ON THE TOPOLOGY OF FIBRATIONS WITH SECTION AND FREE LOOP SPACES

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1. *Introduction*

Let

$$\zeta\colon F \xrightarrow{\;i\;} E \xrightarrow{\;\pi\;} B$$

be a fibration with a section $B \xrightarrow{s} E$. One of the main results of this paper asserts that the differentials on the spherical classes in the Serre spectral sequence for ζ are entirely determined by 'brace products' in ζ.

Brace products for a fibration with section were originally defined by James [**6**]. Given $\alpha \in \pi_p(B)$ and $\beta \in \pi_q(F)$, one can take the Whitehead product $[s_*(\alpha), i_*(\beta)]$ in $\pi_{p+q-1}(E)$. Since $\pi_*([s_*(\alpha), i_*(\beta)]) = 0$, one deduces from the long exact sequence in homotopy associated to ζ that $[s_*(\alpha), i_*(\beta)]$ must lift to a class (unique once the section is chosen)

$$\{\alpha, \beta\} \in \pi_{p+q-1}(F),$$

the so-called *brace product* of α and β. Note that this class depends on the choice of section. The brace product operation then gives a pairing

$$\{\ ,\ \}\colon \pi_p(B) \times \pi_q(F) \longrightarrow \pi_{p+q-1}(F).$$

Let $h\colon \pi_*(X) \longrightarrow H_*(X; \mathbb{Z})$ denote the Hurewicz homomorphism. Our first main observation can now be stated.

THEOREM 1.1. *Let $F \longrightarrow E \longrightarrow S^p$, with $p > 1$, be a fibration with section. Then in the Serre spectral sequence for E (with integral coefficients), the following diagram commutes:*

$$\begin{array}{ccc}
\pi_p(S^p) \otimes \pi_q(F) & \xrightarrow{\{\ ,\ \}} & \pi_{p+q-1}(F) \\
\downarrow & & \downarrow h \\
H_p(S^p, H_q(F)) & & H_{p+q-1}(F) \\
\downarrow \cong & & \downarrow \cong \\
E^p_{p,q} & \xrightarrow{\;d^p\;} & E^p_{0,p+q-1}
\end{array}$$

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For general fibrations $F \longrightarrow E \longrightarrow B$ with section and with B simply connected, similar (long) differentials exist on the product of spherical classes that survive to the $E^p_{p,q}$ term.

REMARKS. The map $\pi_p(B) \otimes \pi_q(F) \longrightarrow H_p(B, H_q(F))$ is of course the composite

$$\pi_q(B) \otimes \pi_q(F) \xrightarrow{h \otimes h} H_p(B) \otimes H_q(F) \xrightarrow{\nu} H_q(F, H_p(B)),$$

where ν is a universal coefficient homomorphism. Note that even though the brace product does depend on the choice of section s, commutativity of the above diagram does not (cf. § 3).

Theorem 1.1 relies in its proof on a beautiful and classical theorem of George Whitehead [**13**] relating the boundary homomorphism in the homotopy long exact sequence of a free loop fibration on a space X to the Whitehead products in X. More precisely, let X be a finite CW complex (based at x_0) and consider the evaluation fibration

$$\Omega^k X \xrightarrow{i} \mathscr{L}^k X \xrightarrow{\text{ev}} X \tag{1.2}$$

where $\mathscr{L}^k X = \text{Map}(S^k, X)$ is the space of all continuous maps from S^k to X (the 'kth free loop space'), and $\Omega^k X$ is the subspace of basepoint-preserving maps. We let $\mathscr{L}^n_f(X)$ denote the component containing a given map f.

THEOREM 1.3 [**13**]. *The homotopy boundary*

$$\partial\colon \pi_p(X) \longrightarrow \pi_{p-1}(\Omega^k_f(X)) \cong \pi_{p+k-1}(X)$$

in the long exact sequence in homotopy associated to

$$\Omega^k_f(X) \longrightarrow \mathscr{L}^k_f(X) \xrightarrow{\text{ev}} X$$

is given (up to sign) by the Whitehead product $\partial\alpha = [\alpha, f]$ for $\alpha \in \pi_p(X)$.

We use this theorem in § 3 to prove Theorem 1.1.

Free loop spaces. A particularly interesting application of Theorem 1.1 occurs for the evaluation fibration (1.2) when the connectivity of X is greater than k. In this case (1.2) admits a section and Theorem 1.1 applies.

In addition to loop sum, the homology ring $H_*(\Omega^k X)$ admits a second homology operation on two variables called the Browder operation and denoted by λ_k. This operation is essential in the calculation of the homology of iterated loop spaces (cf. [**3**] for extensive details). We quickly sketch its construction: first there is an *operad* map

$$\theta\colon S^{k-1} \times \Omega^k X \times \Omega^k X \longrightarrow \Omega^k X$$

given as follows: a map $f \in \Omega^k X$ can be thought of as a map of the closed unit disc in $\mathbb{R}^k$ into X which sends the boundary to a basepoint. If one identifies S^{k-1} with the space of pairs of closed non-overlapping discs in $\mathbb{R}^k$, then to each pair (D_1, D_2) and to $(f, g) \in \Omega^k X$ one associates the map $\theta(f, g)$ which is f on the first disc, g on the second and sends the complement and boundary of $D_1 \sqcup D_2$ to the basepoint. One then defines $\lambda_k(x; y) := \theta_*(\iota_k, x, y) \in H_{|x|+|y|+k-1}(\Omega^k X)$.

Let ρ_k be the map

$$\pi_*(X) \xrightarrow{\mathrm{ad}_k} \pi_{*-k}(\Omega^k X) \xrightarrow{h} H_{*-k}(\Omega^k X)$$

where ad_k is the adjoint isomorphism and h the Hurewicz map. If we identify the spherical classes in $H_p(X)$ with classes (of the same name) in $\pi_p(X)$, then ρ determines a map from the spherical classes in $H_p(X)$ to $H_{p-k}(\Omega^k X)$. The second main observation of this article is the following.

Theorem 1.4. *Let X be k-connected, and let $\beta \in H_j(\Omega^k(X))$ and $\alpha \in H_p(X)$ be two spherical classes. Then in the homology Serre spectral sequence for*

$$\Omega^k X \longrightarrow \mathscr{L}^k X \xrightarrow{\mathrm{ev}} X,$$

the following relation holds:

$$d^p(\alpha \otimes \beta) = \lambda_k(\rho_k(\alpha), \beta).$$

In the case where $X = S^n$ is a sphere and $1 \leqslant k < n$, general arguments show that the spectral sequence collapses at E^2 with mod-2 coefficients (§ 4.2). When n is odd, the same collapse occurs with mod-p coefficients. The case n even is then of greater interest and we show the following.

Let $x \in H_n(S^n)$ be the orientation class and $e \in H_{n-k}(\Omega^k S^n)$ be the infinite cyclic generator representing the class of the inclusion $S^{n-k} \longrightarrow \Omega^k S^n$ which is adjoint to the identity map of S^n. When n is even, let $a \in H_{2n-k-1}(\Omega^k S^n)$ be the torsion-free generator (see § 4).

Corollary 1.5. *Assume that $1 \leqslant k < n$ and n is even. Then in the homology Serre spectral sequence (with integral coefficients) for the fibration*

$$\Omega^k S^n \xrightarrow{i} \mathscr{L}^k S^n \xrightarrow{\mathrm{ev}} S^n,$$

we have

$$d^n_{n,n-k}(x \cdot e) = 2a.$$

Corollary 1.6. *Suppose $1 \leqslant k < n$ and n is even. Then the Poincaré series for $H^*(\mathscr{L}^k S^n; \mathbb{Q})$ is given as follows:*

$$\begin{cases} 1 + (x^n + x^{n-k})/(1 - x^{2n-k-1}) & \text{if } k \text{ is odd,} \\ (1 + x^{3n-k-1})/(1 - x^{n-k}) & \text{if } k \text{ is even.} \end{cases}$$

Corollary 1.7. *Suppose $n > 2$ is even and p is odd. Then in the cohomology Serre spectral sequence for $\mathscr{L}^2 S^n$, the* mod-p *differentials are generated by $d_n(x \cdot e) = x_0$, where $H^*(\Omega^2 S^n; \mathbb{Z}_p)$ is a tensor product of a divided power algebra on generators e, y_i, and an exterior algebra on generators x_i, with* $\dim(x_i) = 2(n-1)p^i - 1 = \dim(y_i) + 1$ *for $i \geqslant 0$.*

Corollary 1.7 has also been obtained by Fred Cohen using configuration space model techniques (cf. [**1**]).

We can carry out similar calculations for $\mathscr{L}^s(W)$ where W is a bouquet of spheres. In this paper we focus on the $s=1$ case and there easily recover the cyclic homology description of J. Jones and R. Cohen [**4**]. More explicitly, let $W=\bigvee_k S^{n_i+1}$ be a bouquet of k spheres, with $n_i>1$, let $a_i\in H_{n_i+1}(W)$ be the class of the ith sphere, and $e_i=\rho(a_i)\in H_{n_i}(\Omega W)=T(e_1,\dots,e_k)$ (where T is the tensor algebra). The map $\rho=\rho_1\colon H_{n_i+1}(W)\longrightarrow H_{n_i}(\Omega W)$ is as defined earlier. We prove the next proposition in § 5.

