Divisor Spaces on Punctured Riemann Surfaces

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Abstract

In this paper, we study the topology of spaces of n-tuples of positive divisors on (punctured) Riemann surfaces which have no points in common (these are the *divisor spaces* as defined in [18], [7]). These spaces arise in connection with spaces of based holomorphic maps from Riemann surfaces to complex projective spaces. We find that there are Eilenberg-Moore type spectral sequences converging to their homology. These spectral sequences collapse at the E^2 term and we essentially obtain complete homology calculations. We also study the homotopy type of certain mapping spaces obtained as a suitable direct limit of the divisor spaces. These mapping spaces, first considered in [18], were studied in a special case in [7] and conjectured to split there. In this paper, we show that the splitting does occur provided we invert the prime two.

§0 Introduction

Let $X = M_g$ be a genus g compact oriented Riemann surface with $g \geq 0$; $M_0 = \mathbb{P}^1$ being the Riemann sphere. For the rest of this paper, we will make use of a preferred basepoint x_0 (or $*$) in M_g. Let $SP^r(X)$ denote the r-fold symmetric product of X (i.e. the space of degree r positive divisors on X). We define the subspace

0.1 $$\mathrm{Div}_{k_1,k_2,\ldots,k_n}(X) \subset SP^{k_1}(X) \times \cdots \times SP^{k_n}(X)$$

to be the set of tuples of positive divisors $(D_{k_1}, \ldots, D_{k_n})$ such that $D_{k_1} \cap \cdots \cap D_{k_n} = \emptyset$. In other words, $\mathrm{Div}_{k_1,k_2,\ldots,k_n}(X)$ is the space of divisors D_{k_i}

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on X, $i = 1, \ldots, n$, of degree k_i, and having no points in common. The relevance of these spaces of divisors to spaces of holomorphic maps is now explained.

First assume $g = 0$. Then by associating to every meromorphic function on $\mathbb{P}^1$ its (disjoint) sets of zeros and poles, we can identify the divisor space $\mathrm{Div}_{k,k}(S^2 - *)$ with the space of degree k based self-holomorphic maps of the Riemann sphere, that is

0.2 $$\mathrm{Div}_{k,k}(S^2 - x_0) = \mathrm{Hol}_k^*(\mathbb{P}^1, \mathbb{P}^1).$$

The space $\mathrm{Hol}_k^*(\mathbb{P}^1, \mathbb{P}^1)$ here consists of rational maps $f(z) = p(z)/q(z), z \in \mathbb{C}$ where p and q are monic polynomials of degree k (hence sometimes the notation Rat_k for $\mathrm{Hol}_k^*(\mathbb{P}^1, \mathbb{P}^1)$). We may also write $\mathrm{Rat}_k(\mathbb{P}^n)$ for $\mathrm{Hol}_k^*(\mathbb{P}^1, \mathbb{P}^n)$ and here too we have the identification ([18])

0.3 $$\mathrm{Div}_{\underbrace{k, k, \ldots, k}_{n+1}}(S^2 - x_0) = \mathrm{Rat}_k(\mathbb{P}^n).$$

When $g \geq 1$, the connection between the divisor spaces Div and spaces of holomorphic maps is less direct and it is given essentially by a classical theorem of Abel. Recall that every Riemann surface M_g embeds, via the Abel-Jacobi map μ ([1]), into its associated Jacobi variety $J(M_g)$ which is a complex g dimensional torus. The map μ extends additively to $SP^k(M_g), \forall k$. A classical theorem of Abel ([1], chap. I) translates to the statement that the space of degree k based holomorphic maps $Hol_k^*(M_g, \mathbb{P}^n)$ is the subspace of $\mathrm{Div}_{k,\ldots,k}(M_g - x_0)$ consisting of $(n+1)$-tuples of divisors with the property that
$$\mu(D_1) = \mu(D_2) = \cdots = \mu(D_{n+1}).$$

Using the spaces $\mathrm{Div}(M_g - x_0)$ as intermediate constructs, Segal was able to prove an interesting stability result for spaces of holomorphic maps on Riemann surfaces. More explicitly, he showed that the natural inclusion $I : \mathrm{Hol}_k^*(M_g, \mathbb{P}^n) \hookrightarrow \mathrm{Map}_k^*(M_g, \mathbb{P}^n)$, obtained by simply forgetting the holomorphic structure, induces a homotopy equivalence through a range increasing with k. These results are greatly extended in [13].

A systematic study of the divisor spaces was initiated in [7] where the authors constructed a homotopy model whose cohomology is related to the homology of the Div spaces via Alexander-Poincaré duality. Starting with that model, we are able to construct a homology spectral sequence of the Eilenberg-Moore type, converging to the homology of the Div spaces, and then show that this spectral sequence collapses at the E^2 term for all $g \geq 0$ and for all n. This then yields our first main theorem

Theorem 0.4: *For field coefficients* $\mathbb{F}$*, we have the following isomorphism*

$$H_*(Div_{\underbrace{k,\ldots,k}_{n}}(M_g - *);\mathbb{F}) \cong Tor^{H_*(SP^\infty(M_g))}_{2nk-*,k}(\mathbb{F}, H_*(SP^\infty(M_g);\mathbb{F})^{\otimes n}).$$

To clarify the statement of the theorem above, we need indicate that there is a bigraded algebra structure on the homology groups of $SP^\infty(X)$ yielding in the appropriate manner the bigrading of the Tor term above. The theorem and the details leading to it are discussed in §4. When $n > 2$, the module structure of $H_*(SP^\infty(M_g))^{\otimes n}$ over $H_*(SP^\infty(M_g))$ is trivial and so the calculations are direct. We write $\mathrm{Div}_{\underbrace{k,\ldots,k}_{n}} = \mathrm{Div}^n_k$. One has for instance (§6)

Corollary 0.5: *Assume* $n > 2$ *and* $g \geq 1$*. Then the rational homology of* $Div^n_k(M_g - *)$ *is the subset of the* $(n+1)$*-graded algebra*

$$\Lambda(e_{1;1},\ldots,e_{2g;1},\ldots,e_{1;n},\ldots,e_{2g;n},E) \otimes \mathbb{Q}(h_1,\ldots,h_{2g})$$

where the grading is assigned as follows: $e_{i;r} \mapsto (1;0,\ldots,1,\ldots,0)$*, with* 1 *in the* $r+1$ *position,* $1 \leq r \leq n, 1 \leq i \leq 2g$*,* $E \mapsto (2n-3;1,\ldots,1)$*,* $h_j \mapsto (2n-2;1,\ldots,1)$*. The multigrading is additive. In this setting,* $H_*(Div^n_k(M_g - *);\mathbb{Q})$ *is given by those elements of multidegree* $(*;i_1,\ldots,i_n)$ *with* $i_j \leq k$*.*

Similar results are obtained mod-p. When $g = 0$, 0.4 takes a quite simple and explicit expression for all n and one recovers the original results of [7] on the homology structure of the Rat spaces (§5). In the case when $n = 2$, the module structure in the Tor term of 0.4 is non-trivial and the calculations are much more tedious.

Remark 0.6: The spectral sequence that is considered in this paper and the resulting collapse are used in [13] to study the spaces $\mathrm{Hol}^*_k(M_g, \mathbb{P}^{n-1})$ themselves. It's not coincidental that there too the homology structure does depend on the cases $n = 2$ and $n > 2$.

We can stabilize the divisor spaces with respect to "collar" inclusions

$$\mathrm{Div}_{k_1,\ldots,k_n}(M_g - *)\longrightarrow \mathrm{Div}_{k_1,\ldots,k_{j-1},k_j+1,k_{j+1},\ldots,k_n}(M_g - *)$$

obtained by first continuously deforming $\mathrm{Div}_{k_1,\ldots,k_n}(M_g - *)$ to $\mathrm{Div}_{k_1,\ldots,k_n}(M_g - U)$ where U is a small neighborhood of $*$, and then adding a chosen point $x \neq * \in U$ to the jth divisor. It is now a theorem of Segal that the direct limit over these inclusions is homotopy equivalent to a component of a known (based) mapping space; i.e.

0.7 $$\lim_{k\to\infty} \mathrm{Div}^n_k(M_g - *) \simeq \mathrm{Map}^*_0(M_g, W_n(\mathbb{P})) \qquad \text{(Segal)}$$

where $W_n(\mathbb{P})$ is the n^{th} fat wedge of the infinite complex projective space $\mathbb{P}$ (or $\mathbb{P}^\infty$) and where Map_0 denotes the component of null-homotopic maps. The fat wedge $W_n(X) \subset X^n$ is the subset consisting of tuples where at least one entry is basepoint (eg. $W_1 = * \in X$ and $W_2 = X \vee X$.) In §7 we establish the existence of a fibration with a section

$$S^{2n-1}\longrightarrow W_n(\mathbb{P}^\infty) \hookrightarrow (\mathbb{P}^\infty)^n$$

which when coupled with the mapping space fibration obtained by mapping the cofibration sequence $\bigvee S^1 \to M_g \to S^2$ into $W_n\mathbb{P}$, yields the fibration

0.8 $$\Omega^2 S^{2n-1}\longrightarrow \mathrm{Map}^*_0(M_g, W_n\mathbb{P})\longrightarrow (S^1)^{2ng} \times (\Omega S^{2n-1})^{2g}.$$

It is now not hard to see (§9) that as a result of 0.4 and 0.7 we have

Proposition 0.9: *The (cohomology) Eilenberg-Moore spectral sequence associated to $\Omega^2 S^{2n-1} \to \mathrm{Map}^*_0(M_g, W_n\mathbb{P}) \to (\Omega W_n\mathbb{P})^{2g}$ and converging to $H^*(\mathrm{Map}^*_0(M_g, W_n\mathbb{P}); \mathbb{F})$ collapses at the E_2 term.*

This leads to the determination of $H^*(\mathrm{Map}^*_0(M_g, W_n\mathbb{P}); \mathbb{F})$ and results there turn out to be consistent with the conjecture of [7] which states that the term $(\Omega S^{2n-1})^{2g}$ in the base of 0.8 ought to split off from the mapping space. More explicitly, and in the relevant case when $n = 2$, [7] state that there should be a decomposition

$$\mathrm{Map}^*(M_g, \mathbb{P} \vee \mathbb{P}) \simeq (\mathbb{Z})^2 \times \Omega(S^3)^{2g} \times Y_g,$$

where Y_g is the total space of a fibration $\Omega^2(S^3)\longrightarrow Y_g\longrightarrow (S^1)^{4g}$. The existence of such a splitting is also very much suggested by results of [3] who prove similar decomposition results for $\mathrm{Map}^*(M_g, S^{2n}), n \geq 1$ (see §7). It turns out however that there is an obstruction to such a decomposition.

In §8 we take up the study of the homotopy type of the mapping space $\mathrm{Map}^*(M_g, W_n(\mathbb{P}))$. The problem there becomes to factor the classifying map associated to 0.8 as follows

$$f^! : (\Omega S^{2n+1} \times (S^1)^{n+1})^{2g} \longrightarrow (S^1)^{2g(n+1)} \longrightarrow \Omega S^{2n+1} \hookrightarrow \Omega S^{2n+1} \times (S^1)^{(n+1)}.$$

A first look at $f^!$ shows that there are essential $\mathbb{Z}_2$ obstructions to such a factorization when $n > 2$. When $n = 2$, a close examination of the Postnikov system of $\mathbb{P}^\infty \vee \mathbb{P}^\infty$ shows that there is a non-zero obstruction to the above decomposition taking the form of a triple Whitehead product

$$[a_1, [a_1, a_2]] \in \pi_4(\mathbb{P} \vee \mathbb{P}) \cong \mathbb{Z}_2.$$

So in all cases we're up against essential $\mathbb{Z}_2$ obstructions and we show that

Proposition 0.10: *There is a splitting after inverting* 2

$$Map_0^*(M_g, W_n\mathbb{P}) \simeq (\Omega S^{2n-1})^{2g} \times Y_{g,n}$$

where $Y_{g,n}$ *is the total space of a fibration* $\Omega^2(S^{2n-1}) \longrightarrow Y_{g,n} \longrightarrow (S^1)^{2ng}$.

§1 The Structure of Symmetric Products

Given a space X, we let $SP^n(X) = X^n/\mathcal{S}_n$ denote the n-th symmetric product of X (here $\mathcal{S}_n$ is the group on n-letters acting by permuting factors). Equivalently, $SP^n(X)$ is the set of all unordered n-tuples $\langle x_1, \dots, x_n \rangle$ of points in X.

Let $*$ be a chosen base point in X, then there are natural inclusions $SP^n(X) \hookrightarrow SP^{n+1}(X)$ which identify $\langle x_1, \dots, x_n \rangle$ with $\langle x_1, \dots, x_n, * \rangle$, and we get the expanding sequence of spaces

$$* \equiv SP^0(X) \subset SP^1(X) \subset \cdots \subset SP^{n-1}(X) \subset SP^n(X) \subset \cdots$$

The direct limit over these inclusions is the infinite symmetric product $SP^\infty(X, *)$ (topologized by the weak topology relative to the union of the $SP^i(X)$.) The pairing

$$\begin{array}{ccc} SP^n(X) \times SP^m(X) & \xrightarrow{\mu} & SP^{n+m}(X) \\ \langle x_1, \dots, x_n \rangle \times \langle y_1, \dots, y_m \rangle & \mapsto & \langle x_1, \dots, x_n, y_1, \dots, y_m \rangle \end{array}$$

turns $SP^\infty(X,*)$ into an abelian and associative monoid with pairing

$$\mu : SP^\infty(X,*) \times SP^\infty(X,*) \longrightarrow SP^\infty(X,*).$$

We often write $\cdot$ (or $+$) for addition in $SP^\infty(X,*)$; that is: $\mu((x,y)) = x\cdot y = x + y = xy$ are all equivalent notations. We use the same notation for the induced Pontryagin product on $H_*(SP^\infty(X);\mathbb{A})$.

Since $SP^\infty(X,*)$ is abelian, it must be a product of Eilenberg-McLane spaces and one actually has

Theorem 1.1: (Dold-Thom) *$\pi_*(SP^\infty(X,*)) \cong H_*(X;\mathbb{Z})$ and hence*

$$SP^\infty(X,*) = \prod K\left(\tilde{H}_i(X;\mathbb{Z}), i\right).$$

For example; $SP^\infty(S^n,*) \simeq K(\mathbb{Z},n)$, $n \geq 1$.

PROPERTIES:

- The finite and infinite symmetric products are covariant functors on the category of pointed topological spaces. If $f : (X,*) \to (Y,*)$ is a map of pairs, then the induced maps on the symmetric products are denoted by $SP^n f : SP^n(X) \longrightarrow SP^n(Y)$ and $SP^\infty f : SP^\infty(X,*) \longrightarrow SP^\infty(Y,*)$.
- $SP^n(-)$ is a homotopy functor. In particular, if cX denotes the cone on X, then $SP^n(cX)$ is contractible for all n.

