GENUS ZERO ACTIONS ON RIEMANN SURFACES

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1. Introduction

In this paper we solve the following problem:

Problem. Which finite groups G admit an action on some compact connected Riemann surface M so that if H is any non-trivial subgroup of G then the orbit surface M/H has genus zero, that is

$$1 \neq H \subseteq G \Longrightarrow M/H = \mathbb{P}^1(\mathbb{C}). \tag{1}$$

Any finite group G will admit infinitely many actions $G \times M \to M$ so that $M/G = \mathbb{P}^1(\mathbb{C})$. It will turn out that very few groups G satisfy (1).

The action $G \times M \to M$ is supposed to be analytic and effective. Thus G is a subgroup of $\mathrm{Aut}(M)$, the group of all analytic automorphisms of M. For any action $G \times M \to M$, if $M/H_1 = \mathbb{P}^1(\mathbb{C})$ for some $H_1 \subseteq G$, then automatically $M/H_2 = \mathbb{P}^1(\mathbb{C})$ if $H_1 \subseteq H_2 \subseteq G$. This means that we can reduce the problem to the consideration of cyclic subgroups $\mathbb{Z}_p \subseteq G$, where p is a prime dividing the order of G.

Our problem thus becomes:

Problem. Which finite groups G admit an action on some compact connected Riemann surface M so that if H is any cyclic subgroup of prime order p then the orbit surface M/H has genus zero, that is

$$M/\mathbb{Z}_p = \mathbb{P}^1(\mathbb{C}),\ \text{for all } Z_p \subseteq G,\ \text{where } p \text{ is a prime dividing } |G|. \tag{2}$$

All Riemann surfaces considered in this paper, with just a few exceptions, will be compact and connected. The exceptions are the complex plane and the upper half plane. The groups being studied in this paper will always be finite.

In our solution of this problem not only do we describe the groups G admitting such actions, but we also determine all possible actions for each group. This amounts to describing all admissible epimorphisms $\theta : \Gamma \to G$, where Γ is a Fuchsian group of signature $(0 \mid n_1, \ldots, n_r)$. See Section (2).

In Section (3) we consider the low genera cases $g = 0, 1$, where g is the genus of M. Other than these exceptions we will usually assume that $g > 1$.

Definition 1.1. A group G is said to have genus zero if there exists a Riemann surface M and an action of G on M satisfying (1), or equivalently (2) . We also say that the action has genus zero.

To explain the significance of the above problem, and to give it some motivation, we recall the definition of a fixed point free linear action on a $\mathbb{C}$ vector space V.

Definition 1.2. A linear action $G \times V \to V$ is said to be fixed point free if

$$S \in G,\ S \neq 1 \Longrightarrow\ S(v) \neq v \text{ for all } v \in V,\ v \neq 0.$$

Now let V be the vector space of holomorphic differentials on M. V is a $\mathbb{C}$ vector space of dimension g, where g is the genus of M. The action of G on M induces a linear action on the vector space V. If $S \in G$ is an element of order p then the induced linear transformation $S^* : V \to V$ also has order p, assuming $g > 1$. It follows that the eigenvalues of S^* are p^{th} roots of unity. In particular $+1$ might be an eigenvalue of T^*. Part of the proof of the Eichler trace formula, see Farkas and Kra [2], states that the dimension of the $+1$ eigenspace of S^* is the genus of M/H, where $H \cong \mathbb{Z}_p$ is the cyclic group of order p generated by S.

Therefore the induced action $G\times V\to V$ is fixed point free if, and only if, $G\times M\to M$ is a genus-zero action. We can impose a metric on V so that the action becomes unitary. Thus the action $G\times M\to M$ has genus zero if, and only if, S^{2g-1}/G is an elliptic space form. This equivalence assumes that $g>1$. Here S^{2g-1} denotes the unit sphere in V.

The converse problem of when a fixed point free linear action $G\times V\to V$ arises from a genus-zero action on some Riemann surface is not considered in this paper.

Groups G which admit fixed point free linear actions have been classified, see Wolf [5]. In particular they must satisfy a strong condition on their Sylow p-subgroups. First recall that the generalized quaternion group $Q(2^n)$ is defined as follows:

Definition 1.3. The generalized quaternion group is the group with the presentation:

$$Q(2^n)=\left\langle A,B\mid A^{2^{n-1}}=1,\ B^2=A^{2^{n-2}},\ BAB^{-1}=A^{-1}\right\rangle . \tag{3}$$

We will always assume $n\geq 3$, since otherwise $Q(2^n)$ is cyclic.

Definition 1.4. We say that a group G satisfies the **Sylow conditions** if the following two conditions hold.

(1) For an odd prime p the Sylow p-subgroups are cyclic.
(2) The Sylow 2-subgroups are either cyclic or generalized quaternion.

The Sylow conditions are equivalent to the following:

Definition 1.5. We say that a group G satisfies the $\mathbf{p^2}$ **conditions** if every subgroup of order p^2 is cyclic, where p is any prime.

Every group G admitting a fixed point free linear action satisfies these conditions. In fact these groups must satisfy the even stronger pq conditions.

Definition 1.6. A group G satisfies the **pq conditions** if every subgroup of order pq is cyclic, where p and q are arbitrary primes.

Let D_{2n} denote the dihedral group of order $2n$. If n is odd then D_{2n} satisfies the p^2 conditions but not the pq conditions. In fact any group G of even order which admits a fixed point free linear action must have exactly one element of order 2, and this element generates the center of G.

To describe our results we need another definition and some notation.

Definition 1.7. We let $G_{m,n}(r)$ denote the group presented as follows:

$$\begin{aligned} \text{generators}: &\quad A,B;\\ \text{relations}: &\quad A^m=1,\ B^n=1,\ BAB^{-1}=A^r;\\ \text{conditions}: &\quad GCD((r-1)n,m)=1 \text{ and } r^n\equiv 1\ (mod\ m). \end{aligned} \tag{4}$$

These groups are precisely the groups having all Sylow subgroups cyclic, see Burnside [1]. To avoid the trivial cases where the group is cyclic we will usually assume that $m>1, n>1$. Note that the conditions imply $r\not\equiv 1\ (mod\ m)$.

If d denotes the order of r modulo m then Zassenhaus [6] proved that $G_{m,n}(r)$ satisfies the pq conditions if, and only if, every prime divisor of d also divides $\dfrac{n}{d}$. He also proved that $G_{m,n}(r)$ admits a fixed point free linear representation if, and only if, the pq conditions hold. The groups $G_{m,n}(r)$ are known as Zassenhaus metacyclic groups (abbreviated to ZM groups).

Let I^* denote the binary icosahedral group. It has order 120 and admits a fixed point free linear representation. I^* is non-solvable, and if G is a non-solvable group admitting a linear fixed point free representation then G contains I^* as a subgroup.

The main results of this paper are contained in the following theorems. In particular the first theorem implies that I^* does not have genus zero, and therefore neither does any non-solvable group admitting a fixed point free linear representation.

Theorem 1.8. *The groups having genus zero are the cyclic groups, the generalized quaternion groups $Q(2^n)$, the polyhedral groups and the ZM groups $G_{p,4}(-1)$, where p is an odd prime.*

The cyclic and polyhedral groups have genus zero because they can act on $\mathbb{P}^1(\mathbb{C})$. Some cyclic groups admit genus zero actions on surfaces of higher genus, but most do not. See Theorems (1.9), (1.11), (5.2), and Corollary (5.1).

