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On the group of automorphisms of cyclic covers of the Riemann sphere

By SADOK KALLEL
Université des Science et Technologies de Lille, 59655 Villeneuve d'Ascq, France.

AND DENIS SJERVE†
University of British Columbia, Vancouver V6T 1Z2, Canada.

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Abstract

This paper uses some recent calculations of Conder and Bujalance (on classifying finite index group extensions of Fuchsian groups with abelian quotient and torsion free kernel) in order to determine the full automorphism groups of some cylic coverings of the line (including curves of Fermat and Lefschetz type). The answer is complete for cyclic covers that branch over three points.

1. *Introduction*

Let C be a Riemann surface of genus $g \geqslant 2$. As is known, the group of automorphisms of such a curve is finite. If the curve is given by an explicit affine equation in $\mathbb{C}^2$, a problem is to determine the symmetry group from the equation. In this paper we answer this problem completely for certain cyclic covers of the line.

A cyclic covering of the line is a curve C with an affine equation in $\mathbb{C}^2$ given by

$$C\colon y^n = (x-e_1)^{a_1}(x-e_2)^{a_2}\cdots(x-e_k)^{a_k}.$$

We only deal with *irreducible* curves (putting a small restriction on n and the a_i's). The special cases when $n=2$ or $n=p$ (an odd prime) are known as hyperelliptic curves or p-elliptic curves (see [**11**]). In [**18**], Namba conjectured that the isomorphism type of a cyclic cover $y^n = f(x)$ is determined entirely by the branching data of the projection $(x,y)\in C\mapsto x$. This has been answered affirmatively for $n=p$ by Nakajo [**17**]. A general discussion of these curves and their properties is given in Section 3.

A cyclic cover C is clearly endowed with an action by the cyclic group $\mathbb{Z}_n$, $(x,y)\mapsto(x,\zeta y)$, where ζ is a primitive nth root of unity. If $\mathrm{Aut}(C)$ is the group of symmetries of C then $\mathbb{Z}_n\subset\mathrm{Aut}(C)$. Our aim then becomes to determine all possible extensions of $\mathbb{Z}_n$ that can occur as automorphism groups of cyclic galois covers. To this end, we rely on techniques in Fuchsian group theory and on pivotal recent calculations of Bujalance, Conder and Cirre ([**3**, **4**]).

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The main result of this paper classifies all group actions on cyclic covers which ramify over three distinct points of $\mathbb{P}^1$ (the choice of these points is immaterial since $\mathrm{PSL}_2(\mathbb{C})$ is 3-transitive). These covers are given by

$$C\colon y^n = x^a(x-1)^b(x+1)^c,\ \text{where } 1 \leqslant a, b, c \leqslant n-1,\ a+b+c \equiv 0\ (mod\ n).$$

We refer to them as cyclic *belyi* covers or generalized Lefschetz curves. A curve given as above is irreducible if $[n, a, b, c] = 1$ and its genus is $g = (2 + n - [n, a] - [n, b] - [n, c])/2$, see Section 3. We use the notation $[r, s]$ for the greatest common divisor of r, s.

Given $n > 3$, we say that (a, b, c) is equivalent to (a', b', c') if there is k prime to n and a permutation τ on 3 letters such that $a' \equiv k\tau(a)\ (mod\ n)$, $b' \equiv k\tau(b)\ (mod\ n)$ and $c' \equiv k\tau(c)\ (mod\ n)$. It is well known (Nielsen) that any two equivalent triples yield isomorphic cyclic covers. So as far as automorphism groups are concerned, only equivalence classes of triples will matter. We can now state our main result.

THEOREM 1. *Let $C\colon y^n = x^a(x-1)^b(x+1)^c$ be a cyclic $\mathbb{Z}_n$ Galois cover of the line with $a+b+c \equiv 0\ (mod\ n)$, $1 \leqslant a, b, c \leqslant n-1$ and $[n, a, b, c] = 1$. We fix $n \geqslant 4$ and let G be the full automorphism group of C. Then G is completely determined by the values of n, a, b and c as follows.*

(i) *We can always assume $a = 1$. For the equivalence classes of triples $(1, b, c)$, G is given according to table 1, where we use the notation $H \ltimes K$ to denote a semi-direct product of H by K and $H : K$ to denote a non-split extension of H by K. In case* C.1 *from the table, p denotes a prime.*

(ii) *For all other equivalence classes, $G = \mathbb{Z}_n$.*

Table 1. *Equivalence classes of curves $y^n = x(x-1)^b(x+1)^c$ and their automorphisms*

	n	b	c	g	$\lvert G\rvert$	G
A.1	odd	1	$n-2$	$\frac{n-1}{2}$	$2n$	$\mathbb{Z}_{2n}$
A.2	even	1	$n-2$	$\frac{n}{2}-1$	$4n$	(central $\mathbb{Z}_2$): D_{2n}
B.1	$n \not\equiv 0(8)$ $n \neq 12$	$b \neq 1, b^2 \equiv 1(n)$	$n-1-b$	$\frac{1}{2}(n-[n, b+1])$	$2n$	$\mathbb{Z}_n \ltimes \mathbb{Z}_2$
B.2	$8 \mid n, n > 8$	$\frac{n}{2}-2$	$\frac{n}{2}+1$	$\frac{n}{2}-1$	$4n$	$(\mathbb{Z}_n \ltimes \mathbb{Z}_2)\colon \mathbb{Z}_2$
B.3	8	2	5	3	96	$(\mathbb{Z}_4 \oplus \mathbb{Z}_4) \ltimes S_3$
C.1	$n > 7,\ n \equiv 1(2)$ $\exists\ p \mid n, p \equiv 1(3)$	$b \neq 1, [b, n] = 1$	b^2	$\frac{n-1}{2}$	$3n$	$\mathbb{Z}_n \ltimes \mathbb{Z}_3$
C.2	7	2	4	3	168	$\mathrm{PSL}(2, 7)$
D.1	12	3	8	3	48	(central $\mathbb{Z}_4$): A_4
E.1	8	3	4	2	48	$\mathrm{GL}(2, 3)$
E.2	12	4	7	4	72	(central $\mathbb{Z}_3$): S_4
E.3	24	4	19	10	144	(central $\mathbb{Z}_6$): S_4

Comments

(1) Note that if none of a, b or c is prime to n (but still assuming $[n, a, b, c] = 1$) then the curve $y^n = x^a(x-1)^b(x+1)^c$ has full automorphism group $\mathbb{Z}_n$. It is also interesting to note that the only cyclic cover with three branch points that is Hurwitz (i.e. with

full group of automorphisms of order $84(g-1)$, where g is the genus) is the Klein curve of genus 3 (case C.2).

(2) A discussion of the group structures and presentations as well as explicit descriptions of some of the various G actions on C is given in Sections 4 and 5. In fact a nice aspect of this work is to give explicit equations for some of the actions.

(3) The curves A.1 and A.2 are hyperelliptic, with the hyperelliptic involution generating the $\mathbb{Z}_2$ factor in their automorphism group. Both curves are of "Fermat type" and are special cases of Theorem 3 below. The curve $y^{2m}=x(x-1)(x+1)^{2m-2}$ in A.2 (here $n=2m$) is isomorphic to $y^{2m}=x^2-1$ and is known as the Accola–Maclachlan surface (see Example 2).

(4) The curve $C\colon y^{2m}=x(x-1)^{m-2}(x+1)^{m+1}$ in B.2 ($n=2m$, $4|m$) is on the other hand known as the Kulkarni surface (with symmetries given in Example 4). Note that the curve in B.3 is a Kulkarni surface as well (corresponding to $m=4$) and has an extra $\mathbb{Z}_3$ action in its automorphism group. The curves B.1–B.3 are related by the fact that $a=1$, and either $b^2\equiv 1$ (*mod* n) or $c^2\equiv 1$ (*mod* n).

(5) Some of the symmetry groups above are given by central extensions of cyclic groups by polyhedral groups. This is true for cases D.1–E.3 for instance. $GL(2,3)$, the group of non-singular 2×2 matrices over $\mathbb{F}_3$, is isomorphic to (*central* $\mathbb{Z}_2$): S_4. The geometry behind this is illustrated in part in Theorem 4 below (with further details and examples given in [**11**]).

(6) The last three curves E.1–E.3 are related as follows. Taking the quotient of the curve $y^{24}=x(x-1)^4(x+1)^{19}$ (case E.3) by $\mathbb{Z}_3\subset$ (central $\mathbb{Z}_6$), we get the curve $y^8=x(x-1)^4(x+1)^3$ (since $19\equiv 3 \pmod 8$) which is the curve in E.1. Its group of automorphisms is then $\mathbb{Z}_2=\mathbb{Z}_6/\mathbb{Z}_3$ extended by S_4 and this is same as $GL(2,3)$. Taking the quotient of the curve E.3 by $\mathbb{Z}_2\subset$ (central $\mathbb{Z}_6$) we get E.2.

(7) The curve E.1 turns out to be *unique* in the sense that there is a unique curve of genus 2 which affords an action by $\mathbb{Z}_8$ (the cyclic action and the genus completely determine the surface; see [**12**]). This also happens for the curve $y^5=x(x-1)$ (see Remark 3).

A curve is called *belyi* if it branches over three points on the line. Such curves, by a remarkable theorem of Belyi, have the property of being isomorphic to curves defined over $\bar{\mathbb{Q}}$ (see [**5**]). A special class of belyi covers are the *Lefschetz* curves (see [**19**]) with equation

$$y^p = x^a(x-1),\ 0<a<p-1,\ \ p \text{ prime}.$$

These were originally studied in [**14**]. In fact we can assume that $1\leqslant a<(p-1)/2$ (i.e. if $a>(p-1)/2$, replace a by $p-a-1$, and if $a=(p-1)/2$, replace a by 1. This doesn't change the isomorphism type of the curve; see Section 5).

As a corollary of the classification in Theorem 1 we are able to recover (in particular) the following calculation of Lefschetz which originally used an impressive mix of rational functions, abelian varieties and divisor theory.

