

SPACES OF PARTICLES ON MANIFOLDS AND GENERALIZED POINCARÉ DUALITIES

by SADOK KALLEL[†]

(Pacific Institute for the Mathematical Sciences & University of British Columbia, Vancouver,
Canada V 6T 1Z2)

[Received 12 July 1999. Revised 10 October 1999]

Abstract

In this paper, we use a ‘local to global’ *scanning* process based on a construction of Segal to unify and generalize interesting results throughout the literature relating multi-configuration spaces to mapping spaces.

1. Introduction and statement of results

There are interesting results throughout the literature relating multi-configuration spaces to mapping spaces (cf. [3, 11, 13, 14, 21, 22, 27, 28]). In this paper, we use a ‘local to global’ *scanning* process based on a construction of Segal to unify and generalize these results.

First of all by a *configuration* on a space X we mean a collection of unordered points on X (they can be distinct or not). A multi-configuration will then mean a tuple of configurations with (possibly) certain relations between them. Of course, more rigorous definitions are to follow.

It has been known by classical work of G. Segal [27], that the space of configurations of *distinct* points in Euclidean space is equivalent in homology to an iterated loop space on a sphere. Later work of D. McDuff extended this result to an arbitrary smooth compact manifold (with boundary) where she showed that the space of configurations of distinct points there is equivalent in homology to a space of sections of an appropriate bundle. A little later, F. Cohen and C.F. Bodigheimer proved a similar result for spaces of configurations of distinct points *with labels* (see [3]).

Both Segal and McDuff extended their ideas to spaces made out of pairs of configurations. While Segal worked with *divisor spaces* made out of pairs of configurations having no points in common on a punctured Riemann surface [28], McDuff dealt with what she coined the *space of positive and negative particles* on a general smooth manifold. Both were able to identify these spaces with some function spaces.

This paper extends and generalizes the work of Segal and McDuff in many directions. It also sets a context in which these types of results can be viewed and interpreted by relating them to more classical aspects of algebraic topology, as well as to some recent problems in Gauge theory and in the theory of holomorphic mapping spaces.

A starting point for us has been to address the following question: which (multi-) configuration spaces can be used to model mapping spaces (and vice-versa). Such considerations have led us to introduce a general class of spaces, the *particle spaces*. Our basic definition is as follows. A *particle*

[†] Present address: Faculté de Mathématiques, Université des Science et Technologie, Lille, France.

space is a multi-configuration space with a partial monoid structure (see Section 2). These particle spaces are derived from functors which take values in a class of partial monoids.

Given a manifold M , the most basic example of a particle space on M is the infinite symmetric product $\text{SP}^\infty(M) = \coprod \text{SP}^n(M)$ (and this is an abelian monoid). Here $\text{SP}^n(M) = \text{Map}(\{1, \dots, n\}, M)$ is the set of all maps of $\{1, \dots, n\}$ into M (or equivalently unordered n points on M). Another standard particle space is the (traditional) configuration space $C_\infty(M) \subset \text{SP}^\infty(M)$ consisting of unordered distinct points of M (or embeddings of finite sets into M). We agree on the following notation: an element ζ in $\text{SP}^n(M)$ can be written both as a formal sum $\sum n_i x_i$, $x_i \in M$, $n_i \in \mathbb{N}$ and $\sum n_i = n$, or as an unordered tuple $\langle x_1, \dots, x_n \rangle$.

Here are further examples of functors and spaces we study in this paper.

- Symmetric product spaces with ‘bounded multiplicity’. Given M as above and an integer $d \geq 1$, we define

$$\text{SP}_d^\infty(M) = \left\{ \sum n_i x_i \in \text{SP}^\infty(M) \mid n_i \leq d \right\}.$$

Of course $\text{SP}_1^\infty = C_\infty$ is the configuration space of distinct points.

- $\text{Par}^\infty(M) = \{(\zeta_1, \dots, \zeta_k) \in \text{SP}^\infty(M)^k \mid \zeta_i \cap \zeta_j = \emptyset, i \neq j\}$. A related space will be the set of k -tuples of configurations j of which are distinct, $j \leq k$.
- $\text{Par}^\infty(M) = \prod^n \text{SP}^\infty(M) / \Delta(\text{SP}^\infty(M))$, where Δ is the submonoid generated by diagonal elements.
- Truncated symmetric products and these refer to $\text{TP}_p^\infty(M) = \text{SP}^\infty(M) / px \sim *$ (here we are identifying px with basepoint $* \in M$).
- Spaces of positive and negative particles of McDuff and these refer to $\text{Par}^\infty(M) = C^\pm(M) = C(M) \times C(M) / \sim$, where \sim is the identification

$$(\zeta_1, \eta_1) \sim_R (\zeta_2, \eta_2) \Leftrightarrow \zeta_1 - \eta_1 = \zeta_2 - \eta_2.$$

- The divisor spaces of Segal studied in connection with the space of holomorphic maps of Riemann surfaces into projective spaces (see [3] and [7]). They are defined as

$$\text{Div}^n(M) = \{(\zeta_1, \dots, \zeta_n) \in \text{SP}^\infty(M)^n \mid \zeta_1 \cap \zeta_2 \cap \dots \cap \zeta_n = \emptyset\}.$$

As is standard, one can define relative particle spaces whereby the functor Par^∞ can be applied to a pair of spaces. If $N \subset M$, then $\text{Par}^\infty(M, N)$ consists (roughly) of all those multiconfigurations in $\text{Par}^\infty(M - N)$ which get identified as they approach N . By restricting to neighbourhoods of points in M (that is, by *scanning*) we can construct maps (at least for the case of parallelizable manifolds M)

$$S : \text{Par}^\infty(M) \longrightarrow \text{Map}(M, \text{Par}^\infty(S^n, *)),$$

where $* \in S^n$ can be chosen to be the north pole.

Note that for a given space M , $\text{Par}^\infty(M)$ is a disconnected partial monoid (with components not very comparable). It turns out that by ‘group completing’ with respect to this partial monoid structure, one obtains a space $\text{Par}(M)$ which is better behaved (and all of whose components are homeomorphic). The functor Par (which we construct in Section 3.3) is the last ingredient we need and we are now in a position to state the main result of this paper.

MAIN THEOREM 1.1 *Let M be an n -dimensional, smooth, compact (possibly with boundary) and connected manifold. Then there is a fibre bundle*

$$\text{Par}^\infty(S^n, *) \longrightarrow E_{\text{Par}^\infty} \longrightarrow M \quad (1.2)$$

with a (zero) section. Choose N to be a closed ANR in M and assume that either $N \neq \emptyset$ or $\partial M \neq \emptyset$. Then there is a homology equivalence (induced by scanning)

$$S_* : H_*(\text{Par}(M - N); \mathbb{Z}) \xrightarrow{\cong} H_*(\text{Sec}(M, N \cup \partial M, \text{Par}^\infty(S^n, *)); \mathbb{Z}),$$

*where $\text{Sec}(M, A, \text{Par}^\infty(S^n, *))$ is the space of sections of 1.2 trivial over A .*

The above theorem has several variants described throughout this paper. An immediate question is to decide when the homology equivalence of Theorem 1.1 can be upgraded to a homotopy equivalence.

THEOREM 1.3 *Let N, M be as in 1.1 and suppose that $\pi_1(\text{Par}(\mathbb{R}^n))$ is abelian; then scanning is a homotopy equivalence*

$$\text{Par}(M - N) \xrightarrow{\cong} \text{Sec}(M, N \cup \partial M, \text{Par}^\infty(S^n, *)).$$

Corollaries and examples

- When M is parallelizable, the bundle of configurations 1.2 trivializes and sections turn into maps into the fibre. One therefore has the equivalence

$$H_*(\text{Par}(M - N); \mathbb{Z}) \xrightarrow{\cong} H_*(\text{Map}(M, N, \text{Par}^\infty(S^n, *)); \mathbb{Z}),$$

where $\text{Map}(M, N, \text{Par}^\infty(S^n, *))$ is the space of maps of M into $\text{Par}^\infty(S^n, *)$ sending N to the canonical basepoint in $\text{Par}^\infty(S^n, *)$. When $N = *$, we write $\text{Map}^*(M, \text{Par}^\infty(S^n, *))$ for the corresponding (based) mapping space.

- (Segal [27]) Let M_g be a genus- g Riemann surface. Then $\text{Div}^2(M_g - *) \simeq \text{Map}^*(M, \mathbf{P} \vee \mathbf{P})$ where $\mathbf{P} = K(\mathbb{Z}, 2)$ is the infinite complex projective space and where Map_c^* is any component of the subspace of based maps (see 7.4).
- $C^\pm(\mathbb{R}^n) \simeq \Omega^n(S^n \times S^n / \Delta)$, where Δ is the diagonal copy of S^n in $S^n \times S^n$ [21].
- Let C be the configuration functor associated to C_∞ . Then [28]

$$H_*(C(\mathbb{R}^n); \mathbb{Z}) \xrightarrow{\cong} H_*(\Omega^n S^n; \mathbb{Z}). \quad (1.4)$$

This result can be generalized as follows. Let $C^{(k)}(\mathbb{R}^n) \subset \prod^k C(\mathbb{R}^n)$ consist of the subspace of *pairwise disjoint* configurations. Then we prove

$$H_*(C^{(k)}(\mathbb{R}^n); \mathbb{Z}) \xrightarrow{\cong} H_*(\Omega^n(\underbrace{S^n \vee \dots \vee S^n}_k); \mathbb{Z}).$$

Another way of extending Segal's result is to consider bounded multiplicity symmetric products $\text{SP}_d^\infty(M)$, $d \geq 1$. In Section 7 we prove

PROPOSITION 1.5 *Scanning S is a homotopy equivalence*

$$\mathrm{SP}_d(\mathbb{R}^n) \xrightarrow{\simeq} \Omega^n \mathrm{SP}^d(S^n)$$

whenever $d > 1$, and a homology equivalence when $d = 1$.

REMARK. The proposition stated above has been obtained independently by M. Guest, A. Kozłowski and K. Yamaguchi [12]. A labelled analog of 1.5 is given in [16] and yields a direct generalization of the May–Milgram model for iterated loop spaces. As is made explicit in Section 7, 1.5 is valid for a larger class of open manifolds or manifolds with boundary.

One main interest in Theorem 1.1 is the way it relates to and generalizes many of the classical dualities on manifolds. The following unifying ‘space-form’ version of the Alexander, Lefschetz and Poincaré dualities is obtained after a detailed analysis of the bundle 1.2 for the case $\mathrm{Par}^\infty = \mathrm{SP}^\infty$.

THEOREM 1.6 *Let M be n -dimensional, smooth and compact, and let N be an ANR in M . Suppose that M is orientable. Then scanning induces a homotopy equivalence*

$$S : \mathrm{SP}^\infty(M - N, *) \xrightarrow{\simeq} \mathrm{Map}_c(M, N \cup \partial M, \mathrm{SP}^\infty(S^n, *)),$$

where Map_c is any component of the space of maps.

COROLLARY 1.7 (Alexander–Lefschetz–Poincaré). *Let M and N be as above, then*

$$\tilde{H}_*(M - N; \mathbb{Z}) \cong H^{n-*}(M, N \cup \partial M, \mathbb{Z}).$$

REMARK. The equivalence in 1.6 has been obtained by P. Gajer [11] based on different ideas.

This work finds its origins in an attempt to construct *configuration space models* for spaces of holomorphic maps on Riemann surfaces M_g . In the past decade and as a result of the increasing ‘rapprochement’ between mathematics and physics, there has been a flurry of activity towards understanding the topology of spaces $\mathrm{Hol}^*(M_g, X)$ of (basepoint preserving) holomorphic functions into various algebraic varieties. It turns out that the particle spaces provide good models for certain spaces of holomorphic maps and we use this in Section 7 to recover the following theorem of Guest [14].