FACTS:

Here are now some known properties of symmetric products that we will be using

- $SP^\infty(X \vee Y,*) \simeq SP^\infty(X,*) \times SP^\infty(Y,*)$
- $\pi_1(SP^n(X))$ is abelian when $n \geq 2$ 1.2
- $SP^n(S^1) \simeq S^1$, $n \geq 1$ 1.3
- $SP^n(S^2) \cong \mathbb{P}^n$, $n \geq 1$. 1.4
- There is a diffeomorphism $SP^n(\mathbb{C}) \cong \mathbb{C}^n$.

To see this last statement, choose a tuple of n points in $\mathbb{C}$, say $(v_1, v_2, \ldots, v_n)$ and associate to it the coefficients of the monic polynomial $(z - v_1)(z - v_2)\cdots(z - v_n)$. This sets up the correspondence between $SP^n(\mathbb{C})$ and $\mathbb{C}^n$ and it's easy to see that it is a diffeomorphism. A straighforward corollary of this is:

Corollary 1.5: *Let M be a 2 dimensional manifold without boundary, then $SP^n(M)$ is a $2n$ dimensional manifold for all n. In particular, $SP^n(M_g)$ is a complex manifold $\forall g$.*

§1.1 The Homology of Symmetric Products

The symmetric products exhibit interesting homological properties. For instance, it was proved by Dold [9] that for X a CW-complex, $H_*(X^n/G)$ only depends on $H_*(X)$ for any subgroup $G \subset \mathcal{S}_n$. The homology groups $H_*(SP^n(X))$ for instance are entirely determined by the homology groups of X. Moreover, we have the following classical splitting result due to Steenrod

Theorem 1.6: (Steenrod) *For X connected and for untwisted coefficients $\mathbb{A}$, we have*

$$\begin{aligned} H_*(SP^n(X);\mathbb{A}) &= \sum_{k=1}^{n} H_*(SP^k(X), SP^{k-1}(X);\mathbb{A}) \\ &= H_*(SP^n(X), SP^{n-1}(X);\mathbb{A}) \oplus H_*(SP^{n-1}(X);\mathbb{A}). \end{aligned}$$

Remark 1.7: The splitting above induces a bigrading on $H_*(SP^\infty(X,*),\mathbb{A})$; for an element $x \in H_*(SP^\infty(X,*),\mathbb{A})$ has bidegree (i,k) iff $x \in H_i(SP^k(X), SP^{k-1}(X),\mathbb{A})$. This evidently implies that $H_*(SP^\infty(X,*),\mathbb{A})$ has the structure of a bigraded algebra. We will write $\deg(x)$ for the *homological* degree of x and $\mathrm{fil}(x)$ for its *filtration degree* k. Notice that

$$\deg(x\cdot y) = \deg(x) + \deg(y), \quad \mathrm{fil}(x\cdot y) = \mathrm{fil}(x) + \mathrm{fil}(y).$$

Remark 1.8: For finitely generated CW-complexes, there is a standard procedure due to Milgram [17] to determine the homology of the symmetric products. This procedure amounts to first determining the bigraded algebra structure of $H_*(SP^\infty A(G,n);\mathbb{A}) = H_*(K(G,n),\mathbb{A})$ for Moore spaces and this can be deduced from Cartan's determination of the homology of Eilenberg-McLane spaces [5].

Knowledge of the homology of symmetric products of Moore spaces can then be used to determine $H_*(SP^\infty(X,*))$ for any finitely generated CW-complex X. More precisely, given such X (arcwise connected), one can recover the homology type of X via a wedge of Moore spaces Y_i and hence the problem reduces to calculating $H_*(SP^\infty(\bigvee Y_i, *))$ as a bigraded algebra. But it's not hard to see that for CW-complexes X and Y (and untwisted coefficients $\mathbb{A}$) there is a bigraded algebra isomorphism:

$$H_*(SP^\infty(X \vee Y, *), \mathbb{A}) \cong H_*(SP^\infty(X, *), \mathbb{A}) \otimes H_*(SP^\infty(Y, *), \mathbb{A}).$$

§1.2 Symmetric Products of Curves

A genus g compact Riemann surface M_g is obtained by attaching a 2-cell, D^2, to a wedge of $2g$-circles via the commutator map. If we denote by $a_1, b_1, \cdots, a_g, b_g$ the generators of $\pi_1(M_g)$ each representing a copy of S^1 in the one skeleton $\underbrace{S^1 \vee \cdots \vee S^1}_{2g} \subset M_g$, then we can write $M_g \simeq (\bigvee^{2g} S^1) \cup_{[a_1,b_1]\cdots[a_g,b_g]} D^2$.

We choose the letters $\{e_i,\ i = 1, \ldots, 2g\}$ to label the homology generators in $H_1(M_g; \mathbb{Z})$. The boundary of the top 2-dimensional class D^2 vanishes (being a commutator) and hence D^2 generates a homology class which corresponds to the orientation class $[M_g]$ (or M for short). In homology we have that $H_*(M_g) \cong H_*(\bigvee^{2g}(S^1) \vee S^2)$ and it follows from 1.4 and 1.8 that

Lemma 1.9: *We have the following bigraded algebra isomorphism*

$$H_*(SP^\infty(M_g, *); \mathbb{Z}) \cong \Lambda(e_1) \otimes \cdots \Lambda(e_{2g}) \otimes \Gamma[M]$$

where $\Gamma[M]$ is the divided power algebra over $\mathbb{Z}$ generated by elements $\gamma_i = \frac{M^i}{i!}$.

Here it is clear that $\deg(e_i) = 1 = \mathrm{fil}(e_i)$ and so the e_i's have bidegree $(1, 1)$; while M has bidegree $(2, 1)$. As a consequence of 1.6 one can check that

Lemma 1.10: *$H_*(SP^n(M_g); \mathbb{Z}) \subset H_*(SP^\infty(M_g, *); \mathbb{Z})$ consists of all elements of bidegree $(*, i), i \leq n$. For instance $H_*(SP^n(M), SP^{n-1}(M); \mathbb{Z})$ has generators of the following type*

$$e_{i_1} \cdots e_{i_r} \gamma_s, \quad r + s = n.$$

Lemma 1.10 describes entirely the homology of $SP^n(M_g)$ for finite n. Notice at this point that 1.1 and then 1.3 imply that

1.11

$$SP^\infty(M_g,*)\simeq K(\mathbb{Z}^{2g},1)\times K(\mathbb{Z},2)\simeq (S^1)^{2g}\times\mathbb{P}\simeq SP^\infty(S^1)^{2g}\times SP^\infty(S^2).$$

We can give an explicit construction of the homotopy equivalence above as follows. First we have the obvious map $SP^\infty(\bigvee^{2g}S^1,*)=SP^\infty(S^1,*)^{2g}\xrightarrow{SP^\infty(i)}SP^\infty(M_g,*)$ induced from the inclusion of the one skeleton $i:\bigvee^{2g}S^1\hookrightarrow M_g$ and sending the wedgepoint to basepoint $*\in M_g$. Next, we can consider the composite

$$\tau : S^1=\partial D^2\longrightarrow\bigvee^{2g}S^1\longrightarrow SP^2(\bigvee^{2g}S^1)\longrightarrow SP^2(M_g).$$

At the level of fundamental groups, τ_* factors through a commutator f_* and since $\pi_1(SP^2(X))$ is abelian (1.2), it follows that $\tau_*([S^1])=0$. The map τ extends to a map from a new disk D'^2

$$\tau : D'^2\longrightarrow SP^2(\bigvee^{2g}S^1)\longrightarrow SP^2M_g.$$

We can draw the following diagram:

$$\begin{array}{ccccc}
D'^2 & & \xrightarrow{\quad\tau\quad} & & SP^2(\bigvee^{2g}S^1)\\
\uparrow & & & & \|\\
S^1 & \xrightarrow{f} & \bigvee^{2g}S^1 & \hookrightarrow & SP^2(\bigvee^{2g}S^1)\\
\downarrow & & & & \downarrow\\
D^2 & \longrightarrow & M_g & \hookrightarrow & SP^2(M_g)
\end{array}$$

which is then seen to give rise to a map

1.12 $$h : S^2=D^2\cup_{S^1}D'^2\longrightarrow SP^2(M_g)$$

and hence to a map $SP^\infty(h):SP^\infty(S^2,*)=\mathbb{P}^\infty\to SP^\infty(M_g,*)$.

Lemma 1.13: *The composite*

$$\mu(SP^\infty(i)\times SP^\infty(h)) : (SP^\infty(S^1,*))^{2g}\times SP^\infty(S^2,*)\longrightarrow SP^\infty(M_g,*)$$

is a homotopy equivalence (here again μ is the monoid addition in $SP^\infty(M_g,*)$).

About divided power algebras: Let A be a commutative graded algebra, and $a\in A$ an even degree element. A *divided power algebra* on a; denoted by

$\Gamma(a)$, is the algebra generated by elements $\gamma_i = \gamma_i(a)$ with relations:

$$\gamma_0 = 1,\ \ \gamma_1 = a;\ \ \gamma_k\gamma_h = \binom{k+h}{h}\gamma_{k+h};\ \text{ and boundary } d\gamma_k = (da)\gamma_{k-1}, k \geq 1.$$

Here the degree of γ_k is determined by the fact that $\deg(\gamma_k) = k\deg(a)$. Over $\mathbb{Z}$, the generators of $\Gamma(a)$ are uniquely defined by the formula $\gamma_k = \frac{a^k}{k!}$. Over $\mathbb{Q}$, everything becomes a unit and hence $\Gamma(a) = \mathbb{Q}[a]$. With mod-$p$ coefficients, $\Gamma(a)$ splits into products of truncated polynomial algebras (see §6).

§2 A Model for The Divisor Spaces

For any space X, and given $k_i \geq 1$ integers, we defined (§0) the divisor space

$$\mathrm{Div}_{k_1,\dots,k_n}(X) = \{(D_{k_i},\dots,D_{k_n}) \mid D_{k_i} \in SP^{k_i}(X),\ D_{k_1}\cap D_{k_2}\cap\cdots\cap D_{k_n} = \emptyset\}.$$

The element $D_k \in SP^k(X)$ can be either represented by an unordered k-tuple of points $\langle x_1,\dots,x_k\rangle$ (the x_j not necessarily distinct), or by a formal sum $\sum n_i x_i$ such that $\sum n_i = k$ and $x_i \neq x_j$ (when X is a curve, these are called positive divisors in the language of algebraic geometry.)

NOTATION: We write $\mathrm{Div}_{k_1,\dots,k_n} = \mathrm{Div}_{k_1,\dots,k_n}(M_g - *)$ and $\mathrm{Div}^n_k(M_g - *) = \mathrm{Div}_{\underbrace{k,\dots,k}_{n}}$.

Let Δ denote the diagonal multiplication

2.1
$$\left(\coprod_{j=1}^{\infty} SP^j(X)\right) \times SP^{k_1}(X)\times\cdots\times SP^{k_n}(X) \longrightarrow \coprod_j SP^{k_1+j}(X)\times\cdots\times SP^{k_n+j}(X)$$

given on points by $\Delta(D, D_1,\dots,D_n) = (D_1 + D,\dots,D_n + D)$. It is clear that the divisor spaces $\mathrm{Div}_{k_1,\dots,k_n}(X)$ include in the product $SP^{k_1}(X) \times \cdots SP^{k_n}(X)$ as the complement of $Im(\Delta)$; that is

Lemma 2.2: $\mathrm{Div}_{k_1,\dots,k_n} = SP^{k_1}(M_g - *) \times \cdots \times SP^{k_n}(M_g - *) - Im(\Delta)$.

Remark 2.3: The inclusion $\mathrm{Div}_{k_1,\dots,k_n}(X) \subset SP^{k_1}(X) \times \cdots \times SP^{k_n}(X)$ is an open embedding and since for curves X the left hand side is a $k_1 + \ldots +$

k_n complex manifold (1.5), it follows that $\mathrm{Div}_{k_1,\dots,k_n}(X)$ is also a complex manifold of dimension $k_1 + \ldots + k_n$.

We can define at this point the quotient space

$$TY_{k_1,\dots,k_n} = SP^{k_1}(M_g)\times \quad\cdots\quad \times SP^{k_n}(M_g)/ \left\{\bigcup_i SP^{k_1}(M_g)\times\cdots SP^{k_i-1}(M_g)\times\cdots\times SP^{k_n}(M_g)\cup Im(\Delta)\right\}$$

which can be thought of as $SP^{k_1}(M_g)\times\cdots\times SP^{k_n}(M_g)/V$ where V is such that $\mathrm{Div}_{k_1,\dots,k_n} = SP^{k_1}(M_g)\times\cdots\times SP^{k_n}(M_g) - V$. One can therefore invoke Poincaré-Alexander duality to write

2.4 $$H^i(\mathrm{Div}_{k_1,\dots,k_n};\mathbb{F}) = H_{2(k_1+\dots+k_n)-i}(TY_{k_1,\dots,k_n};\mathbb{F}).$$

The homology of the quotient space $TY_{k_1,\dots,k_n}$ is not easy to extract when the space is presented under this form. However, we can use a homotopy equivalent construction due to [7] which makes the homological structure much more apparent.

Consider the *twisted* product space

2.5 $$DY^n(X) = (\underbrace{SP^\infty(X,*)\times\cdots\times SP^\infty(X,*)}_{n})\times_t SP^\infty(cX)$$

where t identifies the points

$$(D_1,\cdots,D_n,\quad \langle(t_1,z_1)\cdots(t_k,z_k)\rangle)\sim (D_1+z_i,\dots,D_n+z_i,\langle(t_1,z_1)\cdots(t_{i-1},z_{i-1})(t_{i+1},z_{i+1})\cdots(t_k,z_k)\rangle)$$

whenever $t_i = 0$ (here $D_i\in SP^{k_i}(X)$ for some $k_i\geq 1$ and $(t_j,z_j)\in cX, t_j\in [0,1]$.)

The space $DY^n(X)$ is naturally filtered as follows

$$DY_{k_1,\dots,k_n}(X) = \bigcup_{\substack{i_1+l\leq k_1\\ \vdots \\ i_n+l\leq k_n}} \left(SP^{i_1}(X)\times\cdots\times SP^{i_n}(X)\right)\times_t SP^l(cX).$$

We observe that there is a projection $p : DY_{k_1,\dots,k_n} \longrightarrow SP^{k_1}(X) \times \cdots \times SP^{k_n}(X)/Im(\Delta)$ given by

$$(D_{i_1}, D_{i_2}, \dots, D_{i_n}, \langle (t_1, z_1)\cdots, (t_l, z_l)\rangle) \mapsto (D_{i_1}, D_{i_2}, \dots, D_{i_n}).$$

It is easy to see that p is acyclic, inverse images of points being contractible sets. It follows that p induces an isomorphism in homology and combining this with 2.4 yields

Lemma 2.6: *There is an isomorphism*

$$H^i(Div_{k_1,\dots,k_n}(M_g - *); \mathbb{F}) = H_{2(k_1+\cdots+k_n)-i}(DY_{k_1,\dots,k_n}/\bigcup_i DY_{k_1,\dots,k_i-1,\dots,k_n}; \mathbb{F}).$$

It is the quotient spaces $DY_{k_1,\dots,k_n}/\bigcup_i DY_{k_1,\dots,k_i-1,\dots,k_n}$ that we analyze in this paper.