Proposition 1.8. *In the Serre spectral sequence for $\Omega W\longrightarrow\mathscr{L}W\longrightarrow W$, where $W=\bigvee_k S^{n_i+1}$, the differentials are given by the cyclic operators*

$$d(a_r,e_{i_1}\otimes e_{i_2}\otimes\dots\otimes e_{i_s})$$
$$=e_r\otimes e_{i_1}\otimes\dots\otimes e_{i_s}-(-1)^{|e_r|(|e_{i_1}|+\dots+|e_{i_s}|)}e_{i_1}\otimes e_{i_2}\otimes\dots\otimes e_{i_s}\otimes e_r,$$

where $e_r=\rho(a_r)$ and $|e_j|=n_j$ is the dimension of e_j.

This description yields an effective method for calculating the homology of W with mod-2 coefficients and also rational coefficients. Following some ideas of Roos, and in the case when the spheres are equidimensional, we can show the following.

Proposition 1.9. *Let $W=\bigvee_k S^{n+1}$ and denote by $P(\mathscr{L}W,\mathbb{F})$ the Poincaré series for $H_*(\mathscr{L}W,\mathbb{F})$. Then*

$$P(\mathscr{L}W,\mathbb{Z}_2)=1+(1+z)\left(\sum_{m\geqslant 1}a_m z^{mn}\right)$$

where

$$a_m=\sum_{d\,|\,m}\frac{1}{d}\sum_{e\,|\,d}\mu\left(\frac{d}{e}\right)k^e=\sum_{e\,|\,m}\frac{1}{m}\phi\left(\frac{m}{e}\right)k^e,$$

ϕ being the Euler ϕ-function.

The rational case is slightly different in the case of even spheres.

Proposition 1.10. *Let $W=\bigvee_k S^{n+1}$. Then*

$$P(\mathscr{L}W,\mathbb{Q})=1+(1+z)\left(\sum_{m\geqslant 1}a_m z^{mn}\right)$$

where

$$a_m=\begin{cases}\displaystyle\sum_{e\,|\,m}\frac{1}{m}\phi\left(\frac{m}{e}\right)k^e, & \text{for } n \text{ odd, or } n \text{ even and } m \text{ odd,}\\ \displaystyle\sum_{\substack{d\,|\,m\\ d\text{ even}}}\frac{1}{d}\sum_{e\,|\,d}\mu\left(\frac{d}{e}\right)k^e, & \text{for } n \text{ even and } m \text{ even.}\end{cases}$$

REMARK. As was pointed out to us by N. Dupont, the above calculations recover (in particular) the following beautiful result of Roos and Parhizgar:

$$\dim H^{2n}(\mathscr{L}(S^3 \vee S^3); \mathbb{Q}) = \frac{1}{n} \sum_{i=1}^{n} 2^{(i,n)}$$

where (i, n) is the greatest common divisor of i and n.

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2. *Brace products: examples and properties*

NOTATION AND CONVENTIONS. We often (but not always) identify a map $f\colon S^p \longrightarrow X$ with its homotopy class $[f] \in \pi_p(X)$. We do so when there is no risk of confusion and to ease notation. We also write ad for the adjoint isomorphism

$$\mathrm{ad}_k\colon \pi_{i+k}(X) \xrightarrow{\cong} \pi_i(\Omega^k X).$$

In the introduction we defined brace products for a fibration

$$F \xrightarrow{i} E \xrightarrow{p} B$$

with a section $B \xrightarrow{s} E$. Brace products are related to Whitehead products by the commutative diagram

$$\begin{array}{ccc} \pi_p(B) \otimes \pi_q(F) & \xrightarrow{\{\ ,\ \}} & \pi_{p+q-1}(F) \\ \Big\downarrow{\scriptstyle s \otimes i} & & \Big\downarrow{\scriptstyle i} \\ \pi_p(E) \otimes \pi_q(E) & \xrightarrow{[\ ,\ ]} & \pi_{p+q-1}(E) \end{array} \tag{2.1}$$

The next examples compute the brace product pairing for some classes of fibrations with section.

EXAMPLE 2.2. Let E be a sphere bundle over $B = S^n$ with fiber $F = S^k$ and group $O(k+1)$,

$$S^k \xrightarrow{i} E \xrightarrow{\pi} S^n.$$

This fibration is classified (up to homotopy) by a clutching function

$$\mu\colon S^{n-1} \longrightarrow O(k+1).$$

If E has a section then the group of the bundle reduces to $O(k)$ (because the associated vector bundle does). The map μ factors (up to homotopy) through $S^{n-1} \longrightarrow O(k) \hookrightarrow O(k+1)$, giving a class $\alpha \in \pi_{n-1}O(k)$. Let J be the Hopf–Whitehead construction

$$J\colon \pi_{n-1}(O(k)) \longrightarrow \pi_{n+k-1}(S^k).$$

Finally, let $\iota_n \in \pi_n(E)$ be the class of $s\colon S^n \longrightarrow E$ and $\iota_k \in \pi_k(E)$ be the class of the fiber. Then (up to sign) the following holds.

PROPOSITION 2.3. *In* $\pi_{n+k-1}(S^k)$, $\{\iota_n, \iota_k\} = J\alpha$.

Proof. We will make use of some intermediate results that we prove in § 3. Start with the map $\alpha\colon S^{n-1} \longrightarrow O(k)$. One can think of $O(k)$ as transformations of the closed unit disc D^k. It follows that α adjoins to a map $S^{n-1} \times D^k \longrightarrow D^k$ and, by pinching the boundary of D^k, we get the following commutative diagram:

$$\begin{array}{ccc} S^{n-1} \times D^k & \xrightarrow{\alpha} & D^k \\ \downarrow & & \downarrow \\ S^{n-1} \wedge S^k & \xrightarrow{J} & S^k \end{array}$$

Since $S^{n-1} \wedge S^k = S^{n+k-1}$, the bottom map can indeed be identified with (and actually is) the J homomorphism. Consider the composite

$$\phi\colon S^{n-1} \wedge S^k \xrightarrow{J} S^k \xrightarrow{i} E.$$

We write its adjoint as a map $g\colon S^{n-1} \longrightarrow \Omega^k E$. Notice that the image of g lies in the component containing the fiber inclusion $i\colon S^k \longrightarrow E$ and hence maps into $\Omega^k_{\iota_k} E$. Moreover notice that g, when extended to $\mathscr{L}^k_{\iota_k} E$, is trivial (Lemma 3.4) and hence the map g factors through the fiber ΩE of the inclusion $\Omega^k E \hookrightarrow \mathscr{L}^k E$ as follows:

$$\begin{array}{cccc} S^{n-1} & \xrightarrow{\mathrm{ad}(s)} & \Omega E & \\ \Big\downarrow{=} & & \Big\downarrow & \\ S^{n-1} & \xrightarrow{g} & \Omega^k E & \longrightarrow \mathscr{L}^k E \end{array} \tag{2.4}$$

The adjoint of the top map is $s\colon S^n \longrightarrow E$ and we denote the class of this map by ι_n. According to Lemma 3.2 we must have $\phi = [\iota_k, \iota_n] \in \pi_{n+k-1}(E)$. Both ϕ and the Whitehead product map lift to S^k. Since the lift of ϕ is J and the lift of $[\iota_k, \iota_n]$ is $\{\iota_k, \iota_n\}$, the proof is complete. $\square$

REMARK. Sphere bundles with section can be constructed by taking a vector bundle ζ over $B = S^n$ with fiber $F = \mathbb{R}^k$ and then compactifying fiberwise the unit disc bundle. The new bundle (with fiber S^k) has a canonical cross section (sending each point in S^n to the point at infinity in the fiber).

EXAMPLE 2.5. It is known that the fiber of the inclusion $X \vee X \longrightarrow X \times X$ is $\Sigma(\Omega X \wedge \Omega X)$ (a theorem of Ganea). Taking $X = \mathbb{P} = \mathbb{P}^\infty$, the infinite complex projective space, we find that there is a fibration $S^3 \longrightarrow \mathbb{P} \vee \mathbb{P} \longrightarrow \mathbb{P} \times \mathbb{P}$, and hence, after looping, we obtain a fibration

$$\Omega S^3 \longrightarrow \Omega(\mathbb{P} \vee \mathbb{P}) \longrightarrow S^1 \times S^1 \tag{2.6}$$

with a section given by the composite

$$\begin{aligned} S^1 \times S^1 \longrightarrow \Omega\mathbb{P} \times \Omega\mathbb{P} &\xrightarrow{=} \Omega(\mathbb{P} \vee *) \times \Omega(* \vee \mathbb{P}) \\ &\hookrightarrow \Omega(\mathbb{P} \vee \mathbb{P}) \times \Omega(\mathbb{P} \vee \mathbb{P}) \xrightarrow{*} \Omega(\mathbb{P} \vee \mathbb{P}). \end{aligned}$$

It turns out that (2.6) has trivial brace products since $\Omega(\mathbb{P} \vee \mathbb{P})$ is an H-space and Whitehead products vanish in H-spaces.

EXAMPLE 2.7 (Saaidia). Suppose $F \longrightarrow E \longrightarrow B$ is a fibration with section, and F is a G-space with a G-invariant basepoint. Consider the classifying bundle

$$F \longrightarrow EG \times_G F \longrightarrow BG.$$

This fibration also admits a section and its brace products are identified with the so-called 'secondary Eilenberg invariant' of the fibration E (cf. [**11**]). These invariants are fundamental in the study of the homotopy type of the space of sections of E.

