrightarrow K(\mathbb{Z}, i)$. This represents an element in $H^i(C^{(n+1)})$ and by construction $i^*(\beta) = \alpha$. This shows that i^* is surjective, and hence i_* is injective. ∎

Remark 1.7. In fact a more general theorem of Steenrod ([8], section 2.3) shows that the maps i_* are split embeddings in homology. This statement is valid for any connected simplicial complex X and an elementary illustration can be found in lemma 1.14.

Note that the multiplication $C^{(r)} \times C^{(s)} \longrightarrow C^{(r+s)}$ given by concatenation of points, has known effect in homology. For instance the covering map $\pi : C^n \longrightarrow C^{(n)}$ corresponds to the iterated multiplication $(C^{(1)})^n \longrightarrow C^{(n)}$, and under π the class $b^{\otimes n}$ maps to the orientation class of $C^{(n)}$ (since this is an oriented manifold); that is

$$\pi_* : [C]^{\otimes n} \mapsto n![C^{(n)}]$$

where $n! = \deg\pi$. It is convenient to write in this case $\gamma_n := [C^{(n)}]$. Write

$$\cdot : H_i(C^{(r)}) \otimes H_j(C^{(s)}) \longrightarrow H_{i+j}(C^{(r+s)})$$

the corresponding Pontryagin product. Under this product, we see that $b \cdot b = 2\gamma_2$, and more generally we have

[2] In fact $K(\mathbb{Z}, i)$ is the infinite symmetric product of the sphere S^i.

Proposition 1.8. [8] *$H_*(C^{(n)})$ has generators $\gamma_i = [C^{(i)}]$, e_i, $1 \leq i \leq 2g$, and all products of such $e_{i_1} \cdot \ldots \cdot e_{i_k} \cdot \gamma_j$, $i_1 < i_2 < \ldots < i_k$, of length at most n (i.e. $k + j \leq n$). The classes γ verify $\gamma_i \cdot \gamma_j = \binom{i+j}{i} \gamma_{i+j}$. When $n \geq 2g$, the exterior algebra $E(e_1, \ldots, e_{2g})$ embeds into $H_*((C^{(n)})$.*

Remark 1.9. We do identify all throughout this paper $C^{(i)}$ with its image in $C^{(n)}$ under the basepoint embeddings for $1 \leq i \leq n$. We also identify, as in proposition 1.8, $H_*(C^{(i)})$ with its image in $H_*(C^{(n)})$ so that for example the class $e_1 \cdot e_2 \in H_2(C^{(3)})$ comes from the subspace $H_2(C^{(2)})$. Technically it would have been better to say that if $x \in H_*(C^{(i)})$ then $x \cdot \iota^{(n-i)} \in H_*(C^{(n)})$ where ι is the generator in $H_0(C)$.

Example 1.10. As an illustration, $H_*(C^{(2)})$ has generators e_i (in dimension 1), $b = [C]$ and all two fold products $e_i \cdot e_j, i < j$ (in dimension 2), $e_i \cdot b$ in dimension 3 and then b^2 in dimension four. The Euler characteristic is $\chi(S) = (g-1)(2g-3)$ in agreement with the formula $\binom{2g-2}{2}$ in [11].

There is a slick homotopy argument to see why proposition 1.8 is true. If we denote the direct limit of the inclusions i_n's (lemma 1.6) by the infinite symmetric product $\mathrm{SP}^\infty(C)$ ([3]), then we have a monomorphism $H_*(C^{(n)}) \hookrightarrow H_*(\mathrm{SP}^\infty(C))$. Now $\mathrm{SP}^\infty(C)$ is a very easy space to describe. This is the free abelian monoid on points of C. We then have two maps:
- The map into the Jacobian extends additively to $\mu : \mathrm{SP}^\infty(C) \longrightarrow J(C)$.
- There is a map $p : \mathrm{SP}^\infty(C) \longrightarrow \mathrm{SP}^\infty(S^2)$ induced from the quotient map $C \longrightarrow S^2$.

The crucial observation is now that

Lemma 1.11. *$\mu \times p : SP^\infty(C) \longrightarrow SP^\infty(S^2) \times J(C) \simeq \mathbb{P}^\infty \times (S^1)^{2g}$ is a homotopy equivalence.*

Proof. The composite $\bigvee S^1 \hookrightarrow C \longrightarrow J(C)$ induces an isomorphism in H_1 by construction of the Abel-Jacobi map. On the other hand, the map $C \longrightarrow S^2$ in (1) sends orientation class to orientation class. The main ingredient we need is the theorem of Dold and Thom which asserts that $\pi_*(\mathrm{SP}^\infty(X)) \cong \tilde{H}_*(X; \mathbb{Z})$ for any finite type CW complex (cf. [8], chapter 2). Applying π_* to both sides of $\mathrm{SP}^\infty(C) \longrightarrow \mathrm{SP}^\infty(S^2) \times J(C)$, we obtain the map $H_*(C) \longrightarrow H_*(S^2) \oplus H_*(\bigvee S^1)^{\otimes 2g}$ which by the first two observations is an isomorphism. It follows that $\mu \times p$ is a weak equivalence, and hence a homotopy equivalence. ∎