COROLLARY 1.8 (Guest). *Let X be a projective toric variety (non-singular). The natural inclusions $i_D : \mathrm{Hol}_D(S^2, V) \rightarrow \Omega_D^2 V$ (where D are multidegrees parametrizing the components) induce a homotopy equivalence when D goes to ∞ ; that is,*

$$\lim_{D \rightarrow \infty} \mathrm{Hol}_D(S^2, V) \xrightarrow{\simeq} \Omega_0^2 V,$$

where $\Omega_0^2 V$ is any component of $\Omega^2 V$.

Finally, it is not hard to see that the ideas presented above apply equally well (but in a different context) to obtaining space level descriptions of Spanier–Whitehead duality for any generalized homology theory (cf. Section 15).

THEOREM 1.10 *Let E be a connected Ω spectrum and define the functor $F_E(-) = \Omega^\infty(\mathbf{E} \wedge -)$ on the category of finite CW-complexes. Then for all $X \in CW$, there is a homotopy equivalence*

$$S : F(X) \xrightarrow{\simeq} \mathrm{Map}_*(D(X, k), F(S^k)),$$

where $D(X, k) = S^k - X$ is the Spanier–Whitehead dual of $X \hookrightarrow S^k$.

COROLLARY 1.11 (Spanier–Whitehead duality). *Let h be any homology theory and suppose that $A, B \in S^k$, A and B are n dual. Then there is an isomorphism*

$$h_i(B) \cong h^{n-1-i}(A).$$

Similar results as in 1.10 can be traced back to J. Moore [25] (see also [9]).

2. Particle functors and particle spaces

In this section, we define the Par^∞ spaces associated to a path connected space M . The main ingredient we use is the infinite symmetric product $\text{SP}^\infty(M) = \coprod \text{SP}^n(M)$ (see Section 1). This is an abelian topological monoid and we denote its pairing by $+$. We write the product $\prod^k \text{SP}^\infty(M) = \text{SP}^\infty(M)^k = \text{SP}^\infty(\bigvee^k M)$ interchangeably. Elements of $\text{SP}^\infty(M)^k$ or its quotients (see below) will be referred to as *multiconfigurations*.

DEFINITIONS 2.1

- The support of $\zeta \in \text{SP}^\infty(M)$ is the set of points making up ζ . The support of a multiconfiguration $\vec{\zeta} = (\zeta_1, \dots, \zeta_k) \in \text{SP}^\infty(M)^k$ is the union of the supports of the ζ_i . We say that $\vec{\zeta}$ lies in $A \subset M$ if the support of $\vec{\zeta}$ is in A .
- Given $A \subset \text{SP}^\infty(M)^k$ and $\vec{\zeta} \in \text{SP}^\infty(M)^k$ then $\vec{\zeta} \cap A$ is the subtuple of $\vec{\zeta}$ made out of the points of $\vec{\zeta}$ that are in A .
- A subset $S \subset \prod \text{SP}^\infty(M)$ is a *partial submonoid* if every time two multiconfigurations $\vec{\zeta}$ and $\vec{\eta}$ in S have disjoint supports, then their sum $\vec{\zeta} + \vec{\eta}$ is in S . One can define morphisms between submonoids to be any map that preserves $+$ whenever it is defined. In particular a morphism takes disjoint supports to disjoint supports.
- A quotient partial monoid of $S \hookrightarrow \text{SP}^\infty(M)^k$ is a partial monoid Q , a map $q : S \rightarrow Q$ and for each $A, B \subset X$, $A \sqcup B = \emptyset$, we have a commutative diagram.

$$\begin{array}{ccc} A \times B & \xrightarrow{+} & S \\ \downarrow q \times q & & \downarrow q \\ q(A) \times q(B) & \xrightarrow{+} & Q. \end{array}$$

In words, the pairing downstairs is defined whenever the pairing upstairs is defined.

DEFINITION 2.2 Let \mathcal{C} be the category of spaces and injective maps, and let \mathcal{S} be the category of abelian partial monoids. A (covariant) functor $\text{Par}^\infty : \mathcal{C} \rightarrow \mathcal{S}$ is a *particle functor* if

- $\text{Par}^\infty(X)$ is a submonoid or a quotient of a submonoid of $\text{SP}^\infty(\bigvee^k X)$ for some k ;
- $\forall A, B \subset X \in \mathcal{C}$, $A \cap B = \emptyset$, the partial pairing $+$ gives an identification

$$\text{Par}^\infty(A \sqcup B) = \text{Par}^\infty(A) + \text{Par}^\infty(B).$$

REMARK 2.3 The second property is instrumental in constructing ‘restriction’ maps in Section 3. There should be a general framework where the definition above fits. It suffices to point out that both properties above are needed in the definition. For example the functor F such that $F(M) \subset \text{SP}^\infty(M)^3$ is the partial monoid consisting of triples $(\zeta_1, \zeta_2, \zeta_3)$ with $\deg(\zeta_1) = \deg(\zeta_2) + \deg(\zeta_3)$ satisfies the first condition in 2.2 but not the second (that is, $F(A) + F(B) \subset F(A \sqcup B)$ is proper). This is not a particle functor in our sense.

Notation. We write an element $\vec{\zeta} \in \text{Par}^\infty(M)$ as a tuple $(\zeta_1, \dots, \zeta_n)$ which could either be in $\text{SP}^\infty(M)^k$ or could represent $q^{-1}(\zeta), q : \text{SP}^\infty(M)^k \rightarrow \text{Par}^\infty(M)$.

2.1. Construction of particle spaces

Let S be a topological (partial) monoid and $A \subset S$ any subspace. Then by $S//A$ we mean the identification space

$$S//A = S/a + x \sim x, a \in A \text{ and whenever } a + x \text{ is defined.}$$

LEMMA 2.5 *The product of two particle spaces is a particle space, and the quotient of two particle spaces is again a particle space.*

Proof. If $\text{Par}_1^\infty(M)$ and $\text{Par}_2^\infty(M)$ are two particle spaces with $\text{Par}_2^\infty(M) \subset \text{Par}_1^\infty(M)$, one can verify

$$\begin{aligned} \text{Par}_1^\infty(A \sqcup B) // \text{Par}_2^\infty(A \sqcup B) &= [\text{Par}_1^\infty(A) + \text{Par}_1^\infty(B)] // [\text{Par}_2^\infty(A) + \text{Par}_2^\infty(B)] \\ &= \text{Par}_1^\infty(A) // \text{Par}_2^\infty(A) + \text{Par}_1^\infty(B) // \text{Par}_2^\infty(B). \end{aligned}$$

The second claim follows.

Consider for each $M \in \mathcal{C}$ a map of monoids $f_M : \text{SP}^\infty(M)^m \rightarrow \text{SP}^\infty(M)^n, m, n$ positive integers. We assume that the maps $f_M, M \in \mathcal{C}$ are compatible with inclusions $N \subset M$; that is there are commutative diagrams

$$\begin{array}{ccc} \text{SP}^\infty(N)^m & \xrightarrow{f_N} & \text{SP}^\infty(N)^n \\ \downarrow \subset & & \downarrow \subset \\ \text{SP}^\infty(M)^m & \xrightarrow{f_M} & \text{SP}^\infty(M)^n. \end{array}$$

DEFINITION 2.6 Given a subset $\emptyset \neq A \subset \text{SP}^\infty(M)^n$ we denote by $(A) \in \text{SP}^\infty(M)$ the submonoid

$$(A) = \{a + x, a \in A, x \in \text{SP}^\infty(M)^n\}.$$

The following is easy to verify.

PROPOSITION 2.7 *Let f_M be defined as above for $M \in \mathcal{C}$. Then all of the functors*

- (i) $F(M) = \text{Im}(f_M),$
- (ii) $F(M) = \text{SP}^\infty(M)^n - (\text{Im}(f_M)),$
- (iii) $F(M) = \text{SP}^\infty(M)^n // \text{Im } f_M$ and
- (iv) $F(M) = \text{SP}^\infty(M)^n // \text{SP}^\infty(M)^n - (\text{Im } f_M)$

are particle functors.

EXAMPLE 2.8

- Consider the diagonal map

$$M \xrightarrow{\Delta} M \times M \xrightarrow{+} \text{SP}^2(M)$$

and extend it multiplicatively to a map $f_M : \text{SP}^\infty(M) \rightarrow \text{SP}^\infty(M)$. It is easy to see that the complement of $(\text{Im} f_M)$ is $C_\infty(M)$.

- Take the quotient $C_\infty(M)^2 \subset \text{SP}^\infty(M)^2$ by $\Delta C_\infty(M)$, where Δ is the diagonal $\Delta : C_\infty(M) \rightarrow C_\infty(M) \times C_\infty(M)$. Then this quotient corresponds to McDuff's $C^\pm(M)$ -space.

- Consider the map

$$M \times M \rightarrow \text{SP}^\infty(M)^{\times 3}, (a, b) \mapsto (a, b, a + b)$$

and extend it additively to a map $f_M : \text{SP}^\infty(M)^{\times 2} \rightarrow \text{SP}^\infty(M)^{\times 3}$. Then $\text{Im} f_M$ corresponds to triples of configurations $(\zeta_1, \zeta_2, \zeta_3)$ such that $\zeta_3 = \zeta_1 + \zeta_2$.

- Spaces of pairwise disjoint configurations; $D\text{Div}^n$ (already defined in the introduction) can be described along the lines formulated above. Assume for example that $n = 3$, then $D\text{Div}^3(M)$ is the complement in $\text{SP}^\infty(M)^3$ of $(\text{Im} f_M)$, where f_M is given by

$$f_M : \text{SP}^\infty(M)^3 \rightarrow \text{SP}^\infty(M)^3, (\zeta, \eta, \psi) \mapsto (\zeta + \eta, \zeta + \psi, \eta + \psi).$$

2.2. Some topological properties

Naturally $\text{Par}^\infty(M)$ inherits its topology from $\prod \text{SP}^\infty(M)$ and the topology on $\text{SP}^\infty(X)$ is the weak topology relative to the subspaces $\text{SP}^r(X)$, $r \geq 1$; that is a set $U \subset \text{SP}^\infty(X)$ is closed if and only if $U \cap \text{SP}^r(X)$ is closed for all k .

LEMMA 2.9 *Let Par^∞ be a particle functor and let M be a manifold of dimension $n \geq 1$ such that $\text{Par}^\infty(M) \neq \emptyset$. Then $\text{Par}^\infty(A) \neq \emptyset$ for all open $A \subset M$.*

Proof. Let $(\zeta_1, \dots, \zeta_k) \in \text{Par}^\infty(M)$. Then $S = \{\zeta_1 \cup \dots \cup \zeta_k\}$ is a finite set of points and so there is always an injection of $\tau : S \rightarrow A$. Since Par^∞ is a functor from \mathcal{C} to \mathcal{S} , it follows that there is an induced injection sending $\{(\zeta_1, \dots, \zeta_k)\} \in \text{Par}^\infty(S)$ into $\text{Par}^\infty(A)$ and the lemma follows.

Recall that Par^∞ is a self-functor of the category \mathcal{C} of spaces and injections as morphisms. In particular, Par^∞ takes inclusions to inclusions. Using the isotopy properties of $\text{Par}^\infty(-)$ the following is not hard to establish.

LEMMA 2.10 *Let M be compact with boundary and denote by M^{int} its interior. Then we have a homeomorphism $\text{Par}^\infty(M) \cong \text{Par}^\infty(M^{\text{int}})$.*

DEFINITION 2.11 We let $\mathcal{C}_n \subset \mathcal{C}$ consist of the subcategory of n -dimensional ($n \geq 1$), smooth, connected and compact manifolds.

From the functorial properties of Par^∞ , it is clear that any injective homotopy $h_t : U \rightarrow M$; that is, a homotopy through injective maps, induces a homotopy of particle spaces; $\text{Par}^\infty(h_t) : \text{Par}^\infty(U) \rightarrow \text{Par}^\infty(M)$. Given any two points p and $q \in \text{int}(M)$, M connected, any smooth path between them gives rise to an isotopy from p to q (that is, a smooth homotopy through embeddings). This isotopy can be extended to an ambient isotopy (that is, to an isotopy of M). Generally one can show the following [18].