§2.1 Homotopy invariance

Here we show that the topology of the space DY doesn't depend on the choice of the diagonal approximation Δ. Choose a map $\Delta' : SP^\infty(X) \to SP^\infty(X)^n$ homotopic to Δ and define the corresponding space $DY'(X)$ obtained from $(SP^\infty(X,*))^n \times SP^\infty(cX)$ via the identification

$$\begin{aligned}(D_1, \quad \cdots \quad &, D_n, \langle (t_1,z_1)\cdots(t_k,m_k)\rangle) \\ &\sim \quad (\nu(\Delta'(z_i), (D_1,\cdots,D_n)), \langle (t_1,z_1)\cdots(t_{i-1},z_{i-1})(t_{i+1},z_{i+1})\cdots(t_k,m_k)\rangle)\end{aligned}$$

whenever $t_i = 0$. Here ν is the componentwise symmetric product multiplication.

Lemma 2.7: $DY'(M_g) \simeq DY(M_g)$.

Proof: Denote by $AG(X)$ the free abelian group on points of X or equivalently the group completion of $SP^\infty(X)$. Points of $AG(X)$ have the form $*$ or

$$\{x_1\cdots x_r, y_1^{-1}\cdots y_s^{-1} \mid * \neq x_i, y_j, x_i \neq y_j\}.$$

It is known ([10]) that for connected CW complexes, the inclusion $SP^\infty(X) \hookrightarrow AG(X)$ is a homotopy equivalence.

For simplicity of notation, write $G = AG(M_g)$. The diagonal Δ (resp. Δ') extends in the obvious way to a map $G \to G^n$ and it induces an action $\delta : G \times G^n \to G^n$ (resp. δ') as described previously. We can then consider the associated "completed" model

$$\hat{DY}(M_g) = G^n \times_G AG(cM_g)$$

where G acts on G^n via δ. Similary we can construct $\hat{DY}'(M_g)$ associated to δ'. It is easy to see that the new model $\hat{DY}(M_g)$ is homotopy equivalent to $DY(M_g)$. This follows by considering the map of quasifiberings

$$\begin{array}{ccc} SP^\infty(M_g)^n & \longrightarrow & G^n \\ \downarrow & & \downarrow \\ DY & \longrightarrow & \hat{DY} \\ \downarrow & & \downarrow \\ SP^\infty(\Sigma M_g) & \longrightarrow & AG(\Sigma M_g) \end{array}$$

where the top and bottom maps are homotopy equivalences (similarly $\hat{DY}'(M_g) \simeq DY'(M_g)$).

Since Δ is a diagonal approximation, we have a homotopy $G \times G^n \times I \xrightarrow{\phi} G^n$ where if we write $\phi(g, x, t) = g_t(x)$, the map g_0 corresponds to componentwise multiplication $G \times G^n \to G^n$ and $g_1 = \delta$. The inclusion $i : (G^n \times \{0\}) \times AG(cM_g) \hookrightarrow (G^n \times I) \times AG(cM_g)$ is a G-map (here the action of G on the righthand side is given by $g((x,t), w) = ((g_t(x), t), gw)$), and it is clearly a homotopy equivalence. Since G acts freely on $AG(cM_g)$, i is a map of free G spaces and it is then a theorem of equivariant homotopy [4] that i is actually a homotopy equivalence through G maps. This then descends to an equivalence of quotients and we have

$$\hat{DY}(M_g) = (G^n \times \{1\}) \times_G AG(cM_g) \simeq (G^n \times \{0\}) \times_G AG(cM_g).$$

The same argument shows that $\hat{DY}'(M_g)$ is homotopy equivalent to the righthand side and the lemma is proved. ∎

§3 Stabilization

As indicated in the introduction, there are homotopy inclusions $\mathrm{Div}_k^n = \mathrm{Div}_{k,\dots,k}(M_g - *) \hookrightarrow \mathrm{Div}_{k+1}^n$ defined as follows (and more generally, there are inclusions $\mathrm{Div}_{k_1,\dots,k_i,\dots,k_n} \hookrightarrow \mathrm{Div}_{k_1,\dots,k_i+1,\dots,k_n}$ that raise any degree): Choose a sequence of concentric neighborhoods $\{U_k\}, k \geq 1, U_{k+1} \subset U_k$, around the basepoint $x_0 \in M_g$. For each neighborhood U_k pick an n-tuple of *distinct* points $(x_1^k, \dots, x_n^k) \in U_k - U_{k+1}$. The map

$$\mathrm{Div}_k^n(M_g - U_k) \longrightarrow \mathrm{Div}_{k+1}^n(M_g - U_{k+1}) \tag{3.1}$$

given by sending a configuration $(D_1, \dots, D_n)$ to $(D_1 + x_1^k, \dots, D_n + x_n^k)$ is a closed embedding, and it extends to an open embedding

$$e : \mathrm{Div}_k^n(M_g - U_k) \times (U_k - \bar{U}_{k+1}) \longrightarrow \mathrm{Div}_{k+1}^n(M_g - U_{k+1}).$$

It is not hard to see that $\mathrm{Div}_k^n(M_g - U_k) \cong \mathrm{Div}_k^n$, and so we regard Div_k^n as a codimension $2n$ (real) submanifold of Div_{k+1}^n.

NOTATION: The direct limit of Div_k^n over the embeddings e is denoted by $\mathrm{Div}^n(M_g - *)$.

Consider now the following diagram

$$\begin{array}{ccc}
H_*(\mathrm{Div}_k^n) & \xrightarrow{e_*} & H_*(\mathrm{Div}_{k+1}^n) \\
\downarrow \cong & & \downarrow \cong \\
H^{2kn-*}(\mathrm{Div}_k^n, \partial \mathrm{Div}_k^n) & \xrightarrow{f} & H^{2(k+1)n-*}(\mathrm{Div}_{k+1}^n, \partial \mathrm{Div}_{k+1}^n) \\
\downarrow \cong & & \downarrow \cong \\
H^{2kn-*}(TY_k^n) & \xrightarrow{f} & H^{2(k+1)n-*}(TY_{k+1}^n)
\end{array}.$$

Lemma 3.2: *The map f corresponds to cupping with $a_1 \cup a_2 \cup \cdots \cup a_n$.*

PROOF: Let V be a tubular neighborhood of Div_k^n in Div_{k+1}^n. We have that Div_k^n is a (complex) codimension n submanifold of Div_{k+1}^n and we can identify V with the normal disc bundle to $\mathrm{Div}_k^n \subset \mathrm{Div}_{k+1}^n$. Denote by η the entire normal bundle and let $M(\eta) = V/\partial V$ be the corresponding Thom space.

Note that we can compactify Div_j^n by adding a boundary term $\partial \mathrm{Div}_j^n$ (corresponding to its complement in $SP^j(M_g)^n$.) Poincaré duality and the

Thom isomorphism interlock in the following diagram of isomorphisms

3.3
$$\begin{array}{ccc} H^*(\mathrm{Div}^n_k, \partial \mathrm{Div}^n_k) & \xrightarrow{\cup U} & H^{*+2n}(M(\eta), M(\eta_{|\partial})) \\ \downarrow \cong & & \downarrow \cong \\ H_{2kn-*}(\mathrm{Div}^n_k) & \xrightarrow{=} & H_{2kn-*}(\mathrm{Div}^n_k) \end{array}$$

where U is the Thom class and $M(\eta_{|\partial})$ is the Thom space of the normal bundle η restricted to $\partial\mathrm{Div}^n_k$. Note also that there is an (excision) isomorphism

$$H^*(M(\eta)) = H^*(V, \partial V) \xrightarrow{\cong} H^*(\mathrm{Div}^n_{k+1}, \mathrm{Div}^n_{k+1} - \mathrm{Div}^n_k)$$

which then yields a map

3.4
$$H^*(M(\eta), M(\eta_{|\partial})) \longrightarrow H^*(\mathrm{Div}^n_{k+1}, \partial \mathrm{Div}^n_{k+1}).$$

This is now enough to give a description of the map f, for we have that the Thom isomorphism (given by the top map in 3.3) combines with 3.4 to yield

$$f : H^*(\mathrm{Div}^n_k, \partial\mathrm{Div}^n_k) \xrightarrow{\cong} H^{*+2n}(M(\eta), M(\eta_{|\partial})) \longrightarrow H^{*+2n}(\mathrm{Div}^n_{k+1}, \partial\mathrm{Div}^n_{k+1}).$$

Write the tubular neighborhood V of Div^n_k as

$$V = \mathrm{Div}^n_k \times V_{x^k_1} \times \cdots \times V_{x^k_n}$$

where $V_{x^k_i}$ is a small disc around x^k_i. The Thom class is by definition the orientation class $U \in H^{2n}(V_{x^k_1} \times \cdots \times V_{x^k_n}, \partial(V_{x^k_1} \times \cdots \times V_{x^k_n}))$. Since $H^2(V_{x^k_i}, \partial V_{x^k_i})$ is generated by a_i, it is now clear that

$$\begin{aligned} U = a_1 \dots a_n \in H^2(V_{x^k_1}, \partial V_{x^k_1}) \otimes \quad \cdots \quad & \otimes H^2(V_{x^k_n}, \partial V_{x^k_n}) \\ \hookrightarrow \quad & H^{2n}\left(V_{x^k_1} \times \cdots \times V_{x^k_n}, \partial(V_{x^k_n} \times \cdots \times V_{x^k_n})\right) \end{aligned}$$

and the proof is complete. ∎

Corollary 3.5: (Segal) *The "collar" inclusions* $H_*(Div^n_k) \xrightarrow{e_*} H_*(Div^n_{k+1})$ *are injections.*

§4 The Homology of Divisor Spaces

In this section, we prove our main result (0.4 in the introduction and 4.14 below.) So we start with the model (§2)

$$DY^n(M_g) = (SP^\infty(M_g,*))^n \times_t SP^\infty(cM_g,*)$$

where t is the diagonal twisting described in 2.5. For simplicity, we will write M for M_g and $SP^\infty(M)$ for $SP^\infty(M_g,*)$.

We fix a diagonal approximation

$$\Delta_* : C_*(SP^\infty(M)) \longrightarrow C_*(SP^\infty(M))^{\otimes^n}.$$

This induces an action of $C_*(SP^\infty(M))$ on $C_*(SP^\infty(M))^{\otimes^n}$. On the other hand, the inclusion

$$M \hookrightarrow cM,\ x \mapsto (0,x)$$

induces an action of $C_*(SP^\infty(M))$ on $C_*(SP^\infty(cM))$. Using these actions, a chain complex for DY is given by

4.1 $$C_*(SP^\infty(M))^{\otimes n} \otimes_{C_*(SP^\infty(M))} C_*(SP^\infty(cM))$$

and by lemma 2.7, any other choice of Δ_* yields chain homotopic complexes.

At this point we need describe the module structure of $C_*(SP^\infty(cM))$ over $C_*(SP^\infty(M))$ and for that purpose we need review some constructions.

§4.1 Milgram's Bar Construction

Infinite symmetric products provide models for topological bar constructions and classifying spaces as was observed by Milgram [15-16].

Let X be an associative topological monoid $\mu : X \times X \longrightarrow X$ with μ cellular. We assume that μ has a unit $*$. Let σ^n be the n-simplex which we parametrize as follows

$$\sigma^n = \{(t_1, t_2, \cdots, t_n)|\ 0 \leq t_1 \leq \cdots \leq t_n \leq 1\}.$$

The (acyclic) Milgram's bar construction on X is the space

$$E_T(X) = \coprod_{i=1}^{\infty} X \times \sigma^i \times X^i / \sim$$

with identifications $\sim$ given as follows

$$
\begin{array}{lll}
(i) & (x_0,t_1,\cdots,t_n,x_1,\cdots,x_n)\sim(x_0,t_1,\cdots,\hat{t}_j,\cdots,t_n,x_1,\cdots,\hat{x}_j,x_jx_{j+1},\cdots,x_n) & \text{if } t_j=t_{j+1}\\
(ii) & (x_0,t_1,\cdots,t_n,x_1,\cdots,x_n)\sim(x_0x_1,t_2,\cdots,t_n,x_2,\cdots,x_n) & \text{if } t_1=0\\
(iii) & (x_0,t_1,\cdots,t_n,x_1,\cdots,x_n)\sim(x_0,t_1,\cdots,t_{n-1},x_1,\cdots,x_{n-1}) & \text{if } t_n=1\\
(iv) & (x_0,t_1,\cdots,t_n,x_1,\cdots,x_n)\sim(x_0,t_1,\cdots,\hat{t}_j,\cdots,t_n,x_1,\cdots,\hat{x}_j,\cdots,x_n) & \text{if } x_j=*
\end{array}
$$

Clearly, X acts freely on $E_T(X)$ by multiplying on the left and it turns out that $E_T(X)$ is contractible [15]. This implies that the quotient space $B_T(X)=E_T(X)/X$ is a classifying space for X. The space $B_T(X)$ is also referred to as the *topological bar construction* on X (and we will sometimes write B_X for $B_T(X)$.)

Let $E(A)$ and $B(A)$ denote respectively the *acyclic* and the *reduced* algebraic bar constructions on A. In our case, A will be a differential graded; or DG, algebra. Suppose now that X is an abelian H-space, then $C_*(X)$, the chain complex for X, is a DG algebra. There is a correspondence $\lambda : C_*(B_T(X))\longrightarrow BC_*(X)$ given on generators by

$$\lambda(\sigma^n\times e_1\times\cdots\times e_n)=|e_1|\cdots|e_n|.$$

Both $C_*(B_T(X))$ and $BC_*(X)$ are bigraded and it can be checked that λ is a differential bigraded algebra homomorphism. Actually more is true

Theorem 4.2: (Milgram) *There is an isomorphism of differential bigraded algebras (dba);*

$$C_*(B_T(X))\cong B(C_*(X)),\ C_*(E_T(X))\cong E(C_*(X)).$$

Remark 4.3: We mentioned earlier that there is an interesting connection between infinite symmetric products and the classifying space construction above. Indeed, one can order points

$$\langle (t_1,z_1),\dots,(t_n,z_n)\rangle\in SP^n(\Sigma X)$$

according to the ascending order of the t_i's. However there is an ambiguity whenever $t_i=t_j$ in which case we identify $\langle (t_i,z_i),(t_i,z_j)\rangle$ with $\langle t_i,\langle z_iz_j\rangle\rangle$, where $\langle z_iz_j\rangle$ is the product in $SP^\infty(X)$. Of course when $t_i=0$ or $t_i=1$ we get the basepoint identification (in the suspension). It then follows that when elements of $SP^\infty(\Sigma X)$ are represented in the *normal form* $(t_1 \le t_2 \le \ldots \le t_n, x_1, \ldots, x_n)$, the following homeomorphism becomes apparent

$$SP^\infty(\Sigma X) = B_T(SP^\infty(X)).$$

Corollary 4.4: (Milgram) $C_*(SP^\infty(\Sigma X, *)) \cong_{dba} B\left(C_*(SP^\infty(X,*))\right).$

A similar statement holds for cX; that is $SP^\infty(cX) \cong E_T(SP^\infty(X))$ and there is a dba (i.e. differential bigraded algebra) isomorphism

$$C_*(SP^\infty(cX,*)) \cong_{dba} E\left(C_*(SP^\infty(X,*))\right).$$

It then follows that $C_*(SP^\infty(cM)) \cong EC_*(SP^\infty(M))$ as modules over $C_*(SP^\infty(M))$. Combining this with 4.1 gives

Lemma 4.5: $H_*(DY^n(M); \mathbb{A}) \cong Tor^{C_*(SP^\infty(M,*))}(\mathbb{A}, C_*(SP^\infty(M,*))^{\otimes n}).$

§4.2 The Collapse

The total space for $Tor^{C_*(SP^\infty(M))}(\mathbb{A}, C_*(SP^\infty(M))^{\otimes n})$ is given by

4.6 $$C_*(SP^\infty(M))^{\otimes n} \otimes_{C_*(SP^\infty(M))} E(C_*(SP^\infty(M))$$

We will write $\otimes_{\Delta_*}$ instead of $\otimes_{C_*(SP^\infty(M))}$ for shorthand. Of course, filtering 4.6 by the number of bar degrees yields the classical homology Eilenberg-Moore spectral sequence.