Proof of proposition 1.8. The equivalence $\mathrm{SP}^\infty(C) \xrightarrow{\simeq} J(C) \times \mathbb{P}^\infty$ constructed above is an equivalence of H-spaces since the map is multiplicative. It follows that $H_*(\mathrm{SP}^\infty(C), \mathbb{Z})$ (as a ring under the pontryagin product $\cdot$) is an exterior algebra on $2g$ generators e_i of dimension 1 (the homology of $J(C)$), tensored with a divided power algebra $\Gamma[b] = \mathbb{Z}\{b = \gamma_0, \gamma_1, \ldots, \}$ on a 2-dimensional generator $b = [C]$ (which is the homology of $\mathbb{P}^\infty$). By construction $H_*(C^{(n)})$ includes in this ring as those products of lengths at most n. ∎

1.3 The Map into the Jacobian. The map $C^{(n)} \longrightarrow J(C)$ is injective in cohomology for all $n \geq 2g$. This is because the composite

$$\bigvee^{2g} S^1 \hookrightarrow C \longrightarrow C^{(n)} \xrightarrow{\mu} J(C)$$

is an isomorphism on $H_1(-; \mathbb{Z})$ for all $n \geq 1$, and hence for $n \geq 2g$, the image of μ^* is precisely the exterior algebra on the e_i^* and is an isomorphism onto its image.

Remark 1.12. To see this injectivity statement in a more conceptual level, one can quote a general fact pointed out by Gottlieb [4]. Let $f : M \longrightarrow N$ be a smooth surjective map between oriented closed manifolds, and suppose $F = f^{-1}(y)$ (for some regular value $y \in N$ and hence F

[3] SP^∞ is a homotopy functor and we adopt here the notation of Dold and Thom.

is a closed manifold) is not a boundary. Then $f^*: H^*(N;\mathbb{Z}_2)\longrightarrow H^*(M;\mathbb{Z}_2)$ is injective. Note that in our case when $f=\mu$, F is a projective space and hence cannot be a boundary. Moreover since the spaces in questions have no torsion, Goettlieb's mod-2 statement is true integrally. Note that the Abel-Jacobi map is no longer smooth for $n<2g$.

Proposition 1.13. *$J(C)$ is a homotopy retract of $C^{(n)}$ when $n\geq 2g$.*

By that we mean there are maps $J(C)\longrightarrow C^{(n)}\xrightarrow{\mu} J(C)$ of which composite is homotopic to the identity of $J(C)$. This was obviously the case when $n=+\infty$ according to lemma 1.11, so this is represents a refinement to finite n. The statement is in fact a painless corollary of the following useful but not difficult fact (this is part of the main theorem of [12] with a longer argument of proof based on the theory of hyperplane arrangements).

Lemma 1.14. *There is a homotopy equivalence $\left(\bigvee^k S^1\right)^{(n)}\simeq (S^1)^k$ for $n\geq k\geq 1$.*

Proof. When $k=1$, this is easy since $(S^1)^{(n)}$ can be identified with $(\mathbb{C}^*)^{(n)}$, and the map

$$(\mathbb{C}^*)^{(n)}\longrightarrow\mathcal{H},\ \ \sum z_i\mapsto\prod_{1\leq i\leq n}(z-z_i)$$

is a homeomorphism, where on the left we have identified the space of monic polynomials of degree n with $\mathbb{C}^n$, and those polynomials with roots avoiding the origin with $\mathcal{H}$; the complement of a hyperplane in $\mathbb{C}^n$. Since $\mathcal{H}\cong\mathbb{C}^{n-1}\times\mathbb{C}^*\simeq S^1$, the claim follows in this case. It is amusing to see for instance that $(S^1)^{(2)}$ is precisely the mobius band (cf. [8], chapter 2). We next assume $n\geq k>1$.

The map inducing the homotopy equivalence is simply given by the embedding into the infinite symmetric product

$$\phi:(\bigvee^k S^1)^{(n)}\longrightarrow \mathrm{SP}^\infty(\bigvee^k S^1)=\prod^k \mathrm{SP}^\infty(S^1)\simeq (S^1)^k$$

Here we use the fact that $\mathrm{SP}^\infty(S^1)\simeq S^1$, or more generally that $\mathrm{SP}^\infty(S^n)$ is a model for $K(\mathbb{Z},n)$. Note that the composite $\bigvee^k S^1\longrightarrow(\bigvee^k S^1)^{(n)}\xrightarrow{\phi}(S^1)^k$ is an injection in homology according to lemma 1.6, and an isomorphism on H_1 between two copies of $\mathbb{Z}^k$. Since symmetrization abelianizes fundamental groups as soon as $n\geq 2$, and since H_1 is π_1 made abelian, it follows that ϕ induces an isomorphism in π_1 as well.