This theorem gives a solution to Problem (1) but does not describe the actions involved. To do this we need some more notation. Let $\Gamma(0 \mid n_1, \ldots, n_r)$ denote the abstract group presented by

$$\left\langle X_1, X_2, \ldots, X_r \;\middle|\; X_1^{n_1} = X_2^{n_2} = \cdots = X_r^{n_r} = X_1 X_2 \cdots X_r = 1 \right\rangle. \tag{5}$$

For a more detailed explanation of the notation see Section (2).

If $G \times M \to M$ is an action satisfying $M/G = \mathbb{P}^1(\mathbb{C})$, and the genus of M is g, then there exists a short exact sequence

$$1 \to \Pi \to \Gamma \xrightarrow{\theta} G \to 1, \tag{6}$$

where $\Pi \cong \pi_1(M)$ and $\Gamma = \Gamma(0 \mid n_1, \ldots, n_r)$ for some choice of n_j. We say that the signature of the action is $(0 \mid n_1, \ldots, n_r)$. The signature of the action, together with the epimorphism $\theta : \Gamma \to G$ and the particular realization of Γ as a Fuchsian group, completely determines the action.

Theorem 1.9. *All genus-zero actions of the cyclic group $\mathbb{Z}_{p^e}$, where $p \geq 2$ is any prime, have signature $(0|\underbrace{p, \ldots, p}_{r}, p^e, p^e)$, where r is arbitrary. The genus is $g = \frac{1}{2}r\left(p^e - p^{e-1}\right)$.*

If $e = 1$ this means that the signature of the action is $(0|\underbrace{p, \ldots, p}_{r})$, where $r \geq 2$, and the genus is $g = \frac{1}{2}(r-2)(p-1)$.

Theorem 1.10. *All genus-zero actions of the generalized quaternion group $Q(2^n)$ have signature $(0|\underbrace{2, \ldots, 2}_{r}, 4, 4, 2^{n-1})$, where r is odd. The genus is $g = 2^{n-2}(r+1)$.*

Theorem 2 shows there are infinitely many signatures for genus-zero actions by cyclic $p-$groups. This is not true for other cyclic groups.

Theorem 1.11. *Suppose p, q are distinct primes. Then the genus-zero actions of $\mathbb{Z}_{pq}$ have signature and corresponding genus given by*

(1) *sig*$(\Gamma) = (0 \mid pq, pq)$, *in which case* $g = 0$.
(2) *sig*$(\Gamma) = (0 \mid p, q, pq)$, *in which case* $g = \dfrac{1}{2}(p-1)(q-1)$.
3. *sig*$(\Gamma) = (0 \mid p, p, q, q)$, *in which case* $g = (p-1)(q-1)$.

See Section (4) for the proofs of Theorems (1.9), (1.10) and Section (5) for the proof of Theorem (1.11). These sections also give details of the epimorphisms $\theta : \Gamma \to G$ classifying the actions.

Theorem 1.12. *All genus-zero actions of the ZM group $G_{p,4}(-1)$, where p is an odd prime, have signature $(0 \mid 4, 4, p)$. The corresponding genus is $g = p - 1$.*

This theorem is proved in Section (6). In Section (7) we complete the proof of Theorem (1).

2. Preliminaries

In this section we collect some preliminary material and review some of the material on Riemann surfaces that we need later.

Let $\mathbb{U}$ denote a simply connected Riemann surface, that is $\mathbb{P}^1(\mathbb{C})$, $\mathbb{C}$ or the upper half plane $\mathbb{H}$. A Fuchsian group Γ is any finitely generated discrete subgroup of $PSL_2(\mathbb{R})$, the group of analytic automorphisms of $\mathbb{H}$. By abuse of terminology we shall also call a finitely generated discrete subgroup of $\mathrm{Aut}(\mathbb{U})$ Fuchsian.

To every Fuchsian group Γ we associate a signature $(h \mid n_1, n_2, \ldots, n_r)$, where h is the genus of the orbit surface $\mathbb{U}/\Gamma$ and $n_1, \ldots, n_r$ are the orders of the distinct conjugacy classes of maximal cyclic subgroups of Γ. The n_j are called the periods. In general it is possible that some of the n_j are infinite, but the Fuchsian groups Γ considered in this paper will all have the property that $\mathbb{U}/\Gamma$ is compact, and so the periods n_j will be finite. We use the notation $\operatorname{sig}(\Gamma)$ for the signature of Γ.

The notation $\Gamma(h \mid n_1, \ldots, n_r)$ denotes any Fuchsian group of signature $(h \mid n_1, n_2, \ldots, n_r)$. If $r = 0$ this group is torsion free and we use the notation $\Gamma(h \mid -)$. In fact the Fuchsian groups with $r = 0$ are just the fundamental groups of Riemann surfaces.

Of particular interest to us will be those Fuchsian groups $\Gamma = \Gamma(0 \mid n_1, \ldots, n_r)$ since they play a seminal role in actions $G \times M \to M$ satisfying $M/G = \mathbb{P}^1(\mathbb{C})$. As an abstract group Γ has the presentation (5). The geometry of Γ is spherical, euclidean or hyperbolic according as $\sum_{j=1}^{r} \frac{1}{n_j} > 1$, $= 1$ or < 1. In the hyperbolic case, for any realization of Γ as a Fuchsian group, the X_j are elliptic and so are rotations about vertices $V_j \in \mathbb{H}$. Up to conjugation by elements of $\operatorname{Aut}(\mathbb{U})$ the space of all such realizations is a cell of dimension $2r - 6$.

Let G be a group acting on a Riemann surface M of genus g and let $\mathbb{U}$ be the universal covering space of M. Then there exists a short exact sequence of groups

$$1 \to \Pi \to \Gamma \xrightarrow{\theta} G \to 1, \tag{7}$$

where

1. Γ is a Fuchsian group with signature $(h \mid n_1, n_2, \ldots, n_r)$.
2. $\Pi = \operatorname{Ker}(\theta)$ is a torsion free Fuchsian group with signature $(g \mid -)$.
3. $M = \mathbb{U}/\Pi$ and the action of an element $S \in G$ on M is given by $S[z] = [\gamma(z)]$, where the brackets $[\ ]$ indicate the Π equivalence class of points in $\mathbb{U}$ and $\gamma \in \Gamma$ is any element such that $\theta(\gamma) = S$.
4. The orbit surface M/G has genus h and is naturally isomorphic to $\mathbb{U}/\Gamma$.

The relationship between the genera g, h is given by the Riemann-Hurwitz formula:

$$2g - 2 = |G| \left(2h - 2 + \sum_{j=1}^{r} \left(1 - \frac{1}{n_j} \right) \right). \tag{8}$$

Definition 2.1. The signature of the action of G on M is defined to be the signature of Γ.

Suppose $S \in G$ is an element of order p, where p is a prime, and H is the cyclic subgroup generated by S. Then $\Gamma' = \theta^{-1}(H)$ is a Fuchsian group and there exists a short exact sequence

$$1 \to \Pi \to \Gamma' \xrightarrow{\theta} H \to 1. \tag{9}$$

The signature of Γ' will have the form $(k \mid \underbrace{p, \ldots, p}_{t})$, where t is the number of fixed points of $S : M \to M$.