Proposition 4.7: *There is an embedding* $e : H_*(SP^\infty(M); \mathbb{A}) \hookrightarrow C_*(SP^\infty(M_g), \mathbb{A})$ *inducing an isomorphism*

$$Tor^{H_*(SP^\infty(M_g))}(\mathbb{A}, H_*(SP^\infty(M_g))^{\otimes n}) \cong Tor^{C_*(SP^\infty(M_g))}(\mathbb{A}, C_*(SP^\infty(M_g))^{\otimes n}).$$

From the Cartan-Moore comparison theorem (ref. [14], corollary 7.6.), 4.7 would follow if e is compatible with the module structures; that is if e commutes with the diagonal action

$$\begin{array}{ccc} H_*(SP^\infty(M)) & \xrightarrow{e} & C_*(SP^\infty(M)) \\ \downarrow \Delta_* & & \downarrow \Delta_* \\ H_*(SP^\infty(M))^{\otimes n} & \xrightarrow{\otimes^n e} & C_*(SP^\infty(M))^{\otimes n}. \end{array}$$

So first we record the existence of e as a separate lemma.

Lemma 4.8: *A chain complex for $SP^\infty(M_g)$ can be chosen so that there is an embedding $e : H_*(SP^\infty(M_g);\mathbb{Z}) \hookrightarrow C_*(SP^\infty(M_g),\mathbb{Z})$.*

PROOF: We go back to the standard representation of M_g as $M_g \simeq (\bigvee^{2g} S^1)\cup_{[-]} D^2$ where D^2 is the top 2 cell attached via the commutator map $[-]$ to a bouquet of $2g$ one dimensional S^1's. The i-th copy of S^1 in $\bigvee^{2g} S^1$ represents a one dimensional cell e_i attached trivially to basepoint. If $*$ denotes the product in $SP^\infty(M)$, then it is direct to see that

$$e_{i_1} * e_{i_2} * \ldots * e_{i_n}, i_j \neq i_k, \text{ and } e_i * D^2$$

give genuine cells in $SP^\infty(M)$ (which can be thought of as the cross product cells). We also know that $SP^n(D^2), n \geq 1$ are cells of dimension $2n$ (lemma 1.5). We can consider then the complex $\mathcal{C}$ generated by the different products $e_{i_1} * e_{i_2} * \ldots * e_{i_n} * SP^l(D^2)$ with $i_j \neq i_k$. Since $\partial e_i = 0,\ \ \partial D^2 = 0$, we notice that $\partial\left(e_i * SP^n(D^2)\right) = 0$ and hence elements of $\mathcal{C}$ represent homology classes. As such, it is clear that $n!(SP^n(D^2)) = [M]$ for this is simply equivalent to the statement that the projection map $M^n \to SP^n(M)$ has degree $n!$. It then follows that the $SP^l(D^2)$'s generate a divided power algebra in $\mathcal{C}$. From lemma 1.8 we see that $\mathcal{C} \cong H_*(SP^\infty(M))$ and the embedding of the homology into the chain complex is constructed. ∎

PROOF OF 4.7: The diagonal approximation $\Delta : M \longrightarrow M^n$ can be extended multiplicatively (on each component) to a map $\Delta^\infty : SP^\infty(M) \longrightarrow (SP^\infty(M))^n$, and clearly Δ^∞ is homotopic to the diagonal on $SP^\infty(M)$. We have the following commuting diagram

$$\begin{array}{ccccc}
SP^\infty(M) \times SP^\infty(M) & \xrightarrow{*} & SP^\infty(M) & \xrightarrow{\Delta^\infty} & SP^\infty(M)^n \\
\Big\downarrow{\scriptstyle \Delta^\infty\times\Delta^\infty} & & & & \Big\uparrow{\scriptstyle *\times *} \\
SP^\infty(M)^n \times SP^\infty(M)^n & & \xrightarrow{shuffle} & & SP^\infty(M)^n \times SP^\infty(M)^n
\end{array}$$

where as before $shuff\left((x_1,\ldots,x_n),(y_1,\ldots y_n)\right) = \left((x_1,y_1),\ldots,(x_n,y_n)\right)$. Note that $shuff_*$ is cellular. Now Δ^∞ is already cellular on M by construction and the diagram above shows that Δ^∞_* must be cellular on $\mathcal{C}$. By standard considerations, Δ^∞ extends to a cellular map on all of $SP^\infty(M)$.

The embedding $e : \mathcal{C} \hookrightarrow C_*(SP^\infty(M))$ does commute with the diagonal action and since $\mathcal{C} = H_*(SP^\infty(M_g)$ the proposition follows by Cartan-Moore. ∎

§4.3 The Main Result

Write $\mathcal{A} = H_*(SP^\infty(M))$. We now put a multigrading on $Tor^{\mathcal{A}}(\mathbb{A}, \mathcal{A}^{\otimes n})$ and derive theorem 0.4. To start and using Steenrod's splitting 1.6, the total space for $Tor^{\mathcal{A}}(\mathbb{A}, \mathcal{A}^n)$ takes the form

4.9 $$Tot = \bigoplus H_*(T_{k_1,\dots,k_n}; \mathbb{A}) \otimes_{\mathcal{A}} E(\mathcal{A})$$

where $T_{k_1,k_2,\dots,k_n} = SP^{k_1}(M) \times \cdots \times SP^{k_n}(M)/V$ with

$$V = \bigcup_{1\leq i\leq n} SP^{k_1}(M) \times \cdots SP^{k_i-1} \times \cdots \times SP^{k_n}(M).$$

Since the algebra $\mathcal{A}$ is bigraded then so is $E(\mathcal{A})$. More precisely, let $a_i \in \mathcal{A}$ have bidegree $(\deg(a_i), \mathrm{fil}(a_i))$ as in 1.7, then

$$\text{bidegree } (a_0|a_1|a_2|\dots|a_n|) = \left(n + \sum_{i=0}^{n} \deg(a_i), \sum \mathrm{fil}(a_i)\right)$$

which means that the homological degree of $a_0|a_1|a_2|\dots|a_n| \in E(\mathcal{A})$ is given by $\sum deg(a_i) + n$ while the filtration degree is simply $\sum \mathrm{fil}(a_i)$ (this is not the bar degree). Define $E_{m,k}(\mathcal{A})$ to consist of all elements of bidegree $(m,k) \in E(\mathcal{A})$.

The boundary ∂ in Tot is described as follows

4.10 $$\partial(|a_1|a_2|\dots|a_n|) = \Delta_*(a_1) \otimes |a_2|\dots|a_n| + \partial_B(|a_1|\dots|a_n|)$$

where ∂_B is the reduced Bar differential which in this case is given by

$$\partial_B(|a_1|\dots|a_n|) = (-1)^i \sum_{1}^{n-1} |a_1|\dots|a_{i-1}|a_i a_{i+1}|a_{i+2}|\dots|a_n|.$$

By definition we have that $H_*(Tot, \partial) = Tor^{\mathcal{A}}(\mathbb{A}, \mathcal{A}^n)$ for untwisted coefficients $\mathbb{A}$. Note that $\Delta_* : H_*(SP^r(M)) \to H_*(SP^r(M))^{\otimes n}$ preserves the filtration degree r and hence the total space Tot splits as a sum of subchain complexes

$$\bigoplus_{l_i\le k_i} Tot_{l_1,\dots,l_n} = \bigoplus_{l_i=r_i+j\le k_i} H_*(T_{r_1,\dots,r_n};\mathbb{A})\otimes_{\mathcal{A}} E_{*,j}(\mathcal{A}).$$

The homology $H_*\left(\bigoplus_{l_i\le k_i} Tot_{k_1,\dots,k_n}\right)$ coincides with $H_*(DY_{k_1,\dots,k_n},\mathbb{A})$ and this is a direct summand of $H_*(DY,\mathbb{A})$. On the other hand and by construction $H_*(TY_{k_1,\dots,k_n})$ is a quotient of $H_*(DY_{k_1,\dots,k_n})$ which only sees elements of exact filtration $(k_1,\dots,k_n)$. When passing to this quotient, the diagonal term Δ_* gets *reduced* according to

4.11

$$\Delta_*^{\text{red}} : H_*(SP^r(M))\xrightarrow{\Delta_*}(H_*(SP^r(M)))^{\otimes n}\xrightarrow{q} H_*(SP^r(M),SP^{r-1}(M))^{\otimes n}.$$

We write

$${}^2Tor^{H_*(SP^\infty(M))}(\mathbb{A},H_*(SP^\infty(M))^{\otimes n})$$

for the new tor term where the action is understood to be reduced. This new action induces a new boundary ${}^2\partial$ as in 4.10 and in this case

4.12 $$ {}^2Tot_{k_1,\dots,k_n} = \bigoplus_{r_i+j=k_i} H_*(T_{r_1,\dots,r_n};\mathbb{A})\otimes_{\Delta_*} E_{*,j}(\mathcal{A})$$

is a *subcomplex* of $(Tot,{}^2\partial)$. One then has

4.13

$$H_*(TY_{k_1,\dots,k_n};\mathbb{A})\cong\ 2Tor^{H_*(SP^\infty(M))}_{*,k_1,\dots,k_n}(\mathbb{A},H_*(SP^\infty(M))^{\otimes n}) = H_*\left({}^2Tot_{k_1,\dots,k_n}\right)$$

and these represent the homology classes of exact filtration $(k_1,\dots,k_n)$. When $k_i=k$, we shorten the notation $(*;k,\dots,k)$ to a bidegree notation $(*;k)$ for simplicity. From now on, we drop the upperscript ${}^2Tor$ and write Tor with the understanding that the module action of $H_*(SP^\infty(M))$ on the tensor product is reduced (cf 4.11). The preceding discussion then yields

Theorem 4.14: *For field coefficients $\mathbb{F}$, we have the following isomorphism*

$$H_*(Div_{k,\dots,k}(M_g-*);\mathbb{F})\cong Tor^{H_*(SP^\infty(M_g))}_{2nk-*;k}(\mathbb{F},H_*(SP^\infty(M))^{\otimes n})$$

with the module structure induced from $\Delta_*^{\text{red}} : H_*(SP^\infty(M))\to H_*(SP^\infty(M)^n)$ in 4.11.

PROOF: Apply the duality $H_*(TY_{k,\dots,k};\mathbb{F})\cong H^{2nk-*}(\text{Div}^n_k(M-*);\mathbb{F})$. ∎

Remark 4.15: Note that $H_*(\mathrm{Div}_{k,\ldots,k}(M_g - *);\mathbb{F})$ must vanish beyond the middle dimension $* > nk$; this being a particularity of Stein spaces.

§4.4 An Alternate Description

One could have filtered the space $TY^n = \bigcup DY_{k_1,\ldots,k_n}/\cup_j DY_{k_1,\ldots,k_j-1,\ldots,k_n}$ not by the number of bars but as follows. Write

$$TY^n \quad = \quad \bigcup TY_{k_1,\ldots,k_n} = \bigcup_{r_i+l=k_i} T_{r_1,\ldots,r_n} \times_t (SP^l(cM_g)/SP^{l-1}(cM_g))$$

with filtration pieces $\mathcal{F}^j = \bigcup_{\substack{l\leq j \\ r_i+l=k_i}} (T_{r_1,\ldots,r_n})\otimes_t (SP^l(cM_g)/SP^{l-1}(cM_g))$. The same arguments as in §4.3 can now be expressed in the following form

Proposition 4.16: *There exists a spectral sequence converging to $H_*(TY_{k_1,\ldots,k_n})$ with E^1 term*

$$E^1 = \coprod_{r_i+j=k_i} H_*(T_{r_1,\ldots,r_n};\mathbb{F}) \otimes H_*(SP^j(\Sigma M_g), SP^{j-1}(\Sigma M_g), \mathbb{F}).$$

The spectral sequence collapses at E^1 for $n > 2$.

PROOF: It can easily be checked in light of §4.3 that

$$d_1(c_* \otimes |a_1|\ldots|a_l|) = c_*\Delta_*^{\mathrm{red}}(a_1) \otimes |a_2|\ldots|a_l|$$

where here $a_i \in H_*(SP^\infty M)$ and $|a_1|\ldots|a_l| \in H_*(SP^\infty(\Sigma M)) = H_*(B_T(SP^\infty(M))) = B(H_*(SP^\infty M))$. This yields the first part of the proposition (cf. 4.12). That the spectral sequence collapses when $n > 2$ is a corollary of the fact that Δ_*^{red} vanishes in this case (see lemma 6.2). ∎

Remark 4.17: When $n = 2$, $d_1(|M|) \neq 0$ and there are higher differentials d_{p^i} described in remark 6.16.