The homology of $(S^1)^k$ is obviously exterior on one-dimensional generators $E(e_1,\dots,e_k)$. Let $e_i\in H_1((\bigvee S^1)^{(n)})$ be the generators in dimension one coming from the various factors. Using lemma 1.11, and the fact that ϕ is a multiplicative map, we find that $H_*((\bigvee^k S^1)^{(n)})$ (once identified with its image in the exterior algebra $E(e_1,\dots,e_k)$) is generated by all monomials of the form

$$e_{i_1}\otimes\cdots\otimes e_{i_r}\ ,\ \ \sum i_j\leq n$$

But when $n\geq k$, these account for *all* generators in $E(e_1,\dots,e_k)$ meaning that ϕ is actually an isomorphism in homology. But then the spaces are simple ($(S^1)^k$ being a group even) and so by Whitehead's theorem, the map ϕ is a homotopy equivalence.

Note the one thing we used above is that $H_*(\mathrm{SP}^\infty(\bigvee^k S^1))\cong E(e_1,\dots,e_k)$ not only as abelian groups but also and more importantly as Pontryagin rings. ∎

Proposition 1.13 now follows by setting $k = 2g \leq n$ and observing that the composite[4]

$$J(C) \simeq (\bigvee^{2g} S^1)^{(n)} \xrightarrow{\quad i^{(n)} \quad} C^{(n)} \xrightarrow{\ \mu\ } J(C)$$

is precisely the homotopy equivalence ϕ indicated above, with $i : \bigvee^{2g} S^1 \hookrightarrow C$ is as in (1).

1.4. Intersection Theory on Symmetric Products. Since $M = C^{(n)}$ is a $2n$-dimensional (oriented) manifold, we have an intersection pairing $\bullet$ which relates to cup product via Poincaré duality pd as follows

$$\begin{array}{ccc} H_i(M) \otimes H_j(M) & \xrightarrow{\quad \bullet \quad} & H_{i+j-2n}(M) \\ \Big\downarrow{\scriptstyle pd} & & \Big\downarrow{\scriptstyle pd} \\ H^{2n-i}(M) \otimes H^{2n-j}(M) & \xrightarrow{\quad \cup \quad} & H^{4n-i-j}(M). \end{array}$$

When $i = j = n$ and upon identifying $H_0(M) \cong \mathbb{Z}$, $\bullet$ becomes the intersection from in middle homology. If x is a (co)cycle, we denote by $x^\perp$ its Poincaré dual. For example $C^{(n-i)} \hookrightarrow C^{(n)}$ is a subvariety of codimension $2i$ and we claim that

$$[C^{(n-i)}]^\perp = (b^*)^i \tag{2}$$

To see this we rely on the following principle (which is discussed in [4] for example) : Suppose $f : M \longrightarrow N$ is a (smooth) map between two closed oriented manifolds. Then if a cycle and a cocycle are related by Poincaré duality in N, f^{-1}(cycle) and f^*(cocycle) are related by Poincaré duality in M. Alternatively if X is a cycle on N such that $f^{-1}(X)$ is defined as a cycle, and Y is a cycle on M such that $Y \bullet f^{-1}(X)$ is defined, then in homology

$$f(Y \bullet f^{-1}(X)) = f(Y) \bullet X$$

(the right side beeing automatically defined). Let's apply this principle to the projection $\pi : C^n \longrightarrow C^{(n)}$. The inverse image of the cycle $[C^{(n)}]$ is $\sum [C]^i$ where the sum is over all possible ways to include $[C]^{n-i} := b^{\times n-i}$ into $b^{\times n}$. But the Poincaré dual to $b^{\times n-i}$ is $(b^*)^{\times i}$ inserted in the remaining spots. For example the dual to $b \otimes 1$ in $H_*(C^2)$ is $1 \otimes b^*$. And so we see that $(\pi^{-1}[C^{(n-i)}])^\perp$ coincides with $\pi^*(b^*)^i$, and by the above principle $[\mathrm{SP}^{n-1}]^\perp = (b^*)^i$.

For the sake of the next argument, write e_i and e_{i+g} the symplectic duals in $H_1(C)$ (instead of e_{2i-1} and e_{2i}). We can then show in $H_*(C^{(n)})$ that

$$e_i^\perp \;=\; (b^*)^{n-1} e^*_{i+g} \quad , \quad (b \cdot e_i)^\perp = (b^*)^{n-2} e^*_{i+g}$$

To see the first equality for example, notice that in C^n, $1 \times \cdots 1 \times e_i \times 1 \cdots 1$ is Poincaré dual to $b^* \times \cdots b^* \times e^*_{i+g} \times \cdots \times b^*$. And so

$$[\pi^{-1}(e_i)]^\perp = \Big[\sum 1 \otimes e_i \otimes 1\Big]^\perp = \sum_r (b^*)^r \otimes e^*_{i+g} \otimes (b^*)^{n-r-1} = \pi^*\left((b^*)^{n-1} e^*_{i+g}\right)$$