THEOREM 2. *Let $C\colon y^p=x^a(x+1)$ be a Lefschetz curve and let $G:=Aut(C)$. Then*:

(1) *if $a=1$, G is cyclic of order $2p$*;
(2) *if $p=7$ and $a=2$, then $G=PSL(2,7)$ is the simple group of order* 168;
(3) *if $p\equiv 1$ (mod 3), $p>7$ and $1+a+a^2\equiv 0$ (mod p), then G is the unique non-abelian group of order* $3p$;
(4) *For all other cases, $Aut(C)=\mathbb{Z}_p$.*

It is possible in special cases to solve for the automorphism group of cyclic covers that are not belyi (no systematic technique is known). The situation is particularly interesting for the Fermat curves $F: y^n + x^d = 1$, with $d \leqslant n$. For $d \geqslant 4$, the $\mathbb{Z}_n$ action in this case is not anymore uniformized by a triangle group and hence our methods do not apply directly. There is however a way to get around this by not constraining ourselves to $\mathbb{Z}_n$ but by uniformizing the entire $\mathbb{Z}_d \oplus \mathbb{Z}_n$ action on F. When this is done, triangle groups appear again and the classification of Conder, Bujalance and Cirre [**3**] can be applied to the situation. We summarize our calculations in this case.

THEOREM 3. *Let F be the curve given by $y^n + x^d = 1$, where $4 \leqslant d \leqslant n$, and let $Aut(F)$ be its group of automorphisms.*

(1) *If $d = n$, then $Aut(F) = (\mathbb{Z}_n \oplus \mathbb{Z}_n) \ltimes S_3$ where S_3 is the symmetric group on 3 letters.*
(2) *If d does not divide n then $Aut(F) = \mathbb{Z}_d \oplus \mathbb{Z}_n$.*
(3) *If $d|n$, $d < n$, $Aut(F)$ is the central $\mathbb{Z}_d$ extension by the dihedral group D_{2n}, given by the presentation*

$$Aut(F) = \langle s, t, u \mid s^d = t^n = u^2 = [s, t] = [s, u] = 1, (ut)^2 = s^{-1} \rangle.$$

The special cases $d = 2, 3$ are covered in Examples 7, 8. The automorphism group of the classic Fermat curve $x^n + y^n = 1$ is of course well known (see [**21**] or [**23**]). The involution u in the case $d|n$ is fairly easy to describe. If $n = md$ then $u: (x, y) \mapsto (x/y^m, 1/y)$ is an involution acting on the curve $y^n + x^d + 1 = 0$, which is isomorphic to F.

The ideas we use in the proofs of these theorems are classical. The first basic idea in determining $\mathrm{Aut}(C)$ is to associate to a Galois cover $\beta: C \xrightarrow{G} \mathbb{P}^1$ a short exact sequence of groups. Write $C = \mathbb{U}/\Pi$ where Π is a torsion free (Fuchsian) group acting fixed point freely on $\mathbf{U}$, the upper half plane (see Section 2). Then there is a Fuchsian group $\Gamma \subset PSL_2(\mathbb{R})$ and a short exact sequence

$$1 \longrightarrow \Pi \longrightarrow \Gamma \xrightarrow{\theta} G \longrightarrow 1 \qquad (E)$$

where θ is an epimorphism with torsion free kernel (or **skep** for surface kernel epi). We say that the sequence uniformizes the action. Let $N(\Pi)$ be the normalizer of Π in $\mathrm{PSL}_2(\mathbb{R})$. Then $N(\Pi)$ is itself a Fuchsian group and $\mathrm{Aut}(C) = N(\Pi)/\Pi$. When C is a cyclic cover of the line with group $\mathbb{Z}_n$, then $\Gamma_n := \Gamma$ has a very special form. It has signature $(0 \mid \frac{n}{[n,a_1]}, \ldots, \frac{n}{[n,a_k]})$, see Section 3.

The core of our work consists first of all in analyzing all possible skeps $\theta_n: \Gamma_n \to \mathbb{Z}_n$ and then seeking extensions to larger Fuchsian groups $\theta'_n: \Gamma'_n \to G'$, with $\mathbb{Z}_n \subset G'$ and $\ker(\theta'_n) = \ker(\theta_n) = \Pi$. If Γ' happens to be finitely maximal, then necessarily $G' = \mathrm{Aut}(C)$. This method of extendability through skeps works very well for *triangle groups* Δ with signature $(0 \mid m_1, m_2, m_3)$, and hence the content of the main theorem.

Finally, we observe that a type of stability occurs: when the number of branch points is large enough, the cyclic $\mathbb{Z}_n$ action necessarily normalizes.

THEOREM 4. *Let $C: y^p = f(x)$, where p is a prime and $f(x)$ is a polynomial with r distinct roots, $r > 2p$. Then the automorphism group of C is an extension of $\mathbb{Z}_p$ by a polyhedral group.*

A good example is already illustrated by the Fermat curve $y^n + x^d = 1$, $d|n$, with automorphism group $\mathbb{Z}_d : D_{2n}$.

2. *Group actions and extensions of Fuchsian groups*

Let C be a Riemann surface of genus $g \geqslant 2$. By the uniformization theorem, C is the quotient of a discrete (torsion free) group Π acting fixed-point freely on the upper half plane. The holomorphic structure on C is induced by the quotient map and so depends not only on the abstract isomorphism class of the group Π but also on the way this group embeds in $\mathrm{PSL}_2(\mathbb{R})$.

By a Fuchsian group we mean any discrete subgroup of $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{Isom}\,(\mathbf{U})$, where $\mathbf{U}$ is the upper-half plane. A Fuchsian group with compact quotient has presentation (see [**22**]):

$$\Gamma = \langle x_1, \ldots, x_r; a_1, b_1, \ldots, a_g, b_g \mid x_1^{m_1} = \cdots = x_r^{m_r} = x_1 \cdots x_r [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle.$$

Here g is the genus of $\mathbb{U}/\Gamma$. We encode this in a "signature" $\sigma(\Gamma) = (g \mid m_1, \ldots, m_r)$ and when $g = 0$ we write $\sigma(\Gamma) = (m_1, \ldots, m_r)$. In this case the action of the elliptic element $x_i \in \Gamma$ on $\mathbf{U}$ is given by rotation by $2\pi/m_i$ about a fixed point.

When $r = 0$, the group Γ is torsion free and corresponds to the universal covering group of a Riemann surface. The signature in this case is $\sigma(\Gamma) = (g \mid -)$ and Γ is isomorphic to the fundamental group of the genus g surface $\mathbb{U}/\Gamma$. The group Γ is of genus zero if $g = 0$.

Let Γ be the uniformizing Fuchsian group for the action of G on C, as in the introduction, and let Γ' be another Fuchsian group containing Γ with finite index. We say that the action of G extends to G' if there is a commuting diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi & \longrightarrow & \Gamma & \xrightarrow{\theta} & G & \longrightarrow & 1 \\
 & & \big\downarrow{\scriptstyle =} & & \big\downarrow{\scriptstyle \nu} & & \big\downarrow{\scriptstyle \mu} & & \\
1 & \longrightarrow & \Pi & \longrightarrow & \Gamma' & \xrightarrow{\theta} & G' & \longrightarrow & 1
\end{array}$$

where $C \cong \mathbb{U}/\Pi$ and μ, ν are inclusions. Now clearly $G \subset G' \subset \mathrm{Aut}(C)$. If Γ' is maximal then it must coincide with the normalizer of Π in $\mathrm{PSL}_2(\mathbb{R})$ and hence

Lemma 1. *If Γ' is maximal then $Aut(C) = G'$.*

From this viewpoint, finite extendability of Fuchsian groups is a necessary step in computing automorphism groups of Riemann surfaces. Given an arbitrary Fuchsian group Γ it is usually not possible to find a proper inclusion $\Gamma \subset \Gamma'$ of finite index, in which case $\mathrm{Aut}(C) = G$. The geometry of the fundamental domain of Γ plays a seminal role in the existence or non-existence of such extensions (with an exception for the triangle groups as is explained below). In general, some embeddings of the abstract group Γ as a Fuchsian group will admit extensions and others will not.

It turns out however that for a certain class of Fuchsian groups Γ, every monomorphism $\rho\colon \Gamma \to \mathrm{PSL}_2(\mathbb{R})$ extends to a monomorphism $\rho'\colon \Gamma' \to \mathrm{PSL}_2(\mathbb{R})$ for some Γ' of finite index. Such groups are said to have "non-finitely maximal signature" and a list of them is given below (we only look at genus 0 curves).

Theorem 5 (Greenberg, Singermann). *The only genus zero Fuchsian groups Γ with non-finitely maximal signature are those on Table* 2 (*GS*).

Table 2. *The first column in this table shows* Γ *(or rather its signature), the second an extension* Γ' *of* Γ *and the third the index of the extension. Only the first five extensions are normal*

	Γ	Γ'	$\lvert\Gamma' : \Gamma\rvert$
1	$(n, n, n), n \geqslant 4$	$(3, 3, n)$	3
2	$(n, n, n), n \geqslant 4$	$(2, 3, 2n)$	6
3	$(n, n, m), n \geqslant 3, n + m \geqslant 7$	$(2, n, 2m)$	2
A	$(n, n, n, n), n \geqslant 3$	$(2, 2, 2, n)$	4
B	$(n, n, m, m), n + m \geqslant 5$	$(2, 2, n, m)$	2
4	$(7, 7, 7)$	$(2, 3, 7)$	24
5	$(2, 7, 7)$	$(2, 3, 7)$	9
6	$(3, 3, 7)$	$(2, 3, 7)$	8
7	$(4, 8, 8)$	$(2, 3, 8)$	12
8	$(3, 8, 8)$	$(2, 3, 8)$	10
9	$(9, 9, 9)$	$(2, 3, 9)$	12
10	$(4, 4, 5)$	$(2, 4, 5)$	6
11	$(n, 4n, 4n), n \geqslant 2$	$(2, 3, 4n)$	6
12	$(n, 2n, 2n), n \geqslant 3$	$(2, 4, 2n)$	4
13	$(3, n, 3n), n \geqslant 3$	$(2, 3, 3n)$	4
14	$(2, n, 2n), n \geqslant 4$	$(2, 3, 2n)$	3

We refer to the table as the GS table. A signature (with abstract group Γ) is therefore finitely maximal if for some embedding of Γ in $\mathrm{PSL}_2(\mathbb{R})$, the Fuchsian group so obtained is finitely maximal.