§5 The Rat Spaces

When $g = 0$, we have the homeomorphism described in the introduction

$$\mathrm{Div}_k^{n+1}(S^2 - *) = \mathrm{Hol}_k^*(S^2, \mathbb{P}^n) = \mathrm{Rat}_k(\mathbb{P}^n).$$

Applying theorem 4.14 in this case, we see that the module structure of $H_*(SP^\infty(S^2)) = \Gamma(a)$ on $H_*(SP^\infty(S^2))^n$ is trivial for in this case $\Delta_*(a = [S^2]) = a\otimes 1 + 1\otimes a$ and hence by 4.11, $\Delta_*^{\rm red}(a) = 0$. It follows that

$$Tor^{H_*(SP^\infty(S^2))}(\mathbb{F}, H_*(SP^\infty(S^2))^{n+1}) = \Gamma(a_1,\dots,a_{n+1})\otimes Tor^{\Gamma(a)}(\mathbb{F},\mathbb{F}).$$

We observe that we have an identification

$$Tor^{\Gamma(a)}(\mathbb{F},\mathbb{F}) = H_*(SP^\infty(S^3),\mathbb{F}) = \coprod_i H_*(SP^i(S^3), SP^{i-1}(S^3),\mathbb{F})$$

and one can then write

5.1

$$H_*(TY_k^{n+1}(S^2);\mathbb{F}) \cong \coprod_j \gamma_j(a_1a_2\dots a_{n+1})H_{*-2(n+1)j}(SP^{k-j}(S^3), SP^{k-j-1}(S^3);\mathbb{F}).$$

We can consider the inclusion $\mathrm{Rat}(\mathbb{P}^1)\hookrightarrow \Omega^2S^2$. The space $\Omega^2_0S^2\simeq\Omega^2S^3$ stably splits (Snaith) as an infinite bouquet

$$\Omega^2\Sigma^2S^1 \simeq_s \bigvee_0^\infty D_k$$

where $D_k = F(\mathbb{C},k)\wedge_{\mathcal{S}_k} S^{(k)}$ are the building blocks of the May-Milgram model for S^1 (here $S^{(k)}$ denotes the k fold smash of S^1 with itself). It is known [3] that there is a duality isomorphism

$$H_*(D_k,\mathbb{F}) \cong H^{4k-*}(SP^k(S^3), SP^{k-1}(S^3),\mathbb{F}).$$

The identity 5.1 combines with the duality $H_{2k(n+1)-*}(TY_k^{n+1},\mathbb{F})\cong \tilde H^*(\mathrm{Rat}_k(\mathbb{P}^n),\mathbb{F})$ to yield

$$\begin{aligned} H_*(\mathrm{Rat}_k(\mathbb{P}^n),\mathbb{F}) &= H^{2k(n+1)-*}(TY_k^{n+1},\mathbb{F}) = \bigoplus_j H^{2(n+1)(k-j)-*}(SP^{k-j}(S^3), SP^{k-j-1}(S^3);\mathbb{F}) \\ &= \bigoplus H_{4(k-j)-2(n+1)(k-j)+*}(D_{k-j};\mathbb{F}) = \bigoplus_j H_{*-(2n-2)(k-j)}(D_{k-j};\mathbb{F}). \end{aligned}$$

Proposition 5.2: ($\mathrm{C^2M^2}$) $\quad H_*(\mathrm{Rat}_k(\mathbb{P}^n),\mathbb{F}) \cong H_*(\bigvee_{j=1}^k \Sigma^{(2n-2)j}D_j,\mathbb{F}).$

Corollary 5.3: (Segal) *Let* $Rat_\infty(\mathbb{P}^n)$ *be the direct limit induced from the system of collar inclusions* $Rat_k(\mathbb{P}^n)\longrightarrow Rat_{k+1}(\mathbb{P}^n)$, *then* $H_*(Rat_\infty(\mathbb{P}^n),\mathbb{Z}) \cong H_*(\Omega^2_0 S^{2n+1},\mathbb{Z})$.

Remark 5.4: Cohen and Shimamoto show that the isomorphism in 5.2 is induced from an actual homotopy equivalence $\mathrm{Rat}_k(\mathbb{P}^n)\simeq C_k(\mathbb{C},S^{2n-1})$ whenever $n>1$. We refer to [8] for a definition of the labelled configuration space $C_k(\mathbb{C},S^{2n-1})$ and its relation with $\Omega^2 S^{2n-1}$.

§6 The Positive Genus Case

In this section we determine the full structure of $Tor^{\mathcal{A}}(\mathbb{F},\mathcal{A}^{\otimes n})$ for $g\geq 1$, and for both rational and $\mathbb{Z}_p$ coefficients. We start by making explicit the action of $\mathcal{A}$ on $\mathcal{A}^{\otimes n}$. The algebra $\mathcal{A}=H_*(SP^\infty(M_g))$ acts on $\mathcal{A}^{\otimes n}$ via the prescription

$$x\cdot(c_1\otimes\cdots\otimes c_n)=\nu_*\left(\Delta^{\mathrm{red}}_*(x)\otimes(c_1\otimes\cdots\otimes c_n)\right)$$

where ν is the componentwise symmetric product multiplication and where Δ^{red}_* is as in 4.11. Now recall that $\mathcal{A}$ has generators the 1-dimensional classes $e_i, 1\leq i\leq 2g$ which are primitive, as well as the top orientation class $[M]$.

NOTATION: We denote by $e_{i;r}$ the element $1\otimes\cdots\otimes e_i\otimes\cdots\otimes 1$ where e_i is in the r^{th} position, $1\leq r\leq n$. By $e_{i;r}e_{j;s}$ for $r<s$ we then mean $1\otimes\cdots\otimes e_i\otimes\cdots\otimes e_j\otimes\cdots\otimes 1$.

Lemma 6.1: $\Delta_*([M_g])=\sum_r [M_g]_r+\sum_{j=1}^{g}\sum_{r<s}(e_{2j;r}e_{2j-1;s}-e_{2j-1;r}e_{2j;s})$.

PROOF: There is a natural collapse map from M_g to a wedge of g tori $T_1\vee\cdots\vee T_g$ inducing an isomorphism in H_1 and such that the image of $[M]$ is $\sum[T_i]$. We have that $T_i=S^1\times S^1$ and $H_*(T_i)=\Lambda(e_{2i-1},e_{2i})$. It is easy to see that

$$\Delta_*[T_i]=\sum[T_i]_r+\sum_{r<s}(e_{2i;r}e_{2i-1;s}-e_{2i-1;r}e_{2i;s})$$

and hence by adding these up the lemma follows. ∎

Lemma 6.2: $\Delta_*^{\text{red}}(e_i) = 0$, $\forall 1 \le i \le 2g, n \ge 2$. *On the other hand*

$$\Delta_*^{\text{red}}([M]) = \begin{cases} \sum_{j=1}^{g} \left(e_{2j} \otimes e_{2j-1} - e_{2j-1} \otimes e_{2j}\right), & \text{if } n = 2 \\ 0, & \text{if } n > 2. \end{cases}$$

PROOF: Note that $e_{2j-1} \otimes e_{2j} - e_{2j} \otimes e_{2j-1} \in H_1(M,*)^{\otimes 2} \subset H_2(SP^\infty(M)^2)$ (here $n = 2$) and this is non-trivial in the image of Δ_*^{red}. The rest is a direct consequence of 4.11. ∎

Corollary 6.3: *Suppose* $n > 2$. *Then*

$$\begin{aligned} Tor^{\mathcal{A}}(\mathbb{F}, \mathcal{A}^{\otimes n}) &\cong \mathcal{A}^{\otimes n} \otimes Tor^{\mathcal{A}}(\mathbb{F}, \mathbb{F}) \\ &\cong \mathcal{A}^{\otimes n} \otimes H_*(SP^\infty(\Sigma M_g, *); \mathbb{F}) \\ &\cong \mathcal{A}^{\otimes n} \otimes_i Tor^{\Lambda(e_i)}(\mathbb{F}, \mathbb{F}) \otimes Tor^{\Gamma([M])}(\mathbb{F}, \mathbb{F}). \end{aligned}$$

PROOF: Since both $\Delta_*^{\text{red}}(e_i)$ and $\Delta_*^{\text{red}}([M])$ vanish for $n > 2$, it follows that Δ_*^{red} vanishes on the generators of $\mathcal{A}$ and hence induces a trivial action on $\mathcal{A}^{\otimes n}$ whenever $n > 2$. This gives the first isomorphism. The last two identities are a consequence of the embedding $H_*(SP^\infty(M)) \hookrightarrow C_*(SP^\infty(X))$ (4.7) and of Cartan-Moore; i.e.

$$\begin{aligned} H_*(SP^\infty(\Sigma X); \mathbb{F}) = Tor^{H_*(SP^\infty(M))}(\mathbb{F}, \mathbb{F}) &= Tor^{\Lambda(e_1, \cdots, e_{2g}) \otimes \Gamma([M])}(\mathbb{F}, \mathbb{F}) \\ &= \otimes_i Tor^{\Lambda(e_i)}(\mathbb{F}, \mathbb{F}) \otimes Tor^{\Gamma([M])}(\mathbb{F}, \mathbb{F}). \end{aligned}$$

We now describe $H_*(SP^n(\Sigma M); \mathbb{F})$ for $\mathbb{F} = \mathbb{Q}$ and $\mathbb{F} = \mathbb{Z}_p$.

§6.1 The Homology of $\mathbf{SP^\infty(\Sigma M_g) = B_{SP^\infty(M_g)}}$

The acyclic bar construction for $\Lambda(e)$ over $\mathbb{Z}$ gives rise to a minimal resolution which is generated at each level $B_i(\Lambda(e))$ by elements of the form $|e|\cdots|e|$ or $e|e|\cdots|e|$ (# of bars is i) and has boundary $\partial|e|e|\cdots|e| = e|e|\cdots|e|$. The generators $|e|\cdots|e|$ generate a divided power algebra (under the shuffle product) and it is readily seen that

$$Tor_{*,*}^{\Lambda(e)}(\mathbb{Z}, \mathbb{Z}) = \Gamma(|e|).$$

The case of divided power algebras is harder. When $\mathbb{F} = \mathbb{Q}$, divided power algebras turn into polynomial algebras and so in this case $Tor^{\Gamma(a)}(\mathbb{Q}, \mathbb{Q}) =$

$\Lambda(|a|)$, implying that

6.4 $$H_*(SP^\infty(\Sigma M);\mathbb{Q}) = \mathbb{Q}(|e_1|,\ldots,|e_{2g}|)\otimes\Lambda([M]).$$

When $\mathbb{F}=\mathbb{Z}_p$, we see that $a^p = p!\gamma_p = 0$. Similarly, $\gamma_{p^i}^p$ is also zero. This shows that each γ_{p^i} generates a *truncated polynomial algebra*

$$P_T(a,p) = \mathbb{Z}_p[a,a^2,\cdots,a^{p-1}]/a^p = 0.$$

Lemma 6.5: (Cartan) $\{\gamma_{p^i}, i\geq 0\}$ *generate* $\Gamma(a)$ *as an algebra over* $\mathbb{Z}_p$ *and one has*

$$\Gamma(a)\otimes\mathbb{Z}_p \cong P_T(a,p)\otimes P_T(\gamma_p,p)\otimes\cdots\otimes P_T(\gamma_{p^i},p)\cdots$$

We assume in what follows that a has even degree (for our purpose $a = [M]$). One can construct a minimal resolution for $P_T(a,p)$ (over $\mathbb{Z}_p$) which is generated by elements

$$\{|a^{p-1}|a|\cdots|a^{p-1}|a|, |a||a^{p-1}|a|\cdots|a^{p-1}|a|\}$$

with boundary $\partial\underbrace{|a^{p-1}|a|\cdots|a^{p-1}|a|}_{i} = a^{p-1}\underbrace{|a^{p-1}|a|\cdots|a^{p-1}|a|}_{i-1}$. As an algebra under the shuffle product, it is checked that

Lemma 6.6:

$$Tor^{P_T(a,p)}(\mathbb{Z}_p,\mathbb{Z}_p)\cong\Lambda(|a|)\otimes\Gamma(|a^{p-1}|a|), p>2.$$

When $p=2$, *then* $P_T(a,2)=\Lambda(a)$ *and* $Tor^{P_T(a,2)}(\mathbb{Z}_2,\mathbb{Z}_2)\cong\Gamma[|a|]$.

Remark 6.7: The element $|a|a^{p-1}|$ in the bar construction is known as the *transpotence* of a. It can be checked that $\beta(|\gamma_{p^{i+1}}|) = |\gamma_{p^i}^{p-1}|\gamma_{p^i}|$ where β is the mod-p bockstein.

Remark 6.8: The generators $|e_i|$, $|M|$, and $|\gamma_{p^i}|$ all represent homology classes in $H_*(\Sigma M)\subset H_*(SP^\infty(\Sigma M))$ and this explains why they are referred to as *suspension classes*. All generators in $H_*(SP^\infty(\Sigma M))$ are assigned a bidegree as in remark 1.7 and we find that

Generator	*Bigrading*
$\lvert e_i\rvert$	$(2;1)$
$\lvert M\rvert$	$(3;1)$
$\lvert\gamma_{p^i}\rvert$	$(2p^i+1;p^i)$.

Bidegrees are additive; for example the bidegree of $|\gamma_{p^i}|\gamma_{p^j}|$ is $(2(p^i + p^j) + 2; p^i + p^j)$. If we let $h_{2p^i+1,p^i} = |\gamma_{p^i}|$, we can then write

$$Tor^{\Gamma(a)}(\mathbb{Z}_p, \mathbb{Z}_p) \cong \Lambda(|a|, \ldots, h_{2p^i+1,p^i}, \ldots) \otimes \Gamma(\beta h_{2p+2,p}, \ldots, \beta h_{2p^i+2,p^i}, \ldots).$$

This is the same as noted in §5 as $H_*(SP^\infty(S^3); \mathbb{Z}_p) = H_*(K(\mathbb{Z}, 3); \mathbb{Z}_p)$. Generally we have the following.

Lemma 6.9: *The homology $H_*(SP^n(\Sigma M_g); \mathbb{F})$ is given by those classes of bidegree $(*; i)$ with $i \leq n$, in*

$$\Gamma(|e_1|, \ldots, , |e_{2g}|) \otimes Tor^{\Gamma(a)}(\mathbb{F}, \mathbb{F}).$$

§6.2 Homology Calculations, n > 2

The preceding discussion as well as corollary 6.3 show that for $n > 2$

6.10

$$Tor^{\mathcal{A}}(\mathbb{F}, \mathcal{A}^{\otimes n}) = \Gamma(a_1, \ldots, a_n) \otimes \Gamma(|e_1|, \ldots, |e_{2g}|) \otimes \Lambda(e_1, \ldots, e_{2ng}) \otimes Tor^{\Gamma([M])}(\mathbb{F}, \mathbb{F}),$$

the terms of $(n+1)$-grading $(*, k \ldots, k)$ making up all of $H_{2nk-*}(\mathrm{Div}_k(M - *; \mathbb{F})$.

Proposition 6.11: *Assume $n > 2$ and $g \geq 1$ and consider the algebra*

$$\Lambda(f_{1;1}, \ldots, f_{2ng;1}, \ldots, f_{1;n}, \ldots, f_{2g;n}) \otimes \Lambda(E) \otimes \mathbb{Q}(h_1, \ldots, h_{2g}).$$

This algebra is $(n+1)$-graded according to: $f_{i;r} \mapsto (1; 0, \ldots, 1, \ldots, 0)$ with 1 in the $r+1$ position, $1 \leq r \leq n$, $E \mapsto (2n-3; 1, \ldots, 1)$ and $h_j \mapsto (2n-2; 1, \ldots, 1)$. The multigrading is additive. The homology groups $H_*(Div_k(M_g - *); \mathbb{Q})$ are now given by those elements of multidegree $(*; i_1, \ldots, i_n)$ with $i_j \leq k$.