With a little more care one can show that

Lemma 1.15. $(e_i \cdot e_j)^\perp = \begin{cases} (b^*)^{n-2} e^*_{i+g} e^*_{j+g} & j \neq i + g \ \ mod\ (2g) \\ (b^*)^{n-1} + (b^*)^{n-2} e^*_{i+g} e^*_{j+g}, & j = i + g \ \ mod\ (2g) \end{cases}$

The difference in expression is of course due to the fact that $e_i^* e_j^* = 0$ if $i \not\equiv j + g \mod(2g)$ in $H^2(C) = \mathbb{Z}$, and that $e_i^* e^*_{i+g} = 1$. The following corollary is useful for later and can of course be deduced form [11]

[4]Given $f : X \to Y$, we define its symmetrization $f^{(n)} : X^{(n)} \longrightarrow Y^{(n)}$ in the obvious way $f^{(n)}(\sum n_i x_i) := \sum n_i f(x_i)$.

Corollary 1.16. *The signature of $S := C^{(2)}$ is $\sigma(S) = 1 - g$.*

Proof. We write down the intersection matrix! A basis for $H_2(C^{(2)})$ is given by b and then all possible classes $e_i \cdot e_j$ with $i < j$. We have that $b^\perp = b^*$, $(e_i \cdot e_{i+g})^\perp = b^* - e_i^* e_{i+g}^*$ and that $(e_i \cdot e_j)^\perp = e_{i+g}^* e_{j+g}^* = -e_i^* e_j^*$ if $1 \leq i < j \neq i + g \leq 2g$. Upon identifying $H_0 = H^4 = \mathbb{Z}$ (and $(b^*)^2 = 1$), we see that $b \bullet b = 1$. For $j = i + g$, $s = r + g$, we see that

$$(e_i \cdot e_j) \bullet (e_r \cdot e_s) = (b^* + e_{i+g}^* e_{j+g}^*)(b^* + e_{r+g}^* e_{s+g}^*) = (b^* - e_i^* e_j^*)(b^* - e_r^* e_s^*) = -1$$

according to theorem 1.1. This shows that the self-intersection of $(e_i \cdot e_j)$ is trivial if $i \not\equiv j \bmod (2g)$ and is -1 otherwise. Since there are g generators of the form $e_i e_{i+g}$ with self-intersection -1, and a single generator b with self-intersection $+1$, the claim follows. ∎

Remark 1.17. (Chern Classes). The total Chern class of $C^{(n)}$ is computed in [11], p.332, from which one deduces in particular that

$$c_1 = (n - g + 1)b^* - \theta \quad , \quad c_2 = (n - g + 1)(n - g).\frac{(b^*)^2}{2} - (n - g)b^* \theta + \frac{\theta^2}{2} \tag{3}$$

By using Hirzebruch signature formula $\sigma(S) = \frac{1}{3} < c_1^2 - 2c_2, [S] >$, the fact that $x\theta = g(b^*)^2$ in $C^{(2)}$ and that $< (b^*)^2, [C^{(2)}] >= 1$, we recover corollary 1.16 immediately.

2. The Spherical Class

Generally and for any curve C, one easily constructs an essential map $\alpha : S^2 \longrightarrow C^{(2)}$ as follows. One can see directly (and for any X) that $\pi_1(X^{(n)}) \cong H_1(X; \mathbb{Z})$ for $n \geq 2$ (cf. [8], proposition 2.4 for a very short proof). Now C as a CW complex is a two disk D^2 attached to the bouquet $\bigvee^{2g} S^1$ via a product of commutators in the free group $\pi_1(\bigvee^{2g} S^1)$. The attaching map (on $\partial D^2 = S^1$) becomes null-homotopic when mapping into $C^{(2)}$ since $\pi_1(C^{(2)})$ is abelian. It follows that up to homotopy the basepoint embedding $C \longrightarrow C^{(2)}$ factors through the cofiber S^2 and we have a map

$$C \longrightarrow S^2 \xrightarrow{\alpha} C^{(2)}$$

Lemma 2.1. *$u := \alpha_*([S^2]) = b - \ell$, where $\ell = \sum_{i \leq g} e_{2i-1} \cdot e_{2i} \in H_2(C^{(2)})$.*

Proof: [7] $h_*([S^2])$ is spherical and hence primitive. We will be done if we show that the only primitive classes in dimension 2 are (multiples) of $b - \ell$. The classes of dimension two in $H_*(C^{(n)})$ are b and the products $e_i \cdot e_j$ and so we need determine the diagonal on each. For $b = [C]$ this is fairly direct. Assume first that $C = T$ is a torus; i.e. $T \simeq S^1 \times S^1$, with one-dimensional generators e_1 and e_2. These classes are primitive with $\Delta_*(e_i) = e_i \otimes 1 + 1 \otimes e_i$ and hence

$$\begin{aligned} \Delta_*([T]) &= \Delta_*(e_1 \otimes e_2) = (e_1 \otimes 1 + 1 \otimes e_1)(e_2 \otimes 1 + 1 \otimes e_2) \\ &= [T] \otimes 1 + e_1 \otimes e_2 - e_2 \otimes e_1 + 1 \otimes [T] \end{aligned}$$