Now triangle groups Δ with signature (m_1, m_2, m_3) have the following special properties:

(1) all embeddings of $\Delta(m_1, m_2, m_3)$ in $\mathrm{PSL}_2(\mathbb{R})$ are conjugate;

(2) if $\Delta \subset \Gamma$ is a finite index extension of a triangle subgroup, then Γ is itself a triangle group ([**2**, theorem 10·6·5]).

The first property means that the existence of a finite index extension of $\Delta(m_1, m_2, m_3)$ in $\mathrm{PSL}_2(\mathbb{R})$ does not depend on the way the group embeds. So a triangle group either always extends or never extends.

The embeddings. There is nice geometry behind the embedddings described in the GS table. Often it is possible to deduce inclusions of triangle groups $\Gamma \subset \Gamma'$ by subdividing the triangle associated to Γ into n copies of the triangle associated to Γ', where n is the index of Γ in Γ'. We illustrate this in the case $\Delta(n, 2n, 2n) \hookrightarrow \Delta(2, 4, 2n)$ (see Figure 1).

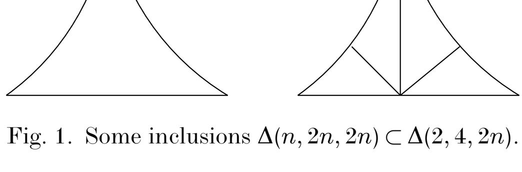

Fig. 1. Some inclusions $\Delta(n, 2n, 2n) \subset \Delta(2, 4, 2n)$.

The left-hand diagram is a triangle with angles $\pi/2n$, $\pi/2n$ (bottom) and π/n (top). We can bisect it by dropping a perpendicular from the top vertex. This gives us 2 copies of a triangle with angles $\pi/2, \pi/2n, \pi/2n$ and illustrates an index 2 inclusion (there are 2 of them): $\Delta(n, 2n, 2n) \subset \Delta(2, 2n, 2n)$. Constructing 2 more perpendiculars gives 4 copies of the triangle with angles $\pi/2, \pi/4, \pi/2n$ and 4 inclusions $\Delta(n, 2n, 2n) \hookrightarrow \Delta(2, 4, 2n)$. In fact, by an appropriate choice of elliptic generators x_1, x_2 and x_3 for $\Delta(n, 2n, 2n)$ and y_1, y_2 and y_3 for $\Delta(2, 4, 2n)$ an embedding $\Delta(n, 2n, 2n) \hookrightarrow \Delta(2, 4, 2n)$ can be described by

$$x_1 \longmapsto y_3^{-1} y_1 y_3^2 y_1 y_3, \quad x_2 \longmapsto y_1 y_2^{-1} y_3 y_2 y_1, \quad x_3 \longmapsto y_3.$$

The following assertion (whose proof we skip) relies on a tedious case-by-case study.

Proposition 1. *Any two embeddings $\Gamma \hookrightarrow \Gamma'$ in the GS table differ by an automorphism of Γ'.*

Corollary. *If $\theta: \Gamma \to G$ extends to $\Gamma' \to G'$ for some embedding $\Gamma \subset \Gamma'$, Γ as in the GS table, then θ also extends for all other embeddings.*

3. *Cyclic Galois coverings of the line*

Consider the affine curve

$$C: y^n = p(x), \text{ where } p(x) \text{ is a polynomial of degree } m.$$

Write $f(x, y) = y^n - p(x)$. The points in $\mathbb{C}^2$ where the curve has singularities (i.e. where $\partial f/\partial x = 0 = \partial f/\partial y$) occur at points $(x, y) = (a, 0)$ where a is a multiple root of $p(x)$. To get a compact Riemann surface from the affine model of C, we need to remove these singularities (by "adding" ramification points), and then compactify in $\mathbb{C}^2 \subset \mathbb{P}^2$ (that is adding points at infinity in a prescribed way). The points at infinity may themselves be singular so this needs to be analyzed as well.

Consider the projection $\pi: C \to \mathbb{C}$, which sends $(x, y) \mapsto x$. Obviously π is going to be regular everywhere but at the roots of p (these are the branch points and their preimages we call the ramification points). Let a be a root of p and write $y^n = (x-a)^k q(x)$. If $k \equiv j$ $(mod\ n)$, then by the birational transformation $(x, y) \mapsto (x, y/(x-a)^l)$, where $k = ln + j$, we see that C is biholomorphic to the curve $y^n = (x-a)^j q(x)$. So we might as well assume to begin with that $C: y^n = (x-a)^k q(x)$ with $k < n$.

In a sufficiently small neighbourhood of a, $q(x)$ is non-zero and by picking an nth root branch we can absorb it with the term in y so that the curve (around the singularity $x = a$) has the equation $w^n = (x-a)^k$ (where $w^n = y^n/q(x)$). Let $l = [n, k]$, the greatest common divisor, and let $n = ln'$, $k = lk'$. Then

$$w^n - (x-a)^k = \prod_{i=0}^{l-1} (w^{n'} - \zeta^i (x-a)^{k'})$$

where ζ is a primitive lth root of unity. Each term $w^{n'} - \zeta^i (x-a)^{k'} = 0$ has a *removable* singularity at $(a, 0)$ because $[n', k'] = 1$; see [**16**, p. 71.] So by adding a point there, locally the sheets of $C': w^{n'} - \zeta^i (x-a)^{k'} = 0$ come together in a smooth way. Going back to C, adding a (ramification) point for each of the terms $w^{n'} - \zeta^i (x-a)^{k'}$ gives us a smooth surface. In total we have added $[n, k]$ such points.

The same technique, after a slight change of variables, allows one to work in the neighbourhood of infinity (see [**16**, p. 73]). Write $C: y^n = c(x-a_1)^{k_1} \cdots (x-a_m)^{k_m}$. Then it can be checked that if $\sum k_i \equiv 0$ $(mod\ n)$, one needs to add n points $x_1, \ldots, x_n$

to C over ∞ in order that the projection $(x,y)\mapsto x$ can be extended to a ramified covering (also denoted by π) from the compactified C to $\mathbb{P}^1=\mathbb{C}\cup\infty$. There is no branching over ∞ in this case. On the other hand if $\sum k_i = qn+r$, $0<r<n$, then one needs to add $[n,r]$ points over infinity to compactify the surface (in which case π is branched over ∞).

The branch point data for $\pi\colon C\to\mathbb{P}^1$ is now totally explicit. At each ramification point over the branch point $(y=0, x=a_i)$, there are $n/[n,k_i]$ sheets coming together (this is the multiplicity of the ramification point). If we assume $\sum k_i\equiv 0\ (mod\ n)$ (to avoid branching over ∞), then we can immediately read off the genus of C from the Riemann–Hurwitz formula

$$2-2g = n\left(2-\sum_{i=1}^m\left(1-\frac{[n,k_i]}{n}\right)\right)\implies g=\frac{1}{2}\left(2+n(m-2)-\sum_{i=1}^m[n,k_i]\right).$$

Note that if all the k_i are prime to n then $g=(n-1)(m-2)/2$ if $\sum k_i\equiv 0\ (mod\ n)$.

PROPOSITION 2. *Consider the compact Riemann surface $C\subset\mathbb{P}^2$ associated to the affine curve $C\colon y^n=(x-a_1)^{k_1}\cdots(x-a_m)^{k_m}$, $m\geqslant 3$, and assume $\sum k_i\equiv 0\ (mod\ n)$. The curve is irreducible if $[n,k_1,\dots,k_m]=1$. Moreover, its genus is given by the formula $g=\frac{1}{2}[2+(m-2)n-\sum[n,k_i]]$. The cyclic action of $\mathbb{Z}_n$ on the curve is uniformized by an exact sequence*

$$1\longrightarrow\Pi\longrightarrow\Gamma\left(\frac{n}{[n,k_1]},\dots,\frac{n}{[n,k_m]}\right)\xrightarrow{\ \theta\ }\mathbb{Z}_n\longrightarrow 1$$

where θ is defined on the elliptic generators $x_i\in\Gamma$ (of order $n/[n,k_i]$) by $\theta(x_i)=T^{k_i}$, T being a generator of $\mathbb{Z}_n$.

Proof. It follows from a theorem of Capelli and Kneser ([**20**, p. 92]) that the binomial $y^n-f(x)$ is irreducible over $\mathbb{C}(x,y)$, and hence over $\mathbb{C}[x,y]$, if $[n,k_1,\dots,k_m]=1$.

Now $\mathbb{U}\to\mathbb{U}/\Gamma=\mathbb{P}^1$ is ramified exactly at the branch points of $\pi\colon C\to\mathbb{P}^1$ (given by the a_i). The periods of Γ coincide with the order of the stabilizers at the ramification points. At each of these points (lying over a_i say), there are $n/[n,k_i]$ sheets coming together and the stabilizer group (at any of these points) is necessarily cyclic (of that order). This yields the assertion about the form of Γ.

Finally, to analyze θ we need to understand how the sheets come together to give the smooth surface C (this is encoded in the monodromy representation of the associated unbranched or étale cover). Let a denote one of the branch points and k the corresponding exponent. Then C is locally given by $y^n=(x-a)^kg(x)$, where $g(a)\neq 0$. Choose a basepoint $x_0\in\mathbb{C}-\{a_1,\dots,a_m\}$ and let w_a denote a smooth simple closed curve based at x_0 and going once around a (chosen so that the other branch points are in its exterior). At the point x_0 we can choose an arbitrary branch y_1 of the multi-valued function $y=f(x)$ and then choose the other branches by $y_j=\zeta^{j-1}y_1$, $1\leqslant j\leqslant n$, where $\zeta=e^{\frac{2\pi i}{n}}$.