The Riemann-Hurwitz formula (8) gives

$$2g - 2 = p \left(2k - 2 + t \left(1 - \frac{1}{p} \right) \right). \tag{10}$$

A comparison between Formulas (8) and (10) then gives a numerical restriction on the possible genera and the number of fixed points.

If $G \times M \to M$ is an action satisfying $M/G = \mathbb{P}^1(\mathbb{C})$ then the signature of Γ will be given by $\operatorname{sig}(\Gamma) = (0 \mid n_1, \ldots, n_r)$, and so Γ will be given abstractly by the presentation in (5). In this case let

$$T_j = \theta(X_j) \text{ and } G_j = \text{ the subgroup of } G \text{ generated by } T_j, 1 \le j \le r.$$

Then

1. $T_1, T_2, \ldots, T_r$ generate G (θ is an epimorphism).
2. $T_1^{n_1} = T_2^{n_2} = \cdots = T_r^{n_r} = T_1 T_2 \cdots T_r = 1$ (θ must preserve the relations in Γ).
3. The order of T_j is n_j, $1 \le j \le r$ (the kernel of θ is torsion free).

The converse is true. In other words if we are given $T_1, T_2, \ldots, T_r$ satisfying these conditions then there is an action $G \times M \to M$ satisfying $M/G = \mathbb{P}^1(\mathbb{C})$ and having signature $(0 \mid n_1, \ldots, n_r)$. It follows that every group G admits infinitely many actions $G \times M \to M$ satisfying $M/G = \mathbb{P}^1(\mathbb{C})$.

Remark 2.2. The key to the study of actions $G \times M \to M$ satisfying $M/G = \mathbb{P}^1(\mathbb{C})$ is the determination of the number of fixed points of the action. Let $S \in G$, $S \neq 1$, and choose any $\gamma \in \Gamma$ such that $\theta(\gamma) = S$. Then a point $[z] \in M$ will be a fixed point of S if, and only if, there exists $\mu \in \Pi$ such that $\mu\gamma(z) = z$. Now the elements of Γ that have fixed points are the conjugates of powers of the elliptic generators. That is we must have

$$\mu\gamma = \delta X_j^{k_j} \delta^{-1}, \text{ where } 1 \leq j \leq r,\ 1 \leq k_j < n_j,\ \delta \in \Gamma,$$

in which case $z = \delta(V_j)$. Thus the fixed points of S are those Π equivalence classes $[\delta(V_j)]$, any $\delta \in \Gamma$, satisfying

$$S = dT_j^{k_j} d^{-1}, \text{ for some } k_j,\ 1 \leq j \leq r, \text{ where } d = \theta(\delta). \tag{11}$$

Condition (11) is especially easy to use if $S \in G, S \neq 1$, is in the center.

Lemma 2.3. *Suppose G is a group admitting an action $G \times M \to M$ satisfying $M/G = \mathbb{P}^1(\mathbb{C})$ and let $S \in G, S \neq 1$, be a central element. Then the fixed points of S are the Π equivalence classes $[\delta(V_j)],\ 1 \leq j \leq r$, any $\delta \in \Gamma$, such that $S \in G_j$.*

Corollary 2.4. *Suppose G is a group admitting an action $G \times M \to M$ satisfying $M/G = \mathbb{P}^1(\mathbb{C})$ and let $S \in G$ be a central element of order p. Then the number of fixed points of S is $|G|{\sum_j}' \dfrac{1}{n_j}$, where the prime indicates we sum only over those j such that $S \in G_j$.*

Proof. According to Lemma (2.3) the fixed points of S are those Π equivalence classes $[\delta(V_j)],\ 1 \leq j \leq r$, any $\delta \in \Gamma$, such that $S \in G_j$. If $S \in G_j \cap G_k$ then

$$\begin{aligned} [\delta(V_j)] = [\epsilon(V_k)] &\iff \mu\delta(V_j) = \epsilon(V_k) \text{ for some } \mu \in \Pi \\ &\iff j = k \text{ and } \theta(\delta) \equiv \theta(\epsilon)(mod\ G_j). \end{aligned}$$

Thus the fixed points of S, for a fixed j with $S \in G_j$, are in one-to-one correspondence with the cosets of G_j in G. ☐

In a similar fashion we can prove the next corollary.

Corollary 2.5. *Suppose G is a group admitting an action $G \times M \to M$ satisfying $M/G = \mathbb{P}^1(\mathbb{C})$. Suppose G has a unique subgroup of order p, where p is a prime, and let S be any element of order p. Then the number of fixed points of S is $|G|{\sum_j}' \dfrac{1}{n_j}$, where the prime indicates we sum over those j so that $p|n_j$.*

The cyclic groups $\mathbb{Z}_{p^e}$ have unique subgroups of orders $p, p^2, \ldots, p^e$ and the generalized quaternion goup $Q(2^n)$ has a unique subgroup of order 2, the subgroup generated by the central element B^2. In fact the only p-groups which contain a unique subgroup of order p are the cyclic groups and the generalized quaternion groups.

Let G denote either of these groups and suppose $S \in G$ is an element of order p. If G acts on a Riemann surface M so that $M/G = \mathbb{P}^1(\mathbb{C})$ then there is a short exact sequence as in (6), and moreover the signature of Γ must have the form

$$\operatorname{sig}(\Gamma) = (0|\underbrace{p, \ldots, p}_{r_1}, \underbrace{p^2, \ldots, p^2}_{r_2}, \ldots, \underbrace{p^e, \ldots, p^e}_{r_e}) \text{ if } G = \mathbb{Z}_p^e. \tag{12}$$

$$\operatorname{sig}(\Gamma) = (0|\underbrace{2, \ldots, 2}_{r_1}, \underbrace{4, \ldots, 4}_{r_2}, \ldots, \underbrace{2^{n-1}, \ldots, 2^{n-1}}_{r_{n-1}}) \text{ if } G = Q(2^n). \tag{13}$$

Not all signatures are realizable since there are restrictions on the r_j that must be satisfied in order that $\theta : \Gamma \to G$ be well defined, onto and have torsion free kernel. See Section (4).

Corollary 2.6. *Let G denote either $\mathbb{Z}_{p^e}$ or $Q(2^n)$, and S an element of order p. Suppose G acts on a Riemann surface M so that $M/G = \mathbb{P}^1(\mathbb{C})$. Then the number of fixed points of S is*

$$p^e \sum_{j=1}^{e} \frac{r_j}{p^j} = \sum_{j=1}^{e} r_j p^{e-j} \text{ if } G = \mathbb{Z}_{p^e} \text{ and } 2^n \sum_{j=1}^{n-1} \frac{r_j}{2^j} = \sum_{j=1}^{n-1} r_j 2^{n-j} \text{ if } G = Q(2^n).$$

Proof. This is an immediate consequence of the last corollary. □

3. Actions on a Surface of Genus 0 or 1

In this section we determine all genus-zero actions on either $\mathbb{P}^1(\mathbb{C})$ or a torus. Such actions on $\mathbb{P}^1(\mathbb{C})$ are very easy to describe since the possible groups are just the finite subgroups of $PSL_2(\mathbb{C})$. These are well known to be either cyclic or polyhedral. In terms of our notation for Fuchsian groups the finite subgroups of $PSL_2(\mathbb{C})$ are:

(1) $\Gamma(0 \mid n, n)$, the cyclic group of order n.
(2) $\Gamma(0 \mid 2, 2, n)$, the dihedral group of order $2n$.
(3) $\Gamma(0 \mid 2, 3, 3)$, the tetrahedral group of order 12.
(4) $\Gamma(0 \mid 2, 3, 4)$, the octahedral group of order 24.
(5) $\Gamma(0 \mid 2, 3, 5)$, the icosahedral group of order 60.