When C is of genus $g \geq 1$, we have a map $C \longrightarrow T_1 \vee \ldots \vee T_g$ which is an isomorphism on H_1 and maps $[C]$ to $\oplus [T_i]$. Combining these facts yields

$$\Delta_*([C]) = [C] \otimes 1 + \sum_{i \leq g} e_{2i-1} \otimes e_{2i} - \sum e_{2i} \otimes e_{2i-1} + 1 \otimes [C]$$

On the other hand by tracing through the commutative diagram

$$\begin{array}{ccccc} C \times C & \xrightarrow{\Delta \times \Delta} & C^2 \times C^2 & \xrightarrow{1 \times T \times 1} & C^2 \times C^2 \\ \big\downarrow \cdot & & & & \big\downarrow \cdot \times \cdot \\ C^{(2)} & & \xrightarrow{\quad\quad\Delta\quad\quad} & & C^{(2)} \times C^{(2)} \end{array}$$

(where $\cdot$ refers to the symmetric product pairing and T is the appropriate shuffle map), we readily determine that

$$\Delta_*(e_i\cdot e_j) = e_i\cdot e_j\otimes 1 + e_i\otimes e_j - e_j\otimes e_i + 1\otimes e_i\cdot e_j$$

Let $\ell = \sum_{1\leq i\leq g} e_{2i-1}\cdot e_{2i}$. By the formulae above $\bar\Delta_*([C]-\ell) = 0$ (where $\bar\Delta$ is the reduced diagonal), hence implying that $[C]-\ell$ is primitive, and by inspection the unique such (up to multiple). To show this multiple is $+1$, and since the embedding $C\hookrightarrow \mathrm{SP}^2(C)$ is homotopic to $g: C\longrightarrow S^2\longrightarrow \mathrm{SP}^2(C)$ we must have $g^*(u^*) = b^*$. But then if $u = k(b-\ell)$, $g^*(u^*) = kg^*(b^*-\ell^*) = kg^*(b^*) = kb^*$. The claim follows. ∎

Lemma 2.2. *We have* $(e_{2i-1}\cdot e_{2i})^* = e^*_{2i-1}e^*_{2i} - b^*$, *with the product on the left being the symmetric product pairing and on the right the cup product.*

Proof: We wish to determine the hom-dual of $e_{2i-1}\cdot e_{2i}$ in $H^2(C^{(n)})$. By an earlier remark, $\pi^*: H^*(C^{(n)})\longrightarrow H^*(C)^{\otimes n}$ is injective, where $\pi: C^n\longrightarrow C^{(n)}$ is the covering projection. The class $e_{2i-1}\cdot e_{2i}$ is the image of classes of the form $\pm 1\otimes\cdots 1\otimes e_r\otimes 1\cdots 1\otimes e_s\otimes 1\cdots\otimes 1$ with $\{r,s\} = \{2i-1, 2i\}$ (it is $+$ if $r<s$ and $-$ if $r>s$). It follows that

$$\pi^*(e_{2i-1}\cdot e_{2i})^* = \sum \pm 1\otimes\cdots 1\otimes e^*_r\otimes 1\cdots 1\otimes e^*_s\otimes 1\cdots\otimes 1,\quad \{r,s\} = \{2i-1,2i\}$$

The sum is over all possible distinct spots e_r and e_s can assume. On the other hand, $\pi^*(e^*_i) = \sum 1\otimes\cdots 1\otimes e^*_i\otimes 1\cdots\otimes 1$ and hence

$$\begin{aligned}
\pi^*(e^*_{2i-1}e^*_{2i}) &= \pi^*(e^*_{2i-1})\pi^*(e^*_{2i})\\
&= \pi^*(e_{2i-1}\cdot e_{2i})^* + \sum 1\otimes\cdots\otimes e^*_{2i-1}e^*_{2i}\otimes 1\cdots\otimes 1\\
&= \pi^*(e_{2i-1}\cdot e_{2i})^* + \sum 1\otimes\cdots\otimes b^*\otimes 1\cdots\otimes 1\\
&= \pi^*(e_{2i-1}\cdot e_{2i})^* + \pi^*(b^*)
\end{aligned}$$

Since π^* is injective we get the desired equality. ∎

Remark 2.3. More generally we can show that $(e^*_{2k_i-1}e^*_{2k_i} - b^*)\cdots(e^*_{2k_c-1}e^*_{2k_c} - b^*)$ is dual to the $2c$-fold product $e_{2k_i}\cdot e_{2k_i}\cdot\cdots\cdot e_{2k_c-1}\cdot e_{2k_c}$. Evidently if the length of this class which is $2c$ exceeds n, then by lemma 1.8, the class is trivial in $H_{2c}C^{(n)}$ and this explains MacDonald's relation $(e^*_{2k_i-1}e^*_{2k_i} - b^*)\cdots(e^*_{2k_c-1}e^*_{2k_c} - b^*) = 0$ for $2c > n$.