Brace products and Samelson products

The commutator map at the level of loop spaces (better known as the Samelson product) is related to the Whitehead product as follows. First write S for the commutator

$$S\colon \Omega(X) \wedge \Omega(X) \longrightarrow \Omega(X), \quad (a, b) \mapsto aba^{-1}b^{-1}.$$

Then the following commutes (up to sign):

$$\begin{array}{ccc} \pi_p(\Omega X) \times \pi_q(\Omega X) & \xrightarrow{\ S\ } & \pi_{p+q}(\Omega X) \\ \Big\downarrow{\scriptstyle \mathrm{ad}\times\mathrm{ad}} & & \Big\downarrow{\scriptstyle \mathrm{ad}} \\ \pi_{p+1}(X) \times \pi_{q+1}(X) & \xrightarrow{[\ ,\ ]} & \pi_{p+q+1}(X) \end{array} \tag{2.8}$$

where ad is the adjoint isomorphism. This fact (originally due to H. Samelson) can be combined with (2.1) to show the following.

LEMMA 2.9. *Let $F \longrightarrow E \longrightarrow B$ be a fibration with section s. There is a homotopy commutative diagram*

$$\begin{array}{ccc} \Omega B \wedge \Omega F & \xrightarrow{\{\ ,\ \}} & \Omega F \\ \Big\downarrow{\scriptstyle \Omega s \wedge \Omega i} & & \Big\downarrow{\scriptstyle \Omega i} \\ \Omega E \wedge \Omega E & \xrightarrow{\ S\ } & \Omega E \end{array}$$

where the upper map (also denoted by $\{\ ,\ \}$) induces James' brace product at the level of homotopy groups.

Proof. The composite

$$\Omega B \wedge \Omega F \xrightarrow{S \circ (\Omega s \wedge \Omega i)} \Omega E$$

is trivial when projected into ΩB (because $\Omega p \circ \Omega i$ is trivial). It then lifts to ΩF as desired. This lift is unique up to homotopy since any two maps differ by a map

$$\Omega B \wedge \Omega F \longrightarrow \Omega^2 B \xrightarrow{\ \partial\ } \Omega F,$$

and this 'boundary' map ∂ is null-homotopic (it is trivial on homotopy groups because of the presence of a section). The rest of the claim follows from (2.8). $\square$

Brace products as obstructions

As pointed out in [**6**], brace products form an obstruction to retracting the total space E into the fiber F. They also represent obstructions to the triviality of certain pull-back fibrations in the Postnikov tower for B (see [**11**]). In what follows we exhibit yet another obstruction expressed in terms of these brace products.

Let $F \xrightarrow{i} E \longrightarrow B$ be a fibration of CW complexes and consider the loop fibration

$$\Omega F \longrightarrow \Omega E \longrightarrow \Omega B. \tag{2.10}$$

Suppose that (2.10) has a section s' and denote by $*$ the loop sum in ΩE. Then the composite

$$\Omega i * s' \colon \Omega F \times \Omega B \xrightarrow{\simeq} \Omega E$$

is a weak homotopy equivalence and hence an equivalence. This trivialization however is not necessarily an H-space map and its failure to be such is measured by the commutator $(\Omega i)s'(\Omega i)^{-1}(s')^{-1}$. We illustrate this with an example.

EXAMPLE 2.11. Consider the Hopf fibering $S^1 \longrightarrow S^3 \longrightarrow S^2$ which can be looped to a fibering

$$\Omega S^3 \longrightarrow \Omega S^2 \longrightarrow S^1.$$

This has an obvious section and as before $S^1 \times \Omega S^3 \xrightarrow{\simeq} \Omega S^2$. Notice that the left-hand side is abelian (since S^3 is a topological group) while the right-hand side ΩS^2 is not. Indeed consider the map $S^1 \longrightarrow \Omega S^2$ and take its self commutator in ΩS^2. This commutator in homotopy is adjoint (by the result of Samelson (2.8)) to the Whitehead product $[\iota_2, \iota_2] = 2\eta \in \pi_3(S^2)$ which is non-zero (here η is the class of the hopf map). Hence ΩS^2 is not abelian and the splitting $\Omega S^2 \simeq S^1 \times \Omega S^3$ is not an H-space splitting.

LEMMA 2.12. *Let $F \longrightarrow E \longrightarrow B$ be a fibration with section s. If the brace products in this fibration vanish identically, then*

$$\theta = \Omega s * \Omega i \colon \Omega B \times \Omega F \xrightarrow{\simeq} \Omega E$$

is an H-space splitting.

Proof. Here $s' = \Omega s$ and Ωi are naturally H-space maps and we need only check that the following diagram homotopy commutes:

$$\begin{tikzcd}
(\Omega B \times \Omega F)^2 \arrow[r, "1\times\chi\times 1"] \arrow[d, "\theta^2"] & (\Omega B)^2 \times (\Omega F)^2 \arrow[r, "*\times *"] & \Omega B \times \Omega F \arrow[d, "\theta"] \\
(\Omega E)^2 \arrow[rr, "*"] & & \Omega E
\end{tikzcd}$$

where $1\times\chi\times 1$ is the shuffle map $(x, a, b, y) \mapsto (x, b, a, y)$. Now the images of Ωs and Ωi commute in ΩE (this follows from Lemma 2.9 and from the fact that the brace products vanish). The claim follows immediately. $\square$

3. *Whitehead's theorem and the proof of Theorem* 1.1

In this section we prove Theorems 1.3 and 1.1 of the introduction. Denote by D^n the closed unit disc in $\mathbb{R}^n$ and by $\partial D^n = S^{n-1}$ its boundary. If $D^n = D^p \times D^q$, we can then write $S^{n-1} = \partial D^n = D^p \times \partial D^q \cup \partial D^p \times D^q$ (where the union is over $\partial D^p \times \partial D^q$). Let $\mathscr{L}^q X = \mathrm{Map}(S^q, X)$ be the space of all maps from S^q to X. We have the following pivotal lemma.

LEMMA 3.1 [**13**, Lemma 3.3]. *Start with a map*

$$\phi\colon S^{p-1} \wedge S^q \longrightarrow X$$

and adjoin it to get $g\colon S^{p-1} \longrightarrow \Omega^q_\alpha X$ *(where* $\Omega^q_\alpha X$ *is some component of* $\Omega^q X$ *containing a representative map* α*). Suppose that* g *extends to a map* $D^p \longrightarrow \mathscr{L}^q_\alpha X$ *and hence gives rise to an element* $\beta \in \pi_p(\mathscr{L}^q_\alpha X, \Omega^q_\alpha X) \cong \pi_p(X)$*. Then*

$$\phi = [\alpha, \beta] \in \pi_{p+q-1}(X).$$

An alternative formulation of this lemma that is better suited to us is as follows.

LEMMA 3.2. *Let* E *be a space and think of* ΩE *as the fiber of* $\Omega^q E \longrightarrow \mathscr{L}^q E$*. Given a composite*

$$\phi\colon S^{p-1} \xrightarrow{\ \beta\ } \Omega E \longrightarrow \Omega^q_\alpha E$$

then necessarily $\mathrm{ad}\,\phi = [\alpha, \mathrm{ad}\,\beta] \in \pi_{p+q-1}E$*.*

Proof. The evaluation fibration in (1.2) extends to the left (by looping) and we get the fibration $\Omega E \longrightarrow \Omega^q_\alpha E \longrightarrow \mathscr{L}^q_\alpha E$. The fact that the map $\phi\colon S^{p-1} \longrightarrow \Omega^q_\alpha E$ factors via β through the fiber ΩE is the same as having an extension diagram

$$\begin{array}{ccc} S^{p-1} & \xrightarrow{\ \phi\ } & \Omega^q_\alpha E \\ \downarrow & & \downarrow \\ D^p & \longrightarrow & \mathscr{L}^q_\alpha E \end{array}$$

such that the element of $\pi_p(\mathscr{L}^q_\alpha E, \Omega^q_\alpha E) \cong \pi_p(E)$ defined by this diagram is the class of $\mathrm{ad}\,\beta$. It follows from Lemma 3.1 that $\mathrm{ad}\,\phi = [\alpha, \mathrm{ad}\,\beta]$. □

THEOREM 3.3 [**13**]. *The homotopy boundary*

$$\partial\colon \pi_p(X) \longrightarrow \pi_{p-1}(\Omega^n_f(X)) = \pi_{p+n-1}(X),$$

in the long exact sequence in homotopy associated to

$$\Omega^q_f(X) \longrightarrow \mathscr{L}^q_f(X) \xrightarrow{\ \mathrm{ev}\ } X,$$

is given (up to sign) by the Whitehead product as follows: let $\alpha \in \pi_p(X)$*; then* $\partial\alpha = \mathrm{ad}[\alpha, f] \in \pi_{p-1}(\Omega^q_f X)$*.*

Proof. A fibration $F \longrightarrow E \longrightarrow B$ extends to the left by $\Omega B \longrightarrow F$, and the boundary homomorphism is given by the induced map in homotopy

$$\pi_p(B) = \pi_{p-1}(\Omega B) \xrightarrow{\partial} \pi_{p-1}(F).$$

Representing $\alpha \in \pi_p(B)$ by the map of the same name, we see that the following commutes:

$$\begin{tikzcd} S^{p-1} \arrow[r, "\mathrm{ad}(\alpha)"] \arrow[d, "="] & \Omega B \arrow[d] \\ S^{p-1} \arrow[r, "\partial\alpha"] & F \end{tikzcd}$$

Letting $B = X$, $F = \Omega_f^q X$ and $E = \mathscr{L}_f^q(X)$, we deduce from Lemma 3.2 that $\mathrm{ad}^{-1}(\partial\alpha) = [\alpha, f]$ and the claim follows. □

We need one more lemma before we can proceed with the proof of Theorem 1.1. Let

$$\zeta\colon F \longrightarrow E \longrightarrow S^n$$

be a fibration with section s, and let $\mu\colon S^{n-1} \longrightarrow \mathrm{Aut}(F)$ be the clutching function. Here $\mathrm{Aut}(F)$ consists of based homotopy equivalences and we denote by $\mathrm{Map}^*(F, E)$ the space of based maps from F into E. There are inclusions

$$\mathrm{Aut}(F) \hookrightarrow \mathrm{Map}^*(F, E) \hookrightarrow \mathrm{Map}(F, E)$$

and we assert that the following lemma holds.