Analytic continuation of any germ y at x_0 once around w_a takes us to ζ^ky, and therefore the cycle decomposition of the monodromy representation at a is given by

$$\pi_a=\left(y_{i_1},\zeta^ky_{i_1},\zeta^{2k}y_{i_1},\dots\right)\times\cdots\times\left(y_{i_{[n,k]}},\zeta^ky_{i_{[n,k]}},\zeta^{2k}y_{i_{[n,k]}},\dots\right)$$

(there are $[n,k]$ such factors). Here the y_{i_r} are representatives of the cycles. let

$\tilde{C}$ denote the unbranched Riemann surface associated to $y^n = \prod^m (x-a_i)^{k_i}$. The unbranched covering $\tilde{C} \to \mathbb{C} - \{a_1, \ldots, a_m\}$ yields a short exact sequence

$$1 \longrightarrow \pi_1(\tilde{C}, \tilde{x}_0) \longrightarrow \pi_1(\mathbb{C} - \{a_1, \ldots, a_m\}, x_0) \xrightarrow{\phi} \mathbb{Z}_n \longrightarrow 1.$$

Now $\pi_1(\mathbb{C} - \{a_1, \ldots, a_m\}) = \langle w_1, \ldots, w_m \mid w_1 \cdots w_m = 1\rangle$. The action of $\phi(w_i)$ on $\tilde{C}$ is determined by what happens over x_0 and this is determined by the monodromy π_{a_i}. Since $\pi_{a_i}(y) = \zeta^{k_i} y$ for any germ y based at x_0, we have $\phi(w_i) = T^{k_i}$.

Remark 1. In [**6**], Harvey has given necessary and sufficient conditions for a skep $\theta\colon \Gamma(m_1, \ldots, m_r) \to \mathbb{Z}_n$ to exist. His main condition is that

$$n = \mathrm{lcm}(m_1, m_2, \ldots, m_r) = \mathrm{lcm}(m_1, \ldots, \hat{m}_i, \ldots, m_r), \ \forall i$$

where $\hat{m}_i$ means deleting the ith entry. The existence of the uniformizing sequence in Proposition 2 shows that if $k_i < n$, $\sum k_i \equiv 0$ $(mod\ n)$ and $[n, k_1, \ldots, k_m] = 1$, then necessarily

$$n = \mathrm{lcm}\left(\frac{n}{[n,k_1]}, \ldots, \frac{n}{[n,k_m]}\right) = \mathrm{lcm}\left(\frac{n}{[n,k_1]}, \ldots, \widehat{\frac{n}{[n,k_i]}}, \ldots, \frac{n}{[n,k_m]}\right)$$

for all $1 \leqslant i \leqslant r$. This can be checked directly of course.

LEMMA 2. *Let $C\colon y^n = \prod_{i=1}^m (x-a_i)^{k_i}$ and let l be prime to n, $k_i' \equiv lk$ $(mod\ n)$. Then C is birationally equivalent to the curve $C'\colon y^n = \prod_{i=1}^m (x-a_i)^{k_i'}$.*

Proof. Consider the curve $C_l\colon y^n = \prod_{i=1}^m (x-a_i)^{lk_i}$ and let $\psi\colon C \to C_l, (x,y) \mapsto (x, y^l)$. We check that this map is 1-1 and hence a biholomorphism. Suppose $(x_1, y_1^l) = (x_2, y_2^l) \in C_l$. Then $x_1 = x_2$ and $y_1^l = y_2^l$. But $(x_1, y_1), (x_2, y_2) \in C$ then implies that $y_1^n = y_2^n$. Now $[l, n] = 1$ and hence there is $k < n$ such that $kl \equiv 1$ $(mod\ n)$. Write $kl = \beta n + 1$, then

$$y_1^{lk} = y_2^{lk} \Longrightarrow y_1^{\beta n} y_1 = y_2^{\beta n} y_2 \Longrightarrow y_1 = y_2.$$

This shows that $\psi\colon C \cong C_l$. Finally, one absorbs all nth powers into the y term to show that $C' \cong C_l$.

Automorphism groups. The Galois group in this case is $G = \mathbb{Z}_n$ and the uniformizing sequence is given by

$$1 \longrightarrow \Pi \longrightarrow \Gamma \xrightarrow{\theta} \mathbb{Z}_n \longrightarrow 1.$$

As explained in Section 2, we seek extensions to $\Gamma' \to G'$ for some finite group G' containing $\mathbb{Z}_n$. We restrict our attention to those Γ, Γ' in the GS Table.

Now a case-by-case study of all such possible extensions (exhausting the GS Table) has already been carried out by Bujalance and Conder. Denote by $x_1, \ldots, x_m$ the generators of $\Gamma(n_1, \ldots, n_m)$ and let $z_1 = \theta(x_1), \ldots, z_m = \theta(x_m)$ be their images in $\mathbb{Z}_n$. Also write T for a generator of $\mathbb{Z}_n$. Their main result is:

THEOREM 6 ([**4**]). *Suppose $\mathbb{Z}_n$ acts on $C = \mathbb{U}/\Pi$. Then, in the following cases the action of $\mathbb{Z}_n$ can be extended to an action of a larger group and so $Aut(C) \neq \mathbb{Z}_n$:*

(1) *$sig(\Gamma) = (n, n, n, n)$, $z_1 = T$, $z_2 = T^a$, $z_3 = T^b$, $z_4 = T^c$ where $abc \equiv 1$ $(mod\ n)$, $a^2 \equiv b^2 \equiv c^2 \equiv 1$ $(mod\ n)$ and $1 + a + b + c \equiv 0$ $(mod\ n)$;*
(2) *$sig(\Gamma) = (n, n, m, m)$, $n + m \geqslant 5$, in which case $\mathbb{Z}_n$ extends to the dihedral group; $D_{2n} = \langle u, v \mid u^2 = v^n = (uv)^2 = 1\rangle$ of order $2n$;*

(3) *$sig(\Gamma) = (n, n, n)$ if $n \geqslant 4$ and $\mathbb{Z}_n$ has an automorphism of order 3 permuting z_1, z_2, z_3;*

(4) *$sig(\Gamma) = (n, n, m)$ where $n \geqslant 3$, $m|n$, and $n + m \geqslant 7$, and either $z_1 = z_2$ or there is a transposition exchanging z_1 and z_2;*

(5) *$sig(\Gamma) = (3, 4, 12)$, with $\{z_1, z_2, z_3\} = \{T, T^3, T^{-4}\}$.*

The converse of this theorem is true if we restrict attention to triangle groups. That is $\mathrm{Aut}(C) = \mathbb{Z}_n$ if the action is uniformized by a triangle group not covered by cases (3), (4) or (5).

The discussion in [**4**] is actually sufficient to give the automorphism group in each situation. Starting with this classification we now seek extensions of

$$\theta\colon \Gamma\left(\frac{n}{[n, k_1]}, \ldots, \frac{n}{[n, k_m]}\right) \longrightarrow \mathbb{Z}_n,\ m = 3, 4$$

where $1 \leqslant k_i < n$, $[n, k_1, \ldots, k_m] = 1$ and $k_1 + \cdots + k_m \equiv 0\ (mod\ n)$. As an example, the next corollary follows immediately from Proposition 2 and Theorem 6.

COROLLARY 1. *Let $C\colon y^n = (x - a_1)^{k_1} \cdots (x - a_4)^{k_4}$ with the above assumptions. Assume in addition that $[n, k_1] = [n, k_2]$ and $[n, k_3] = [n, k_4]$. Then the dihedral group D_{2n} acts on C.*

Finally we repeat the following observation from the introduction.

THEOREM 7. *Let $C\colon y^p = f(x)$, p prime and $f(x)$ a polynomial with r distinct roots, $r > 2p$. Then the automorphism group of C is an extension of $\mathbb{Z}_p$ by a polyhedral group.*

Proof. Let $\nu(\tau)$ denote the number of fixed points of an automorphism $\tau\colon C \to C$. The following result (see [**7**, p. 245]) is crucial. Let $g\colon C \to \mathbb{P}^1$ be a meromorphic map and let $\tau\colon C \to C$ be an automorphism with $\nu(\tau) > 2 \deg g$. Then necessarily g is invariant under τ (i.e. $g \circ \tau = g$).

Suppose $\mathbb{Z}_p = \langle T \rangle$ is the cyclic group of automorphisms acting on C by $T(x, y) = (x, \zeta y)$. Apply the above result to the quotient map $g : C \to C/\langle T \rangle \cong \mathbb{P}^1$ and $\tau = RTR^{-1}$, $R \in \mathrm{Aut}(C)$. The degree of g is p and therefore $RTR^{-1} = T^k$ for some k. Hence $\langle T \rangle$ is normal in $\mathrm{Aut}(C)$ and the result follows by standard covering space theory.

4. *Extensions of triangle groups*

By a "generalized Lefschetz" curve we mean an (irreducible) algebraic curve given by the affine equation

$$L\colon y^n = x^a (x - 1)^b (x + 1)^c,\ 1 \leqslant a, b, c \leqslant n - 1,\ a + b + c \equiv 0\ (mod\ n)$$

(the classic Lefschetz case corresponds to $n = p$, a prime). It has genus

$$g = \frac{1}{2}(2 + n - [n, a] - [n, b] - [n, c])$$

according to Proposition 2. Note that when a, b, c are prime to n, the genus is $g = \frac{1}{2}(n - 1)$. As was discussed in the introduction, generalized Lefschetz curves are belyi with belyi map $\beta: L \to \mathbb{P}^1, (x,y) \to x$, which is branched over $0, 1$ and -1. The $\mathbb{Z}_n$ action is uniformized by a skep

$$\theta: \Delta\left(\frac{n}{[n,a]}, \frac{n}{[n,b]}, \frac{n}{[n,c]}\right) \longrightarrow \mathbb{Z}_n.$$

The main purpose of this section and the next is to analyze the extendability of such skeps. This means looking closely into cases (3), (4) and (5) of Theorem 6. We write z_1, z_2, z_3 for the images of the elliptic generators x_1, x_2, x_3 under $\theta: \Delta(m_1, m_2, m_3) \to \mathbb{Z}_n$.

Case (3). This is the case $\Delta(n/[n,a], n/[n,b], n/[n,c]) = \Delta(n,n,n)$, that is all of a, b, c are prime to n. By Lemma 2 we can choose $a = 1$. The only possible extension in this case is the degree 3 extension of $\Delta(n,n,n) \to \mathbb{Z}_n$ to $\Delta(3,3,n) \to G'$ and this occurs if there exists an automorphism of $\mathbb{Z}_n$ of order 3 cyclically permuting z_1, z_2 and z_3 ([**4**, case N6]). Let τ be such an automorphism and write $\tau(T) = T^k$ for some k prime to n. Then $k^3 \equiv 1$ $(mod\ n)$. On the other hand we know that $z_1 = T, z_2 = T^b$ and $z_3 = T^c$. Since $\tau(z_i) = z_{i+1}$ (index modulo 3) we find that

$$a = 1,\ b = k,\ c \equiv k^2\ (mod\ n),\ [k,n] = [c,n] = 1$$

and so necessarily $1 + k + k^2 \equiv 0$ $(mod\ n)$ (and n is odd).