LEMMA 2.12 *On a manifold $M \in \mathcal{C}_n$, there is an ambient isotopy taking any finite set of interior points to any other set of interior points with the same cardinality.*

COROLLARY 2.13 *Let $N \in \mathcal{C}_n$ be a connected space and assume that $\text{Par}^\infty(N) \subset \text{SP}^\infty(N)^k$. Then $\text{Par}^\infty(N)$ has $\mathbb{Z}^+ \times \cdots \times \mathbb{Z}^+$ components obtained as the intersection of $\text{Par}^\infty(N)$ with $\text{SP}^{n_1}(N) \times \cdots \times \text{SP}^{n_k}(N)$ for all tuples of positive integers (n_1, \dots, n_k) .*

Notation. In the case when $\text{Par}^\infty(M) \subset \text{SP}^\infty(M)^k$ for some k , we index the components as follows:

$$\text{Par}_{m_1, \dots, m_k}^\infty(M) = \text{Par}^\infty(M) \cap \text{SP}^{m_1}(M) \times \cdots \times \text{SP}^{m_k}(M) \subset \prod_{i=1}^k \text{SP}^\infty(M), \quad m_i > 0.$$

In the general case where $\text{Par}^\infty(M)$ is any particle space given as a quotient $q : S \rightarrow \text{Par}^\infty(M)$, $S \subset \text{SP}^\infty(M)^k$, then we define

$$\text{Par}_{m_1, \dots, m_k}^\infty(M) = q(\text{Par}_{m_1, \dots, m_k}^\infty(S)).$$

We will see in 5.14 below that the multidegrees (m_1, \dots, m_k) parametrize maps from $H_n(M; \mathbb{Z})$ into $H_n(\text{Par}^\infty(S^n, *); \mathbb{Z})$.

LEMMA 2.14 *Let $M \in \mathcal{C}_n$, and let $N \subset M$ be an absolute neighbourhood retract. Then $\text{Par}^\infty(M, N)$ is connected.*

Proof. For N as above, there is an open $U \subset M$ containing N and retracting to it via a retraction r . We assume this retraction is injective on $N - U$ (think of a collar). Given a multiconfiguration $\{\zeta_1 \cup \cdots \cup \zeta_k\}$ in $\text{Par}^\infty(M, N)$ (see the note preceding 2.7), we let its support be the set of points making up the ζ_i . If this support lies in U , then the retraction r takes $\{\zeta_1 \cup \cdots \cup \zeta_k\}$ to N and hence to basepoint in $\text{Par}^\infty(M, N)$. Generally if $\vec{\zeta} = \{\zeta_1 \cup \cdots \cup \zeta_k\}$ has support in $M - N$, then there always is an isotopy taking $\vec{\zeta}$ to an element $\vec{\zeta}'$ in U (by Lemma 2.12). Composing this with r gives at the end a path connecting $\{\zeta_1 \cup \cdots \cup \zeta_k\}$ to basepoint and the lemma follows.

EXAMPLE 2.15 Choose a basepoint $* \in M \in \mathcal{C}_n$ which is an interior point. Then $\text{Par}^\infty(M, *)$ is connected. We show in Section 5 that if M is n -connected then so is $\text{Par}^\infty(M, *)$.

3. Particle spaces and cofibrations

3.1. Restrictions and relative constructions

Fix a particle functor Par^∞ and let $M \in \mathcal{C}$ and $* \in N \subset M$ closed. Naturally $\text{SP}^\infty(N)$ is a submonoid of $\text{SP}^\infty(M)$ and we define $\text{SP}^\infty(M, N)$ as the quotient monoid $\text{SP}^\infty(M)/\text{SP}^\infty(N)$ (see Section 2.1). When $N = *$, one can check that $\text{SP}^\infty(M, *)$ is same as taking the direct limit of the inclusions $\text{SP}^n(M) \hookrightarrow \text{SP}^{n+1}(M)$ given by adjoining basepoint $\sum n_i x_i \mapsto \sim n_i x_i + *$.

Suppose $\text{Par}^\infty(M) \subset \text{SP}^\infty(M)^k$ and define

$$\text{Par}^\infty(M, N) = \{\zeta \in \text{SP}^\infty(M, N)^k \mid \zeta \cap (M - N) \in \text{Par}^\infty(M - N)\}.$$

If $\text{Par}^\infty(-)$ is obtained as the quotient of a partial monoid functor $F(M) \subset \text{SP}^\infty(M)^k$, then $\text{Par}^\infty(M, N)$ is obtained as a pushout construction

$$\begin{array}{ccc} F(M) & \longrightarrow & F(M, N) \\ \downarrow & & \downarrow \\ \text{Par}^\infty(M) & \longrightarrow & \text{Par}^\infty(M, N). \end{array}$$

In words, $\zeta \in \text{Par}^\infty(M, N)$ if $\zeta \cap (M - N) \in \text{Par}^\infty(M - N)$ with the additional constraint that as points of ζ tend to N they get identified with basepoint.

REMARK 3.1 Notice that $\text{Par}^\infty(M, N)$ has a canonical basepoint $\zeta = \langle *, *, \dots \rangle$. Observe as well that $\text{Par}^\infty(M, N) \simeq \text{Par}^\infty(M/N, *)$.

LEMMA 3.2 Let $M \in \mathcal{C}_n$, $N \subset M$. Then we have a quotient map $\pi : \text{Par}^\infty(M) \longrightarrow \text{Par}^\infty(M, N)$. If N has boundary ∂N , we get a restriction

$$r : \text{Par}^\infty(M) \longrightarrow \text{Par}^\infty(N, \partial N). \quad (3.3)$$

Proof. We simply need mention that r is a special case of π as applied to the quotient $M \rightarrow M/(M - N)$ and one can check that $\text{Par}^\infty(M, M - N) = \text{Par}^\infty(N, \partial N)$.

REMARK 3.4 We can give an explicit description of π as follows. Let $\vec{\zeta} \in \text{Par}^\infty(M)$. Then since $\text{Par}^\infty(M) = \text{Par}^\infty(N) + \text{Par}^\infty(M - N)$, we can write $\vec{\zeta} = \vec{\zeta}_N + \vec{\zeta}_{M-N}$, where $\vec{\zeta}_N \in \text{Par}^\infty(N)$ and $\vec{\zeta}_{M-N} \in \text{Par}^\infty(M - N)$. The correspondence

$$\vec{\zeta} \mapsto \vec{\zeta}_{M-N}$$

is not continuous. However when post-composed with the quotient map

$$\text{Par}^\infty(M - N) \longrightarrow \text{Par}^\infty(\overline{(M - N)}, \overline{\partial(M - N)}) \cong \text{Par}^\infty(M, \overline{N}) = \text{Par}^\infty(M, N)$$

it becomes so, hence yielding 3.2 (here we use the fact that $\text{Par}^\infty(M - N)$ is homeomorphic to $\text{Par}^\infty(\overline{M - N})$). On the other hand, the correspondence $\vec{\zeta} = \vec{\zeta}_N$ yields the restriction map $r : \text{Par}^\infty(M) \longrightarrow \text{Par}^\infty(N, \partial N)$.

3.2. Behaviour with respect to cofibrations

From now on we restrict attention to the subcategory \mathcal{C}_n , and hence $\text{Par}^\infty : \mathcal{C}_n \rightarrow \mathcal{C}$. Associated to any pair $(M, N) \in \mathcal{C}_n$, ($N \subset M$ is of codimension 0), we have the cofibration sequence

$$N \hookrightarrow M \rightarrow M/N.$$

Using the covariance of Par^∞ with respect to inclusions and using the restriction map constructed early in this section, we can apply Par^∞ to the above sequence and get

$$\text{Par}^\infty(N) \longrightarrow \text{Par}^\infty(M) \longrightarrow \text{Par}^\infty(M, N).$$

More generally, we can start with the cofibration sequence

$$(N, N \cap M_0) \longrightarrow (M, M_0) \longrightarrow (M, N \cup M_0).$$

Adapting the arguments in [3, p. 78] the following can be shown.

PROPOSITION 3.5 Consider the cofibration sequence $(N, N \cap M_0) \longrightarrow (M, M_0) \longrightarrow (M, N \cup M_0)$ with $N \subset M \in \mathcal{C}_n$, $M_0 \subset M$. Suppose that $M_0 \cap N \neq \emptyset$; then

$$\text{Par}^\infty(N, N \cap M_0) \longrightarrow \text{Par}^\infty(M, M_0) \xrightarrow{\pi} \text{Par}^\infty(M, N \cup M_0)$$

is a quasifibration.

Sketch of proof. The submanifold $N \subset M$ being proper and compact, it has non-empty boundary ∂N which we can assume without loss of generality to be connected. The boundary ∂N has a tubular neighbourhood $U_\partial \subset M$ that when restricted to either N or $M - N$ looks like a collar. Let $U = N \cup U_\partial$, then there is an isotopy retraction of $r_t : U \rightarrow N$ which leaves $M - U$ and N invariant [18, Chapter 3]. Consider at this point the subspaces

$$X_{k_1, \dots, k_n} = \{\vec{\eta} \in \text{Par}^\infty(M, N \cup M_0) \mid \vec{\eta} \cap (M - N \cup M_0) \in \text{Par}_{i_1, \dots, i_n}^\infty(M - N \cup M_0), i_j \leq k_j\}$$

(here n is determined by Par) and consider the open sets in $\text{Par}^\infty(M, N \cup M_0)$

$$U_{k_1, \dots, k_n} = \{(\zeta_1, \dots, \zeta_r) \in X_{\vec{k}} \mid (\zeta_1, \dots, \zeta_r) \text{ contains a non-empty subtuple in } \text{Par}^\infty(U)\}.$$

Write $X_{\vec{k}} = X_{k_1, \dots, k_n}$ and similarly $U_{\vec{k}} = U_{k_1, \dots, k_n}$. By construction, we have the following inclusions:

$$X_{<\vec{k}} := \bigcup_i X_{k_1, \dots, k_{i-1}, \dots, k_n} \subset U_{\vec{k}} \subset X_{\vec{k}}.$$

It is easy to see that over $X_{\vec{k}} - X_{<\vec{k}}$ the map $\pi : \text{Par}^\infty(M, M_0) \rightarrow \text{Par}^\infty(M, M_0 \cup N)$ is a product. On the other hand, the isotopy retraction r_t moves ∂N away from itself; $r_1(N) \subset N$, and squeezes the collar U_∂ into N . This is done through a homotopy that is injective on $M - U_\partial$ and so induces a retraction $r_t : U_{\vec{k}} \rightarrow X_{<\vec{k}}$. At $t = 1$, $r_1(U) \subset N$ and we have a lifting to the fibre

$$\tilde{r}_1 : \text{Par}^\infty(N, N \cap M_0) \xrightarrow{\cong} \pi^{-1}(\vec{x}) \rightarrow \pi^{-1}(r_1(\vec{x})) \xrightarrow{\cong} \text{Par}^\infty(N, N \cap M_0). \quad (3.6)$$

The lifting \tilde{r}_1 is referred to as the *attaching map* and is given by addition in $\text{Par}^\infty(N, N \cap M_0)$ of those subtuples in $U - N$ that got shoved inside N by r_t . Since $M_0 \cap N \neq \emptyset$, we can isotope these subtuples in $\text{Par}^\infty(N)$ to a subtuple in $M_0 \cap N$ and hence to basepoint in $\text{Par}^\infty(N, M_0 \cap N)$. This produces a homotopy inverse for \tilde{r}_1 and the proposition follows from a criterion of Dold and Thom [3].

REMARK 3.7 When $N \cap M_0$ is empty, then $\text{Par}^\infty(N)$ does generally split into components. In this case, the attaching map \tilde{r}_1 switches components and it has no homotopy inverse. We deal with this case in the next section.