LEMMA 3.4. *There is an extension diagram*

$$\begin{tikzcd} S^{n-1} \arrow[r, "\mu"] \arrow[d] & \mathrm{Aut}(F) \arrow[r, hook] & \mathrm{Map}^*(F, E) \arrow[d] \\ D^n \arrow[rr] & & \mathrm{Map}(F, E) \end{tikzcd}$$

such that the element $\beta \in \pi_n(\mathrm{Map}(F, E), \mathrm{Map}^*(F, E)) \cong \pi_n(E)$ defined by the diagram corresponds to the class of $s\colon S^n \longrightarrow E$.

Proof. We have the following sequence of fibrations:

$$F \longrightarrow E \longrightarrow S^n \longrightarrow B\,\mathrm{Aut}(F)$$

and the last map classifies the fibration ζ. By looping and letting $S^{n-1} \longrightarrow \Omega S^n$ be the adjoint to the identity map, we get the following diagram:

$$\begin{tikzcd} & \Omega E \arrow[d] & \\ S^{n-1} \arrow[r] & \Omega S^n \arrow[r] & \mathrm{Aut}(F) \end{tikzcd}$$

The lower composite, which we label θ, can be identified with the clutching map

μ. If one has a section $\Omega s\colon \Omega S^n \longrightarrow \Omega E$, then θ factors through ΩE which is the fiber of the mapping $\mathrm{Map}^*(F, E) \longrightarrow \mathrm{Map}(F, E)$. The lemma follows. $\square$

THEOREM 3.5. *There is a commutative diagram*

$$\begin{tikzcd}
\pi_p(B) \otimes \pi_q(F) \arrow[r, "\{\ ,\ \}"] \arrow[d, "h \otimes h"] & \pi_{p+q-1}(F) \arrow[d, "h"] \\
H_p(B, H_q(F)) \arrow[d, "\cong"] & H_{p+q-1}(F) \arrow[d, "\cong"] \\
E^2_{p,q} \arrow[r, "d^p"] & E^2_{0,p+q-1}
\end{tikzcd}$$

where by abuse of notation the bottom map is the long differential d^p defined on those spherical classes that survive to $E^p_{p,q}$.

REMARK 3.6. We first explain why Theorem 3.5 is independent of the choice of section. Suppose $F \xrightarrow{i} E \longrightarrow B$ is as above and assume it has two distinct sections s_1 and s_2. Let $\alpha \in \pi_p(B)$ and $\beta \in \pi_q(F)$. The brace products associated to s_1 and s_2 are given by $\{\alpha, \beta\}_1$ and $\{\alpha, \beta\}_2$ (respectively). Notice that $s_1(\alpha) - s_2(\alpha)$ projects to zero in $\pi_*(B)$ and hence must lift to a class $\alpha_F \in \pi_p(F)$. The difference element $\{\alpha, \beta\}_1 - \{\alpha, \beta\}_2$ is by definition the lift to $\pi_*(F)$ of

$$[s_1(\alpha) - s_2(\alpha), i_*(\beta)] = [i_*(\alpha_F), i_*(\beta)] = i_*[\alpha_F, \beta] \in \pi_*(E).$$

It follows that $\{\alpha, \beta\}_1 - \{\alpha, \beta\}_2 = [\alpha_F, \beta] \in \pi_{p+q-1}(F)$. This Whitehead product in F necessarily maps to zero in $H_*(F)$ by the Hurewicz homomorphism and this is enough to show that the composite $h \circ \{\ ,\ \}$ in the top half of the diagram in Theorem 3.5 is independent of the choice of section as asserted.

Proof of Theorem 3.5. Let $\alpha\colon S^p \longrightarrow B$ represent a class in $\pi_p(B)$. Consider the pullback diagram

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & F \arrow[d] \\
E' \arrow[r] \arrow[d] & E \arrow[d] \\
S^p \arrow[r, "\alpha"] & B
\end{tikzcd}$$

By naturality of the Serre spectral sequence it suffices to prove the theorem for the pull back fibration $F \longrightarrow E' \longrightarrow S^p$. In other words we must prove that the following diagram commutes:

$$\begin{tikzcd}
\pi_p(S^p)\otimes\pi_q(F) \arrow[r, "\{\ ,\ \}"] \arrow[d, "h\otimes h"] & \pi_{p+q-1}(F) \arrow[d, "h"] \\
H_p(S^p, H_q(F)) \arrow[d, "\cong"] & H_{p+q-1}(F) \arrow[d, "\cong"] \\
E^2_{p,q} \arrow[d, "\cong"] & E^2_{0,\,p+q-1} \arrow[d, "\cong"] \\
E^p_{p,q} \arrow[r, "d^p"] & E^p_{0,\,p+q-1}
\end{tikzcd}$$

Now associated to $F \longrightarrow E' \longrightarrow S^p$ is a Wang sequence

$$\dots \longrightarrow H_i(F) \longrightarrow H_i(E') \longrightarrow H_{i-p}(F) \xrightarrow{\tau_*} H_{i-1}(F) \longrightarrow \dots$$

where τ_* is determined in terms of the clutching function of the bundle. Recall that this clutching function is given by a map

$$\mu: S^{p-1}\times F \longrightarrow F$$

whose homotopy class determines the bundle (up to fiber homotopy). Identifying $H_{i-p}(F)$ with $E^2_{p,i-p}$ and $H_{i-1}(F)$ with $E^2_{0,i-1}$, it is not hard to see that $\tau_* = d^p\colon E^2_{p,i-p} \longrightarrow E^2_{0,i-1}$ (see [**13**, p. 332]).

Choose a basepoint $p\in F$. Given $\beta\colon S^q \longrightarrow F$ representing a spherical class (of the same name) in $H_q(F)$; then τ_* can be made explicit as follows. We first have an isomorphism $H_q(F)\cong H_{p+q}(S^p\wedge F)$ and the class β is represented under this isomorphism by a map $S^p\wedge S^q \longrightarrow S^p\wedge F$. Writing

$$D^{p+q} = D^p\times D^q \quad\text{and}\quad \partial D^{p+q} = (D^p\times\partial D^q)\cup(\partial D^p\times D^q),$$

we can represent β as a map of pairs

$$(D^{p+q}, \partial D^{p+q}) \longrightarrow (D^p\times F, D^p\times p\cup\partial D^p\times F).$$

The map on the second component is the boundary map ∂ and it can be prolonged into F,

$$\tau\colon \partial D^{p+q} \xrightarrow{\partial} D^p\times p\cup\partial D^p\times F \longrightarrow F, \tag{3.7}$$

by collapsing $D^p\times p$ to $p\in F$ and sending $\partial D^p\times F = S^{p-1}\times F$ to F via the clutching function μ. (This is possible since $\mu(\partial D^p\times p) = p\in F$.) The composite in (3.7) is a map $S^{p+q-1}\longrightarrow F$ whose Hurewicz image gives a class in $H_{p+q-1}(F)$. This class is exactly $\tau_*(\beta) = d^p(\beta)$.

Note at this point that the map τ gives rise, by restriction, to a map

$$\begin{tikzcd}
\partial D^p\times D^q \arrow[d] & & & \\
S^{p-1}\wedge S^q \arrow[r] & S^{p-1}\wedge F \arrow[r, "\mu"] & F \arrow[r, "i"] & E
\end{tikzcd}$$

The horizontal composite adjoins to a map $\theta\colon S^{p-1} \longrightarrow \Omega^q E$ and the component in which it lies contains the map $\beta\colon S^q \longrightarrow F \longrightarrow E$. By precomposing and using Lemma 3.4, one gets the following extension diagram:

$$\begin{array}{ccccc} S^{p-1} & \longrightarrow & \mathrm{Aut}(F) & \longrightarrow & \Omega^q E \\ \downarrow & & \downarrow & & \downarrow \\ D^p & \longrightarrow & \mathrm{Map}(F, E) & \longrightarrow & \mathscr{L}^q E \end{array}$$

The homotopy class this defines is given by (Lemma 3.4)

$$s(S^p) \in \pi_p(E) \cong \pi_p(\mathscr{L}^q_\beta E, \Omega^q_\beta E).$$

One can now apply Lemma 3.1 directly to obtain

$$i \circ \tau = [s(\alpha), i(\beta)] \quad \text{in } \pi_{p+q-1}(E).$$

Both maps lift to F; the left-hand side lifts to τ and the right-hand side lifts to $\{\alpha, \beta\}\colon S^{p+q-1} \longrightarrow F$. Notice that in homology, the Hurewicz images of $i_* \circ \tau_*$ and $[s(\alpha), i(\beta)]_*$ are zero in $H_{p+q-1}(E)$ (in the first case because of the Wang exact sequence and in the second because of a known property of Whitehead products). It follows by the Wang exact sequence again that the class in the image of $h \circ \{\alpha, \beta\}_*$ in $H_{p+q-1}(F)$ is also in the image of τ_* and by the arguments above it must follow that it is exactly $\tau_*(\beta)$. The theorem follows. □

4. *Spaces of free loops*

As pointed out in the introduction, the previous results apply particularly well to (basepoint-free) mapping spaces from spheres. Consider again the evaluation fibration

$$(4.1) \qquad \Omega^k X \xrightarrow{i} \mathscr{L}^k X \xrightarrow{\mathrm{ev}} X.$$

When the connectivity of X is at least k, (4.1) admits a section (which sends a point in X to the constant loop at that point). Below we use the same name to refer to a spherical class and the homotopy class it comes from. With

$$\rho_k\colon \pi_*(X) \xrightarrow{\mathrm{ad}_k} \pi_{*-k}(\Omega^k X) \xrightarrow{h} H_{*-k}(\Omega^k X)$$

as in the introduction, we prove the following.

