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Configuration Spaces and the Topology of Curves in Projective Space

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To Professor Milgram on his 60th birthday

Abstract. We survey and expand on the work of Segal, Milgram and the author on the topology of spaces of maps of positive genus curves into n-th complex projective space, $n \geq 1$ (in both the holomorphic and continuous categories). Both based and unbased maps are studied and in particular we compute the fundamental groups of the spaces in question. The relevant case when $n = 1$ is given by a non-trivial extension which we fully determine.

§1 Introduction and Statement of Results

The topology of spaces of rational maps into various complex manifolds has been extensively studied in the past two decades. Initiated by work of Segal and Brockett in control theory, and then motivated by work of Donaldson in gauge theory, this study has uncovered some beautiful phenomena (cf. [CM], [Hu], [L]) and brought to light interesting relationships between various areas of mathematics and physics (cf. [Hi], [BM],[BHMM]).

Let C_g (or simply C) denote a genus g (compact) Riemann surface ($\mathbb{P}^1$ when $g = 0$) and let V be a complex projective variety. Both C and V come with the choice of preferred basepoints x_0 and $*$ respectively. The main focus of this paper is the study of the geometry of the space

$$\mathrm{Hol}(C, V) = \{f : C \longrightarrow V, f \text{ holomorphic}\}$$

and more particularly its subspace of basepoint preserving maps

$$\mathrm{Hol}^*(C, V) = \{f : C \longrightarrow V, f \text{ holomorphic and } f(x_0) = *\}$$

The space $\mathrm{Hol}^*(C, V)$ doesn't depend up to homeomorphism on the choice of basepoint when V has a transitive group of automorphisms for example. In that situation, the relationship between the based and unbased (or free) mapping spaces is given by the "evaluation"

$$\mathrm{Hol}^*(C, V) \longrightarrow \mathrm{Hol}(C, V) \xrightarrow{ev} V$$

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which is a holomorphic fibration for V a homogeneous space for example. Here ev evaluates a map at the basepoint of C.

Remark 1.1: By choosing a Riemann surface, we fix the complex structure. We record the link with the theory of "pseudo-holomorphic" curves of Gromov. There one studies the (differential) geometry of spaces of J-holomorphic curves into V, where V is almost complex and J is its almost complex structure. These J-holomorphic curves (of given genus g) correspond when J is integrable to all holomorphic maps of C (with varying complex structure) into V. It is now a theorem of Gromov that for a generic choice of J on V, the space of all such curves (in a given homology class) is a smooth finite dimensional (real) manifold. Let $\mathcal{M}(V,g) = \{(C_g, f) \mid f : C \longrightarrow V \text{ holomorphic}\}$, then there is a projection of $\mathcal{M}(V,g)$, onto the moduli space of curves $\mathcal{M}_g$ sending (C,u) to the isomorphism class of C. The spaces $\mathrm{Hol}(C,V)$ appear then as "fibers" of this projection.

We observe that $\mathrm{Hol}(C,V)$ breaks down into connected components and we write $\mathrm{Hol}_A(C,V)$ for the component of maps f such that $f_*[C] = A$, A a given homology class in $H_2(V)$. When V is simply connected, $[C,V] = [S^2,V] = \pi_2(V) = H_2(V) = \mathbf{Z}^r$ for some rank r and A is completely determined by a multi-degree. The components $\mathrm{Hol}_A(C,V)$ are generally (singular) quasi-projective varieties (see [H2] for example). When $V = \mathbb{P}^n$, we write $\mathrm{Hol}_A(C,\mathbb{P}^n) = \mathrm{Hol}_k(C,\mathbb{P}^n)$ for some $k = \deg A \in \mathbb{Z}$ and these are smooth manifolds as soon as k is big enough (about twice the genus of C). Most of this paper is concerned with the study of these spaces.

Remark 1.2: In physics, spaces of maps between manifolds $\mathrm{Hol}(M,V) \subset \mathrm{Map}(M,V)$ arise in connection with field theory or "sigma models". From the perspective of a physicist, a field on M with values in V is a map $\phi : M \to V$. For example, in the case $M = \mathbf{R}^3$ and $V = \mathbf{S}^3 = \mathbb{R}^3 \cup \{\infty\}$, a map $\phi : \mathbf{R}^3 \to S^3$ could be the field associated to some electrical charge in $\mathbf{R}^3$ (hence vector valued and extending to the point where the charge is located by mapping to ∞). Associated to a field there is an "energy" density or Lagrangian $\mathcal{L}$ (eg. the harmonic measure $\mathcal{L}(\phi) = \frac{1}{2}||d(\phi)||^2$). To an energy density one can in turn associate an "action" which is defined as

$$S[\phi] = \int_M \mathcal{L}(\phi) d\mu(h)$$

where $d\mu(h)$ is the canonical volume measure associated to a metric h on M. Physicists are usually interested in minimizing the action (to determine the dynamics of the system) and hence they are led to study the space of all extrema of this functional. It should be noted that in the case when V is compact, Kahler, a well-known theorem of Eells and Wood asserts that the absolute minima of the energy functional on $\mathrm{Map}(C,V)$ are the holomorphic maps (the critical points here being the harmonic maps).

It has been known since the work of Segal [S] that the topology of holomorphic maps of a given degree $k \in \mathbf{N}$ from C to $\mathbb{P}^n$ compares well with the space of continous maps at least through a range increasing with k. He proved

Theorem 1.3: (Segal) *The inclusion*

$$Hol_k^*(C,\mathbb{P}^n) \hookrightarrow Map_k^*(C,\mathbb{P}^n)$$

induces homology isomorphisms up to dimension $(k-2g)(2n-1)$

A similar statement holds for unbased maps. To simplify notation, we write $\mathrm{Map}_k, \mathrm{Hol}_k, \mathrm{Map}_k^*, \mathrm{Hol}_k^*$ for the corresponding mapping spaces from C into $\mathbb{P}^n$.

When $g=0$ (the rational case) and $n>1$ the homology isomorphism in 1.3 can be upgraded to a homotopy equivalence (cf. [CS]). For $g>0$ however it is not known whether the equivalence in 1.3 holds in homotopy as well; i.e whether the pair $(\mathrm{Map}_k, \mathrm{Hol}_k)$ is actually $(k-2g)(2n-1)$ connected. This is strongly suspected to be true and in this note we give further evidence for this by showing that Hol_k^* and Map_k^* have isomorphic fundamental groups for all n (the relevant case here is $n=1$). In fact we shall show

Theorem 1.4: *Suppose* $k\geq 2g$*, we have isomorphisms*

$$\pi_1(Hol_k^*(C_g,\mathbb{P}^n)) \cong \pi_1(Map_0^*(C_g,\mathbb{P}^n)) \cong \begin{cases} \mathbb{Z}^{2g}, & \text{when } n>1 \\ G, & \text{when } n=1 \end{cases}$$

where G *is a cyclic extension of* $\mathbb{Z}^{2g}$ *by* $\mathbb{Z}$ *generated by classes* $e_1,\dots,e_{2g}$ *and* τ *such that the commutators*

$$[e_i, e_{g+i}] = \tau^2$$

and all other commutators are zero.

Remark[1]: Roughly speaking, the class α can be represented by the one parameter family of maps obtained by rotating roots around poles. Similarly the classes e_i are obtained by rotating roots (or poles) around loops representing the homology generators of C.

Combining this result with a classical result of G. Whitehead (see §2), we deduce

Proposition 1.5: $\pi_1(Map_d(C,S^2))$ *is generated by classes* $e_1,\dots,e_{2g}$ *and* α *such that*

$$\alpha^{2|d|}=1,\ [e_i,e_{g+i}]=\alpha^2$$

and all other commutators are zero. When $d\geq 2g$, $\pi_1(Hol_d(C,\mathbb{P}^1)) \cong \pi_1(Map_d(C,S^2))$.

This description for the continuous mapping space is an earlier result of Larmore-Thomas [LT]. As is clear from 1.5, the components of $\mathrm{Map}(C,\mathbb{P}^n)$ for $n=1$ have different homotopy types. When $n>1$, the fundamental group is however not enough to distinguish between the components. A quick byproduct of our calculations however shows (§8)

[1] J.D.S.Jones has recently informed the author that he has obtained a similar result a few years ago but which he didn't publish.

Proposition 1.6: *$Map_k(C,\mathbb{P}^{2d})$ and $Map_l(C,\mathbb{P}^{2d})$ have different homotopy types whenever l and k have different parity*[2].

The first few sections of this paper are written in a leisurely fashion and are meant in part to survey techniques and ideas in the field most of which carry the deep imprint of Jim Milgram (cf. §4).

In §2 we discuss the rational case (i.e. $g = 0$) and give a short proof of a theorem of Havlicek on the homology of unbased self-morphisms of the sphere. In §3 we introduce a configuration space model (originally given in [K2]) for $\mathrm{Map}^*(C,\mathbb{P}^n)$ and use it to determine the homology of this mapping space. Let $SP^k(M) = M^k/\Sigma_k$ where Σ_k is the cyclic group on k-letters acting by permutations (or equivalently the space of unordered k points on M), and let $SP^k_n(M)$ be the subspace of k points in C no more than n of which are the same ($n \leq k$). When $n = 1$, $SP^k_1(M) = C_k(M)$ is the standard configuration space of k distinct points. When M is a (connected) open manifold or with a (collared) boundary, these spaces can be stabilized (cf. §3) $SP^k_n(M)\longrightarrow SP^{k+1}_n(M)\longrightarrow\cdots$ and we write $SP^\infty_n(M)$ for the (connected) direct limit. The following is discussed in §3

Theorem 1.7 [K2]: *There is a map*

$$SP^\infty_n(C-*)\xrightarrow{\ S\ } Map^*_0(C,\mathbb{P}^n)$$

which is a homotopy equivalence when $n > 1$ and a homology equivalence when $n = 1$.

This model for Map^* in terms of 'configurations of bounded multiplicity' turns out to relate quite well with the corresponding model for Hol^* and we indicate based on this another proof for Segal's result (§5). Notice that 1.4 shows that the homology equivalence S above cannot be upgraded to a homotopy equivalence when $n = 1$ for already both spaces have different fundamental groups. Indeed the braid group $\pi_1(SP^\infty_1(C_g-*)) = \pi_1(C(C_g-*))$ has a presentation quite different from the central extension given in 1.4.