Lemma 2.4. *Suppose α is a differential embedding, and denote by $u^\perp$ the Poincaré dual of $u := \alpha_*[S^2]\subset H_2(C^{(2)})$. Then $u^\perp = (1-g)b^* + \theta$.*

Proof: According to lemma 2.1, $u^\perp = b^\perp - (\sum e_{2i-1}\cdot e_{2i})^\perp$ The claim follows form the calculations in §1.3. Note that in light of lemma 3, the hom-dual and the Poincaré dual of u coincide. ∎

Corollary 2.5. *If α is an embedding, then the euler class of the normal bundle to $\alpha(S^2)$ is $1-g$ where g is the genus of C.*

Proof: One need compute the self-intersection of $\alpha(S^2)$ in $C^{(2)}$. This is of course given by the evaluation $\langle (u^\perp)^2, [C^{(2)}]\rangle$. We again identify $H^4 = \mathbb{Z}$ so that $(b^*)^2 = e^*_{2i-1}e^*_{2i}e^*_{2j-1}e^*_{2j} = 1$ for $i\neq j$. From this we get that $\theta^2 = g(g-1)$ and hence

$$\begin{aligned}
(u^\perp)^2 = [(1-g)b^* + \theta]^2 &= (1-g)^2(b^*)^2 + 2(1-g)b^*\theta + \theta^2\\
&= (1-g)^2 + 2(1-g)g + g(g-1) = (1-g)
\end{aligned}$$

as claimed. ∎

Note that when $\alpha : \mathbb{P}^1 \longrightarrow C^{(2)}$ is a holomorphic embedding, the normal bundle to the image (a Riemann sphere) is a complex line bundle which is according to the above (and as is well-known) isomorphic to $\mathcal{O}(1-g)$. In fact holomorphic embeddings of α gives a characterization of hyperelliptic curves.

Proposition 2.6. *The generator u in $H_2(C^{(2)})$ can be represented by a holomorphically embedded sphere (i.e. a rational curve) if and only if C is hyperelliptic.*

Proof: The "only if" part is straightforward since if $\mathbb{P}^1$ is holomorphically embedded in $C^{(2)}$, then the composite $\mathbb{P}^1 \hookrightarrow C^{(2)} \longrightarrow J(C)$ is trivial necessarily and $C^{(2)}$ carries a g^1_2 [1]; that is C hyperelliptic. Conversely, if C hyperelliptic with a degree two branched covering $\psi : C \longrightarrow \mathbb{P}^1$, then the transfer map $\beta : \mathbb{P}^1 \longrightarrow C^{(2)}$, sending x into the unordered pair of preimages in $\psi^{-1}(x)$, is an embedding of $\mathbb{P}^1$ into $C^{(2)}$. The class $\beta_*[S^2]$ is then a multiple of $u \in H_2(C^{(2)})$. This multiple k is determined by the intersection multiplicity of $\beta(\mathbb{P}^1)$ with the image under π of $p \times C$ in $C^{(2)}$ and this is of course a single point. So if we write $\beta_*[\mathbb{P}^1]^\perp = ku^\perp$, the preceding discussion shows that $ku^\perp b^* = 1$. But $ku^\perp b^* = k[(1-g)b^* + \theta]b^* = k[(1-g)+g] = k$. The claim follows. ∎

2.1. Rational Curves and the Symmetric Square.

The following is very classical (algebraically) but we give it our own spin (topologically). For a hyperelliptic curve $C \xrightarrow{:2} \mathbb{P}^1$, the transfer map $\mathbb{P}^1 \longrightarrow C^{(2)}$ is a holomorphic embedding, and is homologous to the class u. One might then ask whether there are any other rational curves in $C^{(2)}$ of degree $k > 1$? Here C is not necessarily hyperelliptic. We can address this question using very classical geometry. First observe that an algebraic curve X (of genus g) in $S := C^{(2)}$ is (by definition) a divisor on an algebraic surface and hence gives to a line bundle E on S (cf. [6], chapter VI, §1) with the property that $E_{|X} = N_X$ (i.e. the restriction of E to X is the normal bundle of X in S). An application of Riemann-Roch in this situation shows the following *adjunction* formula: let g_X be the genus of X, then

$$g_X = \frac{(X+K)\bullet X}{2} + 1$$

where K is the canonical class of S, and where $\bullet$ the intersection pairing (which is well defined within a divisor class, [6]). For example, when $X = \mathbb{P}^2$ then $K = -3L$, where L the divisor class containing all straight lines in $\mathbb{P}^2$.