THEOREM 4.2. *Let X be k-connected, and let $\beta \in H_j(\Omega^k(X))$ and $\alpha \in H_p(X)$ be two spherical classes. Then in the homology Serre spectral sequence for $\Omega^k X \longrightarrow \mathscr{L}^k X \xrightarrow{\mathrm{ev}} X$, the following identity holds:*

$$d^p(\alpha \otimes \beta) = \lambda_k(\rho_k(\alpha), \beta).$$

Proof. Suppose M is k-connected. Then the evaluation fibration admits a section and the following diagram commutes:

$$\begin{array}{ccc}
\pi_p(M)\otimes\pi_{j+k}(M) & \xrightarrow{[\ ,\ ]} & \pi_{p+j+k-1}(M) \\
\big\downarrow 1\otimes \mathrm{ad}_k & & \big\downarrow \mathrm{ad}_k \\
\pi_p(M)\otimes\pi_j(\Omega^k M) & \xrightarrow{\{\ ,\ \}} & \pi_{p+j-1}(\Omega^k M) \\
\big\downarrow h\otimes h & & \big\downarrow h \\
H_p(M)\otimes H_j(\Omega^k M) & \xrightarrow{d^p} & H_{p+j-1}(\Omega^k M)
\end{array}$$

where d^p is again as described in Theorem 3.5. The bottom half commutes because of Theorem 3.5, while the top half commutes as a result of a theorem of Hansen [**5**]. Notice that the right vertical composite is just ρ_k.

Next we look at the following diagram of Fred Cohen [**3**, p. 215]:

$$\begin{array}{ccc}
\pi_p(M)\otimes\pi_{j+k}(M) & \xrightarrow{[\ ,\ ]} & \pi_{p+j+k-1}(M) \\
\big\downarrow \mathrm{ad}_k\otimes \mathrm{ad}_k & & \big\downarrow \mathrm{ad}_k \\
\pi_{p-k}(\Omega^k M)\otimes\pi_j(\Omega^k M) & & \pi_{p+j-1}(\Omega^k M) \\
\big\downarrow h\otimes h & & \big\downarrow h \\
H_{p-k}(\Omega^k M)\otimes H_j(\Omega^k M) & \xrightarrow{\lambda_k} & H_{p+j-1}(\Omega^k M)
\end{array}$$

This diagram defines the Browder operations for spherical classes and the proof follows by direct comparison of the above two diagrams. □

Remark 4.3. When $k=1$ and X is a suspension, the Browder operation can be described in terms of *commutators* and of the Samelson map

$$\rho\colon\ \pi_*(X)\xrightarrow{\mathrm{ad}}\pi_{*-1}(\Omega X)\xrightarrow{h} H_{*-1}(\Omega X).$$

Let $X=S^n$ (or any suspension will do); then according to [**12**] the image of a Whitehead product under ρ is a commutator in $H_*(\Omega S^n)=H_*(\Omega\Sigma S^{n-1})=T[e]$, where $T[e]$ is a polynomial algebra on one generator e of dimension $n-1$; that is,

$$\rho([x,y])=\rho x*\rho y-(-1)^{p\cdot q}\rho x*\rho y:=[\rho x,\rho y].$$

It then follows from Theorem 4.2 that

$$d(\iota\otimes y)=[\rho(\iota),y],\quad \text{where } \rho(\iota)\in H_{n-1}(\Omega S^n),\ y\in H_j(\Omega S^n)$$

(here y is spherical of course). We see, for instance, that $d(\iota\otimes\rho(\iota))=[\rho(\iota),\rho(\iota)]=0$ if n is odd, and $d(\iota\otimes\rho(\iota))=2$ if n is even. This last fact generalizes to higher free loop spaces.

4.1. *Free loop spaces of spheres* $\mathscr{L}^k S^n$ *for* $1\leqslant k<n$

When $n=1$, 3 or 7, $\mathscr{L}^k S^n$ is an H-space (since S^n is) and so the existence of a section yields a space level splitting for these values of n. Generally and for n

odd, the localised sphere $S^n_{(p)}$ at an odd prime becomes an H-space and hence so is $\mathscr{L}^k S^n_{(p)}$. We therefore have a space level splitting for odd n and, after inverting, for $n=2$. The Serre spectral sequence for (4.1) collapses for odd spheres with $\mathbb{Z}_p$ coefficients (p odd). The case that will occupy us most in this section is that when n is even.

LEMMA 4.4. *Assume $1 \leqslant k < n$. Then under the composite*

$$\rho\colon \pi_{2n-1}S^n \xrightarrow{\text{ad}} \pi_{2n-k-1}(\Omega^k S^n) \xrightarrow{h} H_{2n-k-1}(\Omega^k S^n),$$

the Whitehead square maps as follows:

$$\rho([\iota_n, \iota_n]) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 2x & \text{if } n \text{ is even} \end{cases}$$

(*here x is the infinite cyclic element in $H_{2n-k-1}(\Omega^k S^n; A)$ with n even*).

Proof (sketch for $n = 2q$). Write $\beta_n = [\iota_n, \iota_n]$ and let x be the generator of $H_{n-k}(\Omega^k S^n)$. Then $\rho(\beta_n) = \lambda_k(x, x)$ according to Theorem 4.2. When $n = 2q$, β_{2q} generates an infinite cyclic group in $\pi_{4q-1}(S^{2q}) \cong \mathbb{Z} \oplus \text{torsion}$. It is well known (Serre) that loops on an even sphere split after localizing at any odd prime p,

$$\Omega^k S^{2q} \simeq_{(p)} \Omega^{k-1} S^{2q-1} \times \Omega^k S^{4q-1}.$$

Under this correspondence, it turns out that β_{2q} maps under ρ to the generator in $H_{4q-k-1}(\Omega^k S^{4q-1})$ (mod p). Moreover it is known that $\lambda_k(x, x) = 0$ (mod 2) (cf. [**3**]). Putting these together yields the result. $\square$

The following is Corollary 1.6 of the introduction.

COROLLARY 4.5. *Assume that $1 \leqslant k < n$ and n is even. Then in the Serre spectral sequence for the fibration*

$$\Omega^k S^n \xrightarrow{i} \mathscr{L}^k S^n \xrightarrow{\text{ev}} S^n,$$

the differential $d^n_{n,n-k}$ is given by multiplication by 2 on the torsion-free generator of $H_{2n-k-1}(\Omega^k S^n)$. In particular, $d^n_{n,n-k}$ is an isomorphism with rational coefficients.

Proof. The differential $d^n_{n,n-k}$ is determined according to the first diagram in the proof of Theorem 4.2 by the image of the Whitehead square under the map ρ described in Lemma 4.4. The claim now follows from Lemma 4.4. $\square$

4.2. *Rational and* mod-2 *calculations*

The mod-2 cohomology of $\mathscr{L}^k S^n$, with $k < n$, is completely determined according to the following lemma.

LEMMA 4.6. *The Serre spectral sequence for $\Omega^k S^n \longrightarrow \mathscr{L}^k S^n \longrightarrow S^n$ collapses with* mod-2 *coefficients whenever $k < n$.*

Proof (Fred Cohen). Consider the suspension

$$\Omega^n E\colon \Omega^n S^{n+q} \longrightarrow \Omega^{n+1} S^{n+q+1}$$

and the following induced map of fibrations:

$$\begin{tikzcd}
\Omega^n S^{n+q} \arrow[r, "\Omega^n E"] \arrow[d] & \Omega^{n+1} S^{n+q+1} \arrow[d] \\
\mathscr{L}^n S^{n+q} \arrow[r, "\mathscr{L}^n E"] \arrow[d] & \mathscr{L}^n \Omega S^{n+q+1} \arrow[d] \\
S^{n+q} \arrow[r, "E"] & \Omega S^{n+q+1}
\end{tikzcd}$$

Since ΩS^{n+q+1} is an H-space, then so is $\mathscr{L}^n \Omega S^{n+q+1}$ and consequently we have a splitting

$$\mathscr{L}^n \Omega S^{n+q+1} \simeq \Omega S^{n+q+1} \times \Omega^{n+1} S^{n+q+1}.$$

It is known (cf. [**3**, pp. 228–231]) that the map $\Omega^i E$ is injective in mod-2 homology (for all i) and hence in the diagram above both fiber and base inject in $\mathbb{Z}_2$-homology. The lemma follows. $\square$

Remark. In Proposition 5.3 below, we give an alternative derivation of this fact in the case $k = 1$.

We now use Corollary 4.5 to calculate $H^*(\mathscr{L}^k S^n)$ with rational coefficients. We also give a complete answer mod-p (p odd) for the case of a two-fold loop space. We make use throughout of the following standard fact. Consider the path-loop fibration $\Omega^k S^n \longrightarrow P \longrightarrow \Omega^{k-1} S^n$ for $k < n$. Then

$$H^*(\Omega^k S^n) = \mathrm{Tor}^{H^*(\Omega^{k-1} S^n)}(\mathbb{F}, \mathbb{F}). \tag{4.7}$$

This follows because the Eilenberg–Moore spectral sequence collapses at the E^2 term (cf. [**2**]).