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§2 Preliminaries: The Genus Zero Case

It is customary to write $\mathrm{Rat}(V)$ for the basepoint preserving maps $\mathrm{Hol}^*(S^2,V)$. When $V = \mathbb{P}^n$, these spaces are now very well understood. Let

$$C_k(\mathbb{C},S^{2n-1}) = \coprod_{0\leq i\leq k} F(\mathbb{C},i)\times_{\Sigma_i}(S^{2n-1})^i/\sim$$

[2] it is now verified that all (positive) components of $\mathrm{Map}(C,\mathbb{P}^n)$ have different homotopy type; cf. [KS].

be the standard labeled k-th filtration piece for the May-Milgram model of $\Omega^2 S^{2n+1} \simeq \Omega^2_0(\mathbb{P}^n)$ (cf. [C²M²], [K4], etc). Here $F(\mathbb{C}, i)$ is the set of ordered i-tuples of disctinct points in $\mathbb{C}$ and $\sim$ is a standard basepoint identification identifying an i-tuple with labels to an $i-1$ tuple by discarding a point an its label if the label is at basepoint. Let

$$SP^k(X) = \{\sum k_i x_i, x_i \in X, k_i \in \mathbf{N} \mid x_i \neq x_j \text{ for } i \neq j \text{ and } \sum k_i = k\}$$

be the k-th symmetric product of X (see introduction). Let $SP^k_n(X)$ be the subset of $SP^k(X)$ obtained by restricting to $k_i \leq n$. We then have

Theorem 2.1 ([CS], [K3], [GKY]): *For $k \geq 1$, there are maps and homotopy equivalences*

$$2.1(a) \qquad Rat_k(\mathbb{P}^n) \xleftarrow{\simeq} C_k(\mathbb{C}, S^{2n-1}) \xrightarrow{\simeq} SP^{k(n+1)}_n(\mathbb{C})$$

whenever $n > 1$. When $n = 1$ the spaces are homologous only.

It is interesting to note that no map is known to induce such homology isomorphism between $\mathrm{Rat}_k(\mathbb{P}^n)$ and $SP^{k(n+1)}_n(\mathbb{C})$ $(k > 1)$. When $n > 1$, the left-hand map in 2.1(a) is constructed explicitly in [CS] while the right-hand map is constructed in [K3]. Both configuration space models on either side of 2.1(a) are fairly amenable to calculations and from there the structure of $\mathrm{Rat}_k(\mathbb{P}^n)$ can be made quite explicit (cf. [BM], [C²M²], [K3] and [Ka]).

The space of unbased rational maps is less well understood. The following is due to Havlicek ([H1])

Theorem 2.2: *The Serre spectral sequence for the (holomorphic) fiber bundle*

$$2.2(a) \qquad Rat_k(\mathbf{P}^1) \longrightarrow Hol_k(\mathbf{P}^1) \longrightarrow \mathbf{P}^1$$

has the non-zero differential $d_2(x) = 2k\iota$. The spectral sequence collapses with mod-p coefficients whenever $p = 2$ or p divides k.

We give a short proof for the mod-2 collapse (a general proof and an extension of this result to $\mathrm{Rat}_k(\mathbb{P}^n)$ can be found in [KS]). First of all, to see that 2.2(a) is indeed a fiber bundle, one simply observes that $\mathrm{PSL}_2(\mathbb{C})$ acts transitively on $\mathbb{P}^1$ and if F denotes the stabilizer of a point, then F acts on $\mathrm{Rat}_k(\mathbf{P}^1)$ (by postcomposition). One can see that $\mathrm{Hol}_k(\mathbf{P}^1) = \mathrm{Rat}_k(\mathbf{P}^1) \times_F \mathbb{P}^1$.

Towards the proof of 2.2 (and also in most of §7) we need the following classical result of G. Whitehead. Let X be a based (connected) topological space (with basepoint x_0) and consider the evaluation fibration

$$2.3 \qquad \Omega^n_f X \longrightarrow \mathcal{L}^n_f X \xrightarrow{ev} X$$

where $ev(f) = f(x_0)$, $\mathcal{L}^n_f X = \mathrm{Map}_f(S^n, X)$ is the component of the total mapping space containing a given map $f : S^n \to X$ and $\Omega^n_f X$ the subset of all maps g such that $g(x_0) = f(x_0)$.

Theorem 2.4 [W]: *The homotopy boundary in 2.3 $\partial : \pi_i(X) \to \pi_i(\Omega^n_f X) \cong \pi_{i+n}(X)$ is given (up to sign) by the Whitehead product: $\partial\alpha = [\alpha, f]$.*

Let $\mathcal{L}^2_k S^2 = \mathcal{L}^2_f S^2$ where f is the standard degree k map, and denote by $a \in H^2(S^2)$ the generator. Recall that $\Omega^2 S^2 \simeq \mathbb{Z} \times \Omega^2 S^3$ and so let e be the generator in $H^1(\Omega^2_k S^2) \cong H^1(\Omega^2 S^3)$.

Lemma 2.5: *The Serre spectral sequence for* $\Omega^2_k S^2 \longrightarrow \mathcal{L}^2_k S^2 \longrightarrow S^2$ *has the (homology) differential* $d_2(a) = 2ke$. *It collapses at* E_2 *with mod-2 coefficients.*

PROOF: For a fibration $F \to E \stackrel{p}{\longrightarrow} S^2$ we have the diagram

$$\begin{array}{ccc}
\pi_2(E,F) \cong \pi_2(S^2) & \stackrel{\partial}{\longrightarrow} & \pi_1(F) \\
\downarrow h & & \downarrow h \\
H_2(E,F) & \stackrel{\partial}{\longrightarrow} & H_1(F) \\
\downarrow p_* & & \downarrow \\
H_2(S^2) & \stackrel{\tau}{\longrightarrow} & E^{0,1}
\end{array}$$

where $E^{0,1} \cong H_1(F)$ and τ the transgression. Let $k : S^2 \longrightarrow S^2$ denote multiplication by k. From 2.4 and the diagram above, we deduce that

$$\tau : H_2(S^2) \cong \pi_2(S^2) \xrightarrow{k[\iota,-]} \pi_3(S^2) \xrightarrow{ad} \pi_1(\Omega^2_k S^2) \xrightarrow{h} H_1(\Omega^2_k S^2) = \mathbb{Z}$$

$\iota \in \pi_2(S^2)$ is the generator. As is known for even spheres, $h(ad([\iota,\iota])) = 2$ and this yields the differential.

The mod-2 collapse follows from the following very short argument (which we attribute to Fred Cohen): let $E : S^2 \longrightarrow \Omega S^3$ be the adjoint map. Then there is a map of (horizontal) fibrations

$$\begin{array}{ccccc}
\Omega^2_k S^2 & \longrightarrow & \mathcal{L}^2_k S^2 & \stackrel{ev}{\longrightarrow} & S^2 \\
\downarrow \Omega^2 E & & \downarrow & & \downarrow E \\
\Omega^3_k S^3 & \longrightarrow & \mathcal{L}^2_k \Omega S^3 & \longrightarrow & \Omega S^3
\end{array}$$

The map E is injective in integral homology while $\Omega^2 E$ is injective in mod-2 homology. So mod-2, both fiber and base inject in the bottom fibration which is trivial since ΩS^3 is (homotopy equivalent) to a topological group (and translation of basepoint gives the trivialization). The collapse follows in this case. ∎

PROOF OF 2.2: We have inclusions $\mathrm{Rat}(\mathbb{P}^1) \subset \mathrm{Hol}(\mathbb{P}^1) \subset \Omega^2 S^2$, where $\mathrm{Rat}(\mathbb{P}^1)$ is the subspace of all holomorphic maps sending the north pole ∞ in $\mathbb{P}^1 = \mathbb{C} \cup \infty$ to 1. An element of $\mathrm{Rat}(\mathbb{P}^1)$, say of degree k, is given as a quotient $\frac{p}{q} = \frac{z^k + a_{k-1}z^{k-1} + \cdots + a_0}{z^k + b_{k-1}z^{k-1} + \cdots + b_0}$ where p and q have no roots in common. It is easy to see for example that $\mathrm{Hol}_1(\mathbb{P}^1)$ corresponds to $PSL(2,\mathbb{C})$, the automorphism group of $\mathbb{P}^1$ (which is up to homotopy RP^3), and that

$\mathrm{Rat}_1(\mathbb{P}^1) = \mathbb{C}\times\mathbb{C}^* \simeq S^1$. Consider now the map of fibrations

2.6
$$\begin{array}{ccc} \mathrm{Rat}_k(\mathbb{P}^1) & \hookrightarrow & \Omega^2_k S^2 \\ \downarrow & & \downarrow \\ \mathrm{Hol}_k(\mathbb{P}^1) & \hookrightarrow & \mathcal{L}^2_k S^2 \\ \downarrow & & \downarrow \\ \mathbf{P}^1 & \xrightarrow{=} & S^2 \end{array}$$

According to Segal (theorem 1.3), the top inclusion is an isomorphism in homology up to dimension k. Moreover (cf. [C^{2}M^2], [K1]), $H_*(\mathrm{Rat}_k(\mathbb{P}^1))$ actually injects in $H_*(\Omega^2_0 S^2)$ and it does so in the following nice way. Recall that $\Omega^2_0 S^2 \simeq \Omega^2 S^3 = \Omega^2\Sigma^2 S^1$ and hence it stably splits as an infinite wedge $\bigvee_{j\geq 1} D_j$ where the summands D_j are given in terms of configuration spaces with labels. Stably one finds that

$$\mathrm{Rat}_k(\mathbb{P}^1) \simeq_s \bigvee_{j=1}^{k} D_j.$$

The map of fibers in 2.6 is then an injection and since the bases are same in that diagram, the theorem follows from 2.5 and a spectral sequence comparison argument. ∎

§3 Configuration Space Models and the Higher Genus Case

In this section we describe two configuration space models for each of the mapping spaces $\mathrm{Hol}^*_k(C,\mathbb{P}^n)$ and $\mathrm{Map}^*_k(C,\mathbb{P}^n)$ for $g\geq 1$. A straightforward comparison between both models yields Segal's stability result in §4.