REMARK 3.8 The proposition is also not true if N is not of codimension 0 in M . For example, let $\text{Par}^\infty = C_\infty$ be the functor of disjoint unordered points (see the Introduction). We show in 7.1 that $C_\infty(D^n, \partial D^n) \simeq S^n$. Suppose in this case that $M = D^n$, $N = \{(x_1, \dots, x_{n-1}, 0)\} \subset D^n$ is an n th face and let $M_0 = \partial D^n - N$. If 3.5 were to apply in this case, then we get a quasifibring,

$$C_\infty(D^{n-1}, \partial D^{n-1}) \rightarrow C_\infty(D^n, M_0) \rightarrow C_\infty(D^n, \partial D^n).$$

But since $C_\infty(D^n, M_0)$ is contractible, we would have proved that S^{n-1} is weakly homotopy equivalent to ΩS^n which is obviously false.

PROPOSITION 3.9 *Let M be any $n - 1$ connected finite CW complex ($n > 1$). Then $\text{Par}^\infty(M, *)$ is also $n - 1$ connected.*

Proof. The proof is a standard induction on cells of M . First since M is $n - 1$ connected, it has a CW decomposition with cells starting in dimension n attaching to a basepoint $*$. Let $M^{(i)}$ denote

the i th skeleton of M (here of course $i \geq n$). The inclusion of $M^{(i)}$ into the next skeleton gives a cofibration sequence

$$M^{(i)} \longrightarrow M^{(i+1)} \longrightarrow \bigvee S^{i+1}$$

which yields by 3.5 a quasifibration

$$\text{Par}^\infty(M^{(i)}, *) \longrightarrow \text{Par}^\infty(M^{(i+1)}, *) \longrightarrow \prod \text{Par}^\infty(S^{i+1}, *), \quad i \geq n. \quad (3.10)$$

Suppose that $\text{Par}^\infty(S^n, *)$ is $n - 1$ connected, then $\text{Par}^\infty(M^{(n)}, *) = \prod \text{Par}^\infty(S^n, *)$ is also $n - 1$ connected and the long exact sequence in homotopy attached to 3.10 shows that $\text{Par}^\infty(M^{(n+1)}, *)$ is $n - 1$ connected as well. Proceeding inductively, we can establish the claim as soon as we show that $\text{Par}^\infty(S^n, *)$ is $n - 1$ connected. This is done in 5.2.

3.3. Stabilization

As always, let M be compact (connected) and $A \subset M$ a closed non-empty ANR (typically $A = \partial M$ for example). We can ‘stabilize’ $\text{Par}^\infty(M)$ as follows. Let U be a tubular neighbourhood of A which we assume to retract to A via a retraction r which is injective outside of U . Let $U_{i \in \mathbb{Z}^+}$ be a nested sequence $U_{i+1} \subset U_i \subset U$. The $U_i - U_{i+1}$ being open, $\text{Par}^\infty(U_i - U_{i+1}) \neq \emptyset$ according to Lemma 2.9, and so we can choose $\vec{\eta}_i \in \text{Par}^\infty(U_i - U_{i+1})$. We choose $\vec{\eta}_i$ to be ‘minimal’ in the sense that no smaller subtuple of it lies in $\text{Par}^\infty(U_i - U_{i+1})$. Now notice that we have an inclusion given by summing with $\vec{\zeta}_i$ in the partial monoid structure on $\text{Par}^\infty(M - U_{i+1})$:

$$\text{Par}^\infty(M - U_i) \xrightarrow{+\vec{\zeta}_i} \text{Par}^\infty(M - U_{i+1}).$$

We can now make a definition.

DEFINITION 3.11 For M , A and U as above, we define

$$\text{Par}(M) = \lim_{\vec{\zeta}_i} \left(\text{Par}^\infty(M - U_i) \xrightarrow{+\vec{\zeta}_i} \text{Par}^\infty(M - U_{i+1}) \right).$$

REMARK 3.12 It should be clear that $\text{Par}(M)$ does not depend (up to homeomorphism) on the choice (up to isotopy) of the stabilizing sequence $\vec{\eta}_i$ or of the nested sequence $\{U_i\}$. Further, $\text{Par}(M)$ has homeomorphic components.

REMARK 3.13 We can define $\text{Par}(M, N)$ for pairs (M, N) by taking suitable direct limits over $\text{Par}^\infty(M - U_i, N - U_i \cap N)$. When $A = \partial M$ for example (or a subset of it), $N \cap \partial M \neq \emptyset$, then we can stabilize with respect to a sequence of multic configurations $\{\eta_i\}$ converging to a point $p \in N \cap \partial M$. By a homotopy (again injective in the complement of U) we can retract points of η_j to p and this shows (in this case) that

$$\text{Par}(M, N) \simeq \text{Par}^\infty(M, N).$$

REMARK 3.14 One may observe that if $p \in \partial M \neq \emptyset$, then we can stabilize with respect to a tuple $\vec{\eta}_i$ converging to p (that is the sequence of points making up each η_i converges to p). In this case one can show that

$$\text{Par}_c(M) \simeq \text{Par}^\infty(M, p).$$

DEFINITION. A morphism $\iota : \text{Par}(N) \longrightarrow \text{Par}(M)$ will mean an inclusion such that $\iota(\eta_N) = \eta_M$. The following is a generalization of 3.5.

THEOREM 3.15 Consider a cofibration sequence $N \longrightarrow M \longrightarrow (M, N)$, $N, M \in \mathcal{C}_n$, and assume $\partial N \neq \emptyset$. Then

$$\text{Par}(N) \xrightarrow{i} \text{Par}(M) \xrightarrow{\pi} \text{Par}^\infty(M, N)$$

is a homology fibration.

Proof. The proof amounts to showing that the attaching map $\tilde{r}_1 : \text{Par}(N) \rightarrow \text{Par}(N)$ (see 3.5) is a homology equivalence. According to 3.5, \tilde{r}_1 is obtained by moving particles of N away from $\partial N \subset U$ and then adding a given element $\vec{v} \in \text{Par}^\infty(r_1(U - N)) \subset \text{Par}^\infty(N)$. Now if $\vec{\eta}_i$ is a stabilizing sequence for $\text{Par}(N)$ with respect to some tubular neighbourhood of ∂N , then up to isotopy, we can think of \vec{v} as a subconfiguration of $\vec{\eta}_i$. But $\text{Par}(N)$ is a direct limit over addition of the $\vec{\eta}_j$ and so $+\vec{v}$ necessarily induces a homology isomorphism. The claim follows.

4. Scanning smooth manifolds

The term ‘scanning’ is borrowed from Segal [27]. We assume as usual that M is a smooth and connected n -dimensional manifold.

4.1. Scanning parallelizable manifolds

The scanning process is best pictured when M is parallelizable (that is, M has trivial tangent bundle). Examples of such manifolds are Lie groups or any oriented three dimensional manifolds. Without loss of generality, we restrict attention below to $\text{Par}^\infty(M) = \text{SP}^\infty(M)$ (the more general situation is treated in the exact same way).

DEFINITION 4.1 Let M^n be as above and let N be a closed subset of N . The pair (M, N) is said to be *parallelizable* if $M - N$ is parallelizable. For instance, M is stably parallelizable if $(M, *)$ is parallelizable. Riemann surfaces are examples of stably parallelizable surfaces, as well as compact, oriented, spin four manifolds.

Put a metric on M and consider the unit disc bundle τM lying over M . Let us assume for now that $\partial M = \emptyset$. Via the exponential map we can identify a neighbourhood of every point $x \in M$ with the fibre at x . Denote such a neighbourhood by $D(x) \subset M$. When M is parallelizable, the fibres over τM are canonically identified with a disc D^n and hence one can identify canonically the pairs $(\bar{D}(x), \partial \bar{D}(x))$ for every $x \in M$ with $(D^n, \partial D^n) = (S^n, \infty)$ (where the north pole ∞ is chosen to be the basepoint in S^n).

Given a configuration $\zeta \in \text{SP}^d(M)$ and an $x \in M$, then $\zeta \cap D(x)$ is a configuration on $D(x)$ and its image under the restriction map

$$\text{SP}^\infty(D(x)) \longrightarrow \text{SP}^\infty(\bar{D}(x), \partial \bar{D}(x)) = \text{SP}^\infty(S^n, \infty)$$

is denoted by ζ_x . Notice that the correspondence $\zeta \mapsto \zeta_x$ is now continuous (while the correspondence $\zeta \mapsto \zeta \cap D^n(x)$ was not to begin with). Starting with $\zeta \in \text{SP}^d(M)$, we hence get a map

$$S_d : \text{SP}^d(M) \longrightarrow \text{Map}_d(M, \text{SP}^\infty(S^n, *)), \quad \zeta \mapsto f_\zeta : f_\zeta(x) = \zeta_x.$$

The scanning map S is now given as $\sqcup S_r$.

Scanning manifolds with boundary. We are in the case $\partial M \neq \emptyset$. We can still scan the interior $M - \partial M$ and alter the topology as points tend to ∂M .

Consider the open interior $M^{\text{int}} = M - \partial M$ and let $\text{SP}_\epsilon^r(M^{\text{int}})$ be the subspace of $\text{SP}^r(M)$ consisting of configurations of points that are at least 2ϵ away from the boundary $\partial(M)$. Choose $\zeta \in \text{SP}_\epsilon^r(M^{\text{int}})$. Then by scanning the interior using discs of radius ϵ , it is clear that S_ζ maps x to basepoint for x sufficiently near the boundary. This gives rise to a map

$$\text{SP}_\epsilon^r(M^{\text{int}}) \longrightarrow \text{Map}_r(M/\partial M, \text{SP}^\infty(S^n, *)).$$

As $\epsilon \rightarrow 0$, one obtains in the limit a map

$$S_r : \text{SP}^r(M) \longrightarrow \text{Map}_r(M/\partial M, \text{SP}^\infty(S^n, *)) \simeq \text{Map}_r(M/\partial M, K(\mathbb{Z}, n)).$$

In exactly the same way, one obtains for each parallelizable pair (M, N) a map

$$S : \text{SP}^\infty(M - N) \longrightarrow \text{Map}(M, N \cup \partial M, \text{SP}^\infty(S^n,)),$$

where the right-hand side consists of all based maps sending N into basepoint $* \in \text{SP}^\infty(S^n, *)$.

4.2. Scanning smooth manifolds

The general case of M not necessarily parallelizable and Par^∞ any particle functor is treated similarly. The starting point is again the unit disc bundle τM . Compactifying each fibre yields a bundle

$$S^n \longrightarrow \widehat{\tau M} \longrightarrow M$$

to which we associate the bundle of configurations

$$\text{Par}^\infty(S^n, *) \longrightarrow E_{\text{Par}^\infty} \longrightarrow M \tag{4.2}$$

by applying Par^∞ fibrewise. Note that $\widehat{\tau M}$ has a ‘zero’ section ν sending each $x \in M$ to the compactifying point in the fibre. We label this point by $*$. Clearly, such a section extends to a zero section of $\text{Par}^\infty(S^n, *) \longrightarrow E \longrightarrow M$ also denoted by ν . We denote by $\text{Sec}(M, A, \text{Par}^\infty(S^n, *))$ the space of sections restricting to ν on $A \subset M$.

The exponential map again provides a cover of M by neighbourhoods $\cup_{x \in M} D^n(x)$ with respect to which we can scan. Cutting a neighbourhood $\bar{D}^n \subset M$ yields a cofibration $M - \bar{D}^n \hookrightarrow M \rightarrow (\bar{D}^n, \partial \bar{D}^n)$ and hence we get ‘retraction’ maps

$$\pi_x : \text{Par}^\infty(M) \longrightarrow \text{Par}^\infty(\bar{D}^n(x), \partial \bar{D}^n(x)), \quad \forall x \in M.$$

Starting with an element in $\text{Par}^\infty M$, one can restrict via π_x to neighbourhoods as in 4.1. The elements of $\text{Map}(D^n(x), \text{Par}^\infty(S^n, *))$ are now local sections of (4.2) and one gets the following correspondence.