Note that if $n = p_1^{n_1} \cdots p_k^{n_k}$ is the prime decomposition then $\mathrm{Aut}(\mathbb{Z}_n) \cong \prod_{i=1}^{k} \mathrm{Aut}(\mathbb{Z}_{p_1^{n_i}})$. On the other hand $\mathrm{Aut}(\mathbb{Z}_{p^n}) = \mathbb{Z}_{(p-1)p^{n-1}}$ if p is odd. So if $3 | |\mathrm{Aut}\mathbb{Z}_n|$, then either $9|n$ or there is an odd prime p such that $p|n$ and $p \equiv 1$ $(mod\ 3)$. Since we must have $1 + k + k^2 \equiv 0$ $(mod\ n)$, $9|n$ means $9|(1 + k + k^2)$ which is not possible for any integer k. Therefore, when there exists $p|n$, $p \equiv 1$ $(mod\ 3)$, an index 3 extension becomes possible

$$\begin{array}{ccc} \Delta(n,n,n) & \longrightarrow & \mathbb{Z}_n \\ \downarrow & & \downarrow \\ \Delta'(3,3,n) & \longrightarrow & \mathbb{Z}_n \ltimes \mathbb{Z}_3 \end{array}$$

where $\mathbb{Z}_n \ltimes \mathbb{Z}_3 = G'$ is a *metacyclic* group of order $3n$. In Example 1 below we explicitly construct the desired period 3 automorphism acting on the curve $C: y^n = x(x-1)^k(x+1)^{k^2}$, when n is odd and $1 + k + k^2 \equiv 0$ $(mod\ n)$. This leads to cases C_1 and C_2 of Theorem 1. It turns out that the skep $\Delta'(3,3,n) \to \mathbb{Z}_n \ltimes \mathbb{Z}_3$ can be extended further only in case C_2. See the next section.

Case (4) being the longest to deal with, we first settle case (5).

Case (5). This is the case $\Delta\left(n/[n,a], n/[n,b], n/[n,c]\right) = \Delta(3,4,12)$, where $n = 12$. There is one possible extension ([**4**, case T10])of index 4

$$\begin{array}{ccc} \Delta(3,4,12) & \longrightarrow & \mathbb{Z}_{12} \\ \downarrow & & \downarrow \\ \Delta'(2,3,12) & \longrightarrow & G' \end{array}$$

with G' of order 48 given as a central extension $1 \to \mathbb{Z}_4 \to G' \to A_4 \to 1$, where A_4 is the alternating group. Since $[n,c]$ is prime to 12 we can choose $c = 1$ (by Lemma 2). According to ([**4**, T10]) the extension occurs if $\{z_1, z_2, z_3\} = \{v, v^3, v^{-4}\}$, where v is a generator of $\mathbb{Z}_{12}$. Since $[12,a] = 4$, $[12,b] = 3$, it follows that $a = 4$ or 8 and $b = 3$ or 9. On the other hand $a + b + c \equiv 0$ $(mod\ 12)$ and hence (after permutation) $a = 1, b = 3, c = 8$.

Since $\Delta(2,3,12)$ is maximal, we deduce that $G = \mathrm{Aut}\big(y^{12} = x(x-1)^3(z+1)^8\big)$ has order 48. This leads to case D_1 of Theorem 1.

Case (4). This is the case where a and b are prime to n, so that $\sigma(\Gamma)$ is $(n, n, n/[n,c])$. Note again, according to Lemma 2 we can choose $a=1$. Here $m = n/[n,c]$ and so from the GS Table there are the following subcases to consider: rows 2, 3, 4, 7, 9, 11 and 12. According to Theorem 6, cases 2 and 9 do not admit extensions, and since we have already considered case 4, this leaves only cases 3, 7, 11 and 12. In all cases, for an extension to occur the following condition is necessary: either $z_1 = z_2$, or $z_1 \neq z_2$ and there is an involution of $\mathbb{Z}_n$ exchanging z_1 and z_2. See Theorem 6.

Subcase 4·1. This refers to the index 2 extension in row 3 of the GS Table

$$\begin{array}{ccc} \Delta(n,n,n/[n,c]) & \longrightarrow & \mathbb{Z}_n \\ \downarrow & & \downarrow \\ \Delta'\big(2,n,2n/[n,c]\big) & \longrightarrow & G'. \end{array}$$

We distinguish two cases (see [**4**, N8]).

(i) Suppose $z_1 = z_2$. Then $T^a = T^b$ and hence $a = b = 1$. Thus $c = n-2$ and the curve has the form $C\colon y^n = x^{n-2}(x-1)(x+1)$, after permutation of a,b,c. C admits the involution $(x,y)\mapsto(-x,-y)$ and so $G' = \mathbb{Z}_2 \oplus \mathbb{Z}_n \subset \mathrm{Aut}(C)$. If n is odd then $[n,c] = [n, n-2] = 1$ and thus $\Delta\big(n,n,2n/[n,c]\big) = \Delta(n,n,n)$. The only possible extension is the composite

$$\Delta(n,n,n) \subset \Delta(2,n,2n) \subset \Delta(2,3,2n).$$

But this corresponds to row 2 of the GS Table, and it is known that no extension exists in this case. This leads to case $A.1$ of Theorem 1.

Now assume n is even. Then

$$\Delta(n,n,n/[n,c]) = \Delta(n,n,n/2) \quad \text{and} \quad \Delta(2,n,2n/[n,c]) = \Delta(2,n,n).$$

But $\Delta(2,n,n)$ is not maximal. In fact it leads to the following possible extensions (from the GS Table):

$$\begin{gathered} \Delta(n,n,n/2) \subset \Delta(2,n,n) \subset \Delta(2,4,n) \\ \Delta(4,8,8) \subset \Delta(2,8,8) \subset \Delta(2,3,8) \quad \text{if} \quad n=8. \end{gathered}$$

Each of these cases is considered below.

(ii) Now assume $z_1 \neq z_2$. Then there is an involution τ interchanging z_1 and z_2, say $\tau(T) = T^k$. Since $\tau^2 = 1$, $\tau \neq 1$ we have $k^2 \equiv 1 \ (mod\ n)$, $k \not\equiv 1\ (mod\ n)$. We can take $k = b$, so the curve has the equation $C\colon y^n = x(x-1)^k(x+1)^{n-k-1}$. The group G' is a twisted product of $\mathbb{Z}_2$ with $\mathbb{Z}_n$. More relations can be deduced between n and k (see Example 3). Again there are possible further extensions. Let $m = n/[n,c]$.

$$\begin{gathered} \Delta(n,n,n) \subset \Delta(2,n,2n) \subset \Delta(2,3,2n) \quad \text{if} \quad n=m \\ \Delta(n,n,m) \subset \Delta(2,n,2m) \subset \Delta(2,3,4m) \quad \text{if} \quad n=4m \\ \Delta(2m,2m,m) \subset \Delta(2,2m,2m) \subset \Delta(2,4,2m) \quad \text{if} \quad n=2m \\ \Delta(4,8,8) \subset \Delta(2,8,8) \subset \Delta(2,3,8) \quad \text{if} \quad n=8. \end{gathered}$$

The first possibility corresponds to row 2 of the GS Table, but this extension does not exist (see [**4**]). Each of the remaining cases is dealt with below. Otherwise the extension is maximal. This leads to case $B.1$ of Theorem 1.

Subcase 4·2. This refers to the index 12 extension

$$\begin{array}{ccc} \Delta(4,8,8) & \longrightarrow & \mathbb{Z}_8 \\ \downarrow & & \downarrow \\ \Delta(2,3,8) & \longrightarrow & G'. \end{array}$$

In this case no further extension is possible if $z_1 = z_2$, see ([**4**, T4]). Thus assume $z_1 \neq z_2$. Then we can choose $a = 1$, $b = 5$ and $c = 2$, see ([**4**, T4]), and an extension is possible. The curve is given by $C: y^8 = x(x-1)^2(x+1)^5$. Since $\Delta(2,3,8)$ is maximal the group of automorphisms G' has order 96 (see Example 5). This leads to case $B.3$ in Theorem 1.

Subcase 4·3. We now consider the index 6 extension

$$\begin{array}{ccc} \Delta(n,4n,4n) & \longrightarrow & \mathbb{Z}_{4n} \\ \downarrow & & \downarrow \\ \Delta(2,3,4n) & \longrightarrow & G'. \end{array}$$

According to ([**4**, T8]) we must have $z_1 \neq z_2$ in order to be able to extend further. Moreover $n = 2, 3$ or 6 and $\{z_1, z_2, z_3\} = \{T, T^4, T^{-5}\}$. Up to a permutation we can take $a = 1$, $[4n, b] = 1, [4n, c] = 4$. In fact $c = 4$ since the only element of order n in $\{T, T^4, T^{-5}\}$ is T^4.

(i) If $n = 2$ then $b = 3$ and the curve is $C_1: y^8 = x(x-1)^3(x+1)^4$ with automorphism group G' of order 48. This leads to case $E.1$ in Theorem 1.

(ii) If $n = 3$ then $b = 7$ and the curve is $C_2: y^{12} = x(x-1)^4(x+1)^7$ with automorphism group G' of order 72. This leads to case $E.2$ in Theorem 1.

(iii) If $n = 6$ then $b = 19$ and the curve is $C_3: y^{24} = x(x-1)^4(x+1)^{19}$, with automorphism group G' of order 144. This leads to case $E.3$ in Theorem 1. The automorphism group of $y^{24} = x(x-1)^4(x+1)^{19}$ is given as $central(\mathbb{Z}_6): S_4$ (also described in [**4**] with more on this in [**11**]). On the other hand, it is easy to see that $C_2 = C_3/\mathbb{Z}_2$ and that $C_1 = C_3/\mathbb{Z}_3$ (as quotient surfaces). Since the cyclic group actions are central, the group structures of $\mathrm{Aut}(C_1)$ and $\mathrm{Aut}(C_2)$ follow directly.