First of all, a map $f\in \mathrm{Hol}_k(C,\mathbb{P}^n)$ can be written locally in the form

$$z\mapsto [p_0(z):\dots:p_n(z)]$$

where the $p_i(z)$ are polynomials of degree k each and having no roots in common. The p_i's are not global functions on C (otherwise they would be constant) but rather local maps into $\mathbb{C}$ and hence sections of some line bundle. The roots of each p_i give rise to an element $D_i\in SP^k(C)$ (so called *positive divisor*) and conversely D_i only determines p_i up to a non-zero constant (which will be determined if the maps are based). These D_i's (*the root data*) cannot of course have a root in common and if we base our maps so that basepoint $*$ is sent to $[1:\dots:1]$ say, then none of the D_i's contains the basepoint. Define

$$\mathrm{Div}^{n+1}_k(X) = \{(D_0,\dots,D_n)\in SP^k(X)^{n+1} \mid D_0\cap\dots\cap D_n=\emptyset\},$$

The previous discussion shows the existence of an inclusion

$$\phi: \mathrm{Hol}^*_k(C,\mathbb{P}^n)\longrightarrow \mathrm{Div}^{n+1}_k(C-*)$$

which associates to a holomorphic map its root data. This correspondence has no inverse since it is not true in general that an (n+1) tuple of degree k-divisors with no roots in common gives rise to a (based) holomorphic map out of C. In fact, this can be understood as follows: the quotients

$\frac{p_i(z)}{p_j(z)}$ are meromorphic maps on C and so for $i \neq j$, $D_i - D_j$ must be the divisor of a function on C (i.e. there is $f : C \to \mathbb{C}$ with roots at D_i and poles at D_j). It is not surprising that this last condition is not satisfied for general pairs (D_i, D_j). We say D_i and D_j are *linearly equivalent* if there is indeed a meromorphic function f on C such that (f) :=zeroes of f-poles of $f = D_i - D_j$. Note that when this is the case D_i and D_j have same degree.

Linear equivalence defines an equivalence relation and one denotes by $J(C)$ the set of all linearly equivalent divisors on C of degree 0. There is a map

3.1 $$\mu : SP^n(C)\longrightarrow J(C),\ D \mapsto [D - nx_0]$$

sending a divisor to the corresponding equivalence class. The above discussion then shows the existence of a homeomorphism

$$\left\{\begin{array}{c}\text{A (based) holomorphic}\\ X\longrightarrow\mathbb{P}^r \text{ of degree } k\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c}\text{An } (n+1) \text{ tuple of positive divisors}\\ D_i \text{ on } C - * \mid D_0\cap\cdots\cap D_n = \emptyset\\ \text{deg}D_i = k \text{ and } \mu(D_i) = \mu(D_j)\end{array}\right\}$$

To make this correspondence more precise, we need understand the map μ in 3.1. It turns out that:

• $J(C)$ is a g dimensional complex torus (this is a non-trivial fact) and μ is a multiplicative map. In the case $g = 1$, μ is the identity and if one identifies the curve with $\mathbb{C}/L$ for some lattice L, then the Abel-Jacobi condition $\mu(\sum z_i) = \mu(\sum p_j), z_i, p_i \in C$ is equivalent to $\sum z_i = \sum p_j \mod (L)$ (z for zero and p for pole).

• The preimage of a point $[D] = \mu(D) \in J$ is a projective space. To see this let $\mathcal{L}(D)$ be the set of all holomorphic maps on C such that $(f) + D \geq 0$ (i.e. such that $(f) + D \in \coprod_n SP^n(C)$). This is a $\mathbb{C}$-vector space and since $(\alpha f) = (f)$, we have an identification and a map $\mu^{-1}([D]) = \mathbb{P}\mathcal{L}(D)\longrightarrow SP^n(C)$, $[f] \mapsto (f) + D$. This turns out to be a holomorphic embedding ([G],§4, 4.3).

• The complex dimension of $\mu^{-1}([D])$ is denoted by $r(D)$ and a crucial aspect of the classical theory of algebraic curves is the computation of $r(D)$ as D varies in $SP^n(C)$. A very useful interpretation of this dimension is as follows: *$r(D) \geq r$ if and only for any r points of C, there is a divisor D' with $\mu(D') = \mu(D)$ and passing through these r points.*

• If deg$D = n$, then generically

$$r(D) = max\{0, n - g\}.$$

The exact dimension for every D is determined by Riemann-Roch. This dimension may jump up for certain "special" divisors. However whenever n exceeds $2g-2$, there are no more jumps and r is uniformally given by $n-g$.

It turns out that for k small (i.e. $k < g$), $\text{Hol}^*_k(C, \mathbf{P}^n)$ is very dependent on the holomorphic structure put on the curve (it can naturally be empty). For example if C is hyperelliptic, then

$$\text{Hol}_{2n+1}(C, \mathbb{P}^1) = \emptyset, 2n + 1 \leq g \text{ (see for instance [FK], Chap3)}$$

Also $\mathrm{Hol}_1(C,\mathbb{P}^n)$ is always empty for all positive genus curves. In the range $k\geq 2g-1$, there is however much better behavior (see [KM])

Lemma 3.2: *Assume $k\geq 2g-1$. Then $Hol_k^*(C,\mathbf{P}^n)$ is a $k(n+1)-ng$ complex manifold.*

REMARK 3.3: $\mathrm{Hol}_k^*(C,\mathbf{P}^n)$ has additionally the structure of a quasiprojective variety. Generally, components of $\mathrm{Hol}(C,V)$ do not have a smooth structure (when V is a smooth projective variety). This already fails for $V=G(n,n+k)$ (the Grassmaniann of n planes in $\mathbb{C}^{n+k}$) when $n>1$ (see [Ki]). The "expected" dimension of $\mathrm{Hol}_A(C,V)$ (where A is a fixed homology class in $H_2(V)$ and $f_*[C]=A$ for all $f\in\mathrm{Hol}_A$ can however be computed and at generic smooth points it is given by $c_1.A+n(1-g)$. In our case $c_1=n+1$ ($V=\mathbb{P}^n$) and A is k fold the generator in $H_2(V)$.

§3.1 The configuration space model for $\mathrm{Map}^*(C,\mathbb{P}^n)$

In 1.7 we pointed out to the existence of a model for $\mathrm{Map}(C,\mathbb{P}^n)$ in terms of bounded multiplicity symmetric products $SP^k_n(-)$ defined in §2. More precisely, consider $C-U$ where U is a little open disc around the basepoint $*$ ($C-*$ isotopy retracts onto $C-U$). Let U_k be a nested sequence of neighborhood retracting onto $*$, then one defines the space $SP^\infty_n(C)$ as the direct limit of inclusions $SP^k_n(C-U_k)\longrightarrow SP^{k+1}_n(C-U_{k+1})$, $D\mapsto D+x_k, x_k\in U_k-U_{k+1}$ (here $\{x_k\}$ is a sequence of distinct points converging to $*$).

One can map $SP^\infty_n(C-*)$ to the based mapping space by *scanning*

$$S: SP^\infty_n(C-*)\longrightarrow \mathrm{Map}^*_0(C,\mathbb{P}^n) \tag{3.4}$$

This map is given as follows (details in [K2]): identify canonically each neighborhood $D(x)$ of $x\in C-*$ with a closed disc D^2 (this is possible because $C-*$ is parallelizable). It follows that to every configuration in $SP^\infty_n(C-*)$ one can restrict to $D(x)$ and see a configuration in $SP^\infty_n(D(x),\partial D(x))=SP^\infty_n(D^2,\partial D^2)$ (the relative construction means that configurations are discarded when they get to the boundary). It can be checked that $SP^\infty_n(D^2,\partial D^2)=SP^n(S^2)=\mathbb{P}^n$ and hence 3.4. It turns out that the map S in 3.4. is a homotopy equivalence when $n>1$ or a homology equivalence when $n=1$ (cf. [K2]).

Lemma 3.5: *Assume $g\geq 1$, $k\geq n\geq 1$ and $C'=C-*$. Then the collar inclusion $SP^k_n(C')\to SP^\infty(C')$ induces an epimorphism on H_1 (for $n>1$ it is in fact an isomorphism).*

PROOF: A cohomology class in $H^i(C',\mathbb{Z})$ is represented by (the homotopy class) of a map $C'\longrightarrow K(\mathbb{Z},i)$ and since the target is an abelian topological group, this map extends multiplicatively to

$$C'\xrightarrow{i_1} SP^k_n(C')\xrightarrow{i_2} SP^k(C')\longrightarrow K(\mathbb{Z},i)$$

and hence gives rise to a class in $H^i(SP^k_n(C'))$. The "inclusion" $i_1: C'\to SP^k_n(C')$ (constructed earlier) is then surjective in cohomology and hence injective in homology. Since $\pi_1(SP^k(C'))=H_1(SP^k(C'),\mathbb{Z})=H_1(C';\mathbb{Z})$, the composite $i_2 \circ i_1$ must be an isomorphism on H_1. This then implies that $H_1(SP^k_n(C'))\longrightarrow H_1(SP^k(C'))$ is necessarily surjective. It can be checked that as soon as $n\geq 2$, $\pi_1(SP^k_n(C'))$ abelianizes ([K3]) and from there that $H_1(SP^k_n(C'))\cong H_1(SP^k(C'))=\mathbb{Z}^{2g}$. ∎

Lemma 3.6: *Assume $g\geq 1$. The map $Map^*_k(C,\mathbb{P}^n)\xrightarrow{\alpha} Map^*_k(C,\mathbb{P}^\infty)\simeq (S^1)^{2g}$, induced from post-composition with the inclusion $\mathbb{P}^n\hookrightarrow\mathbb{P}^\infty$, is an isomorphism at the level of H_1 when $n>1$ and a surjection when $n=1$.*

PROOF: Consider the following homotopy diagram

$$\begin{array}{ccc} SP^\infty_n(C-*) & \xrightarrow{\subset} & SP^\infty(C-*)\\ \downarrow & & \downarrow\simeq\\ \mathrm{Map}^*_0(C,\mathbb{P}^n) & \xrightarrow{\alpha} & \mathrm{Map}^*_0(C,\mathbb{P}^\infty)\end{array}$$

The top map is an isomorphism on H_1 (and only a surjection when $n=1$) according to the previous lemma. Since both vertical maps are isomorphisms on H_1 as well (by 3.4), the claim follows. ∎

§4 A Spectral Sequence of Milgram

The following spectral sequence appears in special cases in [BCM], [B], [K1] and [KM] and the main ideas trace back to [M1]. Let X be a space with basepoint $*$ and let $E(X)\hookrightarrow SP^\infty(\bigvee^k X)$ be a submonoid of $SP^\infty(\bigvee^k X)=\prod_k SP^\infty(X)$. Given a map $X\longrightarrow E(X)$, it can always be extended to a map $\nu_0: SP^r(X)\longrightarrow E(X)$ (additively) and then to a map

$$\nu:\coprod_{r\geq 1} SP^r(X)\times E\longrightarrow E,\quad \nu(x,y)=\nu_0(x)+y$$

Our interest is to study the complement $Par(X)=E(X)-\mathrm{Im}(\nu)$ (or *Particle* space). We reserve the notation $C(X)$ for the standard configuration space of distinct points (see 4.1 below). Such generalized families of configuration spaces are studied in [K2].