LEMMA 4.3 *Let $M \in C_n$ and $N \subset M$ a closed ANR. Then scanning yields a map*

$$\text{Par}^\infty(M - N) \longrightarrow \text{Sec}(M, N \cup \partial M, \text{Par}^\infty(S^n, *)).$$

REMARK 4.4 The scanning map constructed above is reminiscent of the little cube construction of Boardman and Vogt and can be shown to be homotopic to the electric field map described in [28]. It also appears in some other forms in some h -principle constructions of Gromov.

5. Proof of the main Theorems 1.1 and 1.3

Notation. Recall that $\text{Par}^\infty(S^n, *)$ has a ‘preferred’ identity $*$. For each pair of spaces (M, N) , we will write $\text{Map}(M, N, \text{Par}^\infty(S^n, *))$ for the space of continuous maps from M into $\text{Par}^\infty(S^n, *)$ which send N to $*$.

PROPOSITION 5.1 *Let D^n be the closed unit disc. For $0 < k \leq n$ we have homotopy equivalences*

$$\text{Par}^\infty(D^n, S^{k-1} \times D^{n-k}) \simeq \Omega^{n-k} \text{Par}^\infty(S^n, *),$$

while for $k = 0$ we have a homology equivalence $H_*(\text{Par}(D^n); \mathbb{Z}) \simeq H_*(\Omega^n(\text{Par}^\infty(S^n, *)); \mathbb{Z})$.

Proof. The proof uses the cofibration sequence described in [3] and follows a quick downward induction (the case $k = n$ being obviously true) which relies heavily on the quasifibration property of Par^∞ (Proposition 3.5). We skip the details as they are analogous to those in [3] for the case $n \geq k > 1$.

When $k = 1$, write $D^n = I \times D^{n-1}$ and consider the cofibration sequence

$$D^n \longrightarrow (D^n, D^{n-1}) \longrightarrow (D^n, S^0 \times D^{n-1}).$$

Upon applying the Par functor we get a diagram of fibrations

$$\begin{array}{ccc} F & \longrightarrow & \Omega^n \text{Par}^\infty(S^n, *) \\ \downarrow & & \downarrow \\ \text{Par}^\infty(D^n, D^{n-1}) & \longrightarrow & PS \\ \downarrow \pi & & \downarrow \\ \text{Par}^\infty(D^n, S^0 \times D^{n-1}) & \xrightarrow{\cong} & \Omega^{n-1}(\text{Par}^\infty(S^n, *)), \end{array}$$

where F is the homotopy fibre for the left-hand side fibration, and the bottom equivalence follows by earlier induction. Now $\text{Par}(D^n, D^{n-1}) \cong \text{Par}^\infty(D^n, D^{n-1})$ is contractible (there is a retraction of D^n onto D^{n-1} which is injective on the complement of a tubular neighbourhood of D^{n-1} and hence $F \simeq \Omega^n \text{Par}^\infty(S^n, *)$. The inclusion of the preimage $\text{Par}(D^n)$ into F is a homology equivalence (since the left-hand side is a homology fibration by 3.15), and the claim follows.

COROLLARY 5.2 $\text{Par}^\infty(S^n, *)$ is $n - 1$ connected.

Proof. We have that $\pi_k(\text{Par}^\infty(S^n, *)) = \pi_0(\Omega^k \text{Par}^\infty(S^n, *))$ and the latter is trivial whenever $k < n$ since $\Omega^k \text{Par}^\infty(S^n, *)$ is identified with $\text{Par}^\infty(D^{n-k} \times D^k, S^{n-k-1} \times D^k)$ which is connected by Lemma 2.14.

EXAMPLE. When $\text{Par}^\infty = C_\infty$, it can be shown that $C_\infty(S^n, *) \simeq S^n$ (see Section 7) and hence for $0 \leq k < n$

$$C_\infty(D^n, D^k \times S^{n-k-1}) \simeq \Omega^k S^n.$$

THEOREM 5.3 *Let $M \in C_n$ and N a closed ANR in M . Suppose either N or ∂M non-empty. Then scanning induces a homology equivalence*

$$S_* : H_*(\text{Par}^\infty(M - N); \mathbb{Z}) \xrightarrow{\cong} H_*(\text{Sec}(M, N \cup \partial M, \text{Par}^\infty(S^n, *)); \mathbb{Z}).$$

Proof. Since Par is an isotopy functor, it follows that $\text{Par}(M - N) = \text{Par}(M - T(N))$, where $T(N)$ is a tabular neighbourhood of N and so without loss of generality we can assume that N is of codimension 0. We consider the case $N \neq \emptyset$ and $\partial M = \emptyset$ (the other cases are treated similarly). Since $\text{Par}(M - N) = \text{Par}(M - \text{int}(N))$, we can assume that $M - N$ is compact and has boundary ∂N . So $M - N$ has a handle decomposition

$$M_0 = D^n \subset M_1 \subset \cdots \subset M_{n-1} = N - N,$$

where the handles we attach have index at most $n - 1$ and all the M_i have boundary $\partial M_i \neq \emptyset$. The proof proceeds by induction on i . Since the number of handles we attach at each stage (finitely many) is immaterial for the arguments below we might as well assume we are only attaching one handle at a time. That is

$$M_i = M_{i-1} \cup H^i, \quad \text{and } \partial M_{i-1} \cap H^i = S^{i-1} \times D^{n-i}.$$

Consider the following two cofibrations:

$$M_{i-1} \longrightarrow M_i \longrightarrow (D^i \times D^{n-i}, S^{i-1} \times D^{n-i}), \quad i < n, \quad (5.4)$$

and the one induced from the handle attachment

$$(H^i, H^i \cap \partial M_i) \longrightarrow (M_i, \partial M_i) \longrightarrow (M_i; H^i \cup \partial M_i) = (M_{i-1}, \partial M_{i-1}).$$

Apply the functor Sec to the second sequence and get the *fibration*

$$\begin{aligned} \text{Sec}(M_{i-1}, \partial M_{i-1}, \text{Par}^\infty(S^n, *)) &\longrightarrow \text{Sec}(M_i, \partial M_i, \text{Par}^\infty(S^n, *)) \\ &\longrightarrow \text{Sec}(H^i, H^i \cap \partial M_i, \text{Par}^\infty(S^n, *)). \end{aligned} \quad (5.5)$$

Since $E_{\text{Par}^\infty(H^i)}$ over H^i is trivial, we can replace $\text{Sec}(H^i, H^i \cap \partial M_i, \text{Par}^\infty(S^n, *))$ by an iterated loop space as follows:

$$\begin{aligned} \text{Map}(H^i, H^i \cap \partial M_i; \text{Par}^\infty(S^n, *)) &= \text{Map}(D^i \times D^{n-i}, \partial D^{n-i}; \text{Par}^\infty(S^n, *)) \\ &= \text{Map}^*(S^{n-i} \times D^i; \text{Par}^\infty(S^n, *)) \\ &= \Omega^{n-i} \text{Par}^\infty(S^n, *). \end{aligned}$$

On the other hand one can apply the functor Par to (5.4) and obtain a quasifibration which maps via scanning into (5.5) as follows:

$$\begin{array}{ccc} \text{Par}(M_{i-1}) & \xrightarrow{S} & \text{Sec}(M_{i-1}, \partial M_{i-1}; \text{Par}^\infty(S^n, *)) \\ \downarrow & & \downarrow \\ \text{Par}(M_i) & \xrightarrow{S} & \text{Sec}(M_i, \partial M_i; \text{Par}^\infty(S^n, *)) \\ \downarrow & & \downarrow \\ \text{Par}^\infty(D^i \times D^{n-i}, S^{i-1} \times D^{n-i}) & \xrightarrow{\cong} & \Omega^{n-i} \text{Par}^\infty(S^n, *). \end{array} \quad (5.6)$$

The bottom map is a homotopy equivalence whenever $1 \leq i \leq n$ by 5.1. When $i = 1$, $M_{i-1} = D^n$ and the top map is a homology equivalence (here $\text{Sec}(M_0, \partial M_0; \text{Par}^\infty(S^n, *))$ is again identified with $\Omega^n \text{Par}^\infty(S^n, *)$). By a standard spectral sequence argument, it follows that the middle map $\text{Par}(M_1) \rightarrow \text{Sec}(M_1, \partial M_1; \text{Par}^\infty(S^n, *))$ is a homology equivalence and one finishes the argument by induction.

REMARK 5.7 The theorem above is not true if both N and ∂M are empty. In that case (M is closed) $\text{Par}(M)$ is not even defined and it does not even hold true that the components of $\text{Map}(M, \text{Par}^\infty(S^n, *))$ are homotopy equivalent.

THEOREM 5.8 *Let N, M be as in 5.2 and suppose that $\pi_1(\text{Par}(\mathbb{R}^n))$ is abelian; then scanning is a homotopy equivalence*

$$\text{Par}(M - N) \xrightarrow{\simeq} \text{Sec}(M, N \cup \partial M, \text{Par}^\infty(S^n, *)).$$

Proof. Consider (5.6) again and the case $i = 1$. When $\pi_1(\text{Par}(\mathbb{R}^n))$ is abelian, the left-hand side in (5.6) becomes a quasifibration (this is explained in 5.9 below) and hence the top map is a weak homotopy equivalence. Since the spaces involved have the homotopy type of CW complexes we get a homotopy equivalence and hence an equivalence in the middle. The rest of the proof is obtained by induction knowing that the bottom map is always a homotopy equivalence when $1 \leq i \leq n - 1$.

5.1. Good functors

The functor Par is *good* if it turns cofibrations $N \rightarrow M \rightarrow M/N$, $N, M \in C_n$ into quasifibrations. A straightforward examination of the proof of 5.3 shows that scanning induces a weak homotopy equivalence

$$\text{Par}(M - N) \xrightarrow{\simeq} \text{Sec}(M, N \cup \partial M, \text{Par}^\infty(S^n, *))$$

whenever Par is good and N and M are as in 5.3 (note that the space of sections has the homotopy type of a CW complex; cf. [2, Lemma 3.5]. the condition needed in Theorem 5.8 is of course slightly weaker.

LEMMA 5.9 *The functor Par is good if it abelianizes fundamental groups; that is if $\pi_1(\text{Par}(M))$ is abelian for any $M \in C_n$.*

Proof. We need show that Par applied to cofibrations yields quasifibrations. This boils down to showing that the attaching maps given by addition of particles (see 3.6) are homotopy equivalences.

These attaching maps which take the form $\text{Par}(M) \xrightarrow{+\zeta} \text{Par}(M)$ are homology equivalences for any twisted coefficients (by construction of $\text{Par}(M)$ as a direct limit over these additions). When $\pi_1(M)$ is abelian, the map $+\zeta$ induces an isomorphism of fundamental groups as well. This implies that the attaching maps must be a homotopy equivalences and the lemma follows.

EXAMPLE 5.10 It is well known that $\pi_1(\text{SP}^\infty(X)) = H_1(X)$ and hence is abelian which shows that SP is ‘good’. The functor C is on the other hand not ‘good’ and the homology equivalence of 5.3 in this case cannot be upgraded in general to a homotopy equivalence.

5.2. Identifying components

The equivalence in 5.3 gives a homology equivalence at the level of components. We identify these components for both $\text{Par}(M - N)$ and the space of sections. For the sake of simplicity we confine ourselves to the case $M - N$ parallelizable.