Corollary 2.7. *Identify $H_2(C^{(2)};\mathbb{Z})$ with $\mathbb{Z}$. If the class k is represented by a rational curve, then necessarily $k = 1$ and C is hyperelliptic.*

Proof. The Poincaré dual of the canonical divisor for $C^{(n)}$ has been computed by MacDonald ([11], p: 334) and it coincides with

$$K^\perp = -c_1(C^{(n)}) = (g-n-1)b^* + \sum e^*_{2i-1}e^*_{2i} = (g-n-1)b^* + \theta \tag{4}$$

Set $n = 2$. Using the fact that $b^* = b^\perp$ and the formula for $u^\perp$, this can be rewritten as $K^\perp = (g-3)b^\perp + u^\perp - (1-g)b^\perp = (2g-4)b^\perp + u^\perp$, and since the Poincaré duality is an isomorphism, we find that

$$K = (2g-4)b + u \quad , \quad K \in H_2(C^{(2)}) \tag{5}$$

Assume X is a holomorphic sphere in $C^{(2)}$ with $[X] = ku \in H_2(C^{(2)})$. Then by the adjuntion formula $0 = (K+X)\bullet X + 2$. Using that $b \bullet u = 1$, $u \bullet u = 1-g$, we write

$$-2 = (K+X)\bullet X = [(2g-4)b + ku]\bullet ku = k(2g-4) + (1+k)k(1-g)$$

This can be rewritten in the form $k(g-1)[1-k] = 2[k-1]$ from which we deduce that the only possibility is when $k = 1$. According to proposition 2.6, this implies the claim. ∎

Remark 2.8. An interesting observation (Mattuck) is that the canonical class K in the case $n = g$ is the union of all the special fibers of $C^{(g)}\longrightarrow J(C)$. When $g=2$, the curve is automatically hyperelliptic and μ has a single exceptional fiber $\mathbb{P}^1$. In this case $K=u$ the spherical class, in agreement with the calculation (5).

3. 2.2. Embedded Spheres in $C^{(2)}$

The question now is whether any other multiple of $u\in H_2(C^{(2)};\mathbb{Z})\cong\mathbb{Z}$, can be realized by a *differentiably* embedded 2-sphere in $C^{(2)}$. This we settle partially as follows.

Proposition 3.1. *Let $\beta : S^2\hookrightarrow C^{(2)}$ be an embedding, with $\beta_*[S^2]=ku$ in $H_2(C^{(2)})=\mathbb{Z}$. If $g=0$ then $k=\pm 1,\pm 2$, and if $k>1$ odd and $g>0$ even, then necessarily $k\equiv\pm 1$ mod (8).*

Remark 3.2. For $g=0$ the answer is known [10] and the following cute result is attributed to Tristam. Let $u\in H_2(\mathbb{P}^2)$ be the generator. Then ku can be represented by an embedded 2-sphere if and only if $|k|<3$. The sufficiency is clear since the conic $\mathbb{P}^1\hookrightarrow\mathbb{P}^2$, $[z,w]\mapsto[z^2:wz:w^2]$ is an embedding of degree 2. There are no other holomorphic embeddings of higher degree since by the degree-genus formula an algebraic curve of genus g in the projective plane must satisfy $g=\frac{1}{2}(k-1)(k-2)$ and so it is rational if and only if $k=1,2$ as pointed out. In the differentiable case, the proof is much less evident and can be seen as a consequence of the theorem of Mrowka-Kronheimer (previously the Thom conjecture) which asserts that any homology class of degree k in $\mathbb{P}^2$ can be smoothly realized by a curve of genus at least $\frac{1}{2}(k-1)(k-2)$.

A condition for embedding a 2-sphere into a four manifold (not necessarily simply connected) was first given by Kervaire and Milnor. It goes as follows. An integral homology class $\beta\in H_2(M,\mathbb{Z})$ is called “characteristic” if it is dual to the Stiefel-Whitney class $w_2(M)$; i.e. if mod-2 reduction followed by Poincaré duality takes β to $w_2(M)$. Equivalently if the intersections verify

$$\beta\bullet\tau\equiv\tau\bullet\tau \bmod(2) \tag{6}$$

for any $\tau\in H_2(M)$. Note that any homology class $\tau\in H_2(M;\mathbb{Z}_2)$ can be represented by an embedded (real) surface (Thom). With integral coefficients the situation is entirely different. The following is the main criterion of Kervaire-Milnor

Theorem 3.3. [9] *Let $\beta\in H_2(M;\mathbb{Z})$ be dual to $w_2(M)$ where M is a closed, connected, oriented differentiable (real) 4-manifold. Then β can be represented by a smoothly embedded 2-sphere only if $\beta\bullet\beta\equiv\sigma(M)$ mod 16.*