EXAMPLES 4.1:

• (i) Let $k=1$ and consider the map $M\longrightarrow E(M)=SP^\infty(M)$, $x\mapsto 2x$. Points in $\mathrm{Im}(\nu)$ are finite sums $\sum n_ix_i$ where $n_i\geq 2$ for at least one index i. In this case

$$Par(M)=C(M)=\{\sum x_i, x_i\neq x_j \text{ for } i\neq j\}$$

Similarly we can map $M\longrightarrow E$ by sending $x\mapsto (n+1)x$ and in this case the discriminant space is the space $SP^\infty_n(M)$ described in 3.4.

• (ii) The divisor space $\mathrm{Div}^n(M)$ introduced in §3 is the particle space obtained as the complement of the diagonal map

$$M\longrightarrow\prod^n SP^\infty(M)=E(M),\ x\mapsto (x,x,\dots,x)$$

• (iii) If $\mu : M \longrightarrow G$ is a map into an abelian group G, then μ can be extended to $SP^\infty(M)$ and one can define

$$E(M,\mu) = \{(D_1,\dots,D_k) \in \prod_k SP^\infty(M) \mid \mu(D_i) = \mu(D_j) \text{ for all } i,j\}$$

This is a submonoid since by construction $\mu(D+D') = \mu(D) + \mu(D')$. The main example we study in this paper is when $M = C$ is a curve and μ is its Abel-Jacobi map.

Consider the following construction

$$DE(M) = E \times_\nu SP^\infty(cM)$$

where cM is the cone on M and where the twisted product $\times_\nu$ means the identification via ν of the points

$$(\vec{\zeta},(t_1,z_1),\dots,(t_l,z_l)) \sim (\vec{\zeta}+\nu(z_i),(t_1,z_1)\dots\widehat{(t_i,z_i)}\dots(t_l,z_l))$$

whenever $t_i = 0$ is at the base of the cone (here *hat* means deletion). The identification above "cones off" the image of ν in E and so we expect that $H_*(DE) = H_*(E/Im\nu)$. Write

$$E_{n_1,\dots,n_k} = E(M) \cap (SP^{n_1}(M) \times \dots \times SP^{n_k}(M))$$

and suppose $\nu : M \longrightarrow E(M)$ lands in $E_{l_1,\dots,l_k}, l_j \geq 1$. We can then consider the filtration of DE by subspaces

$$DE_{k_0,k_1,\dots,k_n}(C_g) = \bigcup_{\substack{i_j+ll_j\leq k_j\\ 1\leq j\leq k}} E_{i_0,i_1,\dots,i_n} \times_\nu SP^l(cC_g)$$

There are (well-defined) projection maps

$$p_{k_0,k_1,\dots,k_n} : DE_{k_0,k_1,\dots,k_n} \longrightarrow E_{k_0,k_1,\dots,k_n}/\{\text{Image}(\nu)\}$$

sending $(v_1,\dots,v_s,(t_1,w_1),\dots,(t_r,w_r))$ to $(v_1,\dots,v_s)$. These maps are acyclic and hence induce isomorphisms in homology (the proof in the case of the standard configuration spaces 4.1 (i) is given in [BCM], lemma 4.6 and relies mainly on the fact that $SP^l(cX)$ is contractible for every l).

Let's assume $l_j = d, 1 \leq j \leq k$ and write $(D)E_{i,\dots,i} = (D)E_i$. Suppose $Par_i(M) = E_i - Im(\nu) \cap E_i$ is an oriented manifold of dimension $N(i)$ (it will be for all cases we consider here otherwise we can use $\mathbb{Z}_2$ coefficients). Then by Alexander-Poincaré duality we have an isomorphism $\tilde{H}^{N(i)-*}(Par_i(M);\mathbb{A}) \cong H_*(E_i/Im(\nu);\mathbb{A})$ for commutative rings $\mathbb{A}$. Combining this with the previous paragraph we get

Proposition 4.2: *There are isomorphisms*

$$\begin{aligned}\tilde{H}^{N(i)-*}(Par_i(M);\mathbb{A}) &\cong H_*(DE_i;\mathbb{A})\\ \tilde{H}^{N(i)-*}(Par_i(M-*);\mathbb{A}) &\cong H_*(DE_i/\bigcup_i DE_{i,\dots,i-1,\dots,i};\mathbb{A})\end{aligned}$$

Let LE_i be the quotient $E_{i,\dots,i}/\bigcup E_{i,\dots,i-1,\dots,i}, i\geq 1$. The map ν is made out of maps

$$\nu : SP^r(M)\times E_i\longrightarrow E_{i+rd}$$

and so we can consider the quotient model

$$QE_k = DE_{k,\dots,k}/\bigcup DE_{k,\dots,k-1,\dots,k} = \bigcup_{i+dj=k} LE_i\times_\nu(SP^j(cM)/SP^{j-1}(cM))$$

One can filter this complex by j (in which case one obtains an analog of the Eilenberg-Moore spectral sequence with E^2 term a Tor term; see [K1], [BCM]) or one can filter according to i by letting

$$\mathcal{F}_r = \bigcup_{\substack{i\geq r\\ i+dj=r}} LE_i\times_\nu (SP^j(cM)/SP^{j-1}(cM))$$

In this case we get the spectral sequence

Theorem 4.3: *There is a spectral sequence converging to $H^{N(k)-*}Par_k(M-*)$ with E^1 term*

$$E^1 = \coprod_{\substack{i+dj=k\\ i\geq 1}} H_*(LE_i;\mathbb{A})\otimes H_*(SP^j(\Sigma M), SP^{j-1}(\Sigma M);\mathbb{A})$$

and where the d^r differentials are obtained from a chain approximation of the identification maps $\nu : SP^j(M)\times LE_i\longrightarrow LE_{i+d}$, $(x,y)\longrightarrow\nu(x)+y$.

Sketch of proof: (see [K1] and [KM] for details) Consider $\mathcal{F}_r$ as described above. Then a chain complex for $\mathcal{F}_r$ is given as

$$\bigoplus C_*(LE_i)\otimes_\nu C_*(SP^l(cM)/SP^{l-1}(cM))$$

Here the symmetric product pairing as well as the identification given by t can be chosen to be cellular. We need determine the homology of this complex. An interesting fact is that one can construct chain complexes

$$C_*(SP^\infty(cM)) = C_*(SP^\infty(M))\otimes C_*(SP^\infty(\Sigma M)) \tag{4.4}$$

where the identification above is given as a bigraded differential algebra isomorphism. This follows from the fact (cf.[M1])) that $C_*(SP^\infty(cM))$ can be identified with the acyclic bar construction on $C_*(SP^\infty(M)$ and that $C_*(SP^\infty(\Sigma M))$ can be identified with the reduced bar construction on $C_*(SP^\infty(M)$. Therefore the boundary on cells of $\mathcal{F}_r$ can be made *explicit*. First a cell in $C_*(SP^l(cM)/SP^{l-1}(cM))$ can be written as

$$c_*\otimes|a_1|\cdots|a_l|$$

(according to the decomposition 4.4) and the boundary decomposes into

$$\partial = \partial c_*\otimes|a_1|\cdots|a_l| + \nu_*(c_*\otimes a_1)\otimes|a_2|\cdots|a_l| + c_*\otimes\partial_B(|a_1|\cdots|a_l|) \tag{4.5}$$

where ∂_B is the usual bar differential. The induced boundary on $\mathcal{F}_r/\mathcal{F}_{r+1}$ (which describes d_0) is given by the first and last term and the homology of the complex $\partial : C_*(\mathcal{F}_r/\mathcal{F}_{r+1})\rightarrow C_*(\mathcal{F}_{r+1}/\mathcal{F}_{r+2})$ is given by the expression in 4.3. Remains to identify the d^r differentials and these are deduced by the middle term in 4.5 according to the filtration term in which they land. ∎

§4.1 An application: Configurations with bounded multiplicity

The homology of $SP^\infty_n(C'), C' = C - *$ can be calculated from the filtration pieces $SP^k_n(C-*)$ as follows. Start by observing that the configuration space $SP^\infty_n(C)$ is the discrimant set in $SP^\infty(C-*)$ of the image of $\nu : SP^\infty(C)\longrightarrow SP^\infty(C)$ which is multiplication by $n+1$ (as described in 4.1 (i)). In this particular case 4.3 takes the form (with field coefficients)
(4.6) *There is a spectral sequence converging to* $H_{2k-*}(SP^k_n(C-*);\mathbb{F})$, $n\geq 1$, *with* $E^1_{i,j}$ *term (to which we refer as* $E^1(k)$ *)*

$$\bigoplus_{\substack{i+(n+1)j=k\\ r+s=*}} H_r(SP^i(C), SP^{i-1}(C);\mathbb{F})\otimes H_s(SP^j(\Sigma C_g), SP^{j-1}(\Sigma C_g);\mathbb{F}).$$

To see this, one simply observes that $SP^k_n(C-*)$ is open in $SP^k(C-*)$ which is a k dimensional complex manifold and then one applies 4.3. We point out that such a spectral sequence was considered in the case of $n=1$ in ([BCM], theorem 4.1) and for $g=0$ and all n in [K3]. The differentials as explained in the proof of 4.3 are induced from a cellular approximation of the maps