LEMMA 5.11 *Let X be a connected topological space, M, N as above, $N \neq \emptyset$. Then all components of $\text{Map}(M, N, \text{Par}^\infty(S^n, *))$ are homotopy equivalent.*

Proof. Let $*$ $\in N \subset M$ and identify $*$ with N in M/N . Pick a map $f \in \text{Map}$ and observe that for a small disc $D \in M - N$, $f(\partial D)$ is null homotopic in $\text{Par}^\infty(S^n, *)$ (since the latter is $n - 1$ connected). So $f|_{\partial D}$ extends out to a map of a sphere S^n and if we denote by $\#$ the connected sum, we have a map

$$\text{Map}^*(M/N, \text{Par}^\infty(S^n, *)) \longrightarrow \text{Map}^*((M/N)\#S^n, \text{Par}^\infty(S^n, *))$$

which takes one component to the next (here of course $(M/N)\#S^n \simeq M/N$). This map is a homotopy equivalence for it can be reverted by attaching another sphere with reverse orientation.

PROPOSITION 5.12 *Let $\overline{M - N}$ be the closure of $M - N$ and let $p \in \partial(\overline{M - N}) \neq \emptyset$. Then*

$$S : \text{Par}^\infty(\overline{M - N}, p) \xrightarrow{S} \text{Map}_0(M, N, \text{Par}^\infty(S^n, *))$$

is a homology equivalence. Here Map_0 stands for the component of null-homotopic maps.

Proof. The stabilization maps $+\vec{\zeta}_i$ constructed in Section 3.3 commute with scanning as follows:

$$\begin{array}{ccc} \text{Par}^\infty(M - U_i) & \xrightarrow{S_i} & \text{Map}(M/U_i, \text{Par}^\infty(S^n, *)) \\ \downarrow +\vec{\zeta}_i & & \downarrow \\ \text{Par}^\infty(M - U_{i+1}) & \xrightarrow{S_{i+1}} & \text{Map}(M/U_{i+1}, \text{Par}^\infty(S^n, *)). \end{array} \quad (5.13)$$

Observe that $\text{Par}^\infty(M - U_i) \cong \text{Par}^\infty(M - N)$ for all i and up to homotopy we have the diagram

$$\begin{array}{ccc} \text{Par}^\infty(M - N) & \xrightarrow{S} & \text{Map}_0(M, N, \text{Par}^\infty(S^n, *)) \\ \downarrow +\vec{\zeta}_i & & \downarrow \\ \text{Par}^\infty(M - N) & \xrightarrow{S} & \text{Map}_0(M, N, \text{Par}^\infty(S^n, *)), \end{array}$$

which yields in the limit the map S and the homology equivalence in 5.12.

REMARK 5.14 Since $\text{Par}^\infty(S^n, *)$ is $n - 1$ connected (Proposition 3.9) it follows that

$$\pi_0 \text{Map}(M, N, \text{Par}^\infty(S^n, *)) = [M/N, \text{Par}^\infty(S^n, *)]_*$$

and hence that the connected components of the corresponding mapping space (and consequently of $\text{Par}(M - N)$) are indexed by maps of $H_n(M, N; \mathbb{Z})$ into $H_n(\text{Par}^\infty(S^n, *))$.

6. Duality on manifolds

As mentioned in Introduction, Theorem 5.3 admits a strengthening when $\text{Par}^\infty = \text{SP}^\infty$. This last functor is a homotopy functor on the one hand, and on the other it takes values in abelian monoids. We start with some standard results.

First we point out that since $K(\mathbb{Z}, n)$ has the homotopy type of an abelian group, so also does the space of maps $\text{Map}(X, K(\mathbb{Z}, n))$ and it is easy to see [26] the following.

THEOREM 6.1 (Thom). *Let X be connected, π an abelian group and $n > 0$. Then*

$$\text{Map}(X, K(\pi, n)) \simeq \prod_{0 \leq i \leq n} K(H^{n-i}(X, \pi), i)$$

and each component is given by the sub-product $1 \leq i \leq n$ in the expression above.

Let $X = K(\mathbb{Z}, n)$ and consider the subspace $\text{Aut}(K(\mathbb{Z}, n)) \subset \text{Map}(K(\mathbb{Z}, n), K(\mathbb{Z}, n))$ of self-homotopy equivalences of $K(\mathbb{Z}, n)$. This is an abelian subgroup and hence is also a product of EM spaces.

PROPOSITION 6.2 *We have the following commutative diagram of inclusions and equivalences:*

$$\begin{array}{ccc} K(\mathbb{Z}, n) \times \text{Aut}(\mathbb{Z}) & \hookrightarrow & K(\mathbb{Z}, n) \times \text{Hom}(\mathbb{Z}, \mathbb{Z}) \\ \downarrow \simeq & & \downarrow \simeq \\ \text{Aut}(K(\mathbb{Z}, n)) & \hookrightarrow & \text{Map}(K(\mathbb{Z}, n), K(\mathbb{Z}, n)) \end{array}$$

Proof. To simplify notation we write $K_n := K(\mathbb{Z}, n)$. From 6.1 and since $H^{n-i}(K_n; \mathbb{Z}) = \mathbb{Z}$ when $i = 0$ and zero otherwise, we get

$$\text{Map}(K_n, K_n) \simeq K(H^0(K_n; \mathbb{Z}), n) \times K(H^n(K_n; \mathbb{Z}), 0) \simeq K_n \times \text{Hom}(\mathbb{Z}, \mathbb{Z})$$

(here of course $H^n(K_n; \mathbb{Z}) = \text{Hom}(H_n(K_n; \mathbb{Z}), \mathbb{Z}) = \text{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$). The equivalence above can be explicitly constructed as follows. We pointed out earlier that $K(\mathbb{Z}, n) \simeq \text{SP}^\infty(S^n, *)$ (this equivalence can be seen in many ways; cf. [10] or [25]) and the abelian monoid structure on $K_n = K(\mathbb{Z}, n)$ is induced from the symmetric product pairing (which we write additively). Given a map $f : \mathbb{Z} \rightarrow \mathbb{Z}$ determined by an integer k , we can consider the k -fold map $S^n \rightarrow S^n$ and extend it out (additively) to a map $(k) : \text{SP}^\infty(S^n, *) \rightarrow \text{SP}^\infty(S^n, *)$ and hence to an element $(k) \in \text{Map}(K_n, K_n)$. On the other hand, K_n maps to the translation elements in $\text{Map}(K_n, K_n)$ and the product map $(x, k) \mapsto T_x + (k)$ induces the equivalence $K_n \times \text{Hom}(\mathbb{Z}, \mathbb{Z}) \rightarrow \text{Map}(K_n, K_n)$. The homotopy inverse sends $f \in \text{Map}(K_n, K_n)$ to $(f(x_0), \deg f)$, where $x_0 \in K_n$ is the basepoint and $\deg f$ is the degree of the induced map at the level of π_n .

Note at this point that since $H_n(S^n) \cong \pi_n(\text{SP}^\infty(S^n, *))$, the map (k) induces multiplication by k at the level of π_n and so (k) is a homotopy equivalence if and only if $k = \pm 1$, in which case multiplication by k is in $\text{Aut}(\mathbb{Z})$. Notice also that an element in K_n acting by translation can be homotoped to the identity and hence the map $T : K_n \rightarrow \text{Map}(K_n, K_n)$ factors through $\text{Aut}(K_n)$. These two facts put together show that the diagram in 6.2 commutes. It remains to show that the left vertical map is an equivalence but it is not hard to see that the right-hand equivalence we just described restricts to $\text{Aut}(K_n)$ and the proposition follows.

REMARK 6.3 We can replace \mathbb{Z} by any abelian group G in 6.2 above and prove similarly that $\text{Aut}(K(G, n)) \simeq K(G, n) \times \text{Aut}(G)$. At the level of simplicial groups, $\text{Aut}(K(G, n))$ is given as a semi-direct product of $\text{Aut}(G)$ and $K(G, n)$ (May [20]). When $G = \mathbb{Z}$, $\text{Aut}(\mathbb{Z}) \cong \mathbb{Z}_2$ and $\text{Aut}(K_n)$ consists of two copies of K_n (consisting respectively of ‘orientation’ preserving and orientation reversing homotopy equivalences).

THEOREM 6.4 *Let $M \in \mathcal{C}_n$. Then the bundle $K(\mathbb{Z}, n) \rightarrow E_{\text{SP}^\infty} \rightarrow M$ is trivial if and only if M is oriented.*

Proof. The bundle E_{SP^∞} is classified by a map $M \rightarrow B\text{Aut}(K(\mathbb{Z}, n))$ and at the level of spaces we get a (trivial) fibration

$$K(\mathbb{Z}, n+1) \rightarrow B(\text{Aut}(K(\mathbb{Z}, n))) \rightarrow B(\text{Aut}(\mathbb{Z})).$$

The classifying map $f : M \rightarrow B\text{Aut}(K(\mathbb{Z}, n))$ lifts to $K(\mathbb{Z}, n+1)$ if and only if the composite $M \rightarrow B(\text{Aut}(\mathbb{Z}))$ is null homotopic or equivalently if the induced map $\pi : \pi_1(M) \rightarrow \text{Aut}(\mathbb{Z})$ is trivial. The action of $\pi_1(M)$ on \mathbb{Z} described by the map ϕ corresponds to the action of $\pi_1(M)$ on $\mathbb{Z} = \pi_n(K(\mathbb{Z}, n))$ in the bundle in 6.4 (this follows directly from the many facts stated in the proof of 6.2). But M being oriented, the tangent bundle τM (and hence its compactified counterpart $\widehat{\tau M}$) is trivial over the 1-skeleton. Consequently, E_{SP^∞} restricted to the one skeleton of M is also trivial and so is the action of $\pi_1(M)$ on the fibre. Namely, $\pi_1(M)$ acts trivially on $\pi_n(K(\mathbb{Z}, n)) = \mathbb{Z}$ and, as indicated above, the map f must lift to a map $\tilde{f} : M \rightarrow K(\mathbb{Z}, n+1)$. Since M is n -dimensional, \tilde{f} is null-homotopic and E_{SP^∞} is trivial.

To prove the other easier direction, suppose E_{SP^∞} is trivial, that is, $E_{\text{SP}^\infty} \simeq K(\mathbb{Z}, n) \times M$. The inclusion $\widehat{\tau M} \subset E_{\text{SP}^\infty}$ composed with projection yields a map of $\widehat{\tau M} \rightarrow K(\mathbb{Z}, n)$ and hence a Thom class in $H^n(\widehat{\tau M}; \mathbb{Z})$. This is equivalent to giving an orientation class for M and the proposition follows.

THEOREM 6.5 *Let $N \hookrightarrow M$ be a closed ANR of a closed, oriented manifold $M \in \mathcal{C}_n$, $n \geq 2$. Then*

$$\text{SP}^\infty(M - N, *) \xrightarrow{\simeq} \text{Map}_0(M, N, \text{SP}^\infty(S^n, *)).$$

Proof. Here of course and, since $E_{\text{SP}^\infty(M-N)}$ is trivial, the space of sections and the space of maps into the fibre coincide. The homotopy equivalence is a consequence of 1.3 (or 5.10.).

Now $\text{SP}^\infty(-)$ is a homotopy functor and more precisely (cf. [10])

$$\text{SP}^\infty(X, *) = \prod_i K(\tilde{H}_i, (X; \mathbb{Z}), i). \quad (6.6)$$

Combining 6.1 with (6.6) we get the equivalence

$$\prod_i K(\tilde{H}_i, (M - N; \mathbb{Z}), i) \simeq \prod_{1 \leq i \leq n} K(H^{n-i}(M/N, \mathbb{Z}), i)$$

from which we easily deduce our main application.

COROLLARY 6.7 (Alexander–Poincaré duality). *Let $N \hookrightarrow M$ be a closed ANR in an orientable manifold M of dimension n . Then $\tilde{H}_i(M - N; \mathbb{Z}) \cong H^{n-i}(M, N; \mathbb{Z})$.*

Similarly, considering the equivalence, $\text{SP}^\infty(M, *) \simeq \text{Map}_c(M, \partial M, K(\mathbb{Z}, n))$ for M compact with boundary yields the following.