Proposition 4.8. *Let* $1 \leqslant k < n$ *and suppose* n *is even. Then the Poincaré series for* $H^*(\mathscr{L}^k S^n; \mathbb{Q})$ *is given as follows:*

$$\begin{cases} 1 + (x^n + x^{n-k})/(1 - x^{2n-k-1}) & \text{if } k \text{ is odd,} \\ (1 + x^{3n-k-1})/(1 - x^{n-k}) & \text{if } k \text{ is even.} \end{cases}$$

Proof. When n is even, one has $H^*(\Omega S^n) = E(e_{n-1}) \otimes \mathbb{Q}(a_{2n-2})$, where $E(e_{n-1})$ is an exterior algebra on an $(n-1)$-dimensional generator. It then follows that

$$\mathrm{Tor}^{E(e_{n-1})}(\mathbb{Q}, \mathbb{Q}) = \mathbb{Q}(e_{n-2}) \quad \text{and} \quad \mathrm{Tor}^{\mathbb{Q}(a_{2n-2})}(\mathbb{Q}, \mathbb{Q}) = E(a_{2n-3}).$$

Iterating these constructions and using (4.7) yields

$$H^*(\Omega^k S^n; \mathbb{Q}) = \begin{cases} \mathbb{Q}(e) \otimes E(a) & \text{for } k \text{ even,} \\ E(e) \otimes \mathbb{Q}(a) & \text{for } k \text{ odd,} \end{cases}$$

where $\deg(e) = n - k$ and $\deg(a) = 2n - k - 1$. Let $\iota \in H_n(S^n)$ be the generator. Then in the Serre spectral sequence for (4.1) with $\mathbb{Q}$ coefficients, the class a hits $e\iota$ and this differential generates all other differentials. When k is odd, one has (up to a unit)

$$d(a^k) = e\iota a^{k-1}, \qquad d(ea^k) = e^2\iota a^{k-1} = 0.$$

The classes that survive are 1, ea^k and τa^k for $k > 0$. This establishes the first claim. When k is even, $H^*(\mathscr{L}^k S^n; \mathbb{Q}) \cong \mathbb{Q}(e)[1, a\iota]$ and this leads to the second assertion. □

Remark. The Poincaré series for $\mathscr{L}S^n$, with n even,

$$(1 + x^n + x^{n-1} - x^{2n-2})/(1 - x^{2n-2}),$$

is well known and is given, for instance, in [**10**].

4.3. *Second fold* (*free*) *loop spaces*

We now determine $H_*(\mathscr{L}^2 S^{2q+2}; \mathbb{F}_p)$ with p odd (the case $p = 2$ having been settled in Lemma 4.6). So recall the description of $\Omega^2 S^{2q+2}$ over the mod-p Steenrod algebra (see [**3**] or [**9**] for a general discussion). We have

$$\Omega^2 S^{2q+2} \simeq_p \Omega S^{2q+1} \times \Omega^2 S^{4q+3}$$

(see the proof of Lemma 4.4), and $H^*(\Omega^2 S^{4q+3})$ is given by

$$H^*(\Omega^2 S^{4q+3}) = E(x_0, x_1, \ldots) \otimes \Gamma(y_1, y_2, \ldots)$$

where $|x_i| = 2(2q+1)p^i - 1$ and $|y_i| = 2(2q+1)p^i - 2$. The action of the Steenrod algebra is given by

$$\beta(y_i) = x_i \quad \text{and} \quad \mathscr{P}^1(y_i^p) = y_{i+1}.$$

Proposition 4.9. *In the* mod-p *cohomology Serre spectral sequence for*

$$\Omega^2 S^{2q+2} \longrightarrow \mathscr{L}^2 S^{2q+2} \longrightarrow S^{2q+2}$$

we have

$$d_{4q+1} x_0 = e \cdot \iota,$$

where e is the generator of $H^{2q}(\Omega^2 S^{2q+2})$ in the fiber and ι is the generator of $H^{2q+2}(S^{2q+2})$ in the base.

Proof. The differential d_{4q+1} is described by Corollary 4.5 and is non-trivial. It can be shown that this is in fact the only non-trivial differential in the spectral sequence. □

5. *The free loop space of a bouquet of spheres*

In this section we illustrate our techniques by calculating the homology (with field coefficients) of $\mathscr{L}(\bigvee_i S^{n_i+1})$ of a finite bouquet of spheres, with $n_i > 0$.

Write $W = \bigvee_i^k S^{n_i+1}$ and consider the free loop fibration

$$\Omega W \longrightarrow \mathscr{L} W \longrightarrow W. \tag{5.1}$$

The image of the orientation class $[S^{n_i+1}]$ in $H_{n_i+1}(W)$ will be denoted by a_i and the inclusion $S^{n_i+1} \hookrightarrow W$ by ι_i. To the a_i correspond by adjointness the $e_i \in H_{n_i}(\Omega W)$. Observe that $W = \Sigma(\bigvee_i^k S^{n_i})$ and so, as is well known (Bott–Samelson),

$$H_*(W) = T(e_1, \ldots, e_k),$$

where $T(e_1, \ldots, e_k)$ is the tensor algebra on the generators e_i. An element $x \in T(e_1, \ldots, e_k)$ is a sum of basic monomials $e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_r}$. Note that x is not spherical in general; however iterated commutators in the e_i are.

LEMMA 5.2 (Samelson). *Under the Samelson map*

$$\rho\colon \pi_*(W) \xrightarrow{\text{ad}} \pi_{*-1}(\Omega W) \xrightarrow{h} H_{*-1}(\Omega W)$$

(*see Remark* 4.3), *the iterated commutator* $[e_{i_1}, [e_{i_2}, [\ldots [e_{i_{r-1}}, e_{i_r}] \ldots]]]$ *is in the image of the iterated Whitehead product* $[\iota_{i_1}, [\iota_{i_2}, [\ldots [\iota_{i_{r-1}}, \iota_{i_r}] \ldots]]]$.

This result is also quoted in Remark 4.3. We are now in a position to make explicit the structure of the differentials in the Serre spectral sequence for (5.1).

PROPOSITION 5.3. *In the Serre spectral sequence for* (5.1), *the differentials are given by*

$$d_{n_i+1}(a_i \otimes x) = [e_i, x] = e_i \otimes x - (-1)^{(n_i)|x|} x \otimes e_i$$

where again $e_i = \rho(a_i)$ *in* $H_{n_i}(\Omega W)$ *and* $x \in H_*(\Omega W)$.

Proof. The result is true for x spherical according to Remark 4.3. Suppose now that $x = e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_r}$ and consider the iterated commutator $[e_{i_1}, [e_{i_2}, [\ldots [e_{i_{r-1}}, e_{i_r}] \ldots]]]$. This being spherical, we get

$$d(a \otimes [e_{i_1}, [e_{i_2}, [\ldots [e_{i_{r-1}}, e_{i_r}] \ldots]]]) = [e, [e_{i_1}, [e_{i_2}, [\ldots [e_{i_{r-1}}, e_{i_r}] \ldots]]]]$$

where $e = \rho(a)$. Writing

$$[e_{i_1}, [e_{i_2}, [\ldots [e_{i_{r-1}}, e_{i_r}] \ldots]]] = \sum_\tau \pm e_{i_{\tau(1)}} e_{i_{\tau(2)}} \ldots e_{i_{\tau(r)}}$$

where τ ranges over the appropriate permutations of $\{1, \ldots, r\}$, we can rewrite this expression as

$$\sum_\tau d(a \otimes e_{i_{\tau(1)}} e_{i_{\tau(2)}} \ldots e_{i_{\tau(r)}}) = \sum_\tau [e, e_{i_{\tau(1)}} e_{i_{\tau(2)}} \ldots e_{i_{\tau(r)}}].$$

Of course we want to show that the above summands correspond. This is essentially forced on us by the symmetry of the situation. We give the detailed argument for the case $r = 2$ (the general case being the same but with more complicated notation). So when $r = 2$,

$$d(a, [e_1, e_2]) = [e, e_1 e_2 - e_2 e_1] = e e_1 e_2 - e e_2 e_1 - e_1 e_2 e + e_2 e_1 e$$

where, to ease notation, we choose $|e_1 e_2|$ to be even and $|a|$ to be odd to get the appropriate signs. We stipulate $e_i \neq e_j$ for $i \neq j$. We know that $d(a, [e_1, e_2]) = d(a, e_1 e_2) - d(a, e_2 e_1)$, and hence one of six things must happen:

(i) $d(a, e_1 e_2) = e e_1 e_2 - e e_1 e_2$ and $d(a, e_2 e_1) = e_1 e_2 e - e_2 e_1 e$;

(ii) $d(a, e_1e_2) = ee_1e_2 + e_2e_1e$ and $d(a, e_2e_1) = ee_2e_1 + e_1e_2e$;

(iii) $d(a, e_1e_2) = ee_1e_2 - e_1e_2e = [e, e_1e_2]$ and $d(a, e_2e_1) = ee_2e_1 - e_2e_1e = [e, e_2e_1]$.

The other three choices are either redundant or easily ruled out. Of course we need to rule out (i) and (ii) to obtain (iii) for the answer.