4.7 $$\nu : SP^j(C)\longrightarrow SP^{(n+1)j}(C),\ x\mapsto (n+1)x$$

More explicitly in this case, a chain complex for a Riemann surface C of genus g is given by $2g$ one dimensional classes (which we label $e_1,\dots,e_{2g}$) and a two dimensional orientation class $a=[C]$. Now the homology of $SP^\infty(C)$ is generated as a ring by the symmetric products of these classes; i.e.

$$H_*(SP^\infty(C)) = E(e_1,\dots,e_{2g})\otimes\Gamma[a]$$

and $H_*(SP^n(C))$ consists of all n-term products in the complex above (here E is an exterior algebra and Γ is divided power algebra; see [K1] for details). It turns out that this homology **embeds** in $C_*(C)$ and so one can think of the product of these classes as cells as well. We can investigate the boundary term 4.5 on these classes. The map ν is given by the composite $C\xrightarrow{\Delta} C^{\times n}\to SP^n(C)$. The primitive classes e_i map to $\sum 1\otimes\cdots\otimes e_i\otimes\cdots\otimes 1$ and hence map into $H_*(C)\subset H_*(SP^{n+1}(C))$. For $n>1$ they clearly vanish in $H_*(SP^{n+1}(C), SP^n(C))$ and so are not seen in the spectral sequence. The class a on the other hand maps via Δ to the class $\sum 1\otimes\cdots\otimes 1\otimes C\otimes 1\otimes\cdots 1 + \sum 1\otimes\cdots e_i\otimes\cdots\otimes e_j\otimes\cdots\otimes 1$ in $H_2(C^{n+1})$. The projection into $H_*(SP^{n+1}(C), SP^n(C))$ vanishes if $n>1$ and is non trivial if $n=1$. More explicitly we have
Lemma 4.8: *When* $n>1$, *all* d^r *differentials vanish and the spectral sequence above collapses at the* E^1 *term. When* $n=1$, *there are higher differentials generated by* $d^1(1\otimes|a|) = 2\sum e_ie_{i+g}$, $1\leq i\leq g$.
EXAMPLE 4.9 ($H_1(SP^k(C-*))$): There are $2g$ one dimensional (torsion free) classes $\tilde e_i$ in $H_1(SP^k_n(C-*))$ corresponding to the classes $e_ia^{k-1}\in E^1_{k,0}$ (which have dimension $2(k-1)+1$). Also and when $n=1$, the class $a^{k-2}|a|$

in E^1 gives rise to a generator in $H_1(SP^k_1(C-*))$ for all $k>1$. But $\partial|a| = 2\sum e_ie_{i+g}$ (according to 4.8) and hence this is a 2-torsion class; i.e.

$$H_1(SP^k_n(C-*);\mathbb{Z}) \cong \begin{cases} \mathbb{Z}^{2g}, & \text{when } n>1, k\geq 1\\ \mathbb{Z}_2\oplus\mathbb{Z}^{2g}, & \text{when } k>n=1\end{cases}$$

This also corresponds to $H_1(\mathrm{Map}^*_0(C,\mathbb{P}^n))$ as will be clear shortly (and as is expected according to 1.4!).

EXAMPLE 4.10: There are inclusions $H_*(SP^k_n(C')) \hookrightarrow H_*(SP^{k+1}_n(C'))$ induced by a map of E^1 terms (in 4.6 above)

$$x\otimes y\mapsto xa\otimes y$$

As a corollary one can easily deduce the following standard fact (for another argument, see [K3] this volume): *There are homology splittings (here $SP^0(C')=*$)*

$$H_*(SP^k_n(C')) \cong \bigoplus_{1\leq i\leq k} H_*(SP^i_n(C'), SP^{i-1}_n(C'))$$

§4.2 The homology of $\mathrm{Map}^*_0(C,\mathbb{P}^n)$

We now determine $H_*(\mathrm{Map}^*(C,\mathbb{P}^n);\mathbb{F})$ for $\mathbb{F}=\mathbb{Z}_2,\mathbb{Z}_p$ as a straightforward application of the calculations in §4.1. Recall that when $g>0$, the surface C_g is described topologically by a 2-disc attached to a bouquet of $2g$ circles via the mapping which wraps around as a product of commutators. More precisely one has a cofibration sequence

4.11 $$S^1 \xrightarrow{f} \bigvee^{2g} S^1 \overset{i}{\hookrightarrow} C_g \xrightarrow{\pi} S^2$$

where $i:\bigvee^{2g}S^1\hookrightarrow C_g$ is the one skeleton inclusion, and the map f is given as a product of commutators $[x_1,x_2][x_3,x_4]\cdots[x_{2g-1},x_{2g}]$ (with the x_i's denoting the generators of $\pi_1(\bigvee_{2g}S^1)=\mathbb{Z}^{*2g}$). Applying the $\mathrm{Map}^*(-,\mathbb{P}^n)$ functor to 4.11 yields the fibration

4.12 $$\Omega^2(\mathbb{P}^n)\longrightarrow \mathrm{Map}^*(C_g,\mathbb{P}^n)\longrightarrow \Omega(\mathbb{P}^n)^{2g}$$

We have $H^*(\Omega(\mathbb{P}^n)^{2g}) = H^*(S^1)^{\otimes 2g}\otimes H^*(\Omega^2S^{2n+1})$. Note that 4.12 has simple coefficients (since the fiber is a loop space). Write again (cf. §2) $H^*(\Omega^2S^3,\mathbb{Z}_p)$ as an exterior $E(x_1,\dots,x_{2p^{i+1}-1},\dots)$ tensor a truncated algebra $P_T(y_{2p-2},\dots,y_{2p^i-2},\dots)$ where the x's and y's are generators in the stated dimensions.

Theorem 4.13: *Assume $g>1$. Then the Serre spectral sequence for 4.12 collapses at E^2 when $n>1$. When $n=1$, the spectral sequence collapses* *with $\mathbb{F}_2$ coefficients but has the mod-p differentials ($p>2$)*

$$\begin{aligned} d_{p^i}(x_{2p^i-1}) &= \frac{1}{p^i}\left(\sum_1^g f_{2i-1}f_{2i}\right)^{p^i} \\ d_{p^i}(y_{2p^{i+1}-2}) &= \left[\frac{1}{p^i}\left(\sum_1^g f_{2i-1}f_{2i}\right)^{(p-1)p^i}\right] x_{2p^i-1}. \end{aligned}$$

where the f_i are the one dimensional generators of $\Omega(\mathbb{P}^1)^{2g} \simeq (S^1)^{2g} \times (\Omega S^3)^{2g}$.

PROOF: This is a counting argument. The homology splitting in 4.10 holds for $k=\infty$ and combining this with 1.7 (and field coefficients) we see that

$$H_*(\mathrm{Map}_k^*(C,\mathbb{P}^n)) \cong \bigoplus_{k\geq 1} H_*(SP_n^k(C'), SP_n^{k-1}(C'))$$

We simply need identify generators. From 4.6, we read off the correspondence between generators in E^∞ and generators in the serre spectral sequence as follows

$$\begin{aligned} e_i &\mapsto f_i \quad \text{and} \quad |a| \mapsto x_1 \\ |e_i|'s &\mapsto \text{generators of } H^2(\Omega S^3)^{2g} \end{aligned}$$

($|e_i|, |a| \in E(2)$ and $e_i \in E(1)$, but they propagate to $E(k)$ after multiplying by suitable powers of a as indicated in 4.10). Now a generates a divided power algebra in $H_*(SP^\infty(C))$ and hence $|a|$ generates an $H_*(K(\mathbb{Z},3))$ in $H_*(SP^\infty(\Sigma C))$ (as is known, the groups $H_*(K(\mathbb{Z},3))$ and $H^*(\Omega^2 S^3)$ are formally "dual" to each other). More precisely, we can construct a correspondence between the two as follows (with mod-p coefficients): let γ_i be the divided power generators in $\Gamma(a)$, then

$$|\gamma_{p^i}| \mapsto x_i, \quad |\gamma_{p^i}^{p-1}|\gamma_p| \mapsto y_i$$

The rest of the proof is now straightforward as the non-zero differentials for $p>2$ are entirely generated by the one in 4.8 (compare [K1]). ∎

§5 Segal's Stability Theorem

In this section we derive Segal's result based on the spectral sequence in 4.3 which in this particular context takes the form

(5.1) *There is a spectral sequence converging to $H^*(Hol_k(C_g,\mathbb{P}^n))$, $k \geq 2g-1$ with E^1 term*

$$_hE^1 = \bigoplus_{\substack{i+j=k\\ i\geq 1}} E^1_{i,j} = \bigoplus_{\substack{i+j=k\\ i\geq 1}} H_{*'}(LE_i;\mathbb{A}) \otimes H_{*''}(SP^j(\Sigma C_g), SP^{j-1}(\Sigma C_g);\mathbb{A}).$$

and identifiable d^1 differential. Here $* = 2(k-g)(n+1)+2g-*'-*''$.

The terms LE_i are constructed out of the Jacobi variety J and its stratifications. Recall that by construction E_i is the pull-back of the diagonal

5.2
$$\begin{array}{ccc} E_i & \longrightarrow & SP^i(C_g)\times\cdots\times SP^i(C_g)\\ \downarrow & & \Big\downarrow \underbrace{\mu\times\cdots\times\mu}_{n+1}\\ J & \xrightarrow{\Delta} & J(C_g)\times\cdots\times J(C_g)\end{array}$$

The image under μ of $SP^i(C)$ in $J(C)$ is denoted by W_i. W_i has complex dimensions i and for for $i\geq g$, μ is surjective and $W_i=J$.

We nee recall at this stage that the Abel-Jacobi map $\mu : SP^i(C_g)\rightarrow J(C_g)$ is an analytic fibration in the "stable" range $i\geq 2g-1$ with fiber $\mathbb{P}^{i-g}$ (that the dimension of the fiber stabilizes is a consequence of Riemann-Roch. That μ is actually a fibration is a theorem of Mattuck [Ma]). So from 5.2 it is easy to deduce that for $i\geq 2g$

5.3
$$H_*(LE_i,\mathbb{A})\cong H_*(J(C_g),\mathbb{A})\otimes H_*(S^{2(i-g)(n+1)};\mathbb{A})$$

(i.e. LE_i is obtained by first pulling back $SP^i(C)^{n+1}$ over $J(C)$ via Δ and then collapsing the fat wedge in the fiber $(\mathbb{P}^{n-i})^{n+1}$.)