Here $\sigma(M)$ is the signature of the intersection form. Proposition 3.1 is now a direct corollary. Proof of Proposition 3.1: Set $M=S$ the second fold symmetric product of C. The class $w_2(S)$ is the reduction mod-2 of the first chern class (for unitary bundles). We've seen (4) that $c_1(S)=(3-g)b^*-\sum e^*_{2i-1}e^*_{2i}$ and hence reducing mod-2, one gets $w_2(S)=b^*+\sum(b^*+e^*_{2i-1}e^*_{2i})$ which is precisely the mod-2 Poincaré dual of $u=\alpha_*([S^2])$ (according to lemma 4). So u is characteristic and so is any odd multiple. There now remains to compute the signature of S and this is given in lemma 1.16. Given then $S^2\longrightarrow C^{(2)}=M$ which corresponds in homology to the class $\beta=ku$, we have that $\beta\bullet\beta=k^2(1-g)$ (this is the euler class of the normal bundle of β if it's an embedding according to corollary 2.5), and hence by the Kervaire-Milnor congruence if β is differentiably embedded (and k odd), then necessarily

$$k^2(1-g)\equiv 1-g \bmod(16)$$

When $k=1$ the sphere always embeds as we previously argued. When $k>1$ and $g=0$, there is no odd k such that the congruence is satisfied (as we know already from Thom's conjecture). For even positive genus, the congruence gives the restriction that $16\mid k^2-1$. Since k odd, this implies that $k\equiv\pm 1$ mod (8). ∎

2.3. The Second Homotopy Group. We don't know if embeddings occur in those cases listed in proposition 3.1, but we suspect the answer is no. Since the homotopy class of α is in $\pi_2(C^{(2)})$ so we have better determine what this group is.

Lemma 3.4. *(anonymous) Suppose C is of genus g, and $n > g$, then $\pi_2(C^{(n)}) \cong \mathbb{Z}$.*

Proof. When $n \geq 2g-1$, a theorem of Mattuck asserts that $C^{(n)}$ fibers over $J(C)$ (the Jacobian) with fiber $\mathbb{P}^{n-g}$. Since $\pi_3(J(C)) = \pi_2(J(C)) = 0$ (being a complex torus), and since $\pi_2\mathbb{P}^i = \mathbb{Z}$ for $i \geq 1$, the claim follows in this range. For $g \leq n \leq 2g-2$, the Abel-Jacobi map μ_n is a quasifibration[5] up to dimension $2(n-g)+1$ according to ([5], theorem 4.1). So for $n > g$, it is a quasifibering up to dimension 3 and so (by definition) there is a short exact sequence

$$\pi_3(J(C)) = 0 \longrightarrow \pi_2(\mathbb{P}^{n-g}) \longrightarrow \pi_2(C^{(2)}) \longrightarrow \pi_2(J(C)) = 0$$

and as in the previous case this gives $\mathbb{Z}$ for an answer. ∎

Now and perhaps surprisingly $\pi_2(C^{(n)})$ for $n \leq g$ is not even necessarily finitely generated (as pointed out to us by "x"). The map $C^{(n)} \longrightarrow J(C)$ in this range has a description of a blowup over various loci in the Jacobian of varying codimension. In particular there is always an exceptional $\mathbb{P}^1 \hookrightarrow C^{(n)}$ and this generates a cyclic group in π_2 (the reason being that the composite

$$\pi_2(\mathbb{P}^1) \xrightarrow{\cong} H_2(\mathbb{P}^1) \hookrightarrow H_2(C^{(n)})$$

is essential), see [2]. There might be however an interesting action of π_1 which, unlike the case $n > g$, is not trivial.

Lemma 3.5. *Suppose C is a curve of genus $g = 2$. Then there is a Laurent polynomial description*

$$\pi_2(C^{(2)}) = \mathbb{Z}[t_i, t_i^{-1}],\ 1 \leq i \leq 4$$

Proof. The curve C being of genus two, it is necessarily hyperelliptic and so it is obtained from its Jacobian (a complex 2-torus) by blowing up a single point. The universal cover of the torus is $\mathbb{C}^2$ and $J(C)$ is the quotient by a lattice $\mathcal{L}$ whose vertices are in one-to-one correspondance with $\mathbb{Z}^4$. It follows that the universal cover $\tilde{X}$ of $C^{(2)}$ is a blowup of $\mathbb{C}^2$ at each and everyone of these vertices. Each exceptional fiber of this blowup being a copy of $\mathbb{P}^1$, it contributes a generator to $H_2(\tilde{X};\mathbb{Z})$. But then

$$H_2(\tilde{X}) = \pi_2(\tilde{X}) = \pi_2(C^{(2)})$$

and so $\pi_2(C^{(2)})$ is infinitely generated indeed. Choose a fundamental domain for the lattice made up of four essential edges mapping to the fundamental group generators of $J(C)$, which we call $t_1, \ldots, t_4$ and which also correspond to the generators of $\pi_1(C^{(2)}) = \mathbb{Z}^4$ since the Abel-Jacobi map is an isomorphism on π_1. Moving from generator to generator of $H_2(\tilde{X})$ through the lattice corresponds in $C^{(2)}$ to multiplication by a word in the t_i's or their inverses. ∎

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