To do this we notice generally that if τ is a permutation on k letters, we can consider the bouquet $W' = \bigvee^k S^{n_{\tau(i)}+1}$ and the (obvious) 'permutation' map $W \longrightarrow W'$. We get an induced loop map $\Omega W \longrightarrow \Omega W'$ and in turn a homology map (which we also denote by τ)

$$\tau\colon T(e_1, \dots, e_k) \mapsto T(e_1', \dots, e_k') = T(e_{\tau(1)}, \dots, e_{\tau(k)});$$

here we have written $e_{\tau(i)}$ for $e_i' = \tau(e_i)$. This map is multiplicative and induces a map of spectral sequences (also written τ). From this we deduce that

$$d(a_{\tau(i)}, e_{\tau(1)}e_{\tau(2)} \dots e_{\tau(r)}) = \tau(d(a_i, e_1e_2 \dots e_r)) \tag{5.4}$$

where by definition $\tau(e_1 \dots e_r) = e_{\tau(1)}e_{\tau(2)} \dots e_{\tau(r)}$. Suppose we are in case (i) and let τ be the transposition permuting 1 and 2 (and leaving other indexes fixed). We then see that $\tau d(a, e_1e_2) = \tau(ee_1e_2 - ee_1e_2) = e'e_2e_1 - e'e_2e_1$. However $d(a, e_2e_1) = d(a, e_{\tau(1)\tau(2)}) = e_1e_2e' - e_2e_1e' \neq \tau d(a, e_1e_2)$. Case (i) cannot happen.

Similarly for case (ii) the same argument as above with $e = e_1$ yields $\tau d(a_1, e_1e_2) = \tau(e_1e_1e_2 + e_2e_1e_1) = e_2e_2e_1 + e_1e_2e_2 \neq d(e_2, e_1e_2)$, implying that (ii) cannot happen either. Case (iii) is the only case that satisfies (5.4), as is easily checked, and the proposition follows for $r = 2$. The general case $r \geqslant 2$ is totally analogous. □

With this description available to us, we can proceed with the calculation of $H_*(\mathscr{L}W)$. The following discussion is valid with any field coefficients $\mathbb{F}$. Write $W = \Sigma X$ and let $V = \widetilde{H}_*(X; \mathbb{F})$. The tensor algebra on V corresponds to $T(V) = T(e_1, \dots, e_k)$. Consider the operator

$$\tau_m\colon V^{\otimes m} \longrightarrow V^{\otimes m},$$

$$e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_m} \mapsto (-1)^{n_{i_1}(n_{i_2}+\dots+n_{i_m})} e_{i_2} \otimes \dots \otimes e_{i_m} \otimes e_{i_1}.$$

The operator τ_m gives an action of the cyclic group $\mathbb{Z}_m$ on $V^{\otimes m}$ and we denote by $V^{(\tau_m)}$ the subspace invariant under this action. Proposition 5.3 then shows that

$$H_*(\mathscr{L}W; \mathbb{F}) \cong \bigoplus_{n \geqslant 0} V^{\otimes n} / _{\mathrm{Im}(1-\tau_n)} \oplus \bigoplus_{n \geqslant 1} \Sigma(V^{(\tau_n)}) \tag{5.5}$$

where the last term is the suspension of $V^{(\tau_n)}$ with degree 1 (compare [**4**]). Clearly $\mathrm{Coker}(1 - \tau_m) \subset H_*(\mathscr{L}W)$ and the kernel of $1 - \tau_m$ is a copy of $\ker(1 - \tau_m)$ suspended one dimension higher. Since $\dim \mathrm{Coker}(1 - \tau_m) = \dim(\ker(1 - \tau_m))$, it follows that

$$P(H_*(\mathscr{L}X)) = 1 + (1 + z)P\left(\bigoplus_{m \geqslant 1} \ker(1 - \tau_m)\right), \tag{5.6}$$

where P is the mod-$\mathbb{F}$ Poincaré series. In what follows we determine $P(\bigoplus_{m \geqslant 1} \mathrm{Ker}(1 - \tau_m))$ for $\mathbb{F} = \mathbb{Q}$ and $\mathbb{Z}_2$.

DEFINITIONS AND NOTATION. (i) We denote by τ the cyclic operator

$$\tau(e_{i_1}\otimes e_{i_2}\otimes\ldots\otimes\ldots e_{i_m}) = e_{i_2}\otimes\ldots\otimes e_{i_m}\otimes e_{i_1},$$

and by τ^d its iterate d times. It is extended to operate additively on all of $V^{\otimes m}$. Note that

$$\tau_m = \begin{cases} \tau & \text{if } n_{i_1} \text{ or } n_{i_2}+\ldots+n_{i_m} \text{ is even,} \\ -\tau & \text{if } n_{i_1} \text{ and } n_{i_2}+\ldots+n_{i_m} \text{ is odd.}\end{cases} \tag{5.7}$$

(ii) A word $x = e_{i_1}\otimes e_{i_2}\otimes\ldots\otimes e_{i_m}\in V^{\otimes m}$ has period d if $\tau^d(x) = x$ and $\tau^i(x)\neq x$ for $i<d$. Such a word must be presented in the form of blocks each of length d and hence necessarily $d\mid m$. For example, $e_1e_2e_3e_1e_2e_3$ for $e_i\neq e_j$ has period $d=3$ (and $m=6$ in this case).

The 'trick' of Roos. Given a word x of period d, consider the element

$$\bar{x} = x+\tau x+\tau^2x+\ldots+\tau^{d-1}x. \tag{5.8}$$

Then $(1-\tau)\bar{x} = x-\tau^dx = 0$. Similarly, consider the sum

$$\bar{\bar{x}} = x-\tau x+\tau^2 x-\ldots(-1)^{d-1}\tau^{d-1}x.$$

In this case we have

$$(1+\tau)\bar{\bar{x}} = \begin{cases} 0 & \text{if } d \text{ is even,} \\ 2x & \text{if } d \text{ is odd.}\end{cases} \tag{5.9}$$

Vice versa, it turns out that any element in $\ker(1-\tau)$ is of the form $\bar{x}$ for some x, and any element in $\ker(1+\tau)$ is of the form $\bar{\bar{x}}$; that is, the following holds.

LEMMA 5.10. *Let $y=\sum_\nu e_{\nu(1)}\otimes\ldots\otimes e_{\nu(m)}\in V^{\otimes m}\subset V$ (the sum over some finite number of permutations ν of $\{1,\ldots,m\}$). Then $\tau(\bar{y})=\bar{y}$ if and only if $\bar{y}$ is a sum of elements of the form $\bar{x}=x+\tau x+\tau^2x+\ldots+\tau^{d-1}x$ for $x\in V^{\otimes m}$ and $d\geqslant 1$.*

Proof. We think of τ as both an operator and a full cyclic permutation. Clearly since $\tau(\bar{x})=\bar{x}$, then for any ν figuring in the expression of $\bar{x}$, there is also a $\nu'=\tau\circ\nu$ in that expression. Since the sum is finite, there is (a smallest) $d_\nu\geqslant 1$ such that $\nu=\tau^{d_\nu}\circ\nu$. The element $x=e_{\nu(1)}\otimes\ldots e_{\nu(m)}$ has order d_ν and $\bar{x}=x+\tau x+\tau^2x+\ldots+\tau^{d_\nu-1}x$ is in the expression for y. We can then look at $y-\bar{x}$ and proceed inductively.

Similarly if $\tau(y)=-y$, then it can be checked that y is a sum of elements of the form $\bar{\bar{x}}=x-\tau x+\tau^2x\ldots-\tau^{d-1}x$ (here d is necessarily even). (To see this one can as a first step reduce mod-2 and then apply the previous lemma.) □

Most of our forthcoming calculations are based on (5.8), (5.9) and Lemma 5.10. In fact, let $x\in V^{\otimes m}$ be of the form $e_{i_1}\otimes e_{i_2}\otimes\ldots\otimes e_{i_m}$ (recall that $|e_i|=n_i$). There are two cases:

(i) the n_i are even (that is, the spheres are odd-dimensional) in which case $\tau_m(x)=\tau(x)$ and by (5.8), x gives rise to an element $\bar{x}$ in the kernel of $1-\tau_m$ (any $x\in T(e_1,\ldots,e_k)$ is necessarily periodic);

(ii) the n_i are not all even, in which case $\tau_m(x)=\pm\tau(x)$ and x gives rise to an element in $\ker(1-\tau_m)$ depending on the parity of d and n_i.

This last situation does not occur with mod-2 coefficients which makes the calculations easier.

Mod-2 *calculations*

When $\mathbb{F} = \mathbb{Z}_2$ the situation simplifies because then $\tau = \tau_m$ in all cases (see (5.7)) and hence by (5.8) any $x = e_{i_1} \otimes \ldots \otimes e_{i_m} \in V^{\otimes m}$ corresponds to an element in the kernel of $1 - \tau_m$ (namely $\bar{x}$). (The same is true when $\mathbb{F} = \mathbb{Q}$ and all spheres are odd.) Since $\bar{x} = \overline{\tau x}$, $\ker(1 - \tau_m)$ is in one-to-one correspondence with orbits of τ acting on $V^{\otimes m}$.

TERMINOLOGY. The operator τ acts on $T(e_1, \ldots, e_k) = \bigoplus_{m \geqslant 1} V^{\otimes m}$ by acting on each $V^{\otimes m}$ by cyclic permutation. An orbit consists then of a monomial $e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_m}$ (for some $m \geqslant 1$) together with all of its cyclic permutations under τ. The period of the orbit is the period of any one of its elements and the dimension of the orbit is the homological dimension of any one of its elements.

Let $f(N)$ be the number of orbits of dimension N of τ acting on $T(e_1, \ldots, e_k)$, and let $W = \bigvee_k S^{n_i+1}$ as above. Then according to (5.6) we have the following.

THEOREM 5.11. $P(H_*(\mathscr{L}W; \mathbb{Z}_2)) = 1 + (1 + z) \sum_{N \geqslant 1} f(N) z^N$.