EXAMPLE 5.4: (the torsion free one-dimensional classes). Note that when $i\geq 2g$, we deduce from 5.3 that there are $2g$ classes in $H_*(LE_k)$ of dimension $2(k-g)(n+1)+2g-1$ which yield $2g$ torsion free generators in $H_1(\mathrm{Hol}^*_k)$.

STABLE CLASSES AND SEGAL'S THEOREM: When $i\geq 2g-1$ (the *stable range*), the terms in ${}_hE^1_{i,j}$ that survive to ${}_hE^\infty$ yield 'stable' classes in $H^*(\mathrm{Hol}^*)$ (and hence in $H_*(\mathrm{Hol}^*)$). The following is shown in ([KM], lemma 8.3)

Proposition 5.5: *The induced map $H_*(Hol^*_k(C_g,\mathbb{P}^n))\longrightarrow H_*(Map^*_k(C_g,\mathbb{P}^n))$ is an isomorphism on the stable classes.*

PROOF (sketch of alternate argument): When $i\geq 2g-1$, E_i is an $(i-g)(n+1)+2g$ complex and maps to $SP^{(i-g)(n+1)+2g}(C)$ (via a factoring of the map $E_i\rightarrow SP^i(C)^{n+1}\xrightarrow{+}SP^{i(n+1)}(C)$ (where $+$ is concatenation). It follows that the stable terms in ${}_hE^1$ (corresponding to $i\geq 2g-1$) map to $E^1(N)$ (see 4.6; here $N=(k-g)(n+1)+2g)$). The main point is that in the stable range the spectral sequence in 5.1 and the one in 4.6 behave identically (same differentials described as in 4.8 above and [KM], lemma 7.8). The stable terms surviving to ${}_hE^\infty$ therefore map isomorphically to $E^\infty(N)$ and so the dual "stable classes" in $H_*(\mathrm{Hol}^*_k)$ have isomorphic image in $H_*(SP^k_n(C-*))$; i.e. in $H_*(SP^\infty_n(C-*))\cong H_*(\mathrm{Map}^*_0(C,\mathbb{P}^n)$. Details omitted. ∎

The unstable terms which appear when $i<2g-1$ in the spectral sequence above form the interesting part and are harder to track down for their existence depends strongly on the geometry of the curve. It follows however that their contribution only starts appearing above a certain range and hence up to that range the homology of $\mathrm{Hol}_k(C_g,\mathbb{P}^n)$ only consists of stable classes and by 5.5 must coincide with that of $\mathrm{Map}_k(C_g, \mathbb{P}^n)$). This is exactly the essence of the stability theorem of Segal and we fill in the details below.

Proposition 5.6: $H_*(LE_i) = 0$ for $* > i(n+1) + 2g$ and for all $1 \leq i \leq 2g$.

PROOF: Define

$$W_i^r = \{x \in \mu(SP^i(C_g)) \mid \mu^{-1}(x) = \mathbb{P}^m, m \geq r\}$$

A well-known theorem of Clifford asserts that in the range $1 \leq i \leq 2g$ the maximum of r is $i/2$; that is $W_i^r = \emptyset$ for $r > \frac{i}{2}$. One can then filter $\mu(SP^i(C)) = W_i$ by the descending filtration

$$W_i^{\frac{i}{2}} \subset \cdots \subset W_i^1 \subset W_i$$

This leads to a spectral sequence converging to $H_*(LE_i)$ with E^1 term

$$\sum_r^{\frac{i}{2}} H_*(W_i^r, W_i^{r+1}) \otimes H_*(S^{2r(n+1)})$$

Since the $W_i^r - W_i^{r+1}$ are algebraic subvarieties of $J(C_g)$ ([ACGH]) we must have that $H_*(W_i^r, W_i^{r+1}) = 0$ for $* > 2g$. This means that each term in the E^1 term above has no homology beyond $2\frac{i}{2}(n+1) + 2g$ and hence

$$H_*(LE_i) = 0 \text{ for } * > 2\frac{i}{2}(n+1) + 2g$$

as asserted. Note that when $i = 2g$, we have that m is uniformly $i - g = g = \frac{i}{2}$ and the proposition is still valid in this case. ∎

Corollary 5.7: (Segal) *The inclusion*

$$Hol_k(C_g, \mathbb{P}^n) \hookrightarrow Map_k(C_g, \mathbb{P}^n)$$

is a homology isomorphism up to dimension $(k - 2g)(2n - 1)$

PROOF: Look in 4.3 at the unstable terms given by

$$H_*(LE_i) \otimes H_{*'}(SP^j(\Sigma C_g), SP^{j-1}(\Sigma C_g)),\ i \leq 2g - 1$$

According to 5.6 these terms vanish for $* > i(n+1) + 2g$. By duality, they contribute therefore no homology to $H_*(\mathrm{Hol}_k(C_g, \mathbb{P}^n); \mathbb{A})$ for

$$* \leq [2(k-g)n + 2k] - [i(n+1) + 2g + 3(k-i)]$$

The expression on the left attains its minimum when $i = 2g$ that is when

$$[2(k-g)n + 2k] - [2g(n+1) + 2g + 3(k-2g)] = (k-2g)(2n-1).$$

So when $* < (k-2g)(2n+1)$ the unstable terms do not contribute to $H_*(\mathrm{Hol}_{k=i+j}(C_g, \mathbb{P}^n); \mathbb{A})$ and the proof now follows from proposition 5.5. ∎

§6 The Fundamental Group

Since the attaching map of the two cell of C_g is given by a product of commutators corresponding to Whitehead products (see 4.11), its suspension must be null-homotopic. This implies that ΣC_g splits as a wedge $\Sigma C_g \simeq \bigvee S^2 \vee S^3$ and one has

6.1 $$\Sigma^i C_g \simeq \bigvee_1^{2g} S^{i+1} \vee S^{i+2},\ i \geq 1.$$

Lemma 6.2: *All components of Map$^*(C_g, X)$ are homotopy equivalent, and $\forall i > 1$, there is an isomorphism of homotopy groups*

$$\pi_i(Map_0^*(C_g, X)) = \bigoplus^{2g} \pi_{i+1}(X)^{2g} \oplus \pi_{i+2}(X).$$

Proof: The statement about components is standard and we denote by $\mathrm{Map}_0^*(C_g, X)$ the component of null-homotopic maps. On the other hand and by definition $\pi_i(\mathrm{Map}_0^*(C_g, X)) = [S^i \wedge C_g, X]_*$. The splitting 6.1 yields (*as sets*) the bijection

$$[S^i \wedge C_g, X]_* = [\bigvee_{2g} S^{i+1} \vee S^{i+2}, X]_* = \pi_{i+2}(X) \oplus \bigoplus^{2g} \pi_{i+1}(X)$$

In order for the decomposition above to induce a group homomorphism, it is necessary that 6.1 be a decomposition as "co-H spaces", that is that there is a map $f_i : \Sigma^i C_g \xrightarrow{f_i} \bigvee_1^{2g} S^{i+1} \vee S^{i+2}$ which commutes with the pinch maps up to homotopy. When $i \geq 2$, $f_i = \Sigma f_{i-1}$ is a suspension and hence is automatically a co-H map. The claim follows. ∎

When $i = 1$, the decomposition in 6.2 is not necessarily a group decomposition. The long exact sequence in homotopy associated to $\Omega^2 X \to \mathrm{Map}^*(C_g, X) \to (\Omega X)^{2g}$ together with 6.2 indicate however that there is a short exact sequence of abelian groups

$$0 \longrightarrow \pi_3(X) \longrightarrow \pi_1(\mathrm{Map}_0^*(C_g, X)) \longrightarrow \pi_2(X)^{2g} \longrightarrow 0$$

When $X = \mathbb{P}^n$, $\pi_2(\mathbb{P}^n) \cong \mathbb{Z}$ and $\pi_3(\mathbb{P}^n) \cong \pi_3(S^{2n+1})$ which is zero when $n > 1$. This shows that

6.3(a) $$\pi_1 \mathrm{Map}_0^*(C, \mathbb{P}^n) = \mathbb{Z}^{2g},\ \text{when } n > 1$$

When $n = 1$, there is a short exact sequence

6.3(b) $$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(\mathrm{Map}_0^*(C, \mathbb{P}^1)) \longrightarrow \mathbb{Z}^{2g} \longrightarrow 0$$

This is a central extension which turns out to be non-trivial as we now show.

First observe that any Riemann surface has meromorphic functions for sufficiently high degrees k (this is certainly true when $k \geq 2g$ and we will assume this is the case throughout the section). Let f be any such function and construct the map $\mathrm{Rat}_1(\mathbb{P}^1) \longrightarrow \mathrm{Hol}_k^*(C, \mathbb{P}^1)$ by post-composition with

f. $\mathrm{Rat}_1(\mathbb{P}^1)$ is the set of pairs of distinct points (a root and a pole) in $\mathbb{C}$ and has the homotopy type of S^1.

Lemma 6.4: *The inclusion* $f^! : Rat_1(\mathbb{P}^1) \longrightarrow Hol^*_k(C,\mathbb{P}^1)$ *induces an injection* $\mathbb{Z} \hookrightarrow \pi_1(Hol^*_k(C,\mathbb{P}^1))$.