This also gives the possible signatures of the actions. Moreover, up to conjugacy in $PSL_2(\mathbb{C})$, each of these groups admits a unique embedding and so a unique genus-zero action on $\mathbb{P}^1(\mathbb{C})$.

Now suppose G acts on a torus M so that (1) is satisfied. It is well known that the group of automorphisms $\mathrm{Aut}(M)$ contains M as a normal subgroup, acting on itself by translations, and that the quotient group is cyclic of order $2, 4$ or 6. In other words there is a short exact sequence

$$1 \to M \to \mathrm{Aut}(M) \to K \to 1, \text{ where } K \cong \mathbb{Z}_2, \mathbb{Z}_4, \text{ or } \mathbb{Z}_6.$$

The elements of the subgroup M act fixed point freely and therefore finite subgroups of M will not satisfy (1). In fact the quotient by any finite subgroup will again be a torus. It follows that G is a subgroup of $\mathbb{Z}_2, \mathbb{Z}_4$ or $\mathbb{Z}_6$, and so $G \cong \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$ or $\mathbb{Z}_6$.

The next theorem summarizes the situation.

Theorem 3.1. *The groups acting on a torus M and satisfying (1) are given as follows:*

$$\begin{aligned} &\mathbb{Z}_2 \text{ with signature } (0 \mid 2,2,2,2).\\ &\mathbb{Z}_3 \text{ with signature } (0 \mid 3,3,3).\\ &\mathbb{Z}_4 \text{ with signature } (0 \mid 2,4,4).\\ &\mathbb{Z}_6 \text{ with signature } (0 \mid 2,3,6).\end{aligned}$$

In all cases the epimorphism $\theta : \Gamma \to G$ is unique up to automorphisms of G.

4. Sylow Group Actions

In this section we prove Theorems (1.9) and (1.10), that is we classify all genus-zero actions of the cyclic groups $\mathbb{Z}_{p^e}$, where $p \geq 2$ is any prime, and the generalized quaternion groups $Q(2^n)$. For each of these groups there are infinitely many possible signatures.

First we consider the case $G = \mathbb{Z}_{p^e}$. Let T be a generator of G and suppose G acts on a Riemann surface M so that $M/G = \mathbb{P}^1(\mathbb{C})$. For the moment we do not assume that the action has genus zero. Then there exists a short exact sequence as in (6) and $\mathrm{sig}(\Gamma)$ is given by (12).

We choose elliptic generators of Γ as follows:

$$X_{j,k} \text{ corresponding to the period } p^j,\ 1 \leq j \leq e,\ 1 \leq k \leq r_j.$$

Then the epimorphism $\theta : \Gamma \to \mathbb{Z}_{p^e}$ is given by:

$$\theta(X_{j,k}) = T^{a_{j,k}p^{e-j}}, \text{ where } 1 \le j \le e,\ 1 \le k \le r_j \text{ and } a_{j,k} \not\equiv 0\ (mod\ p). \tag{14}$$

The conditions on the $a_{j,k}$ guarantee that Π= Ker(θ) is torsion free. To ensure that θ is onto and well defined we also need the conditions:

$$r_e \ge 2 \text{ and } \sum_{j=1}^{e}\sum_{k=1}^{r_j} a_{j,k}p^{e-j} \equiv 0\ (mod\ p^e). \tag{15}$$

Suppose the signature of Π is $(g \mid -)$. Then the Riemann-Hurwitz formula (8) gives

$$2g-2 = p^e\left(-2+\sum_{j=1}^{e} r_j\left(1-\frac{1}{p^j}\right)\right). \tag{16}$$

Let $H = \mathbb{Z}_p$ be the cyclic group of order p generated by some element $S \in \mathbb{Z}_{p^e}$ of order p. According to Corollary (2.6) the number of fixed points of S is

$$t = \sum_{j=1}^{e} r_j p^{e-j}.$$

Using this value of t in Equation (10) and then comparing Equations (10) and (16) we get

$$k = 1-p^{e-1}+\frac{1}{2}\sum_{j=1}^{e} r_j p^{e-j}\left(p^{j-1}-1\right) = \frac{1}{2}\sum_{j=2}^{e-1} r_j p^{e-j}\left(p^{j-1}-1\right)+\frac{1}{2}\left(p^{e-1}-1\right)(r_e-2).$$

All terms on the right hand side of this equation are non-negative since $r_e \ge 2$. Therefore, if the genus of the action is zero, that is if $k = 0$, we conclude that

$$r_1 \ge 0 \text{ is arbitrary, } r_2 = \cdots = r_{e-1} = 0 \text{ and } r_e = 2.$$

If $e = 1$ we interpret this to mean that the signature of the action is $(0 \mid \underbrace{p,\ldots,p}_{r})$, where $r \ge 2$. When $e = 2$ the interpretation is that the signature is $(0 \mid \underbrace{p,\ldots,p}_{r}, p^2, p^2)$, where r is arbitrary.

The calculation of the genus is a simple consequence of the Riemann-Hurwitz formula (8). This concludes the proof of Theorem (1.9).

We can say more about these actions. Recall that $T \in \mathbb{Z}_{p^e}$ is a generator. To simplify notation let the elliptic generators of $\Gamma = \Gamma(0|\underbrace{p,\ldots,p}_{r}, p^e, p^e)$ be denoted by

$$\begin{array}{rcl} X_1,\ldots,X_r & \text{of} & \text{period } p, \\ Y_1, Y_2 & \text{of} & \text{period } p^e. \end{array}$$

Then the epimorphism $\theta : \Gamma \to \mathbb{Z}_{p^e}$ will satisfy

$$\begin{aligned} &\theta(X_j) = T^{a_j p^{e-1}}, \text{ where } a_j \not\equiv 0\ (mod\ p),\ 1 \le j \le r, \\ &\theta(Y_1) = T^{b_1} \text{ and } \theta(Y_2) = T^{b_2}, \text{ where } b_1 \not\equiv 0\ (mod\ p) \text{ and } b_2 \not\equiv 0\ (mod\ p), \\ &p^{e-1}\sum_{j=1}^{r} a_j + b_1 + b_2 \equiv 0\ (mod\ p^e). \end{aligned}$$

Now we prove Theorem (1.10). Let Q denote the generalized quaternion group $Q(2^n)$ and assume $n \ge 4$. The case $n = 3$ will be considered at the end of this section.

Some elementary facts about Q are:

1. The elements of Q are A^j and BA^j, $1 \le j \le 2^{n-1}$.
2. The orders of the elements of Q are $1, 2, 4, \ldots, 2^{n-1}$.
3. There is a unique element of order 2, namely B^2, and it generates the center of Q.

(4) The elements of order 4 are $A^{\alpha 2^{n-3}}$ and BA^a, where α is odd and a is arbitrary.
(5) The elements of order 2^j, where $3 \leq j \leq n-1$, are $A^{\alpha 2^{n-1-j}}$, where α is odd.