COROLLARY 6.8 (Lefschetz–Poincaré duality). *Let M be compact with boundary, of dimension n , and suppose $\text{int}M$ is orientable. Then $H_q(M) \cong H^{n-q}(M, \partial M)$.*

7. Applications

7.1. On theorems of McDuff and Segal

As pointed out in the Introduction, the configuration space functor \mathcal{C}_∞ has been studied in [28] and [21], where special versions of Theorem 1.1 were proved. In this subsection, we extend their results in several directions.

Consider the subspace of $C^{(k)}(M) \subset C(M)^k$ consisting of tuples of configurations which are pairwise disjoint. More explicitly

$$C^{(k)}(M) = \{(\zeta_1, \dots, \zeta_k) \in C(M)^k \mid \zeta_i \cap \zeta_j = \emptyset, i \neq j\}.$$

It is direct to see that $C^{(k)}(M)$ is a particle space.

LEMMA 7.1 *Let $\bigvee^k S^n$ denote the k th wedge, $n \geq 1$. Then $C^{(k)}(S^n, *) \simeq \bigvee^k S^n$.*

Proof. As in [27], we let $C_\epsilon^{(k)}(S^n, *)$ be the open set of $C^{(k)}(S^n, *)$ consisting of multiconfigurations $(\zeta_1, \dots, \zeta_k)$ such that at least $k - 1$ such particles are disjoint from the closed disc U_ϵ of radius $\epsilon > 0$ about the south pole $*$. Notice that there is a radial homotopy, injective on the interior of U_ϵ that expands the north cap U_ϵ over the sphere and takes ∂U_ϵ to $*$. Such an expansion retracts $C_\epsilon^{(k)}(S^n, *)$ to the wedge product $C(S^n, *) \vee \dots \vee C(S^n, *)$. Now since $C^{(k)}(S^n, *)$ is the union of the $C_\epsilon^{(k)}(S^n, *)$ for $\epsilon > 0$, we get that $C^{(k)}(S^n, *) \simeq \bigvee^k C(S^n, *)$.

It remains to show that $C(S^n, *) \simeq S^n$. Here too we consider the subspace

$$C_\epsilon(S^n, *) = \{D \in C(S^n, *) \mid D \cap U_\epsilon = \{\text{at most one point}\}\},$$

where U_ϵ is an epsilon neighbourhood of the north pole (again the south pole corresponds to $*$). Then radial expansion of U_ϵ (N is fixed) maps $(U_\epsilon, \partial U_\epsilon)$ to $(S^n, *)$ (and is injective on U_ϵ hence extending to C). The one point configurations in U_ϵ now produce a homeomorphism $C_\epsilon(S^n, *) \simeq S^n$ and since again $C(S^n, *) = \bigcup_\epsilon C_\epsilon(S^n, *)$ the lemma follows.

Combining Lemma 7.1 with Theorem 5.3 yields the following.

PROPOSITION 7.2 *Let $M \in \mathcal{C}_n$ be a closed manifold and $N \subset M$ such that (M, N) is parallelizable. Then*

$$S_* : H_*(\mathcal{C}^{(k)}(M - N)) \cong H_* \left(\text{Map} \left(M, N, \bigvee^k S^n \right) \right).$$

When $M - N = \mathbb{R}^n$, we can identify $C^{(k)}(\mathbb{R}^n)$ with $C^{(k)}(D^n)$, where D^n is the closed unit disc and so we get the following (see [28] for the case $k = 1$).

COROLLARY 7.3 *Scanning induces a homology isomorphism*

$$S : C^{(k)}(\mathbb{R}^n) \longrightarrow \Omega^n \left(\bigvee^k S^n \right).$$

EXAMPLE 7.4 It can be checked (exactly as in 7.1) that $D\text{Div}^k(S^n, *) \simeq \bigvee^k K(\mathbb{Z}, n)$ and that the following commutes (up to homotopy)

$$\begin{array}{ccc} C^{(k)}(M - N) & \xrightarrow{S} & \text{Map}(M, N, \bigvee^k S^n) \\ \downarrow \subset & & \downarrow \\ D\text{Div}^k(M - N) & \xrightarrow{\simeq} & \text{Map}(M, N, \bigvee^k K(\mathbb{Z}, n)) \end{array}$$

where M and N are as in the statement of Theorem 1.1 and the vertical map in the diagram is induced from the inclusion $S^n \hookrightarrow K(\mathbb{Z}, n)$. The homotopy equivalence at the bottom follows from the fact that $\pi_1(D\text{Div}^k(\mathbb{R}^n))$ is abelian.

7.2. Symmetric products with bounded multiplicities

In this subsection we prove Proposition 1.5 in the Introduction. Recall that

$$\text{SP}_d^\infty(M) = \left\{ \sum n_i x_i \in \text{SP}^\infty(M) \mid n_i \leq d \right\}.$$

We first need the following analogue of 7.1.

LEMMA 7.6 *There is a homotopy equivalence $\text{SP}_d^\infty(S^k, *) \simeq \text{SP}^d(S^k)$.*

Proof. Let $*$ be in S^k and U_ϵ be as in 7.1, and let W_ϵ be the subspace consisting of $\langle x_1, x_2, \dots, x_n \rangle \in \text{SP}_d^\infty(S^k, *)$ such that at most d points in the tuple lie inside U_ϵ . By definition of $\text{SP}_d^\infty(S^k, *)$ each of its elements must fall into a W_ϵ for some ϵ and hence $\text{SP}_d^\infty(S^k, *) \simeq \bigcup_\epsilon W_\epsilon$. Now using the radial retraction of 7.1, it is clear that each $W_\epsilon \simeq \text{SP}^d(S^k)$ and the lemma follows.

THEOREM 7.7 *Let M and N be as in 1.1. Then*

$$S : \text{SP}_d(M - N) \longrightarrow \text{Map}(M, N \cup \partial M, \text{SP}^d(S^n))$$

is a homotopy equivalence whenever $d > 1$ and a homology equivalence when $d = 1$.

Proof. Let $X = M - N$. The claim amounts to showing that $\pi_1(\text{SP}_d^n(X))$ is abelian when $n > 1$ and $d > 1$. We know already (5.10) that $\pi_1(\text{SP}^n(X))$ is abelian for $n > 1$. Since $H_1(\text{SP}^n(X); \mathbb{Z}) \cong H_1(\text{SP}^{n+1}(X); \mathbb{Z})$, it follows that the inclusion $\text{SP}^2(X) \hookrightarrow \text{SP}^n(X)$ for $n \geq 2$ induces an isomorphism in the fundamental group. Consider at this point the commutative diagram

$$\begin{array}{ccc} \text{SP}_d^2(X) & \hookrightarrow & \text{SP}^2(X) \\ \downarrow \subset & & \downarrow \subset \\ \text{SP}_d^n(X) & \hookrightarrow & \text{SP}^n(X). \end{array}$$

Any element $\alpha \in \pi_1(\text{SP}_d^n(X))$ factors through the subset $\text{SP}_d^2(X)$ in $\text{SP}^2(X)$. But for $d > 1$, these last two spaces coincide and since $\pi_1(\text{SP}^2(X))$ is abelian, the claim follows.

Variants of this result are given in [16]. Note that we recover 1.5 in the Introduction by restricting to the case $M = D^n$ the closed unit disc and $N = \emptyset$. In the case $n = 2$, it is well known that $\text{SP}^d(S^2)$ is diffeomorphic to the d th complex projective space \mathbf{P}^d and we obtain the following corollary.

COROLLARY 7.8 *Let M be a compact Riemann surface of genus g with basepoint $*$. Then there is a homotopy equivalence*

$$\text{SP}_d(M_g - *) \simeq \text{Map}^*(M_g, \mathbf{P}^d).$$

7.3. Rational curves on toric varieties and a theorem of Guest

A toric variety V is a projective variety that can be defined by equations of the form ‘monomial in $z_0, \dots, z_n = \text{monomial in } z_0, \dots, z_n$ ’. As an example, consider the quartic

$$M_2 = \{[z_0 : z_1 : z_2 : z_3] \in \mathbf{P}^3 \mid z_2^2 = z_1 z_3\}.$$

A rational curve on V is a holomorphic image of $\mathbf{P}^1 = S^2$ in V and we denote by $\text{Hol}(\mathbf{P}^1, V)$ the space of all holomorphic maps from \mathbf{P}^1 into V . Choosing $x_0 \in \mathbf{P}^1$ and $* \in V$, we let $\text{Hol}^*(\mathbf{P}^1, V)$ be the subspace of $f : \mathbf{P}^1 \rightarrow V$ such that $f(x_0) = *$. It has to be pointed out that the topology of $\text{Hol}^*(\mathbf{P}^1, V)$ could vary with the choice of the basepoint $*$ (unless for example V is homogeneous).

It turns out that for a generic choice of a basepoint $* \in V$, a map $f \in \text{Hol}^*(\mathbf{P}^1, V)$ admits a representation by polynomials. More precisely, given $f : \mathbf{P}^1 \rightarrow V$ holomorphic, the composite

$$\mathbf{P}^1 \xrightarrow{f} V \hookrightarrow \mathbf{P}^n \quad (\text{for some } n)$$

is also holomorphic and so f can be represented by the map $[p_0(z) : \dots : p_n(z)]$, where the $p_i(z)$ satisfy the same set of equations as V and of course have no roots in common. Notice also that when f is basepoint preserving, the p_i can be chosen to be monic. This means that the root data of the p_i determine the map f . For a general toric variety V , which we assume to be non-singular, a rational map $f : \mathbf{P}^1 \rightarrow V$ will have a multidegree D associated to it, where

$$D = (d_1, \dots, d_p) \in \pi_2(V) \cong \bigoplus_{i=1}^p \mathbb{Z}$$

and this multidegree parametrizes components of $\text{Hol}^*(\mathbf{P}^1, V)$. We say that $D \rightarrow \infty$ if all the components d_i tend to infinity.

LEMMA 7.12 *There is a homeomorphism $\text{Hol}_D^*(S^2, V) \cong \text{Par}_D(S^2 - \infty)$ for some particle space $\text{Par}^\infty(S^2 - *)$, sending $f \in \text{Hol}(S^2, V)$ to the roots of the $p_i(z)$, $0 \leq i \leq n$ in its polynomial representation.*

Proof. The proof is direct since if two polynomial representations given by p_i and p'_i , $1 \leq i \leq n$, have root data lying in disjoint sets, then their products $p_i p'_i$ will give rise to another representation describing a new holomorphic map $S^2 \rightarrow V$.

We can up to homeomorphism construct stabilization maps

$$\text{Hol}_D^*(S^2, V) \longrightarrow \text{Hol}_{D+D'}^*(S^2, V) \quad (7.1)$$

as in Section 3.3. This induces stabilization maps at the level of $\text{Par}_D^\infty(S^2 - \infty)$ and the direct limit is a component of $\text{Par}(S^2 - \infty)$ (see Section 3).

THEOREM 7.13 (Guest). *Let X be a projective toric variety (non-singular). The inclusions $i_D : \text{Hol}_D(S^2, V) \rightarrow \Omega_D^2 V$ induce a homotopy equivalence when D goes to ∞ ; that is,*

$$\lim_{D \rightarrow \infty} \text{Hol}_D(S^2, V) \xrightarrow{\lim i_D} \Omega_0^2 V$$

where $\Omega_0^2 V$ is any component.