Proof: With $\mathbb{Q}$ coefficients 6.10 takes the form

$$\mathbb{Q}(a_1, \ldots, a_n) \otimes \mathbb{Q}(|e_1|, \ldots, |e_{2g}|) \otimes \Lambda(e_1, \ldots, e_{2ng}) \otimes \Lambda(|M|).$$

It is now a matter of counting the multidegree $(2nk - *; k, \ldots, k)$ elements. The degree one generators are represented by $a_1^k \ldots e_{i;r} a_j^{k-1} \ldots a_n^k$ in $H_{2nk-1}(TY_k)$, $1 \leq i \leq 2g$ and $1 \leq r \leq n$, and to them correspond the $f_{i;r} \in H_1(\mathrm{Div}_k(M -$

$*);\mathbb{Q})$. Similarly, $|M|$ is Poincaré dual to an element E of the right filtration and of homology degree $2n-3$, whereas h_j are dual to the $|e_j|$'s. Here $\mathbb{Q}(a_1,\ldots,a_n)$ serves as a "calibrating" factor and the calculation follows. ∎

When $\mathbb{F}=\mathbb{Z}_p$, p odd, we can facilitate the counting by dualizing $Tor^{\mathcal{A}}(\mathbb{Z}_p,\mathcal{A}^{\otimes n})$. Divided power algebras turn into polynomial algebras and we get the Tot space

$$\begin{aligned}\mathbb{Z}_p(a_1,\ldots,a_n) &\otimes \Lambda(e_1,\ldots,e_{2ng})\otimes\mathbb{Z}_p\left(|e_1|,\ldots,|e_{2g}|\right)\\ &\otimes \Lambda(|M|,|\gamma_p|,\ldots,|\gamma_{p^i}|\ldots)\otimes\mathbb{Z}_p\left(|M^{p-1}|M|,\ldots,|\gamma_{p^i}^{p-1}|\gamma_{p^i}|,\ldots\right)\end{aligned}$$

Here we ought to write $e_i^*,|e_i|^*,[M]^*$ for the classes above but for the sake of simplicity we leave that out. Our calibration procedure leads generators $e_{i;r}$ as well as

$$\begin{array}{cccc} & h_i & E_i & H_i\\ (n+1)\text{-degree} & (2(n-1);1,\ldots,1) & \left(2(n-1)p^i-1;p^i,\ldots,p^i\right) & \left(2(n-1)p^i;p^i,\ldots,p^i\right).\end{array}$$

Lemma 6.12: *Assume $n>2$ and p odd, then $H_*(Div_k^n(M_g-*);\mathbb{Z}_p)$ is given by those classes in*

$$\otimes_{1\leq r\leq n}\Lambda(e_{1;r},\ldots,e_{2g;r})\otimes\mathbb{Z}_p(h_1,\ldots,h_{2g})\otimes\Lambda(E_1,\ldots,E_j,\ldots)\otimes\mathbb{Z}_p(H_1,\ldots,H_j,\ldots)$$

of multidegree $(*;i_1,\ldots,i_n)$ with $i_j\leq k$.

§6.3 Homology Calculations, n = 2

As noticed in 6.3, the action of $\mathcal{A}$ on $\mathcal{A}^{\otimes 2}$ is not trivial. The Tor term $Tor^{\mathcal{A}}(\mathbb{F},\mathcal{A}\otimes\mathcal{A})$ takes the form

6.13 $\quad\Gamma(a_1,a_2)\otimes\Gamma(|e_1|,\ldots,|e_{2g}|)\otimes Tor^{\Gamma([M])}(\mathbb{F},\Lambda(e_1,\ldots,e_{4g}))$

and the calculation boils down to understanding the term on the far right. The boundary here in the total space $\Lambda(e_1,\ldots,e_{4g})\otimes E(\Gamma[M])$ is generated by $\partial(|M|)=\sum_{i=1}^g e_{2i-1}e_{2i}-e_{2i}e_{2i-1}$ (cf. lemma 6.2). By reordering the e_i's and renaming, we can rewrite it as $\partial(|M|)=\sum_{i=1}^{2g}e_{2i-1}e_{2i}$. Assume for now that $\mathbb{F}=\mathbb{Q}$. We can rewrite 6.13 as follows

$$H_*(\mathcal{W}_g;\mathbb{Q})\otimes\mathbb{Q}(h_1,\cdots,h_{2g})\otimes\mathbb{Q}(a_1,a_2)$$

where $\mathcal{W}_g$ is the complex

$$\Lambda(e_1,\cdots,e_{4g})\otimes\Lambda|M|\xrightarrow{\partial}\Lambda(e_1,\cdots,e_{4g}),\ \ \partial(|M|)=e_1e_2+e_3e_4+\cdots e_{4g-1}e_{4g}.$$

When taking Poincaré duals we see that

Lemma 6.14: *Let $\bar{\mathcal{W}}_g$ denote the complex*

$$\Lambda(e_1,\ldots,e_{4g},f)\to\Lambda(e_1,\ldots,e_{4g}),\ \ \delta f=e_1e_2+\cdots+e_{4g-1}e_{4g}.$$

Then

$$H_*(Div^2_k(M_g-*);\mathbb{Q})\subset H_*(\bar{\mathcal{W}}_g,\mathbb{Q})\otimes\mathbb{Q}(h_1,\ldots h_{2g})$$

consists of elements with tridegrees $(*;i,j), i,j\leq k$, where tridegrees are assigned as follows: $e_{odd}\mapsto(1;1,0), e_{even}\mapsto(1;0,1), f\mapsto(1;1,1), h_i\mapsto(2;1)$.

The complex $\bar{\mathcal{W}}_g$ has been studied in both [2] and [3] and its betti numbers have been completely determined. It is shown there for instance that the map

$$\cup(e_1e_2+\cdots e_{4g-1}e_{4g}):\Lambda(e_1,\cdots,e_{4g})\longrightarrow\Lambda(e_1,\cdots,e_{4g})$$

is injective in degrees $\leq 2g$ and surjective in degrees $\geq 2g$. Moreover, if $\nu(i,g)$ denotes the rank of $H_i(\bar{\mathcal{W}}_g;\mathbb{Q})$, then we have

Lemma 6.15: (BCM) $\nu(i,g)=0$ for $i>4g+1$, and otherwise

$$\nu(i,g)=\begin{cases}\binom{4g}{i}-\binom{4g}{i-2} & \text{for } i\leq 2g\\ \binom{4g}{i-1}-\binom{4g}{i+1} & \text{for } 2g<i\leq 4g+1.\end{cases}$$

Remark 6.16: When $\mathbb{F}=\mathbb{Z}_p$, the boundary terms take the form

$$\begin{aligned}\partial(|\gamma_{p^i}|) &= \frac{1}{p^i}\left(\sum_1^g e_{2i-1}e_{2i}\right)^{p^i},\\ \partial(|\gamma_{p^i}^{p-1}|\gamma_{p^i}|) &= \left[\frac{1}{p^i}\left(\sum_1^g e_{2i-1}e_{2i}\right)^{(p-1)p^i}\right]|\gamma_{p^i}|\end{aligned}$$

These last differentials correspond to the Kudo differential in the Serre spectral sequence associated to the quasi-fibration $SP^\infty(M)^n\longrightarrow DY\longrightarrow SP^\infty(\Sigma M)$. They also describe the d_{p^i} in §4.4.

Example 6.17: As an example, we carry out the calculation for $T = M_1$ a genus 1 surface, $n = 2$ and $p = 2$. The complex at hand can be written as

$$\Lambda(e_1, e_2, e_3, e_4) \otimes \mathbb{Z}_2(|e_1|, |e_2|, |T|, |\gamma_2|, \ldots, |\gamma_{2^i}| \ldots)$$

on generators with tridegrees: $e_1, e_3 \mapsto (1; 1, 0), e_2, e_4 \mapsto (1; 0, 1), |e_i| \mapsto (2; 1, 1), |T| \mapsto (1; 1, 1)$ and $|\gamma_{2^i}| \mapsto (2^{i+1} - 1; 2^i, 2^i)$. The coboundary is given by

$$\delta(|T|) = e_1e_2 + e_3e_4, \ \delta(|\gamma_{2^i}|) = \frac{1}{2^i}(e_1e_2 + e_3e_4)^{2^i}.$$

This implies that $\delta(|\gamma_2|) = e_1e_2e_3e_4$ and $\delta(|\gamma_{2^i}|) = 0$ for $i \geq 2$. We're interested in all elements of tridegree $(*, i, j), i, j \leq k$. For example we find that

Lemma 6.18: *The poincaré series for $H_*(Div_k^2(T - *); \mathbb{F}_2)$, $k = 1, 2$, are given by*

$$P(x) = 1 + 4x + 5x^2 \ (k = 1), \quad P(x) = 1 + 4x + 7x^2 + 9x^3 + 6x^4 \ (k = 2).$$

§7 Homotopy Constructions

We start with a fibration sequence due to Segal, which along with the scanning map also first constructed in [18], constitute the main tool in setting up the correspondence between divisor spaces and mapping spaces. We denote by $\mathbb{P}^n$ the nth complex projective space, and we use interchangeably $\mathbb{P}$ and $\mathbb{P}^\infty$ for the infinite complex space.

§7.1 Fat-Wedge Fibrations

Let $W_n\mathbb{P}$ denote the n^{th} *fat wedge* of $\mathbb{P}^\infty$; that is $W_n\mathbb{P}$ is the subset of $(\mathbb{P}^\infty)^n$ consisting of all n-tuples with at least one entry equal to basepoint in $\mathbb{P}^\infty$ (we sometimes write W_n for $W_n\mathbb{P}$). Of course $W_1 = \{x_0\}$ and $W_2 = \mathbb{P}^\infty \vee \mathbb{P}^\infty$. One has the following

Lemma 7.1: (Segal) *There is a fibration sequence*

$$(S^1)^n \longrightarrow \mathbb{P}^n \longrightarrow W_{n+1}\mathbb{P} \stackrel{\theta}{\longrightarrow} (\mathbb{P}^\infty)^n = B(S^1)^n.$$

The projection θ is described (up to homotopy) in [18],§2. When $n = 1$, this is the folding map (i.e. the restriction of $\theta : \mathbb{P}\times\mathbb{P} \to \mathbb{P}, \theta(a,b) = a\cdot b^{-1}$). In this particular case, the fiber of θ is the total space of the fibration over $\mathbb{P}\vee\mathbb{P}$ induced from the path/loop fibration $S^1 \longrightarrow S^\infty \longrightarrow \mathbb{P}$ and this is seen to be $S^\infty \cup_{S^1} S^\infty \simeq \Sigma S^1 = S^2$.

Lemma 7.2: *There is a homotopy fibration*

$$S^{2n+1} \longrightarrow W_{n+1}\mathbb{P} \hookrightarrow (\mathbb{P}^\infty)^{n+1}$$

PROOF: The proof proceeds by induction. In the case $n = 1$, we have the inclusion $W_2 = \mathbb{P} \vee \mathbb{P} \hookrightarrow \mathbb{P} \times \mathbb{P}$. A general result of Ganea states that the homotopy fiber of the inclusion $X \vee Y \hookrightarrow X \times Y$ is $\Omega(X) * \Omega(Y) \simeq \Sigma(\Omega(X) \wedge \Omega(Y))$ (here $*$ denotes the join product.) In our case, the fiber then becomes $\Sigma(\Omega\mathbb{P} \wedge \Omega\mathbb{P})) \simeq \Sigma(S^1 \wedge S^1) = S^3$.

For $n > 1$, the fiber of $W_{n+1} \hookrightarrow (\mathbb{P}^\infty)^{n+1}$ is given as the total space of the pull-back of the path-loop fibration $(S^1)^{n+1} \to P \to (\mathbb{P}^\infty)^{n+1}$. Write W_{n+1} as the double mapping cylinder

$$W_n \times \mathbb{P} \longleftarrow W_n \times * \longrightarrow (\mathbb{P}^\infty)^n \times *.$$

The fiber of $W_n \times \mathbb{P} \to (\mathbb{P}^\infty)^{n+1}$ is S^{2n-1} by the induction hypothesis, while the fiber of $(\mathbb{P}^\infty)^n \to (\mathbb{P}^\infty)^{n+1}$ is $\Omega\mathbb{P} = S^1$. It follows that the fiber of $W_n \times * \simeq W_n \times S^\infty \to (\mathbb{P}^\infty)^{n+1} = (\mathbb{P}^n) \times \mathbb{P}$ is $S^{2n-1} \times S^1$, and hence the homotopy fiber $W_{n+1}\mathbb{P} \longrightarrow (\mathbb{P}^\infty)^{n+1}$ can be written as the mapping cylinder $S^{2n-1} \times S^1 \times [0,1]/\sim$ with S^1 collapsed at one end and S^{2n-1} collapsed at the other. But this is no other than $S^{2n-1} * S^1 = S^{2n+1}$ and the proof is complete. ∎

Corollary 7.3: *We have the following commutative diagram of fibrations (here h denotes the Hopf map and Δ_{n+1} is the diagonal inclusion)*

$$\begin{array}{ccccc} S^{2n+1} & \xrightarrow{=} & S^{2n+1} & \longrightarrow & * \\ \downarrow h & & \downarrow G & & \downarrow \\ \mathbb{P}^n & \longrightarrow & W_{n+1} & \xrightarrow{\theta} & (\mathbb{P}^\infty)^n \\ \downarrow & & \downarrow & & \downarrow = \\ \mathbb{P}^\infty & \xrightarrow{\Delta_{n+1}} & (\mathbb{P}^\infty)^{n+1} & \longrightarrow & (\mathbb{P}^\infty)^n \end{array}$$

Remark 7.4: Looping 7.2 yields a principal fibration $\Omega(i) : \Omega(S^{2n-1}) \rightarrow \Omega(W_n) \rightarrow \Omega(\mathbb{P}^\infty)^n$ which admits a cross section obtained as follows. Let s_i be the inclusion of $\mathbb{P}^\infty$ into W_n as the i-th factor. Then the composition

$$s : (\Omega\mathbb{P}^\infty)^n \xrightarrow{(\Omega s_1\times\cdots\times\Omega s_n)} (\Omega W_n)^n \xrightarrow{\ *\ } \Omega W_n$$

provides the desired section of 7.2 (where here $*$ is loop multiplication). Naturally this implies that

$$\Omega W_n \simeq (\Omega\mathbb{P}^\infty)^n \times \Omega(S^{2n-1}) \simeq (S^1)^n \times \Omega(S^{2n-1}).$$

This splitting is not an H-space splitting (in case $n = 2$ for instance, the right hand side is abelian while $\Omega(\mathbb{P} \vee \mathbb{P})$ is not). The inclusion $\Omega S^{2n-1} \hookrightarrow \Omega W_n$ is however loop-sum preserving.