Now suppose M is a Riemann surface with an action by Q so that $M/Q = \mathbb{P}^1(\mathbb{C})$. For now we do not assume that the genus of the action is zero. Then there exists a short exact sequence $1 \to \Pi \to \Gamma \xrightarrow{\theta} Q \to 1$, as in (6), and sig($\Gamma$) is given by (13).

We need some notation in order to characterize the epimorphism $\theta : \Gamma \to Q$. Thus let

$$X_{j,k},\ 1 \leq j \leq n-1,\ 1 \leq k \leq r_j,$$

be elliptic generators of Γ chosen to satisfy the relations

(1) $X_{j,k}^{2^j} = 1,\ 1 \leq j \leq n-1,\ 1 \leq k \leq r_j$.
(2) $X_{1,1} \cdots X_{1,r_1} X_{2,1} \cdots X_{2,r_2} \cdots X_{n-1,1} \cdots X_{n-1,r_{n-1}} = 1$.

Then θ must preserve the order of the elliptic generators and therefore we have

(1) $\theta(X_{1,k}) = B^2$, $1 \leq k \leq r_1$.
(2) $\theta(X_{2,k}) = A^{\alpha_{2,k} 2^{n-3}}$ or $BA^{a_{2,k}}$, where $\alpha_{2,k}$ is odd and $a_{2,k}$ is arbitrary.
(3) $\theta(X_{j,k}) = A^{\alpha_{j,k} 2^{n-1-j}}$, where $3 \leq j \leq n-1,\ 1 \leq k \leq r_j$ and the $\alpha_{j,k}$ are odd.

Let the number of $X_{2,k}$ mapping to elements of the form BA^a be r. Then we must have $r > 0$ in order that θ be onto. Moreover, in order that θ be well defined we must have

$$2r_1 + r \equiv 0 \ (mod\ 4). \tag{17}$$

This follows from counting the number of B's in the product relation obtained by applying θ to $X_{1,1} \cdots X_{n-1,r_{n-1}} = 1$. There is also a restriction coming from the power of A, but we do not need it yet.

¿From Equation (17) we see that r must be even, and since $r > 0$, we get $r_2 \geq r \geq 2$.

Now Q has a unique subgroup H of order 2, namely $< B^2 >$, and therefore we need only consider the genus of M/H in order to answer Problem (1). Again according to Corollary (2.6), the number of fixed points of B^2 is given by

$$t = \sum_{j=1}^{n-1} r_j 2^{n-j}.$$

Assume the signature of Π is $(g \mid -)$. Then the Riemann-Hurwitz formulas (8) and (10) give

$$\begin{aligned} 2g-2 &= 2^n \left(-2 + \sum_{j=1}^{n-1} r_j \left(1 - \frac{1}{2^j}\right)\right) \\ &= 2\left(2k - 2 + t\left(1 - \frac{1}{2}\right)\right) \\ &= 4k - 4 + \sum_{j=1}^{n-1} r_j 2^{n-j}. \end{aligned}$$

Assuming $k = 0$ we easily derive the following equation

$$r_2\left(2^{n-1} - 2^{n-2}\right) + r_3\left(2^{n-1} - 2^{n-3}\right) + \cdots + + r_{n-1}\left(2^{n-1} - 2\right) = 2^n - 2. \tag{18}$$

It follows that r_{n-1} must be odd, since the right hand side of Equation (18) is congruent to -2 modulo 4. But then we must have $r_{n-1} = 1$, for otherwise the left hand side would be greater than $2^n - 2$ (this uses $r_2 \geq 2$).

Now it follows that the only solutions of Equation (18) are

$$r_2 = 2, r_3 = \cdots = r_{n-2} = 0 \text{ and } r_{n-1} = 1.$$

If $n = 4$ we interpret this to mean $r_2 = 2$ and $r_3 = 1$.

Thus $r = 2$, and then from Equation (17) we see that r_1 must be odd. This means that the only possible signatures are $(0|\underbrace{2,\ldots,2}_{r_1},4,4,2^{n-1})$, where r_1 is odd. All of these signatures can be realized by genus-zero actions.

If $n = 3$ then the above, suitably interpreted, remains valid. In particular r_1 must be odd and $r_2 = 3$.

The statement about the genus of the action follows easily from the Riemann-Hurwitz formula (8). This concludes the proof of Theorem (1.10).

To simplify the notation let the elliptic generators of Γ be denoted by

$$\begin{array}{rll} X_1,\ldots,X_r & \text{of period} & 2 \\ Y_1,Y_2 & \text{of period} & 4 \\ Z & \text{of period} & 2^{n-1} \end{array}$$

The homomorphism $\theta : \Gamma \to Q(2^n)$ must be onto, respect the relations in Γ and have a torsion free kernel. Therefore, for $n \geq 4$:

$$\theta(X_1) = \cdots = \theta(X_r) = B^2, \theta(Y_1) = BA^{a_1}, \theta(Y_2) = BA^{a_2}, \theta(Z) = A^\alpha$$

where a_1, a_2 are arbitrary, α is odd, r is odd, and $a_2 - a_1 + \alpha \equiv 0\ (mod\ 2^{n-1})$.

There are similar restrictions for $n = 3$.

5. Cyclic Groups

In this section we determine all genus-zero actions for all cyclic groups. In particular we prove Theorem (1.11). Every cyclic group has genus zero since they can act on $\mathbb{P}^1(\mathbb{C})$. Moreover the cyclic groups $\mathbb{Z}_{p^e}$, where $p \geq 2$ is any prime, admit actions of genus zero on Riemann surfaces of arbitrarily high genus. See Theorem (1.9). This will not be true for other cyclic groups.

Now we begin the proof of Theorem (1.11). Thus let $G = \mathbb{Z}_{pq}$, where $p,\ q$ are distinct primes, and choose the presentation

$$G = \langle A, B \mid A^p = 1, B^q = 1, AB = BA\rangle\,.$$

Suppose G acts on a Riemann surface M so that $M/G = \mathbb{P}^1(\mathbb{C})$. Then there is a short exact sequence $1 \to \Pi \to \Gamma \xrightarrow{\theta} G \to 1$, as in (6), where Π is a torsion free Fuchsian group of signature $(g\ |-)$ and

$$\text{sig}(\Gamma) = (0 \mid \underbrace{p,\ldots,p}_{r},\underbrace{q,\ldots,q}_{s},\underbrace{pq,\ldots,pq}_{t}).$$

The Riemann-Hurwitz formula (8) gives

$$2g-2 = pq\left(-2 + r\left(1-\frac{1}{p}\right) + s\left(1-\frac{1}{q}\right) + t\left(1-\frac{1}{pq}\right)\right).$$

Let the elliptic generators of Γ be

$$X_1,\ldots,X_r \text{ of order } p;\ Y_1,\ldots,Y_s \text{ of order } q;\ Z_1,\ldots,Z_t \text{ of order } pq.$$

Let H be the subgroup of G generated by A. Applying Corollary (2.5) we see that there are $rq + t$ fixed points of A. The signature of $\Gamma' = \theta^{-1}(H)$ will have the form $(k|\underbrace{p,\ldots,p}_{rq+t})$ and therefore by (10) we see that

$$2g-2 = p\left(2k-2+(rq+t)\left(1-\frac{1}{p}\right)\right).$$

If $k = 0$ then a comparison of these formulas for $2g-2$ gives $s + t = 2$. The same argument applied to the subgroup generated by B gives $r + t = 2$. The only solutions of these equations are

$$(r, s, t) = (0, 0, 2), (1, 1, 1), (2, 2, 0).$$

All 3 solutions yield actions with genus zero. The genus in each case is computed by the Riemann-Hurwitz formula. This completes the proof of Theorem (1.11).