Starting with k homology classes $e_1, \ldots, e_k$, of respective dimensions $n_1, \ldots, n_k$, and fixing an integer $N \geqslant 1$, we can calculate $f(N)$ as follows. Consider all possible partitions $\mathscr{P}(N)$ of N by elements of $(n_1, \ldots, n_k)$. We write any such partition in the form $[n_{i_1}, \ldots, n_{i_d}]$ with $n_{i_1} + \ldots + n_{i_d} = N$. To each partition $\mathscr{P} = [n_{i_1}, \ldots, n_{i_d}]$, we can let $g(\mathscr{P})$ be the number of orbits made out of elements in the corresponding tuple $(e_{i_1}, \ldots, e_{i_d})$. Then

$$f(N) = \sum_{\mathscr{P} \in \mathscr{P}(N)} g(\mathscr{P}).$$

EXAMPLE. Suppose $W = S^2 \vee S^2 \vee S^4 \vee S^5$ and let us compute the dimension b_4 of $H_4(\mathscr{L}W; \mathbb{Z}_2)$. Here $N = 4$, $n_1 = 1$, $n_2 = 1$, $n_3 = 3$ and $n_4 = 4$. We can check that we have eight different partitions of 4 by integers taken from $\{n_1, n_2, n_3, n_4\}$; that is,

$$[n_1, n_1, n_1, n_1], [n_2, n_2, n_2, n_2], [n_1, n_1, n_1, n_2], [n_1, n_1, n_2, n_2]$$
$$[n_1, n_2, n_2, n_2], [n_1, n_3], [n_2, n_3], [n_4].$$

To the partition $[n_1, n_1, n_1, n_1]$ there corresponds the orbit of $e_1 \otimes e_1 \otimes e_1 \otimes e_1$ (of period 1).

Similarly to $[n_2, n_2, n_2, n_2]$ there corresponds $e_2 \otimes e_2 \otimes e_2 \otimes e_2$.

To $[n_1, n_2, n_2, n_2]$ there corresponds only one orbit represented by $e_1 \otimes e_2 \otimes e_2 \otimes e_2$. The period here is also 4.

To $[n_1, n_1, n_1, n_2]$ there corresponds $e_2 \otimes e_1 \otimes e_1 \otimes e_1$.

To $[n_1, n_1, n_2, n_2]$ there correspond two orbits: $e_1 \otimes e_1 \otimes e_2 \otimes e_2$ and $e_1 \otimes e_2 \otimes e_1 \otimes e_2$. The first has period 4 while the second has period 2.

To $[n_1, n_3]$ there corresponds $e_1 \otimes e_3$ (period 2).

To $[n_2, n_3]$ there corresponds $e_2 \otimes e_3$ (period 2).

To $[n_4]$ there corresponds e_4 (period 1).

For $N = 4$, there are then in total nine orbits and hence nine homology classes (of degree 4). We also need to do the same calculation for $N = 3$ and there we find five classes; so in total

$$H_4(\mathscr{L}W; \mathbb{Z}_2) = (\mathbb{Z}_2)^{14}.$$

Theorem 5.11 can be made totally explicit in the case when the spheres are all of the same dimension. The calculations there take the following form.

PROPOSITION 5.12. *For $W = \bigvee_k S^n$, write*

$$P(\mathscr{L}W, \mathbb{Z}_2) = 1 + (1+z)\left(\sum_{m \geqslant 1} a_m z^{m(n-1)}\right).$$

Then

$$a_m = \sum_{e \mid m} \frac{1}{m} \phi\left(\frac{m}{e}\right) k^e,$$

where ϕ is the Euler ϕ-function.

Proof. Every element in $V^{\otimes m}$ is of degree $N = m(n-1)$. Let $a_{m,d}$ be the number of orbits in $V^{\otimes m}$ of period d. Then $a_m = \bigoplus_{d \mid m} a_{m,d}$. Let $f(d)$ be the number of monomials of period d. Since all monomials in $V^{\otimes m}$ are periodic, we have $\sum_{d \mid m} f(d) = k^m$, and hence by the Möbius inversion formula (see the appendix),

$$f(d) = \sum_{e \mid d} \mu\left(\frac{d}{e}\right) k^e$$

where μ is the Möbius function. It follows that

$$a_{d,m} = \frac{f(d)}{d} = \frac{1}{d} \sum_{e \mid d} \mu\left(\frac{d}{e}\right) k^e.$$

Finally, we can express a_m slightly differently by using some known identities:

$$\begin{aligned} a_m &= \sum_{d \mid m} \frac{1}{d} \sum_{e \mid d} \mu\left(\frac{d}{e}\right) k^e \\ &= \sum_{e \mid m} k^e \sum_{h \mid (m/e)} \frac{1}{eh} \mu(h) \quad \text{(where } h = d/e\text{)} \\ &= \sum_{e \mid m} k^e \frac{1}{m} \left(\sum_{h \mid (m/e)} \frac{m}{eh} \mu(h) \right). \end{aligned}$$

The quantity in parenthesis corresponds to $\phi(m/e)$ according to (A.2) below, and the proposition follows. $\square$

REMARK 5.13. When $k = 1$, it is well known that $\sum_{e\,|\,m} \frac{1}{m}\phi\left(\frac{m}{e}\right) = 1$ and hence in that case $a_m = 1$ for all m. With $\mathbb{Z}_2$ coefficients, we then have

$$P(H_*(\mathscr{L}S^n, \mathbb{Z}_2)) = 1 + (1+z)\left(\sum_{m \geqslant 1} z^{m(n-1)}\right)$$
$$= 1 + (1+z)\left(\frac{1}{1-z^{n-1}} - 1\right) = \frac{1+z^n}{1-z^{n-1}}.$$

But $P(H_*(S^n, \mathbb{Z}_2)) = 1 + z^n$ and $P(H_*(\Omega S^n)) = (1 - z^{n-1})^{-1}$ and so we see that

$$H_*(\mathscr{L}S^n, \mathbb{Z}_2) \cong H_*(S^n, \mathbb{Z}_2) \otimes H_*(\Omega S^n),$$

asserting that the spectral sequence in (5.1) collapses with mod-2 coefficients when $W = S^n$ (as asserted in Lemma 4.6).

Mod-$\mathbb{Q}$ *calculations*

Consider $W = \bigvee_k S^n$ and suppose n is odd. According to (5.7), actions of both τ_m and τ on $V^{\otimes m}$ coincide (for all m) and the same argument as above shows that $P(\mathscr{L}W, \mathbb{Q}) = P(\mathscr{L}W, \mathbb{Z}_2)$.

We are then left with the case when W is the wedge of k even-dimensional spheres. Again we need to determine the rank of $\ker(1 - \tau_m)$. Since in this case $(n-1)$ is odd (corresponding to n_{i_1} in (5.7)), it follows that $\tau_m = \tau$ when $(m-1)$ is even, and $\tau_m = -\tau$ when $(m-1)$ is odd (again by (5.7)). When $(m-1)$ is even, an orbit (of any period d) gives rise to an element in the kernel (cf. (5.8)), and when $(m-1)$ is odd, we get a kernel element only if d is even (cf. (5.8)). This is we have the following.

PROPOSITION 5.14. *As before assume that $W = \bigvee_k S^n$ and n is even. Then*

$$P(\mathscr{L}W, \mathbb{Q}) = 1 + (1+z)\left(\sum_{m \geqslant 1} a_m z^{m(n-1)}\right)$$

where

$$a_m = \begin{cases} \displaystyle\sum_{d\,|\,m} \frac{1}{d} \sum_{e\,|\,d} \mu\left(\frac{d}{e}\right) k^e & \text{if } m \text{ is odd,} \\ \displaystyle\sum_{\substack{d\,|\,m \\ d \text{ even}}} \frac{1}{d} \sum_{e\,|\,d} \mu\left(\frac{d}{e}\right) k^e & \text{if } m \text{ is even.} \end{cases}$$

REMARK 5.15. When $k = 1$, $a_m = 1$ for m odd and $a_m = 0$ for m even (according to (A.1)). In this case one regains the calculation in Proposition 4.8:

$$P(H_*(\mathscr{L}S^n, \mathbb{Q})) = 1 + (1+z)(z^{n-1} + z^{3(n-1)} + \dots)$$
$$= \frac{1 + z^{n-1} + z^n - z^{2(n-1)}}{1 - z^{2(n-1)}}.$$

Appendix (*Möbius Inversion*)

An arithmetic function $f\colon \mathbb{N} \longrightarrow \mathbb{C}$ is said to be multiplicative if $f(n \cdot m) = f(n)f(m)$ for all $n, m \in \mathbb{N}$. It turns out that if f is multiplicative then the function g defined by $g(d) = \sum_{e\,|\,d} f(d)$ is also multiplicative. It is possible to recover $f(d)$ from knowledge of g according to the following inversion formula:

$$g(d) = \sum_{e \mid d} f(e) \iff f(d) = \sum_{e \mid d} \mu\left(\frac{d}{e}\right) g(e).$$

Here $\mu(1) = 1$, $\mu(n) = 0$ if n has a square prime factor, and $\mu(n) = (-1)^r$ if $n = p_1 \ldots p_r$, with $p_i \neq p_j$. A nice discussion of all of this can be found in [**7**]. We simply record the following easily established properties of the Möbius function μ:

(A.1) $$\frac{1}{d} \sum_{e \mid d} \mu\left(\frac{d}{e}\right) = \begin{cases} 1 & \text{if } d = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and if $\phi(d)$ denotes the Euler ϕ-function, then $\sum_{e \mid d} \phi(e) = d$ and hence by Möbius inversion

(A.2) $$\phi(d) = \sum_{e \mid d} \frac{d}{e} \mu(e).$$

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