Subcase 4·4. Finally we look at the extension described in row 12 of the GS Table (index 4)

$$\begin{array}{ccc} \Delta(n,2n,2n) & \longrightarrow & \mathbb{Z}_{2n} \\ \downarrow & & \downarrow \\ \Delta(2,4,2n) & \longrightarrow & G'. \end{array}$$

As usual, $a = 1$, $[2n, b] = 1$ and $[2n, c] = 2$. $\Delta(2,4,2n)$ is maximal except when $n = 4$, but this was treated in subcase 4.2. Thus assume $n \neq 4$.

Two possibilities arise: either $z_1 = z_2$ (and an extension always exists), or $z_1 \neq z_2$ and $4|n$ (see [**4**, T9]).

(i) Suppose $z_1 = z_2$ and hence $a = b = 1$, $c = 2n-2$. Notice that in this case the curve can be brought to the form $y^{2n} = x^{2n-2}(x-1)(x+1)$, and by moving the branch point at $x = 0$ to ∞, it has the affine equation $y^{2n} = x^2 - 1$ ("Accola-Maclachlan" type). This leads to case $A.2$ in Theorem 1.

(ii) Suppose $z_1 \neq z_2$ and hence $b \neq 1$. Then there is k such that $k^2 \equiv 1$ $(mod\ 2n)$ and $k \not\equiv 1$ $(mod\ 2n)$. We can take $k = b$ and thus $c = 2n - 1 - k$. From the fact that $[2n, c] = 2$ it follows that $[2n, k+1] = 2$. But then the congruences above imply that $k = n+1$. The curve in this case is then $y^{2n} = x(x-1)^{n+1}(x+1)^{n-2}$ (known as the

Kulkarni surface). More on this Riemann in Example 4. This leads to case $B.2$ of Theorem 1.

5. *Full automorphism groups of Lefshetz curves*

Based on the calculations in the previous section, we now determine the automorphism groups of the generalized Lefschetz curves

$$L\colon y^n = x^a(x-1)^b(x+1)^c, \quad 1 \leqslant a, b, c \leqslant n-1, \quad a+b+c \equiv 0 \ (mod\ n).$$

First of all, we need to address the question of maximality of the extensions worked out in Section 4. In that section, we analyzed when a skep $\theta : \Delta \to \mathbb{Z}_n$ extends to $\Delta_1 \to G_1$ where the extension $\Delta \subset \Delta_1$ is taken from row 1 of the GS Table (case 3), row 13 (case 5), row 3 (case 4·1), row 7 (case 4·2), row 11 (case 4·3) and row 12 (case 4·4). We now must determine if $\Delta_1 \to G_1$ further extends to $\Delta_2 \to G_2$. Of course this can happen only if Δ_1 is not maximal in the GS Table.

For example, the $\mathbb{Z}_7$ action on the curve $C\colon y^7 = x(x-1)^2(x+1)^4$ is uniformized by $\theta\colon \Delta(7,7,7) \to \mathbb{Z}_7$. According to Case 3 (Section 4), since $1+2+2^2 \equiv 0$ $(mod\ 7)$, there is an index 3 extension (coming from row 1 of the GS Table), $\theta_1 : \Delta(3,3,7) \to G_1$, where G_1 is a group of order $21 = 3 \times 7$ acting on the surface. It turns out that C (of genus 3) is the Klein curve with 168 automorphisms (see Lemma 3). The extra index 8 extension arises from row 6 in the GS Table. What happens in this case is that we have consecutive extensions

$$\Delta(7,7,7) \xrightarrow{r_1} (3,3,7) \xrightarrow{r_6} \Delta(2,3,7)$$

where r_i refers to row i of the GS table. The resulting extension corresponds to r_4 of the GS table. The following is well known.

LEMMA 3. *The only cyclic covering of $\mathbb{P}^1$ of genus 3 with an automorphism group of order* 168 *is the curve $y^7 = x(x-1)^2(x+1)^4$. It is isomorphic to Klein's curve $x^3y + y^3 + x = 0$, with automorphism group $PSL(2,7)$.*

Proof. An extension $\Delta(2,3,7) \to G$ of $\Delta(7,7,7) \to \mathbb{Z}_7$ occurs if and only if $\{z_1, z_2, z_3\} = \{T, T^2, T^4\}$ (or an equivalent triple) for some generator T of $\mathbb{Z}_7$ ([**4**, case T1]). The group G has order 168 in this case and is isomorphic to PSL$(2,7)$. In fact, this shows that if $y^n = x^a(x-1)^b(x+1)^c$ has genus 3 and an automorphism group of order 168, then necessarily $n = 7$, $a = 1, b = 2, c = 4$ (up to permutation). Finally, by a change of variables (see [**13**]) the curve $K\colon x^3y + y^3 + x = 0$ is equivalent to $C'\colon y^7 = x(x-1)^2$, which in turn is equivalent to $C\colon y^7 = x(x-1)^2(x+1)^4$.

The complete list of consecutive extensions from the GS table now reads as follows (row by row):

(i) $\Delta(n,n,n) \xrightarrow{r_1} \Delta(3,3,n) \xrightarrow{r_3} \Delta(2,3,2n)$, $n \geqslant 4$, is equivalent to r_2;
(ii) $\Delta(7,7,7) \xrightarrow{r_1} \Delta(3,3,7) \xrightarrow{r_6} \Delta(2,3,7)$ is equivalent to r_4;
(iii) $\Delta(9,9,9) \xrightarrow{r_1} \Delta(3,3,9) \xrightarrow{r_{13}} \Delta(2,3,9)$ is equivalent to to r_9;
(iv) $\Delta(n,2n,2n) \xrightarrow{r_3} \Delta(2,2n,2n) \xrightarrow{r_3} \Delta(2,4,2n)$, $n \geqslant 3$, is equivalent to to r_{12};
(v) $\Delta(4,8,8) \xrightarrow{r_3} \Delta(2,8,8) \xrightarrow{r_{11}} \Delta(2,3,8)$ is equivalent to to r_7;
(vi) $\Delta(n,n,n) \xrightarrow{r_3} \Delta(2,n,2n) \xrightarrow{r_{14}} \Delta(2,3,2n)$, $n \geqslant 4$, is equivalent to to r_2;
(vii) $\Delta(n,4n,4n) \xrightarrow{r_3} \Delta(2,2n,4n) \xrightarrow{r_{14}} \Delta(2,3,4n)$, $n \geqslant 2$, is equivalent to r_{11};
(viii) $\Delta(4,8,8) \xrightarrow{r_{12}} \Delta(2,4,8) \xrightarrow{r_{14}} \Delta(2,3,8)$ is equivalent to to r_7.

A skep $\Delta(n,n,n)\rightarrow\mathbb{Z}_n$ does not extend to $\Delta(2,3,2n)\rightarrow G$, according to [**4**, N7], and so cases (1) and (6) can be dismissed. The second case above is covered by Lemma 3. Cases (5) and (8) are equivalent and are covered by subcase (4·2). Calculations in [**4**] imply that case (3) not arise. That leaves cases (4) and (7), which are covered by subcases (4·4) and (4·3) resp.

The above discussion together with the calculations in the previous section completely determines the automorphism groups of the cyclic covers $y^n=x^a(x-1)^b(x+1)^c$ and is summarized in Theorem 1. It remains to discuss the structure of the groups in question and this we do in the following list of examples.

Example 1. We discuss the action of $\mathbb{Z}_n\ltimes\mathbb{Z}_3$ on $C\colon y^n=x(x-1)^k(x+1)^{k^2}$, where $1+k+k^2\equiv 0\ (mod\ n)$ (and hence $k^3\equiv 1\ (mod\ n)$). The following argument is adapted from ([**14**, p. 177]). First we consider the *equivalent* curve

$$y^n=(x-1)(x-j)^k(x-j^2)^{k^2}$$

where $j=e^{\frac{2\pi i}{3}}$. Set $1+k+k^2=\alpha n$ and $k^3=\beta n+1$. The $\mathbb{Z}_3$ action is given by

$$S\colon (x,y)\longmapsto (jx, j^\alpha y^k(x-j^2)^{-\beta}).$$

This is well defined (i.e. it does act on C and has period 3). Let T be the order n-transformation. Then $ST=T^kS$. The group generated by S and T is a non-abelian semi-direct product $\mathbb{Z}_n\ltimes\mathbb{Z}_3$. This is case C_1 of Theorem 1.

Example 2. In Case 4·4 of Section 4, the curve $C\colon y^{2n}=x^{2n-2}(x-1)(x+1)$ was found to have an additional $\mathbb{Z}_2$ action (besides the involution $(x,y)\mapsto(-x,y)$). We describe an element of order 4 acting on this Riemann surface. First, C is isomorphic to the surface $C'\colon y^{2n}=x^2-1$. Consider the automorphism

$$u\colon (x,y)\longmapsto\left(\frac{x}{y^n},\frac{\zeta}{y}\right),\quad \zeta^n=-1.$$

It is easy to see that $u^2(x,y)=(-x,y)$ and hence $u^4=1$. Let v be the $\mathbb{Z}_{2n}$ cyclic generator $(x,y)\rightarrow(x,\zeta y)$. The full automorphism of this curve is now given by

$$\langle u,v\mid u^4=v^{2n}=(uv)^2=[u^2,v]=1\rangle.$$

The subgroup generated by u^2 is central and moding out by it we get the dihedral group $D_{4n}=\langle x,v|x^2=v^{2n}=(xv)^2=1\rangle$. This is case A_2 of Theorem 1. See also Example 7.

Example 3. In Case 4·1, Section 4, we established the existence of a $\mathbb{Z}_2$ action on curves of the form $C\colon y^n=x(x-1)^b(x+1)^{n-b-1}$ with $b^2\equiv 1\ (mod\ n)$. We now explicitly describe such an action. First of all, C is isomorphic to $C'\colon y^n=(x+1)^b(x-1)$. Consider the automorphism

$$\begin{aligned}&u\colon (x,y)\longmapsto(-x,(-1)^ly^b(x+1)^{-\beta}),\ \text{where}\\ &b^2=\beta n+1, nl\equiv b+1\ (mod\ 2)\ \text{and}\ l+bl\equiv\beta\ (mod\ 2).\end{aligned}$$

It can be checked that u is a well defined involution, provided $n\not\equiv 0\ (mod\ 8)$. One can also check that $uvu=v^b$, where v is the usual cyclic action $(x,y)\mapsto(x,\zeta y)$. This calculation in fact describes the action of the semi-direct product $\mathbb{Z}_n\ltimes\mathbb{Z}_2=\langle u,v\ |u^2=v^n=1, uvu=v^b\rangle$ on the surface. This is case B_1 of Theorem 1.