PROOF: Observe that the diagram below commutes strictly

6.5
$$\begin{array}{ccc} \mathrm{Rat}_1(\mathbb{P}^1) & \xrightarrow{f^!} & \mathrm{Hol}^*_k(C,\mathbb{P}^1) \\ \big\downarrow{\scriptstyle i_1} & & \big\downarrow{\scriptstyle i_2} \\ \Omega^2_1 S^2 & \xrightarrow{g} & \mathrm{Map}^*_k(C,S^2) \end{array}$$

The vertical maps are inclusions. The bottom map on π_1 is an injection according to 6.3(b) and hence $g \circ i_1$ is also and injection on π_1 (since i_1 on π_1 is an isomorphism between two copies of $\mathbb{Z}$). The composition $i_2 \circ f^!$ is therefore injective on π_1 and hence so is $f^!$ as desired. ∎

Next and from the description of Hol^*_k and $\mathrm{Div}_{k,k}(C-*) = SP^k(C-*)^2 - \Delta$ where Δ is the generalized diagonal consisting of pairs of divisors with (at least) one point in common (see §3), we have the pullback diagram

6.6
$$\begin{array}{ccc} \mathrm{Hol}^*_k(C,\mathbb{P}^1) & \xrightarrow{\phi} & \mathrm{Div}_{k,k}(C-*) \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \mu^2} \\ J(C) & \xrightarrow{\Delta} & J(C)^2 \end{array}$$

where $\pi : \mathrm{Hol}^*_k(C,\mathbb{P}^n) \xrightarrow{p} SP^k(C-*) \xrightarrow{\mu} J(C)$ is the map that sends a holomorphic map to the equivalence class of its divisor of zeros and Δ is the diagonal. The top horizontal map ϕ is an inclusion. We denote by F the homotopy fiber of μ^2 and π.

Lemma 6.7: *The map* $p : Hol^*_k(C,\mathbb{P}^1) \longrightarrow SP^k(C-*)$ *is surjective at the level of fundamental groups.*

PROOF: For $k \geq 2g-1$ the restriction of the Mattuck fibration (cf. §5) to $SP^k(C-*)$ becomes a complex vector bundle

$$\mathbb{C}^{k-g} \longrightarrow SP^k(C-*) \xrightarrow{\mu} J(C)$$

(i.e. the set of linearly equivalent divisors avoiding a point is a hyperplane in $\mathbb{P}^{k-g}$). It follows that μ_* is an isomorphism on $\pi_1(SP^k(C-*))$. Since the fiber of π is connected (see 6.10), it follows that π is surjective at the level of π_1 and hence is p. ∎

Theorem 6.8: *Suppose* $k \geq 2g$. *Then* $\pi_1(Hol^*_k(C,\mathbb{P}^1))$ *is generated (multiplicatively) by classes* $e_1, \dots, e_{2g}$ *and* τ *such that the commutators*

$$[e_i, e_{g+i}] = \tau^2$$

and all other commutators are zero.

We will prove this in a series of lemmas. To begin with, lemmas 6.4 and 6.7 show that there is a sequence

6.9
$$0\longrightarrow\mathbb{Z}\xrightarrow{f_*}\pi_1\mathrm{Hol}^*_k(C,\mathbb{P}^1)\xrightarrow{p_*}\mathbb{Z}^{2g}\longrightarrow 0$$

which is exact at both ends. In 6.13 we show that this sequence maps to 6.3(b) and hence $\mathrm{Im}(f_*)\subset$ Ker (p_*). That Ker(p) $\subset$Im(f_*) follows from 6.11 and the discussion next.

Lemma 6.10: *$\pi_1(Div_{k,k}(C-*)), k>1$ has torsion free generators $a_i, b_j, 1\leq i,j\leq 2g$ and τ. The a_i (resp. b_i) are represented by a root (resp. a pole) going around the generators in a symplectic basis of $H_1(C-*)$, while τ is given by a zero (or pole) going around a pole (resp. a zero).*

Proof: There are a few ways to see that there is a short exact sequence

$$0\longrightarrow\mathbb{Z}\longrightarrow\pi_1(\mathrm{Div}_{k,k}(C-*))\longrightarrow\mathbb{Z}^{4g}\longrightarrow 0$$

with generators having the desired properties.

• By considering the map μ^2 in 6.6 and analyzing the fiber F given by the complement of a hyperplane in $\mathbb{C}^{k-g}\times\mathbb{C}^{k-g}=\mathbb{C}^{2(k-g)}$ (the fiber of μ restricted to $SP^k(C-*)$ is $\mathbb{C}^{k-g}$ as indicated in 6.7). Here $\pi_1(F)=\mathbb{Z}$.

• Mimicking the proof of Jones for the case $C=\mathbb{P}^1$. One considers the subspace U of $\mathrm{Div}_{k,k}(C-*))$ consisting of pairs (D_0,D_1) with the roots of D_1 all distinct. This is the complement of a (real) codimension 2 subspace and there is a surjection $\pi_1(U)\longrightarrow\pi_1(\mathrm{Div}_{k,k}(C-*))$. Now there is a fibration $SP^k(C-Q_{k+1})\longrightarrow\pi_1(U)\longrightarrow\pi_1(F(C-*,k)$ with section (where Q_{k+1} is a set of $k+1$ distinct points). Since $\pi_1(SP^k(C-Q_{k+1}))=H_1(C-Q_{k+1})=\mathbb{Z}^{2g}\times\mathbb{Z}^k$, there is a semi-direct product of $\mathbb{Z}^{2g}\times\mathbb{Z}^k$ with the ordered braid group $B_k(C-*)$ with quotient at least $\mathbb{Z}^{4g}$. A similar argument as in [S], lemma 6.4 implies the answer. ∎

Lemma 6.11: *$\pi_1(Hol^*_k(C,\mathbb{P}^1))$ has generators τ and $e_i, 1\leq i\leq g$ which map under $\phi_*:\pi_1(Hol^*_k(C,\mathbb{P}^1))\hookrightarrow\pi_1(Div_{k,k}(C-*))$ as follows: $\phi_*(\tau)=\tau$ (the generator of the same name) and $\phi_*(e_i)=a_ib_i$. In particular ϕ_* is an injection.*

Proof: Diagram 6.6 induces a map of fibrations and since $\pi_2(J)=0$, a map of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_1(\mathrm{Hol}^*_k(C,\mathbb{P}^1)) & \longrightarrow & \mathbb{Z}^{2g} & \longrightarrow & 0\\
 & & \downarrow{=} & & \downarrow{\phi_*} & & \downarrow{\Delta_*} & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_1(\mathrm{Div}_{k,k}(C-*)) & \longrightarrow & \mathbb{Z}^{4g} & \longrightarrow & 0
\end{array}$$

The map Δ_* is an injection and then so is the middle map. The generator $\tau\in\pi_1(\mathrm{Hol}^*_k)$ constructed in 6.4 corresponds exactly to the generator coming from $\pi_1(F)$ and is described by zeros linking with poles. The rest of the claim follows from the commutativity of the above diagram. ∎

PROOF OF THEOREM 6.8: The generators of π_1 and their images under ϕ having been determined, we next look at the image of the commutators

$$\phi([e_i,e_j]) = a_i \underbrace{b_i a_j} b_j b_i^{-1} \underbrace{a_i^{-1} b_j^{-1}} a_j^{-1}$$

Suppose we know that for $k \geq 2$ and $1 \leq i,j \leq g$

6.12 $$a_i b_j a_i^{-1} b_j^{-1} = \begin{cases} \tau, & |i-j| = g \\ 1, & \text{otherwise} \end{cases} \in \pi_1(\mathrm{Div}_{k,k}(C-*)),$$

then $\phi[e_i,e_j]$ becomes (after transforming the underbraced terms) and for $|i-j| = g$

$$\begin{aligned} \phi([e_i,e_j]) &= a_i \tau a_j b_i b_j b_i^{-1} b_j^{-1} a_i^{-1} \tau a_j^{-1} \\ &= \tau^2 a_j a_i [b_i,b_j] a_i^{-1} a_j^{-1} \\ &= \tau^2 [a_i,a_j] = \tau^2 \end{aligned}$$

This is the claim in 6.8 and therefore the theorem would follow as soon as we prove 6.12.

Consider the sub-divisor space $\mathrm{Div}^2_{k,1}(C-*)) \subset \mathrm{Div}_{k,k}(C-*))$ consisting of pairs $(D_1,q) \in SP^k(C-*) \times (C-*)$ with the property that $q \notin D_1$ (here again $k>1$). There is a projection and a fibration

$$SP^k(C-\{*,q\}) \longrightarrow \mathrm{Div}_{k,1}(C-*) \longrightarrow (C-*)$$

The generators $b_i \in \pi_1(C-*)$ act on the fundamental group of the fiber $\pi_1(SP^k(C-\{*,q\})$ and this action describes the commutator map in 6.12. Since $\pi_1(SP^k(C-\{*,q\})$ is abelian, we can look at the action in homology. This has been already determined in ([C^{2}M^2], prop. 10.12) and is exactly given by 6.12 (b_i is τ_{1i} in their notation). This completes the proof. ∎

Proposition 6.13: *For $k \geq 2g$ and all $n \geq 1$, we have an isomorphism*

$$\pi_1(Hol^*_k(C,\mathbb{P}^n)) \cong \pi_1(Map^*_0(C,\mathbb{P}^n))$$

PROOF: When $n>1$, one can show that $\pi_1(\mathrm{Hol}^*_k(C,\mathbb{P}^n))$ is abelian (this has been proven in [Ki], lemma 3.5). The equivalence in π_1 when $n>1$ becomes an equivalence at the level of H_1 which we know is true by Segal's theorem. In fact when $n>1$, $\pi_1(\mathrm{Hol}^*_k(C,\mathbb{P}^n)) = \pi_1(\mathrm{Map}^*_k(C,\mathbb{P}^n)) \cong \mathbb{Z}^{2g}$ by 6.3(a).

Suppose $n=1$. The proof of 6.11 shows that the sequence in 6.9 is actually exact. We would like to see that 6.9 maps to the exact sequence in 6.3(b). The essential point is to show that the following homotopy commutes

$$\begin{array}{ccc} \mathrm{Hol}^*_k(C,\mathbb{P}^1) & \xrightarrow{\quad p \quad} & SP^\infty(C-*) \\ \Big\downarrow i & & \Big\downarrow \simeq \\ \mathrm{Map}^*_k(C,\mathbb{P}^1) & \longrightarrow (\Omega S^2)^{2g} \longrightarrow & (\Omega\mathbb{P}^\infty)^{2g} \end{array}$$

where again p sends a holomorphic map to the divisor of its zeros, the bottom composite $g : \mathrm{Map}^*_k(C,\mathbb{P}^1) \longrightarrow (\Omega\mathbb{P}^\infty)^{2g}$ is restriction to the one skeleton followed by postcomposition with $S^2 \hookrightarrow \mathbb{P}^\infty$. Since both spaces on the far right are Eilenberg-Maclane spaces $(S^1)^g$, one simply need consider the effect of p and $g\circ i$ on cohomology (recall that i is an isomorphism on H^1 by 1.3). Label generators of $H^1((S^1)^{2g})$ by $f_i, 1\leq i\leq 2g$. As is explicit in theorem 4.13, the $g^*(f_i)$ correspond to the non-trivial torsion free classes in $H^1(\mathrm{Map}^*_k)$ (see 4.13 and 4.9). Similarly the construction of the stable classes in §5 (see example 5.4) implies precisely that $p^*(f_i)$'s are the torsion free generators of $H^1(\mathrm{Hol}^*_k)$. That is $p^* = i^*\circ g^*$ and hence $p\simeq i\circ g$.