Proof. Arguments of Segal and Guest show that in this general case scanning and the inclusion i fits in a homotopy commutative diagram

$$\begin{array}{ccc} \mathrm{Hol}_D^*(S^2, V) & \xrightarrow{i_D} & \mathrm{Map}_D^*(S^2, V) \\ \downarrow & & \downarrow \simeq \\ \mathrm{Par}_c(S^2 - \infty) & \xrightarrow{S} & \mathrm{Map}_c^*(S^2, \mathrm{Par}^\infty(S^2, *)) \end{array}$$

where from above the map $\mathrm{Hol}_D^*(S^2, V) \rightarrow \mathrm{Par}_c(S^2 - \infty)$ can be identified with the map of Hol_D^* into the direct limit of the system in (7.1) (note that Map_c^* denotes any component of $\mathrm{Map}^*(S^2, \mathrm{Par}^\infty(S^2, *))$ and they are all homotopy equivalent by 5.11). The scanning map S at the bottom will be a homotopy equivalence according to 1.3 if we can show that $\pi_1(\mathrm{Par}_c(S^2 - \infty))$ is abelian. It is shown in [5, Corollary 5.6] that $\pi_1(\mathrm{Hol}_D^*(S^2, V))$ is abelian for D consisting of multidegrees (d_1, \dots, d_p) with $d_i \geq 2$. Moreover for D and D' with this property, $\pi_1(\mathrm{Hol}_D^*(S^2, V)) \cong \pi_1(\mathrm{Hol}_{D'}^*(S^2, V))$ hence implying that in the direct limit $\pi_1(\mathrm{Hol}^*(S^2, V))$ is well defined and abelian. The claim now follows.

8. Spanier–Whitehead duality

The ideas of the previous sections can be adapted to prove the Spanier–Whitehead duality for general homology theories h_* and for any finite type CW complex X . The material below is known in some form or another and we include it in this section for completeness.

As a start we denote by CW the category of connected finite CW complexes. For a given $X \in CW$, we let $D(X, k)$ be its Spanier–Whitehead dual (or S -dual). An S -dual always comes equipped with a map $X \wedge D(X, k) \rightarrow S^k$ (see [6]).

Given a connective Ω spectrum $\mathbf{E} = \{E_i, i = 1, \dots\}$, we have that

$$E_0 = \lim_m \Omega^m E_m \equiv \Omega^\infty \mathbf{E}$$

and more generally $E_n = \Omega^\infty(S^n \wedge \mathbf{E})$. We can then associate to \mathbf{E} the functor $F_{\mathbf{E}}$ defined as follows:

$$F_{\mathbf{E}} : X \mapsto F(X) = \Omega^\infty(\mathbf{E} \wedge X).$$

Notice that $F_{\mathbf{E}}(S^n) = E_n$ and also that

$$\begin{aligned} \pi_i(F_{\mathbf{E}}(X)) &= [S^i, \Omega^\infty(X \wedge \mathbf{E})] = \lim_n [S^i, \Omega^n(E_n \wedge X)] \\ &= \lim_n \pi_{i+n}(E_n \wedge X) = h_i(X), \end{aligned}$$

where h_* is the generalized homology theory associated to \mathbf{E} .

THEOREM 8.1 *Let $F = F_{\mathbf{E}}$ for some (omega) spectrum \mathbf{E} . Then for any finite complex X , there is a homotopy equivalence*

$$S : F(X) \xrightarrow{\simeq} \mathrm{Map}^*(D(X, k), F(S^k)).$$

Proof. Let X be a finite CW complex. Then $X \subset S^k$ for some k and $D(\Sigma X, k) = S^k - \Sigma X$. Since X and $D(\Sigma X, k)$ are disjoint, we can consider the map

$$\hat{S} : X \times D(\Sigma X, k) \rightarrow S^{k-1}; (x, y) \mapsto \frac{x - y}{|x - y|} \in S^k.$$

We can assume X to be embedded in the positive quadrant in $\mathbb{R}^n \subset S^k$ with the point at infinity $\infty \in S^k$ adjoined. This means that $\hat{S}(\infty, y) = 1, \forall y \in D(\Sigma X, k)$. On the other hand and since X is compact, it lies in a ball $B \in S^k$. Choose a point $p \in D(\Sigma X, k)$ which is not in B . The map $\hat{S}|_{X \times p}$ extends to $B \times p$ and since B is contractible we get an extension

$$\hat{S} : X \times D(\Sigma X, k) \cup c(X \times p) \longrightarrow S^{k-1},$$

where c denotes the cone construction. It then follows that, up to homotopy, the map \hat{S} gives rise to the map

$$X \wedge D(\Sigma X, k) \longrightarrow S^{k-1}.$$

Suspending both sides yields a map $X \wedge D(X, k) \longrightarrow S^k$ and hence by adjoining a map

$$\hat{S} : X \longrightarrow \text{Map}^*(D(X, k), S^k), \quad (8.2)$$

where the mapping space on the right is pointed. Of course we can compose with the map $i : \text{Map}^*(D(X, k), S^k) \rightarrow \text{Map}^*(D(X, k), F(S^k))$ induced from the ‘identity’ $S^k \rightarrow F(S^k)$. Since F is an infinite loop functor, (8.2) composed with i extends to the desired map

$$S : F(X) \longrightarrow \text{Map}^*(D(X, k), F(S^k)).$$

We show that S is a homotopy equivalence by inducting on cells of X . Let $X^{(i)}$ denote the i th skeleton of X and consider the standard cofibration $X^{(i-1)} \hookrightarrow X^{(i)} \vee S^i$. Applying $\text{Map}^*(-, S^k)$ yields a fibration sequence and a homotopy commutative diagram.

$$\begin{array}{ccc} \prod F(S^{i-1}) & \xrightarrow{\cong} & \prod \Omega^{k-i+1} F(S^k) \\ \downarrow & & \downarrow \\ F(X^{(i-1)}) & \xrightarrow{S} & \text{Map}^*(D(X^{(i-1)}, k), F(S^k)) \\ \downarrow & & \downarrow \\ F(X^{(i)}) & \xrightarrow{S} & \text{Map}^*(D(X^{(i)}, k), F(S^k)). \end{array}$$

The left-hand vertical sequence is a quasifibration since F is a homology theory. The top horizontal map is an equivalence since $\Omega F(S^l) \simeq F(S^{l-1})$ while the bottom map is an equivalence by induction. This then implies that the middle map S is also an equivalence and the proof follows.

COROLLARY 8.3 (Spanier–Whitehead duality). *Let \mathbf{E} be a connective spectrum and let h be the homology theory defined by \mathbf{E} ; that is, $h_*(X) = [S^0, E \wedge X]$. Suppose that $A, B \in S^k$, A and B are n dual. Then there is an isomorphism*

$$h_i(B) \cong h^{n-1-i}(A).$$

Proof. Let \mathbf{E} be a connective spectrum with a unit. We can choose \mathbf{E} to be an Ω spectrum. Indeed if it were not such, then the spectrum representing the generalized homology theory defined by \mathbf{E} still is. And so as far as homology is involved, we could have chosen \mathbf{E} to be an Ω spectrum to start with.

Theorem 8.1 now shows that $F_{\mathbf{E}}(X) \simeq \text{Map}(D(X, k), F_{\mathbf{E}}(S^k))$ and it follows that

$$\begin{aligned} h_i(X) &= \pi_i(F_{\mathbf{E}}(X)) = \pi_i(\text{Map}(D(X, k), F_{\mathbf{E}}(S^k))) \\ &= [S^i \wedge D(X, k), F_{\mathbf{E}}(S^k)] = [D(X, k), \Omega^i E_k] \\ &= [D(X, k), E_{k-i}] = h^{k-i}(D(X, k)). \end{aligned}$$

Here we used the facts that $h_i(X) = \pi_i(F_{\mathbf{E}}(X))$ and $F_{\mathbf{E}}(S^n) \simeq E_n$. This concludes the proof.

Acknowledgements

The author is grateful to Pawel Gajer for commenting on an early version of this paper. He is also grateful to M. Guest, A. Kozłowski and K. Yamaguchi for sending him their preprint [12]. The author would like to express his gratitude to the Fields institute and the organizers of the special homotopy theory program for the 95/96 theme year there; in particular to Paul Selick, Richard Kane, Steve Halperin and Stan Kochman. He would also like to thank Pauline Grant and Becky Sapping. Finally, he wishes to acknowledge the generous support of both the CRM in Montréal and of Professor J. Hurtubise (McGill).

References

1. J. F. Adams, *Infinite loop spaces*, Annals of Mathematics Studies 90, Princeton 1978.
2. J. C. Becker and R. E. Schultz, Equivariant function spaces and stable homotopy theory I, *Comm. Math. Helv.* **49** (1974), 1–34.
3. C. F. Bodigheimer, Stable splittings of mapping spaces, *Algebraic topology, Proc. Seattle, Lecture Notes in Mathematics* 1286, Springer Verlag, Berlin 1985, 174–187.
4. C. F. Bodigheimer, F. R. Cohen and R. J. Milgram, Truncated symmetric products and configuration spaces, *Math. Zeit.* **214** (1993), 179–216.
5. C. P. Boyer, J. C. Hurtubise, B. M. Mann and R. J. Milgram, The topology of rational maps into generalised flag manifolds, *Acta Math.* **173** (1994), 61–101.
6. G. Carlsson and R. J. Milgram, Stable homotopy and iterated loop spaces, *Handbook of algebraic topology* Elsevier, Amsterdam 1995.
7. F. R. Cohen, R. L. Cohen, B. M. Mann and R. J. Milgram, The topology of rational functions and divisors of surfaces, *Acta Math.* **166** (1991), 163–221.
8. F. R. Cohen, Fibration and product decompositions in nonstable homotopy theory, *Handbook of algebraic topology* Elsevier, Amsterdam 1995.
9. F. Cohen and L. Taylor, The homology of function spaces, *Contemp. Math.* (1983), 39–50.
10. A. Dold and R. Thom, Quasifaserungen und unendliche symmetrische produkte, *Ann. Math.* **67** (1958), 239–281.
11. P. Gajer, Poincaré duality and integral cycles, *Compositio Math.* **98** (1995), 193–203.
12. M. A. Guest, K. Kosłowski and K. Yamaguchi, The space of polynomials with roots of bounded multiplicity, *Fund. Math.* **161** (1999), 93–117.
13. M. Guest, On the space of holomorphic maps from the Riemann sphere to the quadric cone, *Quart. J. Math.* **45** (1994), 57–75.
14. M. Guest, The topology of the space of rational curves on a toric variety, *Acta Math.* **174** (1995), 119–145.
15. S. Kallel, Divisor spaces on punctured Riemann surfaces, *Trans. Amer. Math. Soc.* **350** (1998), 135–164.

16. S. Kallel, An analog of the May–Milgram model for configurations with multiplicities, to appear in *Contemp. Math.* Proceedings of the Milgram conference (Aug 99).
17. S. Kallel, Bounded multiplicity polynomials and rational maps, *Ann. Inst. Fourier* (to appear).
18. A. A. Kosinski, *Differentiable manifolds*, Pure and Applied Mathematics 138, Academic Press, London 1993.
19. V. Krushkal and P. Teichner, Alexander duality, gropes and link homotopy, *Geometry and Topology* **1** (1997), 51–69.
20. P. May, *Simplicial objects in algebraic topology*, Van Nostrand, New York 1967.
21. D. McDuff, Configuration spaces of positive and negative particles, *Topology* **14** (1975), 91–107.
22. D. McDuff, Configuration spaces, *K-Theory and Operator Algebras Proc*, Springer Lecture Notes in Mathematics 575, Springer Verlag, Berlin 1975, 88–95.
23. D. McDuff and G. Segal, Homology fibrations and the group completion theorem, *Invent. Math.* **31** (1976), 279–284.
24. J. Milnor, *Differentiable structures*, Lecture Notes, Princeton University, 1961.
25. R. J. Milgram, The bar construction and abelian H-spaces, *Illinois J. Math.* **11** (1967), 249–250.
26. D. Notbohm and L. Smith, Rational homotopy of the space of homotopy equivalences of a flag manifold, *Barcelona Conference on Algebraic Topology*, Lecture Notes in Mathematics, Springer Verlag, Berlin, 1990.
27. G. Segal, The topology of spaces of rational functions, *Acta. Math.* **143** (1979), 39–72.
28. G. Segal, Configuration spaces and iterated loop spaces, *Invent. Math.* **21** (1973), 213–221.
29. G. Segal, Categories and homology theories, *Topology* **13** (1974), 293–312.