Example 4. The Kulkarni surface $y^{2n} = (x+1)^{n+1}(x-1)$, discussed in Case 4·4 (Section 4), has an automorphism group of order $8n = 8g+8$ presented by (see [**4**, T9])

$$\langle u, v \mid u^4 = v^{2n} = (uv)^2 = u^2vu^2v^{n-1} = 1\rangle \approx (\mathbb{Z}_{2n} \ltimes \mathbb{Z}_2) : Z_2.$$

The subgroup generated by $\{u^2, v\}$ is the semi-direct product $\mathbb{Z}_{2n} \ltimes \mathbb{Z}_2$.

Example 5. The genus 3 curve $C: y^8 = x(x-1)^2(x+1)^5$ discussed in Case 4·2 (case B·3) has a group of automorphisms which we claim is a split extension of $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ by S_3. This group is a permutation group on 12 letters, generated by the cycles

$$(1,4)(2,7)(3,10)(5,8)(6,11)(9,12)$$
$$(1,10,9,5)(2,4,11,3,7,12,6,8)$$
$$(1,2,3)(4,5,6)(7,8,9)(10,11,12).$$

See [**4**, T4]. Running MAGMA on this group shows that there is one single normal subgroup H of order 16 with quotient a non-abelian group of order 6 (necessarily S_3). H is abelian with 7 subgroups of order 4. It is then necessarily $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ as claimed.

Classical Lefschetz curves. Now let C be given as a prime Galois cover of the sphere. Such curves are necessarily given by the equation $y^p = f(x)$ (see [**13**]) and if C is belyi it is isomorphic to the curve

$$y^p = x^a(x-1)^b(x+1)^c, \quad a+b+c \equiv 0 \ (mod\ p), \quad 1 \leqslant a, b, c \leqslant p-1.$$

Note that p being prime is crucial in deriving this form of the equation. For example $\mathbb{Z}_{12}$ acts on the curve $x^3 + y^4 = 1$ with quotient $\mathbb{P}^1$ and its equation is not amenable (via birational transformations) to the form above. In fact by moving one of the branch points to $\infty \in \mathbb{P}^1$ we can get the following biholomorphic model

LEMMA 4. *A Lefschetz curve is birationally equivalent to one with equation*

$$y^p = x^a(x+1), \ 1 \leqslant a < \frac{p-1}{2}.$$

Proof. By Lemma 2, $C: y^p = x^a(x-1)^b(x+1)^c$ is isomorphic to $y^p = x^{a'}(x-1)(x+1)^{c'}$ (to see this raise to a power l such that $lb \equiv 1$ $(mod\ p)$ and then reduce a, c modulo p). Note that $a' + c' + 1 = p$, and so by the change of variables $X = x^{-1}$, $Y = \eta y x^{-1}$, $\eta^p = -1$, we obtain the isomorphic curve $Y^p = (X-1)(X+1)^{c'}$. This is equivalent to the curve $y^p = x^{c'}(x+1)$.

So without loss of generality, a Lefschetz curve has the form $C_a: y^p = x^a(x+1)$ for some a, $1 \leqslant a \leqslant p-1$. Applying the change of variables $Y = yx^{-1}, X = x^{-1}$, we find that C_a is isomorphic to C_{p-a-1}, and hence one can choose $1 \leqslant a \leqslant (p-1)/2$. It remains to see that $y^p = x(x+1)$ is isomorphic to $y^p = x^{\frac{p-1}{2}}(x+1)$. According to Lemma 2, $y^p = x^{p-2}(x+1)(x-1)$ is isomorphic to $y^p = x(x-1)^{\frac{p-1}{2}}(x+1)^{\frac{p-1}{2}}$ (since $((p-1)/2)(p-2) \equiv 1$ $(mod\ p)$), and hence to $y^p = x^{\frac{p-1}{2}}(x-1)^{\frac{p-1}{2}}(x+1)$. Another change of variables leads to the equation $y = (x-1)^{\frac{p-1}{2}}(x+1)$, which is what we wanted to prove.

Remark 3. Notice that according to the above representation there is a *unique* Lefschetz curve when $p = 5$ (namely $y^5 = x(x+1)$). In fact it is the unique surface of genus $(p-1)/2 = 2$ which admits a $\mathbb{Z}_5$ action! See [**19**]. Indeed if C has genus 2 and

$\mathbb{Z}_5$ acts on it, then $C/\mathbb{Z}_5 \cong \mathbb{P}^1$ and so $C: y^5 = f(x)$. It has exactly 3 branch points and so must be Lefschetz, and hence of the form above. Its group of automorphism is in fact $\mathbb{Z}_{10}$ (see below).

THEOREM 8. *Let C be a Lefschetz curve isomorphic to $y^p = x^a(x+1), 1 \leqslant a < (p-1)/2$. Let $G = Aut(C)$. Then*:

(1) *if $a = 1$, G is cyclic of order $2p$;*
(2) *if $p \equiv 1 \ (mod\, 3)$, $p > 7$ and $1 + a + a^2 \equiv 0 \ (mod\, p)$, then G is the unique non-abelian group of order $3p$;*
(3) *if $p = 7$ and $a = 2$, then $G = PSL(2,7)$ is the simple group of order 168;*
(4) *for all other cases, $Aut(C) = \mathbb{Z}_p$.*

Proof. We simply read off the possibilities from the classification table in Theorem 1 when $n = p$ is a prime (only cases $A.1, B.1, C.1, C.2$ apply). In that table, representative curves have equations of the form $C: y^p = x(x-1)^b(x+1)^c$, or $y^p = x^a(x+1)$, where $ac \equiv 1 \ (mod\ p)$. Without loss of generality we can assume $1 \leqslant a < (p-1)/2$. We can rule out case B.1 right away as it would imply $b = p-1, c = 0$.

A.1 Here $a = 1$ and *Aut C* $\approx \mathbb{Z}_{2p}$. Indeed $C: y^p = x(x+1)$ has the obvious $\mathbb{Z}_2$ action, $(x, y) \mapsto (-x-1, y)$, commuting with the $\mathbb{Z}_p$ action.

C.1 Here $p \equiv 1 \ (mod\ 3)$, $p > 7$ and $1 + a + a^2 \equiv 0 \ (mod\ p)$. For all such curves $\mathbb{Z}_3$ acts in a twisted fashion (see Example 1).

C.2 Here $p = 7$, $a = 2$ and the surface is Klein's curve $y^7 = x^2(x+1)$ (Example 6).

Example 6. The surface $y^7 = x^2(x+1)$ is isomorphic to Klein's curve $x^3y + y^3z + z^3x = 0$, written in projective coordinates (see Lemma 3). The action of $\mathbb{Z}_3$ is given by permuting x, y and z in a 3-cycle. An automorphism of order 7 is given by $x \mapsto \zeta x, y \mapsto \zeta^4 y$ and $z \mapsto \zeta^2 z$, where ζ is a primitive 7th root of unity. The action of $\mathbb{Z}_2$ is a bit more involved and is described for instance in [**1**].

Remark 4. The representation of a Lefschetz curve as $y^p = x^a(x+1)$ with $1 \leqslant a < (p-1)/2$ is generally not unique. When $p = 7$ it is true however that there only two isomorphism classes of Lefschetz curves: the Klein surface $y^7 = x^2(x+1)$ and the curve $y^7 = x(x+1)$.

PROPOSITION 3. *Two Lefschetz curves $y^p = x^a(x+1)$ and $y^p = x^b(x+1)$, $1 \leqslant a, b < (p-1)/2$ are isomorphic if and only if one of the following is true: $a = b$, $ab + b + 1 \equiv 0 \ (mod\ p)$, $ab + a + 1 \equiv 0 \ (mod\ p)$, $a + b + ab \equiv 0 \ (mod\ p)$ or $ab \equiv 1 \ (mod\ p)$.*

The proof is an easy check using Lemma 2, whose converse is true in the prime case. Note that the count of isomorphism classes of Lefschetz surfaces is given in [**19**] with an extension for more general p-elliptic curves in [**11**].

6. *Curves of Fermat type*

These are curves of the form

$$F: y^n + x^d = 1, \quad 1 < d \leqslant n.$$

The affine curves are smooth. When $d = n$ the projective curve is smooth as well and has genus $(n-1)(n-2)/2$. However, when $d < n$ there is branching over ∞, and a little calculation shows that in this case $g = \frac{1}{2}(2 - d - [d, n] + (d-1)n)$. Notice that there is an obvious action of $\mathbb{Z}_d \oplus \mathbb{Z}_n$ on F, with each cyclic action having quotient the Riemann sphere.

Fermat curves are a special class of cyclic covers of the Riemann sphere, but our prior techniques for the study of $\mathrm{Aut}(F)$ don't apply directly when $d \geqslant 4$ (because the map $F \to F/\mathbb{Z}_n$ has branching over more than 3 points). One way to get around this problem is to consider instead the quotient map

$$\beta\colon F \xrightarrow{\mathbb{Z}_d \oplus \mathbb{Z}_n} \mathbb{P}^1.$$

This turns out to be a belyi map. To see this consider the $\mathbb{Z}_n$ quotient given by the map $(x, y) \mapsto x$. This is a regular n-fold covering branched over $B = \{x \mid x^d = 1\}$ and also over $x = \infty$ when $d < n$. The map $f\colon \mathbb{P}^1 \to \mathbb{P}^1, x \mapsto x^d$, is branched over $0, \infty$ and maps all of B to 1. Therefore the composite β

$$\begin{array}{ccccc} \beta\colon F & \xrightarrow{\psi} & \mathbb{P}^1 & \xrightarrow{f} & \mathbb{P}^1 \\ (x, y) & \longrightarrow & x & \longrightarrow & x^d \end{array}$$

is branched only over $0, 1$ and ∞ and hence is belyi. As can easily be seen this composite is exactly the quotient map $F \to F/\mathbb{Z}_d \oplus \mathbb{Z}_n$. Observe that $|\beta^{-1}(0)| = n, |\beta^{-1}(1)| = d$. The branching over ∞ is different. By the change of variable $z = 1/x$, we can bring this equation (locally at $x = \infty,\ z = 0$) to the form $v^n = z^{n-d}(z^d - 1) = u^m$, where $v = zy$ and $m = n - d$ ([**16**, p. 73]). As explained in Section 3, one needs $[m, n] = [d, n]$ ramification points over ∞ for the $\mathbb{Z}_n$ quotient $\psi\colon F \to \mathbb{P}$ and just one ramification point for f. This means that over ∞ there are $[d, n]$ ramification points for β. Therefore, we can uniformize the action of $G = \mathbb{Z}_d \oplus \mathbb{Z}_n$ on F by the triangle group $\Delta(\frac{nd}{n}, \frac{nd}{d}, \frac{nd}{[d,n]}) = \Delta(d, n, (n, d))$.