Now and upon applying π_1 to the terms of the diagram above as well as to those in 6.5 we get a map from 6.9 down to 6.3(b). Since the end terms of the sequences are isomorphic, it follows by the five lemma that the middle terms are isomorphic; i.e. $\pi_1(\mathrm{Hol}^*_k)\cong \pi_1(\mathrm{Map}^*_k)$. ∎

Corollary 6.14: *$\pi_1(Map^*_0(C,\mathbb{P}^n))$ is generated by classes $e_1,\dots,e_{2g}$ and α such that the commutators $[e_i,e_{g+i}]=\alpha^2$ and all other commutators are zero.*

NOTE: Proposition 6.13 is not sufficient to imply that Segal's homology equivalence 5.7 can be upgraded to a homotopy equivalence. A detailed study of the actions of π_1 on the universal covers is needed.

§7 Spaces of Unbased Maps

There is an important distinction between $\mathrm{Map}^*(C,X)$ and $\mathrm{Map}(C,X)$. For one thing, while the topology of the based mapping space doesn't vary with the degree, this is no longer true in the unbased case. Let $\mathrm{Map}_f(C_g,X)$ denote the component containing a given map f, $x_0\in C_g$ the basepoint and consider again the "evaluation" fibration

7.1 $$\mathrm{Map}^*_f(C_g,X)\longrightarrow \mathrm{Map}_f(C_g,X)\xrightarrow{ev} X\ ,\ ev(g)=g(x_0)$$

Associated to 7.1 is a long exact sequence of homotopy groups

$$\to \pi_i\mathrm{Map}^*_f(C_g,X)\to \pi_i\mathrm{Map}_f(C_g,X)\to \pi_i(X)\xrightarrow{\partial}\pi_{i-1}\mathrm{Map}^*_f(C_g,X)\to$$

Suppose X is simply connected. Notice that given a map $f: C_g\longrightarrow X$, f is null on $\bigvee S^1$ and hence factors (up to homotopy) as in

$$f: C_g\longrightarrow S^2\xrightarrow{\bar f} X$$

One then has the following diagram of vertical fibrations over X

7.3 $$\begin{array}{ccccc}
\Omega^2_{\bar f}X & \longrightarrow & \mathrm{Map}^*_f(C_g,X) & \longrightarrow & (\Omega X)^{2g}\\
\downarrow & & \downarrow & & \downarrow\\
\mathcal{L}^2_{\bar f}X & \longrightarrow & \mathrm{Map}_f(C_g,X) & \longrightarrow & \mathrm{Map}(\bigvee S^1,X)\\
\downarrow{\scriptstyle ev} & & \downarrow{\scriptstyle ev} & & \downarrow{\scriptstyle ev}\\
X & \xrightarrow{=} & X & \xrightarrow{=} & X
\end{array}$$

inducing a diagram of homotopy groups and boundary homomorphisms

$$\begin{array}{ccccc}
\pi_i(X) & \xrightarrow{=} & \pi_i(X) & \xrightarrow{=} & \pi_i(X) \\
\downarrow \partial_1 & & \downarrow \partial & & \downarrow \partial_2 \\
\begin{array}{c}\pi_{i-1}(\Omega^2(X)) \\ =\pi_{i+1}(X)\end{array} & \longrightarrow & \pi_{i-1}(\mathrm{Map}^*_f(C_g, X)) & \longrightarrow & \begin{array}{c}\pi_{i-1}(\Omega_{fi}X)^{2g} \\ =\pi_i(X)^{2g}\end{array}
\end{array}$$

Proposition 7.4: *Assume X simply connected, $f : C_g \longrightarrow X$ and $\bar{f}$ such that $C_g \xrightarrow{\pi} S^2 \xrightarrow{\bar{f}} X$ (up to homotopy). Assume $i > 2$. Then the homotopy boundary*

$$\partial : \pi_i(X) \longrightarrow \pi_{i-1}(Map^*_f(C_g, X)) = \pi_i(X)^{2g} \oplus \pi_{i+1}(X)$$

for the fibration $Map^*_f(C_g, X) \longrightarrow Map_f(C_g, X) \xrightarrow{ev} X$ factors through the term $\pi_{i+1}(X)$ and is given (up to sign) by the Whitehead product: $\partial\alpha = [\alpha, \bar{f}]$.

PROOF: The sequence $0 \to \pi_{i+1}(X) \to \pi_{i-1}(\mathrm{Map}^*_f(C_g, X)) \to \pi_i(X)^{2g} \to 0$ splits canonically (according to 6.2) and with respect to that splitting one has $\partial = \partial_1 + \partial_2$. By Whitehead's theorem we have that $\partial_1\alpha = [\alpha, \bar{f}]$ while $\partial_2\alpha = 0$. The proof follows. ∎

REMARK: When $i = 2$ the situation has to be handled differently for $\pi_1(\mathrm{Map}^*_f(C_g, X))$ does not necessarily split as in 7.4.

Lemma 7.5: $\pi_1(\mathrm{Map}_d(C, \mathbb{P}^n)) \cong \mathbb{Z}^{2g}$ when $n > 1$.

PROOF: Consider the Hopf fibration $S^{2n+1} \longrightarrow \mathbb{P}^n \longrightarrow \mathbb{P}^\infty$ and the induced fibration

$$\mathrm{Map}(C, S^{2n+1}) \longrightarrow \mathrm{Map}_d(C, \mathbb{P}^n) \longrightarrow \mathrm{Map}_d(C, \mathbb{P}^\infty) \simeq (S^1)^{2g} \times \mathbb{P}^\infty$$

Since $\pi_1(\mathrm{Map}(C, S^{2n+1})) = 0$ when $n > 1$, the result follows. ∎

We next address the case $n = 1$ and analyze 7.3 when $X = \mathbb{P}^1$. Let $f : S^2 \longrightarrow X$ be a degree d map. The boundary term in the left vertical fibration $\pi_1(\Omega^2_f X) \xrightarrow{\partial} \pi_1(\mathcal{L}^2_f(X))$ is given according to 7.4 by multiplication by $2d$. On the other hand, the right vertical fibration has a section and hence we get the diagram of fundamental groups

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_1(\mathrm{Map}^*_d) & \longrightarrow & \mathbb{Z}^{2g} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow = & & \downarrow \\
0 & \longrightarrow & \mathbb{Z}_{2d} & \longrightarrow & \pi_1(\mathrm{Map}_d) & \longrightarrow & \mathbb{Z}^{2g} & \longrightarrow & 0
\end{array}$$

Corollary 7.6: (Larmore-Thomas) *$\pi_1(Map_d(C, S^2))$ is generated by classes $e_1, \dots, e_{2g}$ and α such that*

$$\alpha^{2|d|} = 1, \ [e_i, e_{g+i}] = \alpha^2$$

All other commutators are zero. In particular $Map_d(C, \mathbb{P}^1)$ and $Map_{d'}(C, \mathbb{P}^1)$ have different homotopy types whenever $d \neq \pm d'$.

Corollary 7.7: *For $d \geq 2g$, $\pi_1(Hol_d(C, \mathbb{P}^1))$ is generated by $e_1, \dots, e_{2g}$ and α with $\alpha^{2|d|} = 1$, $[e_i, e_{g+i}] = \alpha^2$ and all other commutators are zero.*

PROOF: This is equivalent to showing that $\pi_1(\mathrm{Hol}_d(C,\mathbb{P}^1))$ corresponds to $\pi_1(\mathrm{Map}_d(C,S^2))$ in that range but this follows from a direct comparison of the evaluation fibrations for both mapping spaces and theorem 1.4. ∎

In the case of maps into $\mathbb{P}^n$, $n\geq 2$, the fundamental group is not enough to distinguish between connected components. Proposition 7.4 still implies

Proposition 7.8: $\mathrm{Map}_k(C_g,\mathbb{P}^{2d})$ and $\mathrm{Map}_l(C_g,\mathbb{P}^{2d})$ have different homotopy types whenever $l\not\equiv k(2)$.

PROOF: The long exact sequence for the evaluation fibration gives again

$$\cdots\pi_{2d+1}\mathbb{P}^{2d}\xrightarrow{\partial}\pi_{2d}(\mathrm{Map}_k^*)\rightarrow\pi_{2d}(\mathrm{Map}_k)\rightarrow\pi_{2d}(\mathbb{P}^{2d})\xrightarrow{\partial}\cdots$$

where Map_k^* stands for $\mathrm{Map}^*(C_g,\mathbb{P}^{2d})$. According to 7.4 the sequence above becomes

$$\longrightarrow\pi_{2d+1}\mathbb{P}^{2d}=\mathbb{Z}\langle\eta\rangle\xrightarrow{k[\iota,\eta]}\pi_{2d+2}(\mathbb{P}^{2d})=\mathbb{Z}_2\rightarrow\pi_{2d}(\mathrm{Map}_k)\rightarrow 0$$

where ι is the generator of $\pi_2(\mathbb{P}^{2d})$. The Whitehead product pairing for complex projective spaces

$$\pi_{2m+1}(\mathbb{P}^m)\otimes\pi_2(\mathbb{P}^m)\longrightarrow\pi_{2m+2}(\mathbb{P}^m)$$

is worked out in [P]. The result there is that the Whitehead product is zero if m is odd and non-zero if m is even. Since we are in the case $m=2d$, the map $\mathbb{Z}\langle\eta\rangle\xrightarrow{k[\iota,\eta]}\mathbb{Z}_2$ is in fact multiplication by k and the proposition follows right away. ∎

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