It is simple enough to describe the epimorphism θ in each of these 3 cases.

(1) $\theta(Z_1) = A^a B^b$ and $\theta(Z_2) = A^{-a}B^{-b}$, where $1 \leq a \leq p-1$ and $1 \leq b \leq q-1$.
(2) $\theta(X_1) = A^a,\ \theta(Y_1) = B^b$ and $\theta(Z_1) = A^{-a}B^{-b}$, where $1 \leq a \leq p-1$ and $1 \leq b \leq q-1$.
(3) $\theta(X_1) = A^a,\ \theta(X_2) = A^{-a},\ \theta(Y_1) = B^b$ and $\theta(Y_1) = B^{-b}$, where $1 \leq a \leq p-1$ and $1 \leq b \leq q-1$.

Corollary 5.1. *Suppose G is a cyclic group $\mathbb{Z}_n$ with at least 3 distinct primes dividing n. Then G does not admit a genus-zero action on a Riemann surface of positive genus.*

Proof. If G did admit such an action so would $\mathbb{Z}_{pqr}$, where p, q, r are distinct primes dividing n. According to Theorem (1.11) we would have

$$2^{\alpha}(p-1)(q-1) = 2^{\beta}(p-1)(r-1) = 2^{\gamma}(q-1)(r-1),$$

where α, β, γ are either 0 or -1. This is not possible. □

The next theorem is proved in a similar fashion, but requires a tedious case-by-case analysis. We omit the details.

Theorem 5.2. *$\mathbb{Z}_{p^2q}$, where p, q are distinct primes, does not admit a genus-zero action on a Riemann surface of positive genus.*

6. Zassenhaus Metacyclic Groups

In this section we determine which ZM groups $G_{m,n}(r)$ have genus zero and then prove Theorem (1.12). First note that $G_{m,2}(-1)$, where m is odd, is the dihedral group D_{2m} and therefore acts on $\mathbb{P}^1(\mathbb{C})$. Thus $G_{m,2}(-1)$, for m odd, has genus zero. By a routine, but lengthy computation, it is possible to show that the only ZM groups acting on $\mathbb{P}^1(\mathbb{C})$ are $G_{m,2}(-1)$.

Let G be a ZM group G other than $G_{m,2}(-1)$. Then, according to Theorem (3.1), any genus-zero action must be on a surface of genus $g > 1$. Thus the pq-conditions, see Definition (1.6), must hold.

Therefore a necessary condition for G to admit an action of genus zero is that m, n have at most 2 primes in their prime power factorizations. See Corollary (5.1). Moreover, according to Theorem (5.2), both m and n must be either prime powers or a product of distinct primes.

On the other hand, if our goal is to show that a certain G does not admit an action of genus zero then we may assume m is a prime p since G contains the ZM subgroup generated by A^k and B, where k is chosen so that the order of A^k is p. If the subgroup does not admit a genus-zero action neither will G. There are then 2 cases to consider. First we could have $n = q^e$ and secondly we could have $n = q_1q_2$, where q_1, q_2 are distinct primes.

Assume $n = q_1q_2$. If d is the order of r modulo p then there are 3 choices for d, namely $d = q_1,\ q_2$ or q_1q_2. The possibilities for $\dfrac{n}{d}$ are then $q_2,\ q_1$ or 1 respectively. However, the pq conditions are equivalent to the statement that every prime divisor of d also divides $\dfrac{n}{d}$. Thus there are no choices of d satisfying the pq conditions and so $n = q^e$.

Therefore we first consider the case of a ZM group G where $m = p$ and $n = q^e$, where p, q are primes. The numerical restrictions on $m,\ n$, see (1.7), imply that $p,\ q$ are distinct primes. Moreover we may assume that $e \geq 2$ since otherwise G would be cyclic. In fact we will start with the special case of a ZM group G with $m = p$ and $n = q^2$, where p and q are distinct odd primes. The case where one of the primes is 2 will be treated later in this section. The group G will then satisfy the conditions in Definition (1.6) if, and only if, $d = q$.

In other words we are considering the group G presented by

$$\begin{array}{ll} \text{generators} & : A, B \\ \text{relations} & : A^p = 1,\ B^{q^2} = 1,\ BAB^{-1} = A^r,\ p, q \text{ distinct odd primes} \\ \text{conditions} & : GCD((r-1)q, p) = 1,\ r^q \equiv 1\ (mod\ p),\ \text{and } r \not\equiv 1\ (mod\ p). \end{array} \tag{19}$$

Lemma 6.1. *The ZM group in (19) does not admit a genus-zero action.*

Proof. By induction we easily prove that

$$\left(A^iB^j\right)^k = A^i(1+r^j+r^{2j}+\cdots+r^{(k-1)j})B^{jk}.$$

Thefore the possible orders of elements of G are $1,\ p,\ q,\ q^2$ and pq. In particular the order of the element AB^q is pq. Let $H\cong\mathbb{Z}_{pq}$ be the subgroup generated by AB^q.

Assume G admits an action of genus zero on some Riemann surface M. Then there exists a short exact sequence $1\to\Pi\to\Gamma\xrightarrow{\theta}G\to 1$ as in (6), where $\Pi=\Gamma(g\mid -)$ for some $g\geq 0$, and

$$\operatorname{sig}(\Gamma)=(0\mid \underbrace{p,\ldots,p}_{r},\underbrace{q,\ldots,q}_{s},\underbrace{q^2,\ldots,q^2}_{t},\underbrace{pq,\ldots,pq}_{u}),$$

for some choice of non-negative integers $r,\ s,\ t,\ u$. These integers must be chosen so that $\theta:\Gamma\to G$ is a well defined epimorphism with torsion free kernel. In particular this means that

$$t\geq 2,\ \text{and if } t=2 \text{ then } r+s+u>0.$$

The restriction $t\geq 2$ follows from the fact that $G_{ab}\cong\mathbb{Z}_{q^2}$, and therefore there is an epimorphism $\Gamma\to\mathbb{Z}_{q^2}$.

The Riemann-Hurwitz formula becomes

$$\begin{aligned}2g-2 &= pq^2\left(-2+r\left(1-\frac{1}{p}\right)+s\left(1-\frac{1}{q}\right)+t\left(1-\frac{1}{q^2}\right)+u\left(1-\frac{1}{pq}\right)\right)\\ &= -2pq^2+rq^2(p-1)+spq(q-1)+tp(q^2-1)+uq(pq-1)\\ &= (t-2)pq^2+(r+s+u)pq^2-rq^2-spq-tp-uq.\end{aligned}$$

The action of the subgroup $H\cong\mathbb{Z}_{pq}$ on the Riemann surface M has genus-zero and therefore, according to Theorem (1.11), there are only 3 possibilities for g:

$$g=0,\ g=\frac{1}{2}(p-1)(q-1) \text{ or } g=(p-1)(q-1).$$

We will analyze each case separately and conclude that the group G does not admit an action of genus zero.