In summary, the action of $\mathbb{Z}_d \oplus \mathbb{Z}_n$ on F is uniformized by an exact sequence

$$1 \longrightarrow \Pi \longrightarrow \Delta(d, n, (n, d)) \xrightarrow{\theta} \mathbb{Z}_d \oplus \mathbb{Z}_n \longrightarrow 1, \tag{6·1}$$

where $\Pi \cong \pi_1(F)$. To determine $\mathrm{Aut}(F)$ (for all d and n), we need to study the existence and extendability of skeps $\Delta(d, n, (n, d)) \xrightarrow{\theta} \mathbb{Z}_d \oplus \mathbb{Z}_n$. This in part has been carried out in a recent preprint [**3**].

In Examples 7 and 8 we consider the cases $d = 2$ and 3, but for now we assume $d \geqslant 4$. If $d \not| n$ then $1 < d < n < (d, n)$ and an examination of the GS table reveals that the skep $\Delta(d, n, (d, n)) \xrightarrow{\theta} \mathbb{Z}_d \oplus \mathbb{Z}_n$ is not extendable. The only relevant cases are 13 and 14, but these correspond to $d = 3$ and $d = 2$ resp. Therefore, if $d \not| n$ then $\mathrm{Aut}(F) = \mathbb{Z}_d \oplus \mathbb{Z}_n$.

Thus we are seeking extensions of the skep $1 \to \Pi \to \Delta(d, n, n) \to \mathbb{Z}_d \oplus \mathbb{Z}_n \to 1$ for those cases where $d|n, d \geqslant 4$. This case-by-case study has been carried out in [**3**]. It turns out that extensions occur for cases (1), (2) and (3) of the GS table, and no others. We tacitly assume $4 \leqslant d \leqslant n$ and $d|n$ in what follows.

LEMMA 5 (Extension E.1). *Any skep $\Delta(d, n, n) \xrightarrow{\theta} \mathbb{Z}_d \oplus \mathbb{Z}_n$ admits an index 2 extension $\Delta(2, n, 2d) \to G'$. If we set $\theta(x_1) = S, \theta(x_2) = T$ then G' has the presentation*

$$\begin{aligned} &\langle S, T, U \mid S^d = T^n = 1, ST = TS, U^2 = 1, USU = S, UTU = S^{-1}T^{-1}\rangle \\ &\quad \approx (\mathbb{Z}_d \oplus \mathbb{Z}_n) \ltimes \mathbb{Z}_2. \end{aligned}$$

Proof. The abelianization of $\Delta(d,n,n)$ is $\mathbb{Z}_d \oplus \mathbb{Z}_n$ and θ is the abelianization homomorphism up to an automorphism of $\mathbb{Z}_d \oplus \mathbb{Z}_n$. The extension we are seeking has the form

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi & \longrightarrow & \Delta(d,n,n) & \xrightarrow{\theta} & \mathbb{Z}_d \oplus \mathbb{Z}_n & \longrightarrow & 1 \\ & & \Big\downarrow{\scriptstyle =} & & \Big\downarrow & & \Big\downarrow & & \\ 1 & \longrightarrow & \Pi & \longrightarrow & \Delta(2,n,2d) & \xrightarrow{\theta'} & G' & \longrightarrow & 1. \end{array}$$

We can choose elliptic generators y_1, y_2, y_3 of $\Delta(2,n,2d)$ so that

$$x_1 = y_1^2, \quad x_2 = y_2, \quad x_3 = y_3y_2y_3, \quad y_3^2 = 1, \quad y_1y_2y_3 = 1.$$

Set $U = \theta'(y_3)$. If the above diagram exists then compatibility requires $USU = S, UTU = S^{-1}T^{-1}$, so G' has the stated presentation. Conversely, if G' has the given presentation then the extension exists.

LEMMA 6 (Extension E.2, see [**3**]). *Any skep* $\Delta(n,n,n) \to G = \mathbb{Z}_n \oplus \mathbb{Z}_n$ *extends to* $\Delta(3,3,n)$.

Making use of these calculations we can prove.

THEOREM 9. *Let F be the curve given by $y^n + x^d = 1$, where $4 \leqslant d \leqslant n$, and let $Aut(F)$ be its group of automorphisms.*

(1) *If d does not divide n, then $Aut(F) = \mathbb{Z}_d \oplus \mathbb{Z}_n$.*
(2) *If $d = n$, $Aut(F)$ is the semi-direct product of $\mathbb{Z}_n \oplus \mathbb{Z}_n$ with the symmetric group S_3.*
(3) *If $d|n$, $d < n$, $Aut(F)$ is the semi-direct product of $\mathbb{Z}_d \oplus \mathbb{Z}_n$ by $\mathbb{Z}_2$. A presentation is*

$$Aut(F) = \langle S, T, U \mid S^d = T^n = 1, ST = TS, U^2 = 1, USU = S, UTU = S^{-1}T^{-1}\rangle.$$

Proof. If $d = n$ then the conditions for the extensions E.1 and E.2 are satisfied and an extension of index 6 always occurs (see Remark 5). Indeed S_3 acts on $x^n + y^n = 1$ (by permuting x, y and z in the equivalent projective equation $x^n + y^n + z^n = 0$) and so it follows that $(\mathbb{Z}_n \oplus \mathbb{Z}_n) \ltimes S_3$ acts on F as the full automorphism group. The only other case to consider is the one where $d|n$, $1 < d < n$, but this is covered by Lemma 5.

Remark 5. The index 6 extension of $\Delta(n,n,n) \to \mathbb{Z}_n \oplus \mathbb{Z}_n$ to $\Delta(2,3,2n) \to (\mathbb{Z}_n \oplus \mathbb{Z}_n) \ltimes S_3$ is a combination of E.1 and E.2.

Example 7. In this example we consider the curve $F: y^n + x^2 = 1$. By counting ramification orders we see that the genus is given by

$$g = \begin{cases} (n-2)/2 & \text{if } n \text{ is even,} \\ (n-1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

The action of $\mathbb{Z}_2 \oplus \mathbb{Z}_n$ is uniformized by $\Delta(2,n,2n) \to \mathbb{Z}_2 \oplus \mathbb{Z}_n \cong \mathbb{Z}_{2n}$ if n is odd and by $\Delta(2,n,n) \to \mathbb{Z}_2 \oplus \mathbb{Z}_n$ if n is even. In the odd case the only possible extension comes from case 14 in the GS Table, but according to [**4**] no such extension is possible.

Thus assume n is even. Then, there are 2 cases to consider in the GS table, namely cases 3 and 11 (for $n = 8$). The possible extensions of the skep $\Delta(2,n,n) \to \mathbb{Z}_2 \oplus \mathbb{Z}_n$ are to $\Delta(2,4,n) \to G'$ and $\Delta(2,3,8) \to G'$ respectively. The last case is impossible (see

[**3**]) and the first case always exists (Lemma 5). In this case $G' \cong (\mathbb{Z}_2 \oplus \mathbb{Z}_n) \ltimes \mathbb{Z}_2$ has the presentation

$$G' = \langle a, b, u \mid a^n = 1, b^n = 1, (ab)^2 = 1, ab = ba, u^2 = 1, uau = b, ubu = a\rangle.$$

This agrees with part 3 of Theorem 9 .

Example 8. Now consider the curve $F\colon y^n + x^3 = 1$ of genus

$$g = \begin{cases} n-2 & \text{if } n \equiv 0 \ (mod\ 3) \\ n-1 & \text{otherwise.} \end{cases}$$

The $\mathbb{Z}_3 \oplus \mathbb{Z}_n$ action on F is uniformized by $\Delta(3,n,n) \to \mathbb{Z}_3 \oplus \mathbb{Z}_n$ if $n \equiv 0\ (mod\ 3)$ and by $\Delta(3,n,3n) \to \mathbb{Z}_3 \oplus \mathbb{Z}_n \cong \mathbb{Z}_{3n}$ if $n \not\equiv 0\ (mod\ 3)$.

If $n \equiv 0\ (mod\ 3)$ the possible extensions arise from cases 3, 11 and 12 of the GS Table. Cases 11 and 12 do not extend (see [**3**]). Case 3 concerns the possibility of extending $\Delta(3,n,n) \to \mathbb{Z}_3 \oplus \mathbb{Z}_n$ to $\Delta(2,6,n) \to G'$. This is possible, and in fact $G' \cong (\mathbb{Z}_3 \oplus \mathbb{Z}_n) \ltimes \mathbb{Z}_2$ has the presentation

$$\langle a, b, u \mid a^n = b^n = (ab)^3 = 1, ab = ba, u^2 = 1, uau = b, ubu = a\rangle.$$

This follows from Lemma 5.

Now assume $n \not\equiv 0\ (mod\ 3)$. The only possible extension of $\Delta(3,n,3n) \to \mathbb{Z}_3 \oplus \mathbb{Z}_n \cong \mathbb{Z}_{3n}$ comes from case 13 of the GS table: namely an extension to $\Delta(2,3,3n) \to G'$. This extension exists if, and only if, $n = 4$. In this case G' is a central extension of $\mathbb{Z}_4$ by the alternating group A_4. A presentation for G' is

$$G' \cong \left\langle u_1, u_2 \mid u_1^2 = u_2^3 = [u_1, (u_1u_2)^3] = 1 \right\rangle.$$

The central subgroup $\mathbb{Z}_4$ is generated by $(u_1u_2)^3$. See [**4**] for the details.

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