Lemma 7.5: *Consider 7.2; $S^{2n-1} \xrightarrow{G} W_n \hookrightarrow (\mathbb{P}^\infty)^n$, and let a_i denote the homotopy class of the ith inclusion $S^2 = \mathbb{P}^1 \hookrightarrow 1^{i-1} \times \mathbb{P}^1 \times 1^{n-i} \hookrightarrow (\mathbb{P}^\infty)^n$. Then G is an iterated Whitehead product*

$$G = [\cdots [[a_1, a_2], a_3] \ldots], a_n] \, .$$

Remark 7.6: One can apply the functor $\mathrm{Map}^*(M_g, -)$ to 7.3 and obtain a new diagram of fibrations. It is not hard to see that $\mathrm{Map}^*_c(M_g, \mathbb{P}^\infty) \simeq (S^1)^{2g}$ where Map^*_c is any component of Map^*. Indeed and since the attaching map of the two disc in M_g maps into $\bigvee^{2g} S^1$ as a commutator, it follows that its suspension is null. This implies that

$$\begin{aligned}\mathrm{Map}^*(M_g, \mathbb{P}^\infty) &\simeq \mathrm{Map}^*(\Sigma M_g, K(\mathbb{Z}, 3)) \simeq \mathrm{Map}^*(S^3 \vee \bigvee^{2g} S^2, K(\mathbb{Z}, 3)) \\ &\simeq \mathbb{Z} \times \coprod^{2g} \Omega^2(K(\mathbb{Z}, 3)) \simeq \mathbb{Z} \times (S^1)^{2g}.\end{aligned}$$

One therefore gets the diagram

$$\begin{array}{ccccc}
\mathrm{Map}^*(M_g, S^{2n+1}) & \xrightarrow{=} & \mathrm{Map}^*(M_g, S^{2n+1}) & \longrightarrow & * \\
\Big\downarrow h & & \Big\downarrow G & & \Big\downarrow \\
\mathrm{Map}^*(M_g, \mathbb{P}^n) & \longrightarrow & \mathrm{Map}^*(M_g, W_{n+1}) & \longrightarrow & (S^1)^{2ng} \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow = \\
(S^1)^{2g} & \longrightarrow & (S^1)^{2(n+1)g} & \longrightarrow & (S^1)^{2ng}
\end{array}$$

When $n = 1$, we know that $\mathrm{Map}^*(M_g, S^3) \simeq (\Omega^2 S^3) \times (\Omega S^3)^{2g}$ and that $\mathrm{Map}^*(M_g, S^2)$ splits as $(\Omega S^3)^{2g} \times X_g$ for some total space $\Omega^2(S^3) \rightarrow X_g \rightarrow (S^1)^{2g}$ (see [3],§11; or [7],§7). This however still is not enough to conclude any splitting for $\mathrm{Map}^*(M_g, \mathbb{P}^\infty \vee \mathbb{P}^\infty)$ (see §8.2.)

THE SAMELSON PRODUCT: This is standard [6] but we review it briefly. Given a loop space $\Omega(X)$, we denote by $S : \Omega(X) \times \Omega(X) \longrightarrow \Omega(X)$ the commutator map $S(f,g) = f * g * f^{-1} * g^{-1}$. The map S is null homotopic when either f or g is constant at basepoint and hence it descends to a map $S : \Omega(X) \wedge \Omega(X) \longrightarrow \Omega(X)$. The Samelson product

$$\langle , \rangle : \pi_p(\Omega(X)) \otimes \pi_q(\Omega(X)) \longrightarrow \pi_{p+q}(\Omega(X))$$

is defined to be the composite $S^p \wedge S^q \xrightarrow{\alpha\wedge\beta} \Omega(X) \wedge \Omega(X) \xrightarrow{S} \Omega(X)$.

Theorem 7.7: (Samelson) *Consider the suspension $E : X \longrightarrow \Omega\Sigma X$ and the induced map $ad : X \wedge X \longrightarrow \Omega\Sigma X$ given by $S \circ (E \wedge E)$. Then if x and y are primitive, we have*

$$ad_*(x \otimes y) = x \otimes y - (-)^{|x||y|} y \otimes x \in H_*(\Omega\Sigma X) \cong T(H_*(X)).$$

§7.2 Segal's Scanning Map

We can now describe the map

$$S : \mathrm{Div}_k^n(M_g - *) \longrightarrow \mathrm{Map}^*_{\vec{0}}(M_g, W_n\mathbb{P}) \qquad (7.8)$$

where $\mathrm{Map}^*_{\vec{0}}(M_g, W_n\mathbb{P})$ refers to the subspace of based, null-homotopic maps (or equivalently based maps of multidegree $\vec{0} = (0, \dots, 0)$) in $\mathrm{Map}_{\vec{0}}(M_g, W_n) \subset \mathrm{Map}_{\vec{0}}(M_g, (\mathbb{P}^\infty)^n)$.

Fix $r > 0$ (r small) and let $D_r(x) \subset M_g$ be the disc of radius r around the point $x \in M_g$. Since $M_g - *$ is parallelizable, one can identify canonically the pair $(\bar{D}(x), \partial\bar{D}(x))$ with (S^2, ∞) where the north pole ∞ is chosen to be the basepoint in S^2. To a given positive divisor $D \in SP^r(M_g)$ and to any $x \in M_g$, we can associate the divisor $D^x \in SP^\infty(S^2, \infty) = \mathbb{P}$ made out of points of $D \cap D_r(x)$ and extended out by basepoints; i.e.

$$D^x = \langle D \cup D_r(x), \infty, \ldots \rangle.$$

Let $(D_1, \ldots, D_n) \in \mathrm{Div}_k^n(M_g - *)$, then one defines

7.9 $$S : M_g \longrightarrow \mathrm{Div}^n(S^2, \infty),\ x \mapsto (D_1^x, \ldots, D_n^x)$$

where here $\mathrm{Div}^n(S^2, \infty) \subset (\mathbb{P}^\infty)^n$ consists of all n-tuples of divisors of which supports do not have a point in common (here the support of $D = \sum n_i z_i \in SP^\infty(S^2, \infty)$ is the set of $z_i \neq \infty$). One should probably point out the important difference in topology between $\mathrm{Div}^n(S^2, \infty)$ and $\mathrm{Div}^n(S^2 - \infty)$.

As was observed in [18], we can let Q_ϵ be the open subset of $\mathrm{Div}^n(S^2, \infty)$ consisting of n-tuples of divisors such that (at least) one such divisor, say D_i, is disjoint from the closed disk of radius ϵ about the origin (south pole). Then radial expansion defines a deformation retraction of Q_ϵ into W_n (more precisely in this case, the ith component of $(\mathbb{P}^\infty)^n$ gets retracted to ∞). This shows that $Q_\epsilon \simeq W_n$ and since $\mathrm{Div}^n(S^2, \infty) = \bigcup_{\epsilon > 0} Q_\epsilon$ we get

Lemma 7.10: $\mathrm{Div}^n(S^2, \infty) \simeq W_n\mathbb{P}$.

It is clear that 7.9 has multidegree $(k, \ldots, k)$ and hence when combined with 7.10 it yields a map $S : \mathrm{Div}_k^n(M_g - *) \longrightarrow \mathrm{Map}^*_{(k,\ldots,k)}(M_g, W_n\mathbb{P})$. Since all components of the mapping space are homotopy equivalent we obtain the map 7.8. Note that the stabilization process of §3 yields a (homotopy) commutative diagram

$$\begin{array}{ccc} \mathrm{Div}_k^n(M_g - *) & \longrightarrow & \mathrm{Map}^*_{\vec{k}}(M_g, W_n\mathbb{P}) \\ \downarrow & & \downarrow \\ \mathrm{Div}_{k+1}^n(M_g - *) & \longrightarrow & \mathrm{Map}^*_{\vec{k+1}}(M_g, W_n\mathbb{P}), \end{array}$$

and in the direct limit we obtain

Theorem 7.11: (Segal) *$S : Div^n(M_g - *) \longrightarrow Map^*_0(M_g, W_n\mathbb{P})$ is a homotopy equivalence.*

§8 The Splitting

Recall that associated to M_g, we have the cofibration sequence

8.1 $$S^1 \xrightarrow{f} \vee_1^{2g} S^1 \longrightarrow M_g \longrightarrow S^2 \longrightarrow \vee_1^{2g} S^2,$$

with f given as a product of commutators $[x_1, x_2][x_1, x_2] \cdots [x_{2g-1}, x_{2g}] \in \pi_1(\vee_1^{2g} S^1)$. Applying the functor $\mathrm{Map}^*(-, X)$ to 8.1 yields the fibration sequence

$$\Omega^2 X \longrightarrow \mathrm{Map}^*(M_g, X) \longrightarrow (\Omega X)^{2g} \xrightarrow{f^!} \Omega X.$$

The map $f^!$ classifies $\Omega^2 X \longrightarrow \mathrm{Map}^*(M_g, X) \longrightarrow (\Omega X)^{2g}$ and since $f^! = [x_1, x_2]^! \cdots [x_{2g-1}, x_{2g}]^!$ is described in terms of commutators, it is natural to suspect that obstructions to the nullity of $f^!$ lie in various Whitehead products. This is indeed the case.

Write $X = W_n$. To analyze $f^!$, it is enough to consider one commutator at a time say $[x_1, x_2]^!$. This we write as the following composition

$$(\Omega W_n)^2 \xrightarrow{\Delta} \big((\Omega W_n)^2\big)^2 \xrightarrow{id^2 \times \chi^2} \big((\Omega W_n)^2\big)^2 \longrightarrow (\Omega W_n)^4 \xrightarrow{*^4} \Omega W_n$$

where χ is the inverse map with respect to the loop sum, $\chi(f)(t) = f(1-t) = f^{-1}(t)$.

In §7 we saw that we had a map $\Omega W_n \xrightarrow{\pi} (S^1)^n$ (and a splitting $\Omega W_n \simeq \Omega S^{2n-1} \times (S^1)^n$). The composite

$$(\Omega W_n)^2 \xrightarrow{[x_1,x_2]^!} \Omega W_n \xrightarrow{\pi} (S^1)^n$$

is a commutator in an abelian group and hence it is trivial. It follows that $\pi f^!$ is also homotopy trivial and hence $f^!$ factors (up to homotopy)

$$f^! : (\Omega S^{2n-1} \times (S^1)^n)^{2g} \longrightarrow \Omega S^{2n-1} \hookrightarrow \Omega W_n.$$

QUESTION: Does $f^!$ factor further through $(S^1)^{2ng}$;

8.2 $$f^! : (\Omega S^{2n-1} \times (S^1)^n)^{2g} \longrightarrow (S^1)^{2ng} \longrightarrow \Omega S^{2n-1} \hookrightarrow \Omega W_n.$$

In their study of $\mathrm{Div}_k^2(M_g - *)$, [7] only needed to consider the case $n = 2$ and the question above was conjectured to be true.

We can analyze the obstruction to factoring $f^!$ as in 8.2 as follows. Start with

$$\left(\Omega S^{2n-1}\times(S^1)\right)^2 \xrightarrow{(\Omega G*e)^2} \Omega(W_n)^2 \xrightarrow{[x_1,x_2]^!} \Omega(W_n)$$

where $e: S^1\to W_n$ is anyone of the Ωs_i described in 7.5. Letting e_1 and e_2 (resp. ΩG_1 and ΩG_2) the maps of S^1 (resp. ΩS^{2n-1}) into the first and second copies of ΩW_n, one can write (up to sign)

$$[x_1,x_2]^1\left((e_1*\Omega G_1)\times(e_2*\Omega G_2)\right)\mapsto e_1*\Omega G_1*e_2*\Omega G_2*\chi(\Omega G_1)*\chi(e_1)*\chi(\Omega G_2)*\chi(e_2).$$

Suppose the image of ΩG and the image of e commute. Then we can rewrite the above as follows

$$\begin{aligned}(e_1,e_2,\Omega G_1,\Omega G_2) &\mapsto (e_1*e_2*[\Omega G_1*\Omega G_2*\chi(\Omega G_1)*\chi(\Omega G_2)]*\chi(e_1)*\chi(e_2))\\ &= e_1*e_2*\{\Omega G_1,\Omega G_2\}*e_1^{-1}*e_2^{-1}.\end{aligned}$$

If we suppose further that ΩG_1 and ΩG_2 commute in ΩW_n then $f^!$ would factor as desired through $(S^1)^{2gn}$. This then shows that the desired factorization 8.2 happens under the following conditions:
- ΩG and e commute in ΩW_n
- ΩS^{2n-1} is homotopy abelian.

The second condition is true after inverting 2. Indeed, an odd sphere is an H space after inverting 2 at which point the loop space becomes abelian. To address the validity of the first condition, we restrict our attention to the commutator

$$\Omega S^{2n-1}\times S^1 \xrightarrow{\{\Omega G,e\}} \Omega(W_n\mathbb{P}).$$

Observe that $\Omega S^{2n-1}=\Omega\Sigma(S^{2n-2})\simeq J(S^{2n-2})$ where $J(S^{2n-2})$ is the James construction on S^{2n-2} corresponding to the free monoid generated by points of S^{2n-2}. The commutator map can therefore be reduced to $S^{2n-2}\times S^1\xrightarrow{\{\Omega G,e\}}\Omega(W_n\mathbb{P})$ and from there, one can use the correspondence between the Samelson and Whitehead products [6] to write

8.3 $$ad\{\Omega G,e\}=[G,a]\in\pi_{2n}(W_n\mathbb{P})=\pi_{2n}S^{2n-1}=\mathbb{Z}_2$$

where $a=ade:\Sigma S^1=S^2\hookrightarrow\mathbb{P}\hookrightarrow W_n$.

In either case then, it follows that the obstructions to factoring $f^!$: $(W_n\mathbb{P})^{2g}\to\Omega S^{2n-1}\times(S^1)^n$ through $(S^1)^n$ are $\mathbb{Z}_2$ obstructions. We have proved the following.

Proposition 8.4: *The following splits after inverting* 2

$$Map^*(M_g, W_n\mathbb{P}) \simeq (\mathbb{Z})^n \times \Omega(S^{2n-1})^{2g} \times Y_{g,n}$$

where $Y_{g,n}$ *is the total space of a (principal) fibering* $\Omega^2(S^{2n-1}) \longrightarrow Y_{g,n} \longrightarrow (S^1)^{2gn}$.

§8.1 The obstruction when n = 2

When $n > 2, n \neq 4, 8$, it is clear that 8.4 is best possible in this case. However when $n = 2$, one can hope to relax the localization condition there for in this case ΩS^3 is homotopy abelian (S^3 being a group) and the first obstruction discussed earlier is not essential. We show however that the second obstruction 8.3 is.

Let G be as in lemma 7.5. We know that the homotopy class of G is represented by $[a_1, a_2]$ and hence $[G, a_1]$ corresponds to $[[a_1, a_2], a_1]$. We show that this triple Whitehead product generates $\pi_4(\mathbb{P} \vee \mathbb{P}) \cong \mathbb{Z}_2$.