(1) Assume $g=0$. Then necessarily

$$r\left(1-\frac{1}{p}\right)+s\left(1-\frac{1}{q}\right)+t\left(1-\frac{1}{q^2}\right)+u\left(1-\frac{1}{pq}\right)<2.$$

The left hand side will be larger than what we get by putting $p=q=3$ and using $t=2$. This leads to the inequality $6r+6s+8u<2$, and so $r=s=u=0$. But then θ will not be an epimorphism.

(2) Assume $g=\frac{1}{2}(p-1)(q-1)$. Then the Riemann-Hurwitz equation becomes

$$(t-2)pq^2+(r+s+u)pq^2-rq^2-spq-tp-uq=(p-1)(q-1)-2,$$

which can be rewritten in the form

$$(t-2)p(q^2-1)+rq^2(p-1)+(s+1)pq(q-1)+(u-1)q(pq-1)=p-1.$$

¿From this equation it follows that $u=0$, for otherwise the left hand side would be greater than $p-1$. Recall that $t\geq 2$. Thus the equation becomes

$$(t-2)p(q^2-1)+rq^2(p-1)+(s+1)pq(q-1)-q(pq-1)=p-1,$$

which is equivalent to

$$(t-2)p(q^2-1)+rq^2(p-1)+spq(q-1)=(p-1)(q+1).$$

Now it is simple to see that there are no solutions.

(3) The last case to consider is $g=(p-1)(q-1)$. By arguments similar to the second case there are no solutions of the Riemann-Hurwitz equation.

□

Next we show that a ZM group of odd order does not admit actions of genus zero. Note that by the above considerations we need only prove this when $m = p$ and $n = q^e$, where p and q are distinct odd primes and $e > 2$.

That is suppose G is the ZM group presented by

$$\begin{array}{ll} \text{generators} & : A, B \\ \text{relations} & : A^p = 1,\ B^{q^e} = 1,\ BAB^{-1} = A^r,\ e > 2 \\ \text{conditions} & : GCD((r-1)q, p) = 1, r^{q^e} \equiv 1\ (mod\ p). \end{array} \tag{20}$$

Lemma 6.2. *A ZM group of odd order does not admit genus-zero actions.*

Proof. We need only show that the group G presented in (20) does not have genus zero. The order d of r modulo p will be q^f for some f, $1 \le f < e$. If $f = 1$ then A commutes with $B^q, \ldots, B^{q^{e-1}}$, and since $e > 2$, it follows that G has a subgroup isomorphic to $\mathbb{Z}_{pq^2}$, namely the subgroup $< A, B^q >$. But this contradicts Theorem (5.2).

Thus suppose $f > 1$. Then consider the subgroup G_1 generated by A and $B_1 = B^{q^{f-1}}$. This is a ZM group with presentation

$$A^p = 1,\ B_1^{q^{e-f+1}} = 1,\ B_1AB_1^{-1} = A^{r_1},\ \text{where } r_1 = r^{q^{f-1}}.$$

But now the order of r_1 is q, and again we arrive at a contradiction. Therefore a ZM group of odd order does not admit actions of genus zero. □

The last case to consider is the case of a ZM group G of even order. Thus G has a presentation as in (4), where one of m, n is even. In fact the conditions in (4) imply that m is odd and n is even. According to Corollary (5.1) and Theorem (5.2) the only possibilities for $m,\ n$ are

$$\begin{aligned} m &= p^e \text{ or } pq,\ \text{where } p,\ q \text{ are odd primes and } p \neq q, \\ n &= 2^f \text{ or } 2q' \text{ where } q' \text{ is an odd prime.} \end{aligned}$$

If $n = 2q'$ then $d = 2,\ q'$ or $2q'$. In all cases the pq conditions can not be satisfied. Therefore $n = 2^f$. Moreover we must have $f \ge 2$ to satisfy the pq conditions. In fact $d = 2^j$ for some j, $1 \le j \le f - 1$.

Now assume $m = pq$. Then the subgroup generated by A and $B^{2^{f-1}}$ is cyclic of order $2pq$. This contradicts Corollary (5.1) and so $m = p^e$. But now Theorem (5.2) implies that $e = 1$. Finally we must have $f = 2$ and $d = 2$. In other words the only possible ZM group satisfying (1) is $G_{p,4}(-1)$.

Next we show that $G = G_{p,4}(-1)$ admits genus-zero actions. The following statements are easy to prove.

(1) The order of G is $4p$ and the orders of the elements of G are $1,\ 2,\ 4,\ p$ and $2p$.
(2) There is a unique subgroup of order 2, namely the subgroup generated by B^2.
(3) There is a unique subgroup of order p, namely the subgroup generated by A.

Assume there is a genus-zero action $G \times M \to M$, where M has genus g. Then there is a short exact sequence

$$1 \to \Pi \to \Gamma \xrightarrow{\theta} G \to 1$$

as in (6), where $\Pi = \Gamma(g \mid -)$ and

$$\text{sig}(\Gamma) = (0 \mid \underbrace{2, \ldots, 2}_{r}, \underbrace{4, \ldots, 4}_{s}, \underbrace{p, \ldots, p}_{t}, \underbrace{2p, \ldots, 2p}_{u}).$$

The Riemann-Hurwitz formula becomes

$$\begin{aligned} 2g - 2 &= 4p\left(-2 + r\frac{1}{2} + \frac{3s}{4} + t\left(1 - \frac{1}{p}\right) + u\left(1 - \frac{1}{2p}\right)\right) \\ &= -8p + 2pr + 3ps + 4t(p-1) + 2u(2p-1). \end{aligned}$$

Applying Corollary (2.5) we see that

(1) The number of fixed points of B^2 is $2pr + ps + 2u$.
(2) The number of fixed points of A is $4t + 2u$.

Now applying the Riemann-Hurwitz formula (10) to each of these cases gives

$$\begin{aligned} 2g-2 &= 2\left(-2+(2pr+ps+u)\frac{1}{2}\right) \\ &= p\left(-2+(4t+u)(1-\frac{1}{p})\right) \end{aligned}$$

Working with these equations we see that the only solution is

$$r = 0, s = 2, t = 1, u = 0.$$

This completes the proof of Theorem (1.12).

To describe the epimorphism $\theta : \Gamma \to G$ associated to the genus-zero action let the elliptic generators of Γ be X, Y and Z, chosen so that

$$X^4 = Y^4 = Z^p = XYZ = 1.$$

Then all admissible epimorphisms $\theta : \Gamma \to G$ are given by

$$\begin{aligned} &\theta(X) = B^{\epsilon}A^{a},\ \theta(Y) = B^{-\epsilon}A^{b},\ \theta(Z) = A^{c}, \\ &\text{where } a, b \text{ are arbitrary},\ 1 \le c \le p-1, \\ &\text{and } -a+b+c \equiv 0\ (mod\ p). \end{aligned}$$

We conclude this section with an example illustrating the action when $a = 1, b = 0, c = 1$ and $\epsilon = 1$. In this case the action of Γ on the Poincaré disc can be described as follows. Consider the triangle with vertices E, F and G and angles $\alpha = \pi/p$, $\beta = \pi/4$ and β (respectively) centered at the origin of the hyperbolic disc. Let X and Y be rotations by an angle $2\pi/4$ about the vertices F and G respectively, and let Z be rotation about E with angle $2\pi/p$. A fundamental domain for Γ is a copy of the shaded triangle $(p, 4, 4)$ together with its reflection along the side EG. See Figure 1.