We start by considering the first few stages of the Postnikov decomposition for $X = \mathbb{P} \vee \mathbb{P}$. Notice that

$$\pi_1(X) = 0,\ \pi_2(X) \cong \mathbb{Z} \times \mathbb{Z},\ \pi_3(X) \cong \mathbb{Z} \text{ and } \pi_4(X) \cong \mathbb{Z}_2.$$

and hence

$$\begin{array}{ccccc}
K(\mathbb{Z}_2, 4) & \longrightarrow & X_4 & & \\
 & & \downarrow & & \\
K(\mathbb{Z}, 3) & \xrightarrow{i} & X_3 & \xrightarrow{k^5} & K(\mathbb{Z}_2, 5)\,. \\
 & & \downarrow & & \\
\mathbb{P} \vee \mathbb{P} & \xrightarrow{f_2} & \mathbb{P} \times \mathbb{P} & \xrightarrow{k^4} & K(\mathbb{Z}, 4)
\end{array}$$

where f_2 is the inclusion $\mathbb{P} \vee \mathbb{P} \hookrightarrow \mathbb{P} \times \mathbb{P}$. The fiber of f_2 is S^3 and so

$$\tau(\iota_3) = k^4, \quad \iota_3 \in \mathbb{Z} \cong H^3(S^3, \pi_3(X))$$

where τ is the transgression. Since $H^*(\mathbb{P} \times \mathbb{P}) \cong \mathbb{Z}[a_1, a_2]$, with a_1 and a_2 being the dual cohomology classes to the 2-dimensional generators corresponding to the inclusions $S^2 \hookrightarrow \mathbb{P} \hookrightarrow \mathbb{P} \times \mathbb{P}$, and since

$$H^*(\mathbb{P} \vee \mathbb{P}) \cong \mathbb{Z}[a_1, a_2]/(a_1 a_2),$$

it follows that the class $a_1 a_2$ must be hit by the transgression and hence $k^4 = a_1 a_2$.

Lemma 8.5: *Let γ be the class in $H^5(X_3, \mathbb{Z}_2)$ that restricts to $Sq^2(\iota_3)$, $\iota_3 \in H^3(K(\mathbb{Z}, 3), \mathbb{Z}_2)$. Then γ is non-zero, and $k^5 = \gamma$.*

PROOF: Since d_4 corresponds to the transgression in this case, we have $d_4(\iota_3) = \tau(\iota_3) = a_1 a_2$. Recall that

$$H^*(K(\mathbb{Z}, 3), \mathbb{Z}_2) = \mathbb{F}_2[\iota_3, Sq^2(\iota_3), (\iota_3)^2, Sq^4 Sq^2(\iota_3), \ldots, Sq^{2^i} \cdots Sq^4 Sq^2(\iota_3), \ldots].$$

A quick inspection of the E_4 quadrant shows that the d_4 differential vanishes on all homology generators in the fiber but ι_3. Since $d_4(\iota_3) = a_1 a_2$, and since the classes a_1 and a_2 survive (and their powers), it follows that

$$E_5 = H^*(\mathbb{P} \times \mathbb{P}) \otimes \mathbb{F}_2[Sq^2(\iota_3), \iota_3^2, Sq^4 Sq^2(\iota_3), \ldots].$$

Since ι_3 transgresses, then so does $Sq^2(\iota_3)$. We then have

$$d_6(Sq^2(\iota_3)) = Sq^2(d_4(\iota_3)) = Sq^2(a_1 a_2).$$

But $Sq^2(a_1 a_2)$ is already hit by d_4 as the following application of the Cartan formula (with $Sq^1(a_i) = 0$ in $\mathbb{P}$) shows

$$Sq^2(a_1 a_2) = Sq^2(a_1) a_2 + a_1 Sq^2(a_2) = (a_1 + a_2) a_1 a_2 = d_4(a_1 + a_2)\iota_3.$$

It follows that $d_6(Sq^2(\iota_3)) = 0$ and that $Sq^2(\iota_3)$ survives to E_∞. Since it is the only class in $H^5(X_3, \mathbb{Z}_2)$, it must be the image of the transgression $\tau(H^4(K(\mathbb{Z}_2, 4), \mathbb{Z}_2))$ in the next stage of the Postnikov tower. This proves the lemma. ∎

Now consider the pullback diagram

$$\begin{array}{ccc}
K(\mathbb{Z}, 3) & \xrightarrow{=} & K(\mathbb{Z}, 3) \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\
E & \xrightarrow{j} & X_3 \\
\downarrow & & \downarrow \\
\mathbb{P}^\infty & \xrightarrow{i_1} & \mathbb{P}^\infty \times \mathbb{P}^\infty \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle a_1 a_2} \\
K(\mathbb{Z}, 4) & \xrightarrow{=} & K(\mathbb{Z}, 4)
\end{array}$$

where i_1 is the inclusion of $\mathbb{P}$ into the first factor. The pull back of the k-invariant a_1a_2 under i_1 is trivial (since $i_1^*(a_2) = 0$). The induced total space is then $E = K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 2)$

Lemma 8.6: $j^*(\gamma) = Sq^2(\iota_3) + \iota_3 \cup a_1$.

PROOF: In the fibration

$$K(\mathbb{Z}, 3) \xrightarrow{i} X_3 \xrightarrow{p} \mathbb{P} \times \mathbb{P},$$

$\tau(Sq^2(\iota_3)) = a_1a_2(a_1 + a_2)$ means that there is a class $\beta \in \mathcal{C}^5(X_3)$ (where $(\mathcal{C}, \delta)$ is a cochain complex) such that $i^*([\beta]) = Sq^2(\iota_3)$ (it's the only time here that we differentiate between a cochain x and its cohomology class $[x]$) and that

$$\delta(\beta) = p^*(a_1a_2(a_1 + a_2)) = p^*(a_1a_2) \cup p^*(a_1 + a_2).$$

The cochain β is chosen modulo $p^*(\mathbb{P} \vee \mathbb{P})$. Now, the pull back $i^*(j^*(\beta))$ must correspond to $Sq^2(\iota_3)$ and hence $j^*(\beta) - Sq^2(\iota_3) \in ker(i^*) = \{0, i_3 \cup a_1\}$. By the choice of β modulo $p^*(\mathbb{P} \vee \mathbb{P})$, we must then have that $j^*(\beta) = Sq^2(\iota_3)$.

On the other hand, and since ι_3 transgresses to a_1a_2 it follows also that $p^*(a_1a_2) = \delta(\iota_3)$. (Here we're thinking of ι_3 as some cochain in X_3 mapping onto $\iota_3 \in H^3(K(\mathbb{Z}, 3))$ under the epimorphism $i^* : \mathcal{C}^3(X_3) \longrightarrow \mathcal{C}^3(K(\mathbb{Z}, 3))$. It then follows that

$$\delta(\beta) = \delta(\iota_3) \cup (a_1 + a_2) = \delta(\iota_3(a_1 + a_2))$$

and hence that $\delta(\beta + \iota_3(a_1 + a_2)) = 0$. Moreover $i^*(\beta + \iota_3(a_1 + a_2)) = Sq^2(\iota_3)$. This shows that we can choose $\gamma \in H^5(X_3)$ to be equal to $\beta + \iota_3(a_1 + a_2)$. We have therefore that

$$j^*(\gamma) = j^*(\beta) + j^*(\iota_3(a_1 + a_2)) = Sq^2(\iota_3) + \iota_3 \cup a_1$$

and the lemma follows. ∎

Lemma 8.7: $[[a_1, a_2], a_1]$ *generates* $\pi_4(\mathbb{P} \vee \mathbb{P}) \cong \mathbb{Z}_2$.

PROOF: Consider the composite map

$$S^3 \times S^2 \xrightarrow{f = \iota_3 \times \iota_2} E = K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 2) \xrightarrow{j} X_3.$$

The pull back of γ via $f \circ j$ is

$$f^*(j^*(\gamma)) = f^*(Sq^2(\iota_3)) + f^*(\iota_3 \cup a_1) = f^*(\iota_3) \cup f^*(a_1) = \kappa_3 \cup \kappa_2$$

where κ_i is the generator of $H^i(S^i)$. It follows that the composite $S^3 \times S^2 \longrightarrow X_3 \longrightarrow K(\mathbb{Z}_2, 5)$ is not zero and hence $S^3 \times S^2$ does not lift to the next stage, X_4, of the Postnikov resolution since the latter $K(\mathbb{Z}_2, 4) \longrightarrow X_4 \longrightarrow X_3$ has k-invariant γ.

The fact that $S^3 \times S^2$ doesn't lift to X_4 implies that the class of the Whitehead product $[\kappa_3, \kappa_2] \in \pi_3(S^3 \vee S^2)$ has image the non-zero generator in $\pi_4(E_2) = \pi_4(\mathbb{P} \vee \mathbb{P})$ (here κ_i also denotes the generating class in $\pi_i(S^i)$. The image of κ_2 is a_1 by construction while the image of $\kappa_1 \in \pi_3(S^3)$ is the Whitehead product $[a_1, a_2] \in \pi_3(\mathbb{P} \vee \mathbb{P})$ according to the diagram

$$\begin{array}{ccc} S^3 & \longrightarrow K(\mathbb{Z}, 3) \longrightarrow & X_3 \\ \downarrow & & \downarrow \\ \mathbb{P} \vee \mathbb{P} & \longrightarrow & \mathbb{P} \times \mathbb{P}. \end{array}$$

This concludes the proof. Note that the non-zero generator $[[a_1, a_2], a_1]$ must correspond to $[a_1, a_2] \circ \eta$ where η is the Hopf map $S^4 \longrightarrow S^3$. ∎

§9 Relation with Mapping Spaces

In this section, we prove the following easy consequence of our previous study of the divisor spaces

Proposition 9.1: *The Eilenberg-Moore spectral sequence associated to the fibration*

$$\Omega^2 S^{2n-1} \longrightarrow Map_0^*(M_g, W_n\mathbb{P}) \longrightarrow (S^1)^{2ng} \times (\Omega S^{2n-1})^{2g}$$

collapses at $E_2 = Tor_{H^*(\Omega S^{2n-1})}(\mathbb{F}, H^*((S^1)^{2ng} \times (\Omega S^{2n-1})^{2g}))$.

PROOF: Consider first the case $n = 2$. In this case, the classifying map being null homotopic on ΩS^3 it follows that the action of $H^*(\Omega S^3)$ on $H^*(\Omega S^3)^{2g}$ is trivial and that

$$* \qquad E_2 = H^*(\Omega S^3)^{2g} \otimes Tor_{H^*(\Omega S^3)}(\mathbb{F}, H^*(S^1)^{4g}).$$

We write $H^*(S^1)^{4g} = \Lambda(e_1, \ldots, e_{4g})$ and $H^*(\Omega S^3) = \Gamma(a)$. We should point out that in the EMSS, the bar degrees are subtracted from the total degree of resolution elements rather than added (compare with §1). For instance in this case $\deg|a| = 2 - 1 = 1$. To understand the module structure of $\Gamma(a)$ on $\Lambda(e_1, \ldots, e_{4g})$, we need to know first about the ring structure of $H_*(\Omega(\mathbb{P} \vee \mathbb{P}))$.

Lemma 9.2: (C^2M^2) *Let $e_1, e_2 \in H_1(\Omega(\mathbb{P} \vee \mathbb{P}))$ be the generators corresponding to the inclusions of $\mathbb{P}$ into the first and second factors of $\mathbb{P} \vee \mathbb{P}$ (respectively), and let a represent the class in the Hurewicz image of the generating sphere in $H_2(\Omega(\mathbb{P} \vee \mathbb{P}); \mathbb{Z})$ coming from $\pi_2(\Omega S^3)$. Then if $T()$ denotes the tensor algebra, we have*

$$H_*(\Omega(\mathbb{P} \vee \mathbb{P}); \mathbb{Z}) \cong T(e_1, e_2, a)/(e_1^2 = e_2^2 = 0, e_1e_2 + e_2e_1 = a)$$

PROOF: Since all of $e_1, e_2, \langle e_1, e_2 \rangle$ and $[e_1, e_2]$ are maps of spheres, we will use the same notation for the maps and the corresponding spherical classes they generate. That $e_1^2 = e_2^2 = 0$ follows trivially from the homology of S^1. Since the inclusion $G : S^3 \longrightarrow \mathbb{P} \vee \mathbb{P}$ is given by $G = [\Sigma e_1, \Sigma e_2]$ (lemma 8.5), it then follows that $a = adG = \langle ad\Sigma e_1, ad\Sigma e_2 \rangle = \langle e_1, e_2 \rangle : S^2 \to \Omega S^3 \to \Omega(\mathbb{P} \vee \mathbb{P})$. By the theorem of Samelson 7.6, we must have that $\langle e_1, e_2 \rangle = e_1e_2 + e_2e_1 = a$ as desired. ∎

Going back to the proof of 9.1, we can look at the effect of the commutator $[x_1, x_2]^!$ at the level of homology on $H_*(S^1 \times S^1)$. We have

$$\begin{aligned} [x_1, x_2]^!(e_{ij} \otimes e_{kl}) &= (*) \times (id^2 \times \chi^2)\Delta_*(e_{ij} \otimes e_{kl}) \\ &= (*)(e_{ij}e_{kl} \otimes 1 + e_{ij} \otimes e_{kl} - e_{kl} \otimes e_{ij} + 1 \otimes e_{ij}e_{kl}) \end{aligned}$$

and using the relations in 9.2 above, we see that $[x_1, x_2]^!(e_{ij} \otimes e_{kl}) = 0, i = k$ or $j = l$, and that

$$[x_1, x_2]^!(e_{11} \otimes e_{22}) = [x_1, x_2]^!(e_{12} \otimes e_{21}) = e_1 \otimes e_2 + e_2 \otimes e_1 = a \in \mathbb{Z}[a] = H_*(\Omega S^3)$$

This then defines the map $f^!$ completely. In cohomology, it follows that $f^*(a) = \sum_1^{2g} e_{2i+1}e_{2i}$ which implies that the action of a on $\Lambda(e_1, e_2, \ldots, e_{4g})$ is given by multiplication with $\sum_1^{2g} e_{2i+1}e_{2i}$. In this case $(*)$ takes the form

9.3 $$\Gamma(h_1, \ldots, h_{2g}) \otimes Tor_{\Gamma[a]}(\mathbb{F}, \Lambda(e_1, \ldots, e_{4g})).$$

This already makes up for the homology $H_*(\mathrm{Div}^2(M_g - *);\mathbb{F})$ (cf. §6.2) and hence in light of Segal's homotopy equivalence 7.10, this must give the entire homology of $\mathrm{Map}_0^*(M_g, \mathbb{P}\vee\mathbb{P})$ and $E_2 = E_\infty$.

The case $n > 2$ is simpler for $f^* : H^*(\Omega(W_n\mathbb{P})\longrightarrow H^*\left((\Omega W_n\mathbb{P})^{2g}\right)$ is trivial and hence the E_2 term (*) takes the form

9.4 $$E_2 = H^*(\Omega S^{2n-1})^{2g}),\mathbb{F}) \otimes H^*((S^1)^{2ng};\mathbb{F}) \otimes H^*(\Omega^2 S^{2n-1};\mathbb{F}).$$

Here too results of §6 show that 9.4 accounts for all classes in $H^*(\mathrm{Map}_0^*(M_g, W_n);\mathbb{F})$ and the EMSS must then collapse at E^2. This completes the proof. ∎

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