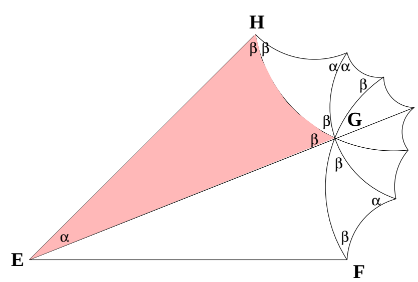

Figure 1.

The translates under the rotations X, Y and Z of this fundamental domain tessellate the hyperbolic disc. Each rotation corresponds to the composition of two reflections along a pair of sides. Fig. 1 shows four copies of the fundamental domain corresponding to four rotations about the vertex G.

Recall that G acts on a Riemann surface M of genus $p-1$ and one has the following tower of quotients and group actions

$$\begin{tikzcd} \mathbb{U} \arrow[d, "\Pi"] \arrow[dd, bend right=60, "\Gamma"'] \\ M \arrow[d, "G"] \\ \Sigma \end{tikzcd}$$

A fundamental domain for Π consists of a choice of $4p$ copies of the fundamental domain of Γ (here $|G| = 4p$). An explicit choice can be obtained from the fundamental domain for Γ (Figure 1) by rotating p times about the vertex E. See Figure 2. The surface M is obtained from this fundamental domain by making certain boundary identifications.

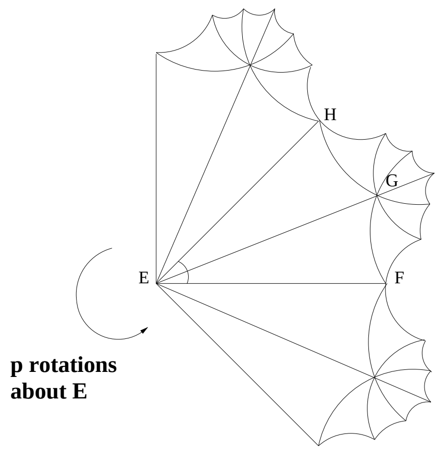

Figure 2.

Let $H \subset G$ be the subgroup of order p generated by A. Then there is a short exact sequence $1 \longrightarrow \Pi \longrightarrow \Gamma' \longrightarrow H \longrightarrow 1$, where $[\Gamma : \Gamma'] = [G : H] = 4$ and $\mathbb{U}/\Gamma' = M/H$. A fundamental domain of Γ' in $\mathbb{U}$ will consist of four copies of the fundamental domain for Γ, and this is depicted in Figure 1. Rotation by $2\pi/p$ about every second vertex in Fig. 1 represents a generator of H, see Fig. 3. The quotient $\mathbb{U}/\Gamma'$ is obtained by identifying the sides as depicted, and is easily seen to be $\mathbb{P}^1(\mathbb{C})$.

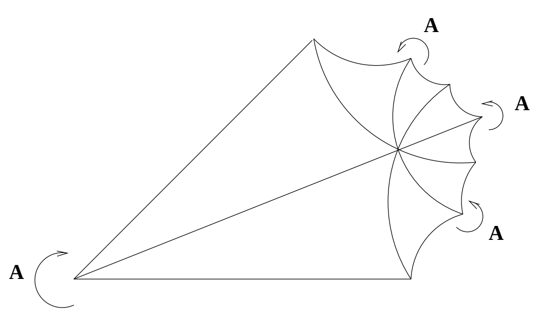

Figure 3.

The same analysis can be carried out for the unique subgroup of order 2, $\langle B^2 \rangle \subset G$. In this case the fundamental domain consists of $2p$ copies of the fundamental domain of Γ and the identifications are given as follows (Figure 4).

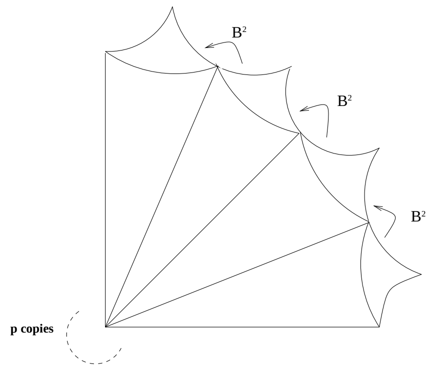

Figure 4.

Contiguous pairs of sides on the boundary are identified by B^2. The quotient surface is again $\mathbb{P}^1(\mathbb{C})$, as expected.

7. The General Case

In this section we complete our analysis of groups G satisfying (1) by establishing Theorem (1). There are two possibilities to consider, either G is solvable or it is non-solvable.

First we suppose G is a finite non-solvable group satisfying the pq conditions, see Definition (1.6). We want to show that G does not admit actions of genus zero. A typical example of such a group is the binary icosahedral group I^*, see Wolf [5]. In fact any such G must contain I^* as a subgroup and therefore we need only show that I^* does not admit an action of genus zero.

The binary icosahedral group has order 120 and contains elements of orders $1, 2, 3, 4, 5, 6$ and 10. By applying Theorem (1.11) we see that if I^* did admit an action of genus zero on some Riemann surface M the genus of M would have to be 2. Moreover the signature of the action would be

$$\mathrm{sig}(\Gamma) = (0 \mid \underbrace{2, \ldots, 2}_{r}, \underbrace{3, \ldots, 3}_{s}, \underbrace{4, \ldots, 4}_{t}, \underbrace{5, \ldots, 5}_{u}, \underbrace{6, \ldots, 6}_{v}, \underbrace{10, \ldots, 10}_{w}).$$

Applying the Riemann-Hurwitz Formula (8) with $g = 2$ gives the diophantine equation

$$30r + 40s + 45t + 48u + 50v + 54w = 121.$$

It is routine to show that this equation does not have any non-negative integral solutions.

This proves the following theorem.

Theorem 7.1. *If G admits an action of genus zero then G is a solvable group.*

Now suppose G is a solvable group satisfying the pq conditions. According to the table on page 179 of [5] there are 4 types of such groups G, (denoted I, II, III and IV), and every type contains some ZM subgroup $G_{m,n}(r)$. According to Theorem (1.12) m must be an odd prime p, $n = 4$ and $r = -1$ if there is to be a genus-zero action. This rules out types III and IV.

Type I is just the ZM group $G_{p,4}(-1)$ and type II is the group G with the following presentation:

$$\begin{aligned} \text{generators :} \quad & A, B, R; \\ \text{relations :} \quad & A^p = 1,\ B^4 = 1,\ BAB^{-1} = A^{-1}, \\ & R^2 = B^2,\ RAR^{-1} = A^l,\ RBR^{-1} = B^{-1}; \\ \text{conditions :} \quad & l \equiv \pm 1\ (mod\ p). \end{aligned}$$

In fact we must have $l = -1$ since otherwise the subgroup $\langle\ A, R\ \rangle$ would be cyclic of order $4p$, contradicting Theorem (5.2). But now $(RAB)^4 = A^4$ and therfore RAB has order $4p$. Again this is a contradiction.

This completes the proof of Theorem (1).

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